APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 23, no. 1 (2022), 31-43
doi:10.4995/agt.2022.16165

Some fixed point results for enriched nonexpansive type mappings in Banach spaces

Rahul Shukla and Rajendra Pant

Department of Mathematics & Applied Mathematics, University of Johannesburg Kingsway Campus, Auckland Park 2006, South Africa (rshukla.vnit@gmail.com, pant.rajendra@gmail.com)

Communicated by S. Romaguera

Abstract

In this paper, we introduce two new classes of nonlinear mappings and present some new existence and convergence theorems for these mappings in Banach spaces. More precisely, we employ the Krasnosel'skiĭ iterative method to obtain fixed points of Suzuki-enriched nonexpansive mappings under different conditions. Moreover, we approximate the fixed point of enriched-quasinonexpansive mappings via Ishikawa iterative method.

2020 MSC: *47H10; 47H09.*

Keywords: *nonexpansive mapping; enriched nonexpansive mapping; Banach space.*

1. Introduction

Let $\mathcal{C}$ be a nonempty subset of a Banach space $(\mathcal{B}, \|.\|)$. A mapping $\xi : \mathcal{C} \to \mathcal{C}$ is said to be nonexpansive if

$$\|\xi(\vartheta) - \xi(\nu)\| \leq \|\vartheta - \nu\|$$

for all $\vartheta, \nu \in \mathcal{C}$. Bruck [5] observed that apart from being an obvious generalization of the contraction mapping, nonexpansive mappings are important due to their connection with the monotonicity methods. Perhaps, nonexpansive mappings belong to the first class of nonlinear mappings for which fixed point theorems were obtained by using the geometric properties of the underlying Banach spaces rather than the compactness assumptions (see fixed point theorems

Received 31 August 2021 – Accepted 24 November 2021

due to Browder [3], Göhde [12] Kirk [15]). This class of mappings also appears in applications as transition operators for initial value problems (of differential inclusion), accretive operators, monotone operators, variational inequality problems and equilibrium problems. A number of extensions and generalizations of nonexpansive mappings in different directions have been considered by many mathematicians in the literature to enlarge the class of nonexpansive mappings, see [11, 17, 8, 25, 20, 21, 24] (see also the references therein).

In 2008, Suzuki [25] introduced a new type of mapping which is more general than nonexpansive mapping, as follows.

Definition 1.1 ([25])**.** Let $\mathcal{C}$ be a nonempty subset of a Banach space $(\mathcal{B}, \|.\|)$. A mapping $\xi : \mathcal{C} \to \mathcal{C}$ is said to satisfy condition (C) if for all $\vartheta, \nu \in \mathcal{C}$

$$\frac{1}{2}\|\vartheta - \xi(\vartheta)\| \leq \|\vartheta - \nu\| \text{ implies } \|\xi(\vartheta) - \xi(\nu)\| \leq \|\vartheta - \nu\|.$$

A mapping satisfying condition (C) is also known as Suzuki type generalized nonexpansive mapping.

Recently, Berinde [1] introduced the following class of nonlinear mappings.

Definition 1.2. Let $(\mathcal{B}, \|.\|)$ be a Banach space. A mapping $\xi : \mathcal{B} \to \mathcal{B}$ is said to be b-enriched nonexpansive mapping if there exists $b \in [0, \infty)$ such that for all $\vartheta, \nu \in \mathcal{B}$

$$\|b(\vartheta - \nu) + \xi(\vartheta) - \xi(\nu)\| \leq (b + 1)\|\vartheta - \nu\|. \tag{1.1}$$

It is shown that every nonexpansive mapping ξ is a 0-enriched mapping. It is interesting to note that both these classes of mappings, Suzuki type nonexpansive and b-enriched nonexpansive mappings are independent. A couple of examples below illustrate these facts.

Example 1.3 ([25])**.** Let $\mathcal{C} = [0, 3]$ be a subset of $\mathbb{R}$ endowed with the usual norm. Define $\xi : \mathcal{C} \to \mathcal{C}$ by

$$\xi(\vartheta) = \begin{cases} 0, & \text{if } \vartheta \neq 3, \\ 1, & \text{if } \vartheta = 3. \end{cases}$$

Then ξ satisfies condition (C). However at $\vartheta = 2.5$ and $\nu = 3$

$$\begin{aligned} \|b(\vartheta - \nu) + \xi(\vartheta) - \xi(\nu)\| &= \|b(2.5 - 3) + (0 - 1)\| \\ &= b(0.5) + 1 > b(0.5) + 0.5 = (b + 1)|\vartheta - \nu| \end{aligned}$$

and ξ is not b-enriched nonexpansive mapping for any $b \in [0, \infty)$.

Example 1.4 ([1])**.** Let $\mathcal{C} = \left[\frac{1}{2}, 2\right] \subset \mathbb{R}$ and $\xi : \mathcal{C} \to \mathcal{C}$ be a mapping defined as $\xi(\vartheta) = \frac{1}{\vartheta}$. Then $F(\xi) = \{1\}$ and ξ is a $\frac{3}{2}$-enriched nonexpansive mapping. On the other hand at $\vartheta = 1$ and $\nu = \frac{1}{2}$, we have

$$\frac{1}{2}\|1 - \xi(1)\| = 0 \leq \frac{1}{2} = \left\|1 - \frac{1}{2}\right\|$$

and

$$\left\|\xi\left(\frac{1}{2}\right)-\xi(1)\right\| = |2-1| = 1 > \frac{1}{2} = \left\|\frac{1}{2}-1\right\|.$$

Thus ξ is not a mapping satisfying condition (C).

Now an interesting question arises that does there exists a class of mappings, which contains both the b-enriched nonexpansive mappings and Suzuki-type generalized nonexpansive mappings? Herein, we answer this question, affirmatively. Indeed, we introduce a new class of mappings, namely, Suzuki-enriched nonexpansive mapping.

