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New topologies between the usual and Niemytzki

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Abstract

We use the technique of Hattori to generate new topologies on the closed upper half plane which lie between the usual metric topology and the Niemytzki topology. We study some of their fundamental properties and weaker versions of normality.

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1. Notations and basic definitions

We use the technique of Hattori [6, 13] to generate new topologies on the closed upper half plane which lie between the usual metric topology and the Niemytzki topology. We study some of their fundamental properties and weaker versions of normality. We denote an order pair by $\langle x, y\rangle$, the set of real numbers by $\mathbb{R}$, the natural numbers by $\mathbb{N}$, and the rationals by $\mathbb{Q}$. Let $X = \{ \langle x, y\rangle \in \mathbb{R}^2 : y \geq 0 \}$ and $P = \{ \langle x, y\rangle \in \mathbb{R}^2 : y > 0 \}$, so the x-axis is $L = X \setminus P$. Denote the usual metric topology on X by $\mathcal{U}$ and the Niemytzki topology on X by $\mathcal{N}$. For every $\langle x, 0\rangle \in L$ and $r \in \mathbb{R}$, $r > 0$, let $D(\langle x, 0\rangle, r)$ be the set of all points of P inside the circle of radius r tangent to x-axis at $\langle x, 0\rangle$ and let $D_r(\langle x, 0\rangle) = D(\langle x, 0\rangle, r) \cup \{\langle x, 0\rangle\}$. For every $\langle x, y\rangle \in X$ and $r > 0$, let $U_r(\langle x, y\rangle)$ be the set of all points of X inside the circle of radius r and centered at $\langle x, y\rangle$. Recall that the Niemytzki topology $\mathcal{N}$ on X is generated by the following neighborhood

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system: For every $\langle x, 0\rangle \in L$, let $\mathcal{B}(\langle x, 0\rangle) = \{ D_r(\langle x, 0\rangle) : r > 0 \}$. For every $\langle x, y\rangle \in P$, let $\mathcal{B}(\langle x, y\rangle) = \{ U_r(\langle x, y\rangle) : r > 0 \}$. Observe that P as a subspace of X with the usual metric topology coincides with P as a subspace of X with the Niemytzki topology.

Definition 1.1. Let A be a non-empty proper subset of the x-axis L. For each $\langle a, 0\rangle \in A$, let $\mathcal{B}(\langle a, 0\rangle) = \{ U_r(\langle a, 0\rangle) : r > 0 \}$, where $U_r(\langle a, 0\rangle)$ is the set of all points of X inside the circle of radius r and centered at $\langle a, 0\rangle$. For each $\langle a, b\rangle \in P$, let $\mathcal{B}(\langle a, b\rangle) = \{ U_r(\langle a, b\rangle) : r > 0 \}$. So, the points in $A \cup P$ will have the same local base as in $(X , \mathcal{U})$. For each $\langle c, 0\rangle \in L \setminus A$, let $\mathcal{B}(\langle c, 0\rangle) = \{ D_r(\langle c, 0\rangle) : r > 0 \}$. So, the points in $L \setminus A$ will have the same local base as in $(X , \mathcal{N})$. We call the topology on X generated by the neighborhood system $\{ \mathcal{B}(\langle x, y\rangle) : \langle x, y\rangle \in X \}$ *the H-generated* topology on X from $\mathcal{U}$ and $\mathcal{N}$ and denote it by $\mathcal{U}_A\mathcal{N}$. We call X with this H-generated topology an *H-space* and denote it by $(X , \mathcal{U}_A\mathcal{N})$.