On the other hand, to check that a given mapping belongs to any of the classes of nonexpansive type mappings can not be an easy task. Keeping this point in mind to make task easier, Diaz and Metcalf [7] considered the following class of mappings known as quasinonexpansive mapping

Definition 1.5. A mapping $\xi : \mathcal{C} \to \mathcal{C}$ is said to be quasinonexpansive if for all $\vartheta \in \mathcal{C}$ and $\vartheta^\dagger \in F(\xi) \neq \varnothing$,

$$\|\xi(\vartheta) - \vartheta^\dagger\| \leq \|\vartheta - \vartheta^\dagger\|$$

where $F(\xi)$ is the set of all fixed points of ξ.

It is well known that a nonexpansive mapping with a fixed point is quasinonexpansive. However the converse need not to be true. Again, it is interesting to see that the classes of b-enriched nonexpansive mappings and that of quasi-nonexpansive mappings are independent in nature, see [23]. Keeping this in mind, we generalize the class of quasinonexpansive mappings in the sense of b-enriched nonexpansive mappings. In particular, we introduce a new class of mappings namely enriched-quasinonexpansive mappings. This class of mappings properly contains both quasinonexpansive mappings and b-enriched nonexpansive mappings.

Motivated by Berinde [1, 2], Suzuki [25], Diaz and Metcalf [7] and others, we introduce two new nonlinear classes of mappings in the setting of Banach spaces and establish some existence and convergence theorems for these classes of mappings. We ensure the existence of fixed points for Suzuki-enriched nonexpansive mappings in Banach spaces under certain assuptions. We employ Ishikawa iterative method to approximate the fixed points of enriched-quasinonexpansive mappings and obtain some weak and strong convergence theorems. Our results complement, extend, and generalize certain results from [1, 2, 25, 7, 18, 9].

2. Preliminaries

Definition 2.1 ([10])**.** A Banach space $\mathcal{B}$ is said to be uniformly convex if for every $\varepsilon \in (0, 2]$ there is some $\delta > 0$ so that, for any $\vartheta, \nu \in \mathcal{B}$ with $\|\vartheta\| = \|\nu\| = 1$, the condition $\|\vartheta - \nu\| \geq \varepsilon$ implies that $\left\|\frac{\vartheta+\nu}{2}\right\| \leq 1 - \delta$.

Definition 2.2 ([19])**.** Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space. A space $\mathcal{B}$ satisfies Opial property if, for every weakly convergent sequence $\{\vartheta_n\}$ with weak limit $\vartheta \in \mathcal{B}$ it holds:

$$\liminf_{n\to\infty} \|\vartheta_n - \vartheta\| < \liminf_{n\to\infty} \|\vartheta_n - \nu\|$$

for all $\nu \in \mathcal{B}$ with $\vartheta \neq \nu$.

All finite dimensional Banach spaces and all Hilbert spaces satisfy the weak-Opial property. Spaces ℓ^p $(p \in (1,\infty))$ are Opial spaces but $L_p(\in (1,\infty),\ p \neq 2)$ spaces are not. [10].

Definition 2.3 ([22])**.** The mapping $\xi : \mathcal{C} \to \mathcal{C}$ with $F(\xi) \neq \varnothing$ satisfies Condition (I) if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0,\infty)$ such that $\|\vartheta - \xi(\vartheta)\| \geq f(d(\vartheta, F(\xi)))$ for all $\vartheta \in \mathcal{C}$, where $d(\vartheta, F(\xi)) = \inf\{\|\vartheta - \nu\| : \nu \in F(\xi)\}$.

Let $\mathcal{C}$ be a convex subset of a Banach space $\mathcal{B}$ and $\xi : \mathcal{C} \to \mathcal{C}$ a mapping. The following iterative method is known as the Krasnosel'skiĭ iterative method (see [16]):

$$\begin{cases} \vartheta_1 \in \mathcal{C} \\ \vartheta_{n+1} = \alpha\vartheta_n + (1-\alpha)\xi(\vartheta_n) \end{cases} \tag{2.1}$$

where $\alpha \in (0,1)$.

Lemma 2.4. *Let $\mathcal{C}$ be a nonempty convex subset a Banach space $\mathcal{B}$. Let $\xi : \mathcal{C} \to \mathcal{C}$ be a mapping, define $S : \mathcal{C} \to \mathcal{C}$ as follows:*

$$S(\vartheta) = (1-\lambda)\vartheta + \lambda\xi(\vartheta)$$

for all $\vartheta \in \mathcal{C}$ and $\lambda \in (0,1)$. Then $F(S) = F(\xi)$.

Definition 2.5. Let $\mathcal{C}$ be a nonempty subset of a Banach space $\mathcal{B}$. A mapping $\xi : \mathcal{C} \to \mathcal{C}$ is said to be compact if $\xi(\mathcal{C})$ has a compact closure.

Lemma 2.6 ([27, p. 484])**.** *Let $\mathcal{B}$ be a uniformly convex Banach space. If two sequences $\{\vartheta_n\}$, $\{\nu_n\}$ in $\mathcal{B}$ such that $\limsup_{n\to\infty} \|\vartheta_n\| \leq \theta$, $\limsup_{n\to\infty} \|\nu_n\| \leq \theta$, $\lim_{n\to\infty} \|\alpha_n\vartheta_n + (1-\alpha_n)\nu_n\| = \theta$, where $\{\alpha_n\} \subseteq [\eta_1, \eta_2] \subset [0,1]$ and $\theta \geq 0$. Then $\lim_{n\to\infty} \|\vartheta_n - \nu_n\| = 0$.*

Lemma 2.7. *Let $\mathcal{B}$ be a uniformly convex Banach space and $\mathcal{C}$ a nonempty closed convex subset of $\mathcal{B}$. Let $S : \mathcal{C} \to \mathcal{C}$ be a quasinonexpansive mapping with $F(S) \neq \varnothing$. For given $\vartheta_1 \in \mathcal{C}$, for all $n \in \mathbb{N}$, $\gamma_n, \delta_n \in [c,d]$ with $c, d \in (0,1)$, we can define a sequence $\{\vartheta_n\}$ (Ishikawa iterative method [13]) as follows:*

$$\begin{cases} \nu_n = (1-\gamma_n)\vartheta_n + \gamma_n S(\vartheta_n) \\ \vartheta_{n+1} = (1-\delta_n)\vartheta_n + \delta_n S(\nu_n), \end{cases} \tag{2.2}$$

Then we have the followings:

(1) *$\lim_{n\to\infty} \|\vartheta_n - z\|$ exists for all $z \in F(S)$.*

(2) $\lim_{n\to\infty} \|\vartheta_n - S(\vartheta_n)\| = 0$.