Observe that if $A = \varnothing$, then $\mathcal{U}_A\mathcal{N}$ is the Niemytzki topology, if A is the x-axis L, then $\mathcal{U}_A\mathcal{N}$ is the usual topology. From now on, when we consider X with an H-generated topology $\mathcal{U}_A\mathcal{N}$ we are assuming that A is a non-empty proper subset of the x-axis L. Let us interchange the local bases in Definition 1.1 as follows: Let A be a non-empty proper subset of the x-axis L. For each $\langle a, 0\rangle \in A$, let $\mathcal{B}(\langle a, 0\rangle) = \{ D_r(\langle a, 0\rangle) : r > 0 \}$. For each $\langle a, b\rangle \in P$, let $\mathcal{B}(\langle a, b\rangle) = \{ U_r(\langle a, b\rangle) : r > 0 \}$. So, the points in $A \cup P$ will have the same local base as in $(X , \mathcal{N})$. For each $\langle c, 0\rangle \in L \setminus A$, let $\mathcal{B}(\langle c, 0\rangle) = \{ U_r(\langle c, 0\rangle) : r > 0 \}$, where $U_r(\langle c, 0\rangle)$ is the set of all points of X inside the circle of radius r and centered at $\langle c, 0\rangle$. So, the points in $L \setminus A$ will have the same local base as in $(X , \mathcal{U})$. We call the topology on X generated by the neighborhood system $\{ \mathcal{B}(\langle x, y\rangle) : \langle x, y\rangle \in X \}$ *the H-generated* topology on X from $\mathcal{N}$ and $\mathcal{U}$ and denote it by $\mathcal{N}_A\mathcal{U}$. It is clear that $\mathcal{U}_A\mathcal{N} = \mathcal{N}_{(L\setminus A)}\mathcal{U}$ for any subset A of the x-axis L.

2. Some Fundamental Properties

Observe that for any non-empty proper subset A of the x-axis we have $\mathcal{U} \subseteq \mathcal{U}_A\mathcal{N} \subseteq \mathcal{N}$. Thus $(X , \mathcal{U}_A\mathcal{N})$ is T_0, T_1, Hausdorff, completely Hausdorff, and connected. To show complete regularity of $(X , \mathcal{U}_A\mathcal{N})$ we use Frink's theorem [4] which is the following characterization, see also [3, 1.5.G]:

Theorem 2.1 (O. Frink)**.** *A space X is completely regular if and only if there exists a base $\mathfrak{B}$ for X satisfying the following two conditions:*

(1) *For every $x \in X$ and every $U \in \mathfrak{B}$ that contains x there exists $V \in \mathfrak{B}$ such that $x \notin V$ and $U \cup V = X$.*
(2) *For any $U, V \in \mathfrak{B}$ satisfying $U \cup V = X$, there exist $U', V' \in \mathfrak{B}$ such that $X \setminus V \subseteq U'$, $X \setminus U \subseteq V'$, and $U' \cap V' = \varnothing$.*

Theorem 2.2. *Every H-space $(X , \mathcal{U}_A\mathcal{N})$ is Tychonoff.*

Proof. Denote the base of the neighborhood system $\{\, \mathcal{B}(\langle x,y\rangle) : \langle x,y\rangle \in X \,\}$ by $\mathcal{B}$. Let $\mathcal{W} = \{X \setminus \overline{G} : G \in \mathcal{B}\}$. Define $\mathfrak{B} = \mathcal{B} \cup \mathcal{W}$. Since $\mathcal{B} \subset \mathfrak{B}$, then $\mathfrak{B}$ is a base. We show that $\mathfrak{B}$ satisfies the conditions of Theorem 2.1. Observe that for each $G, G' \in \mathcal{B}$ with $G \subset G'$ we have $G \subset \overline{G} \subset G'$. Let $\langle x,y\rangle \in X$ be arbitrary, and $U \in \mathfrak{B}$ be arbitrary with $\langle x,y\rangle \in U$. Pick $K \in \mathfrak{B}$ such that $\langle x,y\rangle \in K \subset U$. We have two cases:

(1) $U \in \mathcal{B}$. Then we can let $V = X \setminus \overline{K}$. Therefore $\langle x,y\rangle \notin V$ and $U \cup V = X$.
(2) $U \in \mathcal{W}$. Put $U = X \setminus \overline{G}; G \in \mathcal{B}$. Here K has two cases:
  (a) $K \in \mathcal{B}$. Then we can let $V = X \setminus \overline{K}$.
  (b) $K \in \mathcal{W}$, where $K = X \setminus \overline{G}'; G' \in \mathcal{B}$ with $G \subset G'$. Then we can let $V = G'$. Therefore in each case $\langle x,y\rangle \notin V$ and $U \cup V = X$.

So the first condition of Theorem 2.1 is satisfied.