Proof. From (2.2)

$$\begin{aligned}\|\vartheta_{n+1} - z\| &\leq (1-\delta_n)\|\vartheta_n - z\| + \delta_n\|S(\nu_n) - z\| \\ &\leq (1-\delta_n)\|\vartheta_n - z\| + \delta_n\|\nu_n - z\| \\ &\leq (1-\delta_n)\|\vartheta_n - z\| + \delta_n\{(1-\gamma_n)\|\vartheta_n - z\| + \gamma_n\|S(\vartheta_n) - z\|\} \\ &\leq (1-\delta_n)\|\vartheta_n - z\| + \delta_n\|\vartheta_n - z\| = \|\vartheta_n - z\|.\end{aligned}$$

Hence the sequence $\{\|\vartheta_n - z\|\}$ is monotone nonincreasing and $\lim_{n\to\infty}\|\vartheta_n - z\|$ exists. Let

$$\lim_{n\to\infty}\|\vartheta_n - z\| = r > 0. \tag{2.3}$$

Since, S is a quasinonexpansive mapping

$$\limsup_{n\to\infty}\|S(\nu_n) - z\| \leq \limsup_{n\to\infty}\|\nu_n - z\| \leq \lim_{n\to\infty}\|\vartheta_n - z\| = r \tag{2.4}$$

and

$$\lim_{n\to\infty}\|(1-\delta_n)(\vartheta_n - z) + \delta_n(S(\nu_n) - z)\| = \lim_{n\to\infty}\|\vartheta_{n+1} - z\| = r. \tag{2.5}$$

From (2.3), (2.4), (2.5) and Lemma 2.6, we have

$$\lim_{n\to\infty}\|\vartheta_n - S(\nu_n)\| = 0 \tag{2.6}$$

Again

$$\begin{aligned}\|\vartheta_{n+1} - z\| &\leq (1-\delta_n)\|\vartheta_n - z\| + \delta_n\|S(\nu_n) - z\| \\ &\leq (1-\delta_n)\|\vartheta_n - z\| + \delta_n\|\nu_n - z\|\end{aligned}$$

which implies

$$\frac{\|\vartheta_{n+1} - z\| - \|\vartheta_n - z\|}{\delta_n} \leq \|\nu_n - z\| - \|\vartheta_n - z\|.$$

Since $\delta_n \in [c, d]$

$$\frac{\|\vartheta_{n+1} - z\| - \|\vartheta_n - z\|}{d} \leq \|\nu_n - z\| - \|\vartheta_n - z\|.$$

Thus

$$r \leq \liminf_{n\to\infty}\|\nu_n - z\|$$

From (2.4), we get

$$\lim_{n\to\infty}\|\nu_n - z\| = r = \lim_{n\to\infty}\|(1-\gamma_n)(\vartheta_n - z) + \gamma_n(S(\vartheta_n) - z)\| \tag{2.7}$$

From (2.3), (2.7) and Lemma 2.6, we get

$$\lim_{n\to\infty}\|\vartheta_n - S(\vartheta_n)\| = 0.$$

□

Lemma 2.8. *Let $\mathcal{B}$ be a uniformly convex Banach space and $\mathcal{C}$ a nonempty closed convex subset of $\mathcal{B}$. Let $S : \mathcal{C} \to \mathcal{C}$ be a quasinonexpansive mapping with $F(S) \neq \varnothing$. Then $F(S)$ is closed.*

Lemma 2.9 (Demiclosedness principle, [4]). *Let $\mathcal{B}$ be a uniformly convex Banach space, $\mathcal{C}$ a closed convex subset of $\mathcal{B}$ and $\xi : \mathcal{C} \to \mathcal{C}$ a mapping with a fixed point. Suppose $\{\vartheta_n\}$ is a sequence in $\mathcal{B}$ such that $\{\vartheta_n\}$ converges weakly to ϑ and $\lim_{n\to\infty} \|\vartheta_n - \xi(\vartheta_n)\| = 0$. Then $\xi(\vartheta) = \vartheta$. That is, $I - \xi$ is demiclosed at zero.*

3. Suzuki-enriched nonexpansive mapping

In this section, we introduce the following new class of mappings:

Definition 3.1. Let $(\mathcal{B}, \|.\|)$ be a Banach space and $\mathcal{C}$ a nonempty subset of $\mathcal{B}$. A mapping $\xi : \mathcal{C} \to \mathcal{C}$ is said to be Suzuki-enriched nonexpansive mapping if there exists $b \in [0, \infty)$ such that for all $\vartheta, \nu \in \mathcal{C}$

$$\frac{1}{2(b+1)}\|\vartheta - \xi(\vartheta)\| \le \|\vartheta - \nu\| \text{ implies} \\ \|b(\vartheta - \nu) + \xi(\vartheta) - \xi(\nu)\| \le (b+1)\|\vartheta - \nu\|. \tag{3.1}$$

It can be seen that every Suzuki-nonexpansive mapping ξ is a Suzuki-enriched nonexpansive mapping with $b = 0$.

Theorem 3.2. *Let $\mathcal{B}$ be a Banach space and $\mathcal{C}$ a nonempty compact convex subset of $\mathcal{B}$. Let $\xi : \mathcal{C} \to \mathcal{C}$ be a mapping satisfying (3.1). For given $\vartheta_1 \in \mathcal{C}$, define a sequence $\{\vartheta_n\}$ in $\mathcal{C}$ by*

$$\vartheta_{n+1} = (1-\lambda)\vartheta_n + \lambda\xi(\vartheta_n) \tag{3.2}$$

for all $n \in \mathbb{N}$, where $\lambda \in \left[\frac{1}{2(b+1)}, \frac{1}{b+1}\right)$. Then $F(\xi) \neq \varnothing$ and $\{\vartheta_n\}$ strongly converges to a point in $F(\xi)$.