Let $U, V \in \mathfrak{B}$ be arbitrary satisfying $U \cup V = X$. Then one and only one of the following cases is satisfied:

(1) $U \in \mathcal{B}$ and $V \in \mathcal{W}$, where $V = X \setminus \overline{G}; G \in \mathcal{B}$ with $G \subset U$. Pick $K \in \mathcal{B}$ with $G \subset K \subset U$. Let $U' = K$ and $V' = X \setminus \overline{K}$. Therefore $X \setminus V \subset U'$, $X \setminus U \subset V'$, and $U' \cap V' = \varnothing$.
(2) $U \in \mathcal{W}$ and $V \in \mathcal{B}$, where $U = X \setminus \overline{G}; G \in \mathcal{B}$ with $G \subset V$. Pick $K \in \mathcal{B}$ with $G \subset K \subset V$. Let $V' = K$ and $U' = X \setminus \overline{K}$. Therefore $X \setminus V \subset U'$, $X \setminus U \subset V'$, and $U' \cap V' = \varnothing$.
(3) $U \in \mathcal{W}$ and $V \in \mathcal{W}$, put $U = X \setminus \overline{K}$, $V = X \setminus \overline{G}$ where $K, G \in \mathcal{B}$ with $K \cap G = \varnothing$. Write $K = B_{r_1}(\langle x,y\rangle)$ and $G = B_{r_2}(\langle a,b\rangle)$. Since $X = U \cup V$, then $\overline{K} \cap \overline{G} = \varnothing$. Since $\overline{K}$ and $\overline{G}$ are both closed, then the distance between them are positive say $\delta > 0$. Let $\epsilon_1 = r_1 + \frac{\delta}{4}$ and $\epsilon_2 = r_2 + \frac{\delta}{4}$. Let $V' = B_{\epsilon_1}(\langle x,y\rangle)$ and $U' = B_{\epsilon_2}(\langle a,b\rangle)$. Therefore $X \setminus V \subset U'$, $X \setminus U \subset V'$, and $U' \cap V' = \varnothing$.

So, in all cases, the second condition of Theorem 2.1 is satisfied. Thus $(\, X \,,\, \mathcal{U}_A\mathcal{N} \,)$ is Tychonoff. □

Any H-space $(\, X \,,\, \mathcal{U}_A\mathcal{N} \,)$ is first countable just by taking for each $\langle x,y\rangle \in A \cup P$ the countable local base $\mathcal{B}'(\langle x,y\rangle) = \{\, U_{\frac{1}{n}}(\langle x,y\rangle) : n \in \mathbb{N} \,\}$ and for each $\langle x,0\rangle \in L \setminus A$ the countable local base $\mathcal{B}'(\langle x,0\rangle) = \{\, D_{\frac{1}{n}}(\langle x,0\rangle) : n \in \mathbb{N} \,\}$. Any H-space $(\, X \,,\, \mathcal{U}_A\mathcal{N} \,)$ is separable as $(\mathbb{Q} \times \mathbb{Q}) \cap P$ is a countable dense subset. For the second countability, we have the following theorem:

Theorem 2.3. *Let $(\, X \,,\, \mathcal{U}_A\mathcal{N} \,)$ be an H-space. The following are equivalent:*

(1) *$L \setminus A$ is countable.*
(2) *$(\, X \,,\, \mathcal{U}_A\mathcal{N} \,)$ is second countable.*
(3) *$(\, X \,,\, \mathcal{U}_A\mathcal{N} \,)$ is metrizable.*

Proof.
(1)⇒(2) Assume that $L \setminus A$ is countable. Since $A \cup P$ as a subspace is metrizable, let $\mathcal{W}$ be a countable base for $A \cup P$. Let $\mathfrak{B} = \{\, \mathcal{W}, \mathcal{B}'(\langle x,0\rangle) : \langle x,0\rangle \in$

$L \setminus A\}$, then $\mathfrak{B}$ is a countable base for $(X, \mathcal{U}_A\mathcal{N})$ because $L \setminus A$ is countable.

(2)⇒(3) Assume that $(X, \mathcal{U}_A\mathcal{N})$ is second countable. Since $(X, \mathcal{U}_A\mathcal{N})$ is also T_3, see Theorem 2.2, and any T_3 second countable space is metrizable [3], result follows.

(3)⇒(2) Assume that $(X, \mathcal{U}_A\mathcal{N})$ is metrizable. Since it is separable and any metrizable separable space is second countable [3, 4.1.16], result follows.