Proof. By the definition of mapping ξ, we have

$$\frac{1}{2(b+1)}\|\vartheta - \xi(\vartheta)\| \le \|\vartheta - \nu\| \text{ implies} \\ \|b(\vartheta - \nu) + \xi(\vartheta) - \xi(\nu)\| \le (b+1)\|\vartheta - \nu\|. \tag{3.3}$$

for all $\vartheta, \nu \in \mathcal{C}$. Take $\mu = \frac{1}{b+1} \in (0,1)$ and put $b = \frac{1-\mu}{\mu}$ in (3.3) then the above inequality is equivalent to

$$\frac{1}{2}\mu\|\vartheta - \xi(\vartheta)\| \le \|\vartheta - \nu\| \text{ implies} \\ \|(1-\mu)(\vartheta - \nu) + \mu(\xi(\vartheta) - \xi(\nu))\| \le \|\vartheta - \nu\|. \tag{3.4}$$

Define the mapping S as follows:

$$S(\vartheta) = (1-\mu)\vartheta + \mu\xi(\vartheta) \text{ for all } \vartheta \in \mathcal{C}.$$

Thus

$$\|S(\vartheta) - \vartheta\| = \mu\|\xi(\vartheta) - \vartheta\|. \tag{3.5}$$

Then from (3.4), we get

$$\frac{1}{2}\|\vartheta - S(\vartheta)\| \le \|\vartheta - \nu\| \text{ implies } \|S(\vartheta) - S(\nu)\| \le \|\vartheta - \nu\|$$

for all $\vartheta, \nu \in \mathcal{C}$. Thus S is a mapping satisfying condition (C). Thus all the assumptions of [25, Theorem 2] are satisfied and S has a fixed point in $\mathcal{C}$. From Lemma 2.4, $F(S) = F(\xi) \neq \varnothing$. Next, for given $\vartheta_1 \in \mathcal{C}$ and any $\lambda \in \left[\frac{1}{2}, 1\right)$ consider the sequence

$$\vartheta_{n+1} = (1-\lambda)\vartheta_n + \lambda S(\vartheta_n). \tag{3.6}$$

From [25, Theorem 2], $\{\vartheta_n\}$ strongly converges to a fixed point of S. But $F(S) = F(\xi)$ and

$$(1-\lambda)\vartheta + \lambda S(\vartheta) = (1-\lambda\mu)\vartheta + \lambda\mu\xi(\vartheta)$$

for all $\vartheta \in \mathcal{C}$. Since $\lambda \in \left[\frac{1}{2}, 1\right)$ and $\mu = \frac{1}{b+1}$. This implies that $\lambda\mu \in \left[\frac{1}{2(b+1)}, \frac{1}{b+1}\right)$. Therefore for any $\lambda \in \left[\frac{1}{2(b+1)}, \frac{1}{b+1}\right)$, the sequence $\{\vartheta_n\}$ defined by (3.8) strongly converges to a point in $F(\xi)$. □

Theorem 3.3. *Let $\mathcal{B}$ be a Banach space with the Opial property. Let $\mathcal{C}$ be a nonempty weakly compact convex subset of $\mathcal{B}$ and $\xi : \mathcal{C} \to \mathcal{C}$ a mapping satisfying (3.1). For given $\vartheta_1 \in \mathcal{C}$, define a sequence $\{\vartheta_n\}$ in $\mathcal{C}$ by*

$$\vartheta_{n+1} = (1-\lambda)\vartheta_n + \lambda\xi(\vartheta_n) \tag{3.7}$$

for all $n \in \mathbb{N}$, where $\lambda \in \left[\frac{1}{2(b+1)}, \frac{1}{b+1}\right)$. Then $F(\xi) \neq \varnothing$ and $\{\vartheta_n\}$ weakly converges to a point in $F(\xi)$.

Proof. Following the same proof technique as in Theorem 3.2, we can define a mapping $S : \mathcal{C} \to \mathcal{C}$ as follows:

$$S(\vartheta) = \left(1 - \frac{1}{b+1}\right)\vartheta + \frac{1}{b+1}\xi(\vartheta) \text{ for all } \vartheta \in \mathcal{C}$$

and S is a mapping satisfying condition (C). Then all the assumptions of [23, Theorem 5] are satisfied, hence $\{\vartheta_n\}$ weakly converges to a fixed point of S. But $F(S) = F(\xi)$. This completes the proof. □

Theorem 3.4. *Let $\mathcal{B}$ be a uniformly convex in every direction (or UCED) Banach space and $\mathcal{C}$ a nonempty weakly compact convex subset of $\mathcal{B}$. Let $\xi : \mathcal{C} \to \mathcal{C}$ be a a mapping satisfying (3.1). Then ξ admits a fixed point in $\mathcal{C}$.*

Proof. Following largely the proof of Theorem 3.2, we can define a mapping S satisfying condition (C). Thus all the assumptions of [25, Theorem 5] are satisfied and it is guaranteed that S has at least one fixed point. From Lemma 2.4, $F(S) = F(\xi) \neq \varnothing$. □

Theorem 3.5. *Let $\mathcal{B}$ be a UCED Banach space and $\mathcal{C}$ a nonempty weakly compact convex subset of $\mathcal{B}$. Let G be a family of commuting mappings on $\mathcal{C}$ satisfying (3.1). Then G has a common fixed point.*

Proof. Following the same proof technique of [25, Theorem 6] one can get the desired result. □

Theorem 3.6. *Let $\mathcal{B}$ be a uniformly convex Banach space whose dual $\mathcal{B}^*$ has the Kadec-Klee property. Let $\mathcal{C}$ be a nonempty bounded closed convex subset of $\mathcal{B}$ and $\xi : \mathcal{C} \to \mathcal{C}$ a mapping satisfying (3.1). For given $\vartheta_1 \in \mathcal{C}$, define a sequence $\{\vartheta_n\}$ in $\mathcal{C}$ by*

$$\vartheta_{n+1} = (1-\lambda)\vartheta_n + \lambda\xi(\vartheta_n) \tag{3.8}$$

for all $n \in \mathbb{N}$, where $\lambda \in \left[\frac{1}{2(b+1)}, \frac{1}{b+1}\right)$. Then $F(\xi) \neq \varnothing$ and $\{\vartheta_n\}$ weakly converges to a point in $F(\xi)$.

Proof. Following the same proof technique as in Theorem 3.2, we can define a mapping $S : \mathcal{C} \to \mathcal{C}$ as follows:

$$S(\vartheta) = \left(1 - \frac{1}{b+1}\right)\vartheta + \frac{1}{b+1}\xi(\vartheta) \text{ for all } \vartheta \in \mathcal{C}$$

and S is a mapping satisfying condition (C). Then all the assumptions of [14, Theorem 11] are satisfied, hence $\{\vartheta_n\}$ weakly converges to a fixed point of S. But $F(S) = F(\xi)$. This completes the proof. □

We can obtain the following results due to consequence of Theorem 3.6.

Corollary 3.7. *Let $\mathcal{B}$ be a uniformly convex Banach space having Fréchet differentiable norm. Let $\mathcal{C}$, ξ and $\{\vartheta_n\}$ be same as in Theorem 3.6. Then $F(\xi) \neq \varnothing$ and $\{\vartheta_n\}$ weakly converges to a point in $F(\xi)$.*

4. Enriched-quasinonexpansive mapping

Now, we introduce the following new class of mappings:

Definition 4.1. Let $(\mathcal{B}, \|.\|)$ be a Banach space and $\mathcal{C}$ a nonempty subset of $\mathcal{B}$. A mapping $\xi : \mathcal{C} \to \mathcal{C}$ is said to be b-enriched quasinonexpansive mapping if there exists $b \in [0, \infty)$ such that for all $\vartheta \in \mathcal{C}$ and $\nu \in F(\xi) \neq \varnothing$

$$\|b(\vartheta - \nu) + \xi(\vartheta) - \nu\| \leq (b+1)\|\vartheta - \nu\|. \tag{4.1}$$

Remark 4.2.

- It can be seen that every quasinonexpansive mapping is a 0-enriched quasinonexpansive mapping.
- Every b-enriched nonexpansive mapping with a fixed point is b-enriched quasinonexpansive mapping but the converse need not be true.

We consider the following examples, see [6, Example 6.23].

Example 4.3. Let $\mathcal{B} = \ell^\infty$ and $\mathcal{C} := \{\vartheta \in \ell^\infty : \|\vartheta\|_\infty \leq 1\}$. Define $\xi : \mathcal{C} \to \mathcal{C}$ by

$$\xi(\vartheta) = (0, \vartheta_1^2, \vartheta_2^2, \vartheta_3^2, \dots)$$

for $\vartheta = (\vartheta_1, \vartheta_2, \vartheta_3, \dots) \in \mathcal{C}$. Then it can be seen that ξ is continuous from $\mathcal{C}$ into $\mathcal{C}$ with $p = (0, 0, \dots)$ and $F(\xi) = \{p\}$. Furthermore,

$$\begin{aligned} \|\xi(\vartheta) - p\|_\infty = \|\xi(\vartheta)\|_\infty &= \|(0, \vartheta_1^2, \vartheta_2^2, \vartheta_3^2, \dots)\|_\infty \\ &\leq \|(\vartheta_1, \vartheta_2, \vartheta_3, \dots)\|_\infty = \|\vartheta\|_\infty = \|\vartheta - p\|_\infty \end{aligned}$$

for all $\vartheta \in \mathcal{C}$. Thus, ξ is quasi-nonexpansive mapping and hence 0-enriched quasinonexpansive mapping. However, ξ is not enriched-nonexpansive for any $b \in [0, \infty)$. For, if $\vartheta = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \dots)$ and $\nu = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$, it is clear that $\vartheta, \nu \in \mathcal{C}$. Furthermore, for any $b \in [0, \infty)$

$$\begin{aligned} \|b(\vartheta - \nu) + \xi(\vartheta) - \xi(\nu)\|_\infty &= \left\|\left(\frac{b}{4}, \frac{4b+5}{16}, \frac{4b+5}{16}, \dots\right)\right\|_\infty \\ &= \frac{4b+5}{16} > \frac{b+1}{4} = (b+1)\left\|\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots\right\|_\infty \\ &= (b+1)\|\vartheta - \nu\|_\infty. \end{aligned}$$

Proposition 4.4. *Let $\xi : \mathcal{C} \to \mathcal{C}$ be a Suzuki-enriched nonexpansive mapping with any $b \in [0, \infty)$ and $F(\xi) \neq \varnothing$. Then ξ is a b-enriched quasinonexpansive mapping for any $b \in [0, \infty)$.*

Proof. Let $z \in F(\xi)$ and $\vartheta \in \mathcal{C}$. Since $\frac{1}{2(b+1)}\|z - \xi(z)\| = 0 \leq \|\vartheta - z\|$, we have

$$\|b(\vartheta - \nu) + \xi(\vartheta) - z\| \leq (b+1)\|\vartheta - z\|.$$

□

In the above proposition the inclusion is strict, the following illustrative example [25, Example 2] verifies this fact.