(2)⇒(1) Assume that $(X, \mathcal{U}_A\mathcal{N})$ is second countable. Suppose that $L \setminus A$ is uncountable. Since any basic open set $D_r(\langle x,0\rangle)$ of each element $\langle x,0\rangle$ in $L \setminus A$ does not contain any element from the x-axis other than $\langle x,0\rangle$ itself and any basic open set $U_r(\langle x',0\rangle)$ of each element $\langle x',0\rangle$ in A cannot be contained in $D_r(\langle x,0\rangle)$, we conclude that $(X, \mathcal{U}_A\mathcal{N})$ cannot be second countable which is a contradiction.

□

$(X, \mathcal{U}_A\mathcal{N})$ need not be normal, for example, if $A = \{\langle x,0\rangle : x < -1$ or $0 < x\}$, then $X \setminus (A \cup P)$ is closed uncountable discrete subspace of $(X, \mathcal{U}_A\mathcal{N})$. Since $(X, \mathcal{U}_A\mathcal{N})$ is also separable, then by Jones' Lemma [7], $(X, \mathcal{U}_A\mathcal{N})$ cannot be normal. Since any interval C in $\mathbb{R}$ must contain a closed bounded interval $[a,b]$, we conclude the following theorem:

Theorem 2.4. *If $L \setminus A$ contains a set of the form $C \times \{0\}$, where C is an interval in $\mathbb{R}$, then $(X, \mathcal{U}_A\mathcal{N})$ cannot be normal.*

Since any metrizable space is normal, Theorem 2.3 gives the following:

Theorem 2.5. *If $L \setminus A$ is countable, then $(X, \mathcal{U}_A\mathcal{N})$ is normal.*

For each $n \in \mathbb{N}$, Let $W_n = \mathbb{R} \times [0,n)$. Then $X \subseteq \bigcup_{n\in\mathbb{N}} W_n$ and for all $n \in \mathbb{N}$, $W_n \in \mathcal{U}_A\mathcal{N}$. So the family $\mathcal{W} = \{W_n : n \in \mathbb{N}\}$ is a countable open cover of X which has no finite subcover. Thus $(X, \mathcal{U}_A\mathcal{N})$ is neither compact nor countably compact.

Theorem 2.6. *For a non-empty proper subset A of L, $(X, \mathcal{U}_A\mathcal{N})$ is not locally compact.*

Proof. Let $\langle x,0\rangle \in L \setminus A$ be arbitrary and let G be any open neighborhood of $\langle x,0\rangle$. There exists an $i \in \mathbb{R}$ such that $D_i(\langle x,0\rangle) \subset \overline{D_i(\langle x,0\rangle)} = \overline{D_i(\langle x,0\rangle)}^{\mathcal{U}} \subseteq G$. Observe that $\overline{D_i(\langle x,0\rangle)}^{\mathcal{U}}$ is closed in $(X, \mathcal{U}_A\mathcal{N})$. Now, the circumference of $\overline{D_i(\langle x,0\rangle)}^{\mathcal{U}}$ contains a sequence of points which converges to $\langle x,0\rangle$ in the usual topology, but this sequence can have no accumulation point in $(X, \mathcal{U}_A\mathcal{N})$ as the open neighborhood $D_j(\langle x,0\rangle)$, where $j > i$, satisfies that $D_j(\langle x,0\rangle)$ does not contain any member of the sequence. Thus $\overline{D_j(\langle x,0\rangle)}^{\mathcal{U}}$ is not countably compact, hence not compact. □

The first result about the Lidelöfness of an H-space is a corollary of Theorem 2.3.