Example 4.5. Let $\mathcal{C} = [0, 3] \subset \mathbb{R}$ and $\xi : \mathcal{C} \to \mathcal{C}$ be a mapping defined as

$$\xi(\vartheta) = \begin{cases} 0, & \text{if } \vartheta \neq 3 \\ 2, & \text{if } \vartheta = 3. \end{cases}$$

Then $F(\xi) = \{0\}$ and ξ is a b-enriched quasinonexpansive mapping for any $b \in [0, \infty)$. On the other hand at $\vartheta = 3$ and $\nu = 4$, $\frac{1}{2(b+1)}\|3 - \xi(3)\| = \frac{1}{2(b+1)} \leq 1 = \|3 - 2\|$, we have

$$\|b(\vartheta - \nu) + \xi(\vartheta) - \xi(\nu)\| = \|b(3-2) + \xi(3) - \xi(2)\| = (b+2) > (b+1) = (b+1)\|\vartheta - \nu\|$$

and ξ is not a Suzuki-enriched nonexpansive mapping for any $b \in [0, \infty)$.

For some fix $\vartheta_1 \in \mathcal{C}$, the Ishikawa iterative method can be defined as follows [13]:

$$\begin{cases} \nu_n = (1 - \beta_n)\vartheta_n + \beta_n \xi(\vartheta_n) \\ \vartheta_{n+1} = (1 - \alpha_n)\vartheta_n + \alpha_n \xi(\nu_n), \end{cases} \tag{4.2}$$

where $\{\beta_n\}$ and $\{\alpha_n\}$ are sequences in $[0, 1]$.

Theorem 4.6. *Let $\mathcal{B}$ be a uniformly convex Banach space and $\mathcal{C}$ a nonempty closed convex subset of $\mathcal{B}$. Let $\xi : \mathcal{C} \to \mathcal{C}$ be a b-enriched quasinonexpansive mapping and ξ satisfies Condition I. For given $\vartheta_1 \in \mathcal{C}$, for all $n \in \mathbb{N}$, $\alpha_n \in (c, d)$, $\beta_n \in \left(\frac{c}{b+1}, \frac{d}{b+1}\right)$ with $c, d \in (0, 1)$, define a sequence $\{\vartheta_n\}$ as follows:*

$$\begin{cases} \nu_n = (1 - \beta_n)\vartheta_n + \beta_n \xi(\vartheta_n) \\ \vartheta_{n+1} = (1 - \alpha_n)\vartheta_n + \alpha_n \left[\left(1 - \frac{1}{b+1}\right)\nu_n + \frac{1}{b+1}\xi(\nu_n)\right]. \end{cases}$$

Then $\{\vartheta_n\}$ strongly converges to a point in $F(\xi)$.

Proof. By the definition of b-enriched quasinonexpansive mapping, we have

$$\|b(\vartheta - \nu) + \xi(\vartheta) - \nu\| \leq (b+1)\|\vartheta - \nu\| \tag{4.3}$$

for all $\vartheta \in \mathcal{C}$ and $\nu \in F(\xi)$. Take $\mu = \frac{1}{b+1} \in (0,1)$ and put $b = \frac{1-\mu}{\mu}$ in (4.3), then the above inequality is equivalent to

$$\|(1-\mu)(\vartheta - \nu) + \mu(\xi(\vartheta) - \nu)\| \leq \|\vartheta - \nu\|. \tag{4.4}$$

Define the mapping S as follows:

$$S(\vartheta) = (1-\mu)\vartheta + \mu\xi(\vartheta) \text{ for all } \vartheta \in \mathcal{C}. \tag{4.5}$$

From Lemma 2.4, $F(S) = F(\xi)$. Then from (4.4), we get

$$\|S(\vartheta) - \nu\| \leq \|\vartheta - \nu\|$$

for all $\vartheta \in \mathcal{C}$ and $\nu \in F(S)$. Thus $S : \mathcal{C} \to \mathcal{C}$ is a quasinonexpansive mapping. For given $\vartheta_1 \in \mathcal{C}$, for all $n \in \mathbb{N}$, $\gamma_n, \alpha_n \in [c,d]$ with $c, d \in (0,1)$, we can define a sequence $\{\vartheta_n\}$ as follows:

$$\begin{cases} \nu_n = (1-\gamma_n)\vartheta_n + \gamma_n S(\vartheta_n) \\ \vartheta_{n+1} = (1-\alpha_n)\vartheta_n + \alpha_n S(\nu_n), \end{cases} \tag{4.6}$$

From Lemma 2.7, $\lim_{n\to\infty} \|\vartheta_n - z\|$ exists for all $p \in F(S)$. Thus $\lim_{n\to\infty} d(\vartheta_n, F(S))$ exists. Since ξ satisfies Condition I

$$\|\vartheta_n - \xi(\vartheta_n)\| = (b+1)\|\vartheta_n - S(\vartheta_n)\| \geq f(d(\vartheta_n, F(\xi))) = f(d(\vartheta_n, F(S))).$$

From Lemma 2.7, $\lim_{n\to\infty} \|\vartheta_n - S(\nu_n)\| = 0$. Thus $\lim_{n\to\infty} d(\vartheta_n, F(S)) = 0$. Following largely the proof of [26, Theorem 3], we can choose a subsequence $\{\vartheta_{n_j}\}$ of $\{\vartheta_n\}$ such that

$$\|\vartheta_{n_j} - p_j\| \leq \frac{1}{2^j}$$

for all $j \in \mathbb{N}$, where $\{p_j\} \subseteq F(\xi)$. It can be easily seen that $\{p_j\}$ is a Cauchy sequence and strongly converges to a point p in $F(\xi)$, since $F(\xi)$ is closed. Therefore $\{\vartheta_n\}$ strongly converges to $p \in F(\xi)$. Using the definition of S, we have

$$\begin{cases} \nu_n = (1-\beta_n)\vartheta_n + \beta_n\xi(\vartheta_n) \\ \vartheta_{n+1} = (1-\alpha_n)\vartheta_n + \alpha_n \left[\left(1 - \frac{1}{b+1}\right)\nu_n + \frac{1}{b+1}\xi(\nu_n)\right] \end{cases}$$

where $\beta_n = \frac{\gamma_n}{b+1}$. This completes the proof. □

Theorem 4.7. *Let $\mathcal{B}$ be a uniformly convex Banach space and $\mathcal{C}$ a nonempty closed convex subset of $\mathcal{B}$. Let $\xi : \mathcal{C} \to \mathcal{C}$ be a continuous and b-enriched quasinonexpansive mapping with $F(\xi) \neq \varnothing$. For given $\vartheta_1 \in \mathcal{C}$, define a sequence $\{\vartheta_n\}$ in $\mathcal{C}$ by*