Theorem 2.7. *If $L \setminus A$ is countable, then the H-space $(X, \mathcal{U}_A\mathcal{N})$ is Lindelöf.*

Theorem 2.8. *If $A' = \{x : \langle x,0\rangle \in A\}$ is dense in $(\mathbb{R},\mathcal{U})$, then $(X, \mathcal{U}_A\mathcal{N})$ is Lindelöf.*

Proof. Let $\mathcal{W} = \{W_\alpha : \alpha \in \Lambda\}$ be any open cover for X. For each $\langle x,0\rangle \in A$, there exists an $\alpha_x \in \Lambda$ such that $\langle x,0\rangle \in W_{\alpha_x}$. Thus for each $\langle x,0\rangle \in A$ there exists $r_x > 0$ such that $\langle x,0\rangle \in U_{r_x}(\langle x,0\rangle) \subseteq W_{\alpha_x}$. Since A' is dense in $(\mathbb{R},\mathcal{U})$, then $\cup_{x\in A'} U_{r_x}(\langle x,0\rangle)$ covers L. Now, for each $\langle x,y\rangle \in P$, there exists $W_{\alpha_{\langle x,y\rangle}} \in \mathcal{W}$ such that $\langle x,y\rangle \in W_{\alpha_{\langle x,y\rangle}}$. This means that there exists $r_{\langle x,y\rangle} > 0$ such that $\langle x,y\rangle \in U_{r_{\langle x,y\rangle}}(\langle x,y\rangle) \subseteq W_{\alpha_{\langle x,y\rangle}}$. Thus $\mathcal{W}' = \{U_{r_{\langle x,y\rangle}}(\langle x,y\rangle), U_{r_x}(\langle x,0\rangle) : \langle x,0\rangle \in A, \langle x,y\rangle \in P\}$ is an open cover for $(X,\mathcal{U})$. Since $(X,\mathcal{U})$ is Lindelöf , then there exists a countable subcover from $\mathcal{W}'$. Then $\mathcal{W}$ has a countable open refinement . Therefore, $(X, \mathcal{U}_A\mathcal{N})$ is Lindelöf. □

Theorem 2.9. *$(X, \mathcal{U}_A\mathcal{N})$ is not Lindelöf if and only if $L \setminus A$ contains a set of the form $C \times \{0\}$, where C is an interval in $\mathbb{R}$.*

Proof. ($\Rightarrow$) Assume that $(X, \mathcal{U}_A\mathcal{N})$ is not Lindelöf. Suppose that $L \setminus A$ does not contain any set of the form $C \times \{0\}$, where C is an interval in $\mathbb{R}$. So, for each $a, b \in \mathbb{R}; a < b$, there exists $x \in A$ such that $a < x < b$. This gives that $A' = \{x : \langle x,0\rangle \in A\}$ is dense in $(\mathbb{R},\mathcal{U})$. Then by Theorem 2.8 $(X, \mathcal{U}_A\mathcal{N})$ is Lindelöf which is a contradiction.

($\Leftarrow$) Assume that $L \setminus A$ contains a set of the form $C \times \{0\}$, where C is an interval in $\mathbb{R}$. Let $J = [a, b] \subseteq C$. Let $\epsilon = \frac{b-a}{4}$. Then

$$X \subseteq (\cup_{\langle x,y\rangle \in (A\cup P)\setminus([a,b]\times[0,\frac{3\epsilon}{2}))} U_{\frac{\epsilon}{2}}(\langle x,y\rangle)) \cup (\cup_{\langle x,0\rangle \in L\setminus A} D_\epsilon(\langle x,0\rangle)).$$

So, $\{U_{\frac{\epsilon}{2}}(\langle x,y\rangle), D_\epsilon(\langle x,0\rangle) : \langle x,y\rangle \in (A \cup P) \setminus ([a,b] \times [0, \frac{3\epsilon}{2}); \langle x,0\rangle \in L \setminus A\}$ is an open cover of X which has no countable subcover because J is uncountable and for each $\langle x,0\rangle \in J \times \{0\}$ the set $D_\epsilon(\langle x,0\rangle)$ contains no other element of J except $\langle x,0\rangle$. □

3. Other Properties of H-spaces

Recall that a topological space X is called *C-normal* [1] (*CC-normal* [9], *L-normal* [11], *S-normal* [10]) if there exists a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f\restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each compact (countably compact, Lindelöf, separable) subspace $A \subseteq X$. A topological space X is called *C_2-paracompact* [12] if there exists a T_2 paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f\restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. A topological space (X, τ) is called *submetrizable* if there exists a metric d on X such that the topology τ_d on X generated by d is coarser than τ, i.e., $\tau_d \subseteq \tau$, see [5]. Since submetrizabilty implies both C-normality [1] and C_2-paracompactness [12], we conclude that any H-space $(X, \mathcal{U}_A\mathcal{N})$ is both C-normal and C_2-paracompact being submetrizable by the usual metric. By the theorem "*If X is T_3 separable L-normal and of countable tightness, then X is normal.*" [11, 1.6], we obtain the following theorem:

Theorem 3.1. *$(X, \mathcal{U}_A\mathcal{N})$ is normal if and only if it is L-normal.*

As $(X, \mathcal{U}_A\mathcal{N})$ is always separable we conclude the following theorem:

Theorem 3.2. *$(X, \mathcal{U}_A\mathcal{N})$ is normal if and only if it is S-normal.*

Now, to study the CC-normality of an H-space, we start with the CC-normality of the Niemytzki plane. We do this by three steps.

Lemma 3.3. *A subset C of the Niemytzki plane X is countably compact if and only if $C \cap L$ is finite and C is closed and bounded in X considered with its usual metric topology.*

Proof. Assume that C is a countably compact subspace of the Niemytzki plane X. Suppose that $C \cap L$ is infinite. Pick a countably infinite subset $D = \{\langle d_n, 0\rangle : n \in \mathbb{N}\} \subseteq C \cap L$. For each $n \in \mathbb{N}$, consider the basic open neighborhood $D_1(\langle d_n, 0\rangle)$ of $\langle d_n, 0\rangle$. For each $\langle x, y\rangle \in C \cap P$, consider $U_{\frac{y}{2}}(\langle x, y\rangle)$ and let $U = (\bigcup_{\langle x,y\rangle \in C\cap P} U_{\frac{y}{2}}(\langle x, y\rangle)) \bigcup (\bigcup_{\langle x,0\rangle \in (C\cap L)\setminus D} D_1(\langle x, 0\rangle))$. Then the countable open cover $\{U, D_1(\langle d_n, 0\rangle) : n \in \mathbb{N}\}$ of C has no finite subsover which is a contradiction. Now, Assume that C is countably compact and $C \cap L$ is finite. Suppose that C is either not closed in X considered with its usual metric topology or not bounded. Since in a metrizable space, a subspace is countably compact if and only if it compact [3, 4.1.17]. Also, in the usual metric space, a subspace is compact if and only if it is closed and bounded [3, 3.2.8]. We conclude that C is not countably compact in X considered with its usual metric topology. Since the usual metric topology $\mathcal{U}$ is coarser than the Niemytzki topology $\mathcal{N}$ we conclude that C is not countably compact in the Niemytzki plane X which is a contradiction.

Conversely, assume that $C \cap L$ is finite and C is closed and bounded in X considered with its usual metric topology. Let $\mathcal{W}$ be any countable open cover for C. Then $\mathcal{W}$ is a countable open cover for $C \cap P$. Since C is closed and bounded in X considered with its usual metric topology, then $C \cap P$ is closed and bounded in P considered with its usual metric topology. So, $C \cap P$ is compact in P. Pick a finite $W_1, ..., W_n \in \mathcal{W}$ such that $C \cap P \subseteq \bigcup_{i=1}^{n} W_i$. Since $C \cap L$ is finite, pick for each $\langle x, 0\rangle \in C \cap L$ a member $W_x \in \mathcal{W}$ such that $\langle x, 0\rangle \in W_x$. Then $\{W_1, ..., W_n, W_x : x \in C \cap L\}$ is a finite subcover. Therefore, C is countably compact. $\square$

Lemma 3.4. *Let C be a subspace of the Niemytzki plane X. C is countably compact if and only if C is compact.*

Proof. Let C be any countably compact subspace in the Niemytzki plane X. By Lemma 3.3, $C \cap L$ is finite and C is closed and bounded in X considered with its usual metric topology, hence $C \cap P$ is compact in X with its usual metric topology [3, 3.2.8]. Let $\mathcal{G}$ be any open cover for C consisting of basic open sets. Since $C \cap L$ is finite, pick $G_x \in \mathcal{G}$ such that $\langle x, 0\rangle \in G_x$. Since $C \cap P$ is compact, pick a finite subcover $\mathcal{G}'$ of $\mathcal{G}$ which covers $C \cap P$. Then $\mathcal{G}' \cup \{G_x : \langle x, 0\rangle \in C \cap L\}$ is a finite subcover of $\mathcal{G}$. Thus C is compact. The other direction is clear. $\square$