$$\vartheta_n = \xi^n_{\alpha,\beta}(\vartheta_1), \qquad \xi_{\alpha,\beta} = (1-\alpha)I + \alpha\left[\left(1 - \frac{1}{b+1}\right)I + \frac{1}{b+1}\xi\right][(1-\beta)I + \beta\xi]$$

for all $n \in \mathbb{N}$*, where* $\alpha \in (0,1)$*,* $\beta \in \left[0, \frac{1}{b+1}\right)$ *and* I *is an identity mapping. Then* $\{\vartheta_n\}$ *strongly converges to a point in* $F(\xi)$ *if and only if* $d(\vartheta_n, F(\xi))) \to \infty$ *as* $n \to \infty$*.*

Proof. Following the same proof technique as in Theorem 3.2, we can define a mapping $S : \mathcal{C} \to \mathcal{C}$ as follows:

$$S(\vartheta) = \left(1 - \frac{1}{b+1}\right)\vartheta + \frac{1}{b+1}\xi(\vartheta) \text{ for all } \vartheta \in \mathcal{C}$$

and S is a quasinonexpansive mapping. For given $\vartheta_1 \in \mathcal{C}$, $\alpha \in (0,1)$, $\gamma \in [0,1)$, we can define a sequence $\{\vartheta_n\}$ as follows:

$$\vartheta_n = S^n_{\alpha,\gamma}(\vartheta_1), \qquad S_{\alpha,\gamma} = (1-\alpha)I + \alpha S[(1-\gamma)I + \gamma S]$$

Using the definition of S, we have

$$\vartheta_n = \xi^n_{\alpha,\beta}(\vartheta_1), \qquad \xi_{\alpha,\beta} = (1-\alpha)I + \alpha\left[\left(1 - \frac{1}{b+1}\right)I + \frac{1}{b+1}\xi\right][(1-\beta)I + \beta\xi]$$

where $\beta = \frac{\gamma}{b+1} \in \left[0, \frac{1}{b+1}\right)$. Since ξ is continuous, S is continuous. Then all the assumptions of [9, Theorem 3.1] are satisfied, hence $\{\vartheta_n\}$ strongly converges to a fixed point of S. But $F(S) = F(\xi)$. This completes the proof. □

Theorem 4.8. *Let* $\mathcal{B}$ *be a uniformly convex Banach space and* $\mathcal{C}$ *a nonempty closed convex subset of* $\mathcal{B}$*. Let* $\xi : \mathcal{C} \to \mathcal{C}$ *be a* b*-enriched quasinonexpansive mapping and* $I - \xi$ *is demiclosed at zero. For given* $\vartheta_1 \in \mathcal{C}$*, for all* $n \in \mathbb{N}$*,* $\alpha_n \in (c,d)$*,* $\beta_n \in \left(\frac{c}{b+1}, \frac{d}{b+1}\right)$ *with* $c, d \in (0,1)$*, define a sequence* $\{\vartheta_n\}$ *as follows:*

$$\begin{cases} \nu_n = (1-\beta_n)\vartheta_n + \beta_n\xi(\vartheta_n) \\ \vartheta_{n+1} = (1-\alpha_n)\vartheta_n + \alpha_n\left[\left(1 - \frac{1}{b+1}\right)\nu_n + \frac{1}{b+1}\xi(\nu_n)\right]. \end{cases}$$

Then $\{\vartheta_n\}$ *weakly converges to a point in* $F(\xi)$*.*

Proof. We can define a sequence $\{\vartheta_n\}$ as in (4.6). Since space $\mathcal{B}$ is uniformly convex, $\mathcal{B}$ is reflexive. By the reflexiveness of $\mathcal{B}$ there exists a subsequence $\{\vartheta_{n_j}\}$ of $\{\vartheta_n\}$ such that $\{\vartheta_{n_j}\}$ weakly converges to some $p \in \mathcal{C}$. By Lemma 2.7, $\lim_{n\to\infty} \|\vartheta_n - S(\nu_n)\| = 0$ and

$$\lim_{n\to\infty} \|\vartheta_n - \xi(\vartheta_n)\| = 0.$$

From the demiclosedness principle of $I - \xi$ we have

$$p \in \omega_w(\vartheta_n) \subset F(\xi).$$

Thus, to prove that $\{\vartheta_n\}$ weakly converges to a fixed point of ξ, it suffices to show the unique weak limit for each subsequences of $\{\vartheta_n\}$, that is, $\omega_w(\vartheta_n)$ (cluster points (ω-limit) set of a sequence $\{\vartheta_n\}$) is a singleton. Arguing by contradiction, assume that $\{\vartheta_n\}$ does not converge weakly to p, i.e., take $p, q \in \omega_w(\vartheta_n)$ and let $\{\vartheta_{n_j}\}$ and $\{\vartheta_{m_j}\}$ be subsequences of $\{\vartheta_n\}$ such that $\vartheta_{n_j} \rightharpoonup p$

and $\vartheta_{m_j} \rightharpoonup q$, respectively. If $p \neq q$, and standard application of Opial's property gives us the following contradiction:

$$\begin{aligned} \lim_{n\to\infty} \|\vartheta_n - p\| &= \lim_{j\to\infty} \|\vartheta_{n_j} - p\| < \lim_{j\to\infty} \|\vartheta_{n_j} - q\| \\ &= \lim_{n\to\infty} \|\vartheta_n - q\| = \lim_{j\to\infty} \|\vartheta_{n_j} - q\| \\ &< \lim_{j\to\infty} \|\vartheta_{n_j} - p\| = \lim_{n\to\infty} \|\vartheta_n - p\|. \end{aligned}$$