Theorem 3.5. *The Niemytzki plane is CC-normal.*

Proof. Let $Y = X$ with its usual metric topology. Consider the identity function $id : X \longrightarrow Y$. Since the usual metric topology $\mathcal{U}$ is coarser than the Niemytzki topology $\mathcal{N}$, then $id : X \longrightarrow Y$ is continuous, hence any restriction function of it is continuous. Let C be any countably compact subspace of X. By Lemma 3.4, C is compact in the Niemytzki plane, hence $id_{|_C} : C \longrightarrow id(C) = C$ is a homeomorphism, see [3, 3.1.13]. $\square$

We use similar ideas to show that any H-space is CC-normal.

Lemma 3.6. *A subset C of an H-space $(\, X \,, \mathcal{U}_A\mathcal{N}\,)$ is countably compact if and only if C satisfies the following two conditions:*

1. *C is closed and bounded in $(\, X \,, \mathcal{U}\,)$.*
2. *Any infinite subset of $C \cap (L \setminus A)$ has an accumulation point in $C \cap A$ in L with its usual metric topology.*

Proof. Assume that C is countably compact in an H-space $(\, X \,, \mathcal{U}_A\mathcal{N}\,)$. Suppose that C is either not closed in X considered with its usual metric topology or not bounded. Since in a metrizable space, a subspace is countably compact if and only if it compact [3, 4.1.17]. Also, in the usual metric space, a subspace is compact if and only if it is closed and bounded [3, 3.2.8]. We conclude that C is not countably compact in $(\, X \,, \mathcal{U}\,)$. Since the usual metric topology $\mathcal{U}$ is coarser than the H-topology $\mathcal{U}_A\mathcal{N}$ we conclude that C is not countably compact in the H-space $(\, X \,, \mathcal{U}_A\mathcal{N}\,)$ which is a contradiction. Now, assume that C is a countably compact subspace of an H-space $(\, X \,, \mathcal{U}_A\mathcal{N}\,)$ and C is closed and bounded in $(\, X \,, \mathcal{U}\,)$. Suppose that there exists a countably infinite subset $D = \{\, \langle d_n, 0\rangle : n \in \mathbb{N}\,\}$ of $C \cap (L \setminus A)$ which has no accumulation point in $C \cap A$ with respect to L with its usual metric topology. For each $\langle x, 0\rangle \in C \cap A$, fix $r_x > 0$ such that $U_{r_x}(\langle x, 0\rangle)$ satisfies $U_{r_x}(\langle x, 0\rangle) \cap D = \varnothing$. Let

$$U = \Big(\bigcup_{\langle x,y\rangle \in C\cap P} U_{\frac{y}{2}}(\langle x, y\rangle)\Big) \bigcup \Big(\bigcup_{\langle x,0\rangle \in C\cap((L\setminus A)\setminus D)} D_1(\langle x, 0\rangle)\Big) \bigcup \Big(\bigcup_{\langle x,0\rangle \in C\cap A} U_{r_x}(\langle x, 0\rangle)\Big).$$

Then the countable open cover $\{\, U, D_1(\langle d_n, 0\rangle) : n \in \mathbb{N}\,\}$ of C has no finite subover which is a contradiction.

Conversely, let C be any subset of X satisfies the two conditions. Let $\mathcal{W}$ be any countable open cover for C. By condition (1), $C \cap (P \cup A)$ is closed and bounded in the metrizable space $P \cup A$. So, $C \cap (P \cup A)$ is compact in $P \cup A$. Pick a finite subcover $\mathcal{W}'$ of $\mathcal{W}$ for $C \cap (P \cup A)$. If $C \cap (L \setminus A)$ is finite, there exist $W_1, ..., W_n \in \mathcal{W}$ such that $C \cap (L \setminus A) \subseteq \bigcup_{i=1}^{n} W_i$. Then $\mathcal{W}' \cup \{\, W_1, ..., W_n \,\}$ is a finite subcover of $\mathcal{W}$ covers C. Now, if $C \cap (L \setminus A)$ is infinite. For each $\langle x, 0\rangle \in C \cap (L \setminus A)$ there exists $W_x \in \mathcal{W}$ with $\langle x, 0\rangle \in W_x$. Since $\mathcal{W}$ is countable, pick a countable subset $E = \{\, \langle x_n, 0\rangle : n \in \mathbb{N}\,\} \subseteq C \cap (L \setminus A)$ such that $C \cap (L \setminus A) \subseteq \bigcup_{n\in\mathbb{N}} W_{x_n}$. By condition (2), pick $\langle y, 0\rangle \in C \cap A$ such that $\langle y, 0\rangle$ is an accumulation point of E in L with its usual metric. Then the open set $W_y \in \mathcal{W}'$ with $\langle y, 0\rangle \in W_y$ covers all elements of E except possibly finitely many elements say $\langle x_{n_1}, 0\rangle, ..., \langle x_{n_m}, 0\rangle$. For each $i \in \{1, ..., m\}$ pick $W_i \in \mathcal{W}$

such that $\langle x_{n_i}, 0\rangle \in W_i$. Then $\mathcal{W}' \bigcup \{ W_{n_i} : i \in \{1, ..., m\} \}$ is a finite subcover of $\mathcal{W}$ covers C. Therefore, C is countably compact. □

Lemma 3.7. *Let C be a subspace of an H-space $(X , \mathcal{U}_A\mathcal{N})$. C is countably compact if and only if C is compact.*

Proof. Let C be any countably compact subspace in an H-space $(X , \mathcal{U}_A\mathcal{N})$. By Lemma 3.6, C is closed and bounded in X considered with its usual metric topology, hence $C \cap (P \cup A)$ is compact in X with its usual metric topology [3, 3.2.8]. Let $\mathcal{G}$ be any open cover for C consisting of basic open sets. Since $C \cap (P \cup A)$ is compact, pick a finite subcover $\mathcal{G}'$ of $\mathcal{G}$ which covers $C \cap (P \cup A)$. In particular, $\mathcal{G}'$ covers $C \cap A$. Pick $\langle x_1, 0\rangle, ..., \langle x_n, 0\rangle \in C \cap A$ such that $C \cap A \subseteq \bigcup_{i=1}^{n} U_{r_i}(\langle x_i, 0\rangle)$. By Lemma 3.6, $\bigcup_{i=1}^{n} U_{r_i}(\langle x_i, 0\rangle)$ covers all possible accumulation points of $C \cap (L \setminus A)$ in L with its usual metric topology. Thus $\bigcup_{i=1}^{n} U_{r_i}(\langle x_i, 0\rangle)$ covers all points of $C \cap (L \setminus A)$ except possibly finitely many points, say $\langle y_1, 0\rangle, ..., \langle y_m, 0\rangle \in C \cap (L \setminus A)$, because if $(C \cap (L \setminus A)) \setminus (\bigcup_{i=1}^{n} U_{r_i}(\langle x_i, 0\rangle))$ is infinite, then any countably infinite subset of it will not have an accumulation point in $C \cap A$ in L with its usual metric and this contradicts the countable compactness of C, see Lemma 3.6. For each $j \in \{ 1, ..., m \}$ pick $G_j \in \mathcal{G}$ such that $\langle y_j, 0\rangle \in G_j$. Then $\mathcal{G}' \cup \{ G_j : j \in \{ 1, ..., m \} \}$ is a finite subcover for $\mathcal{G}$ covers C. Thus C is compact. □

Theorem 3.8. *Any H-space $(X , \mathcal{U}_A\mathcal{N})$ is CC-normal.*

Proof. Let $Y = X$ with its usual metric topology. Consider the identity function $id : X \longrightarrow Y$. Since the usual metric topology $\mathcal{U}$ is coarser than the H-topology $\mathcal{U}_A\mathcal{N}$, then $id : X \longrightarrow Y$ is continuous, hence any restriction function of it is continuous. Let C be any countably compact subspace in $(X , \mathcal{U}_A\mathcal{N})$. By Lemma 3.7, C is compact in the H-space $(X , \mathcal{U}_A\mathcal{N})$, hence $id_{|_C} : C \longrightarrow id(C) = C$ is a homeomorphism, see [3, 3.1.13]. □

Conjecture: Under the cases that $(X , \mathcal{U}_A\mathcal{N})$ is not normal, is it κ-normal [8, 14]? quasi-normal [15]?

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