This completes the proof □

Theorem 4.9. *Let $\mathcal{B}$ be a uniformly convex Banach space and $\mathcal{C}$ a nonempty closed convex subset of $\mathcal{B}$. Let $\xi : \mathcal{C} \to \mathcal{C}$ be a compact b-enriched quasinonexpansive mapping and $I - \xi$ is demiclosed at zero. For given $\vartheta_1 \in \mathcal{C}$, $\alpha_n \in (c, d)$, $\beta_n \in \left(\frac{c}{b+1}, \frac{d}{b+1}\right)$ with $c, d \in (0, 1)$, define a sequence $\{\vartheta_n\}$ as follows:*

$$\begin{cases} \nu_n = (1-\beta_n)\vartheta_n + \beta_n \xi(\vartheta_n) \\ \vartheta_{n+1} = (1-\alpha_n)\vartheta_n + \alpha_n \left[\left(1 - \frac{1}{b+1}\right)\nu_n + \frac{1}{b+1}\xi(\nu_n)\right]. \end{cases}$$

Then $\{\vartheta_n\}$ strongly converges to a point in $F(\xi)$.

Proof. From the proof of Theorem 3.2, we can define a quasinonexpansive mapping S (as in (4.5)). We can define a sequence as in (2.2). From Lemma 2.7, $\lim_{n\to\infty} \|\vartheta_n - S(\nu_n)\| = 0$ and

$$\lim_{n\to\infty} \|\vartheta_n - \xi(\vartheta_n)\| = 0. \tag{4.7}$$

Since the range of $\mathcal{C}$ under ξ is contained in a compact set, there exists a subsequence $\{\xi(\vartheta_{n_j})\}$ of $\{\xi(\vartheta_n)\}$ strongly converges to $\vartheta^\dagger \in \mathcal{C}$. By (4.7), the subsequence $\{\vartheta_{n_j}\}$ strongly converges $\vartheta^\dagger$. By the demiclosedness principle $\xi(\vartheta^\dagger) = \vartheta^\dagger$, and $\{\vartheta_n\}$ strongly converges to a point $\vartheta^\dagger$ in $F(\xi)$. □

Acknowledgements. *The authors are thankful to the reviewer and the editor for their constructive comments. The first author acknowledges the support from the GES 4.0 fellowship, University of Johannesburg, South Africa.*

References

[1] V. Berinde, Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces, Carpathian J. Math. 35 (2019), 293–304.

[2] V. Berinde, Approximating fixed points of enriched nonexpansive mappings in Banach spaces by using a retraction-displacement condition, Carpathian J. Math. 36 (2020), 27–34.

[3] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci. U.S.A. 54 (1965), 1041–1044.

[4] F. E. Browder, Convergence theorems for sequences of nonlinear operators in Banach spaces, Math. Z. 100 (1967), 201–225.

[5] R. E. Bruck, Asymptotic behavior of nonexpansive mappings, Fixed points and nonexpansive mappings (Cincinnati, Ohio, 1982), vol. 18, Contemp. Math., pages 1–47. Amer. Math. Soc., Providence, RI, 1983.

[6] C. Chidume, Geometric properties of Banach spaces and nonlinear iterations, vol. 1965 Lecture Notes in Mathematics, Springer-Verlag London, Ltd., London, 2009.

[7] J. B. Diaz and F. T. Metcalf, On the set of subsequential limit points of successive approximations, Trans. Amer. Math. Soc. 135 (1969), 459–485.

[8] J. García-Falset, E. Llorens-Fuster and T. Suzuki, Fixed point theory for a class of generalized nonexpansive mappings, J. Math. Anal. Appl. 375 (2011), 185–195.

[9] M. K. Ghosh and L. Debnath, Convergence of Ishikawa iterates of quasi-nonexpansive mappings, J. Math. Anal. Appl. 207 (1997), 96–103.

[10] K. Goebel and W. Kirk, Topics in metric fixed point theory, vol. 28, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1990.

[11] K. Goebel and W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35 (1972), 171–174.

[12] D. Göhde, Zum Prinzip der kontraktiven Abbildung, Math. Nachr. 30 (1965), 251–258.

[13] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44 (1974), 147–150.

[14] S. H. Khan and T. Suzuki, A Reich-type convergence theorem for generalized nonexpansive mappings in uniformly convex Banach spaces, Nonlinear Anal. 80 (2013), 211–215.

[15] W. A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly 72 (1965), 1004–1006.

[16] M. A. Krasnosel'skiĭ, Two remarks on the method of successive approximations, Uspehi Mat. Nauk (N.S.) 10 (1955), 123–127.

[17] E. Llorens-Fuster and E. Moreno Gálvez, The fixed point theory for some generalized nonexpansive mappings, Abstr. Appl. Anal. 2011, Art. ID 435686, 15 pp.

[18] M. Maiti and M. K. Ghosh, Approximating fixed points by Ishikawa iterates, Bull. Austral. Math. Soc. 40 (1989), 113–117.

[19] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967), 591–597.

[20] R. Pandey, R. Pant, V. Rakočević and R. Shukla, Approximating fixed points of a general class of nonexpansive mappings in Banach spaces with applications, Results Math. 74 (2019), Paper No. 7, 24 pp.

[21] R. Pant and R. Shukla, Some new fixed point results for nonexpansive type mappings in Banach and Hilbert spaces. Indian J. Math. 62 (2020), 1–20.

[22] H. F. Senter and W. G. Dotson, Jr., Approximating fixed points of nonexpansive mappings, Proc. Amer. Math. Soc. 44 (1974), 375–380.

[23] R. Shukla and R. Pant, Some new fixed point results for monotone enriched nonexpansive mappings in ordered Banach spaces, Adv. Theory Nonlinear Anal. Appl. 5 (2021), 559–567.

[24] R. Shukla and A. Wiśnicki, Iterative methods for monotone nonexpansive mappings in uniformly convex spaces, Adv. Nonlinear Anal. 10 (2021), 1061–1070.

[25] T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl. 340 (2008), 1088–1095.

[26] K. K. Tan and H. K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993), 301–308.

[27] E. Zeidler, Nonlinear functional analysis and its applications. I. Fixed-point theorems, Springer-Verlag, New York, 1986.

