

## Fixed point results of enriched interpolative Kannan type operators with applications

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### ABSTRACT

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*The purpose of this paper is to introduce the class of enriched interpolative Kannan type operators on Banach space that contains the classes of enriched Kannan operators, interpolative Kannan type contraction operators and some other classes of nonlinear operators. Some examples are presented to support the concepts introduced herein. A convergence theorem for the Krasnoselskii iteration method to approximate fixed point of the enriched interpolative Kannan type operators is proved. We study well-posedness, Ulam-Hyers stability and periodic point property of operators introduced herein. As an application of the main result, variational inequality problem is solved.*

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## 1. INTRODUCTION AND PRELIMINARIES

Let  $(X, d)$  be a metric space and  $T : X \rightarrow X$ . We denote the set  $\{x \in X : Tx = x\}$  of fixed points of  $T$  by  $Fix(T)$ . Solving a fixed point problem of an operator  $T$ , denote by  $FPP(T)$  is to show that the set  $Fix(T)$  is nonempty. Define  $T^0 = I$  (the identity map on  $X$ ) and  $T^n = T^{n-1} \circ T$ , called the  $n^{th}$ -iterate of  $T$  for  $n \geq 1$ . The most simplest iteration procedure to approximate the solution of a fixed point equation  $Tx = x$  is the method of successive approximations (or Picard iteration) given by

$$(1.1) \quad x_n = T^n x_0, \quad n = 1, 2, \dots,$$

where  $x_0$  is an initial guess in domain of an operator  $T$ .

A sequence  $\{x_n\}_{n=0}^{\infty}$  in  $X$  given by (1.1) is called a  $T$ -orbital sequence around  $x_0$ . The collection of all such sequences is denoted by  $O(T, x_0)$ . If there exists  $x^*$  in  $X$  such that  $Fix(T) = \{x^*\}$  and the Picard iteration associated with  $T$  converges to  $x^*$  for any initial guess  $x_0$  in  $X$ , then  $T$  is called a Picard operator (see [1, 7, 40]).

If there exists  $k \in [0, 1)$  such that for any  $x, y \in X$ , we have

$$(1.2) \quad d(Tx, Ty) \leq k d(x, y).$$

Then  $T$  is called a Banach contraction operator which, if defined on a complete metric space, is a classical example of a continuous Picard operator. Thus, it was natural to ask the question whether in the framework of an complete metric space, a discontinuous operator satisfying somewhat similar contractive conditions is a Picard operator. This was answered in an affirmative by Kannan [26] in 1968.

An operator  $T$  on  $X$  is called Kannan contraction operator if there exists  $a \in [0, 0.5)$  such that for any  $x, y \in X$ , we have

$$(1.3) \quad d(Tx, Ty) \leq a\{d(x, Tx) + d(y, Ty)\}.$$

Kannan contraction operator defined on a complete metric space is an example of a discontinuous Picard operator ([25], [26]).

Subrahmanyam [41] proved that a metric space  $X$  is complete if and only if every Kannan contraction operator on  $X$  has a fixed point. Moreover, Connell [18] gave an example of an incomplete metric space  $X$  on which every Banach contraction operator has a fixed point. This shows that the Banach contraction operators do not characterize the completeness of their domain. As long as contractive conditions are concerned, the classes of Banach and Kannan contraction operators are incomparable ([27]), but the class of Kannan contraction operators attracted the attention of several mathematicians because of its connection with a characterization of its metric completeness.

Kannan's theorem has been generalized in different ways by many authors to extend the limits of metric fixed point theory in different directions.

Karapinar introduced a new class of Kannan type operators called interpolative

Kannan type operators and proved a fixed point result for such operators in the setup of complete metric spaces ([28]).

An operator  $T : X \rightarrow X$  is called an interpolative Kannan type if there exists  $a \in [0, 1)$  such that for all  $x, y \in X \setminus \text{Fix}(T)$ , we have

$$(1.4) \quad d(Tx, Ty) \leq a[d(x, Tx)]^\alpha [d(y, Ty)]^{1-\alpha},$$

where  $\alpha \in (0, 1)$ .

The main result in [28] is stated as follows.

**Theorem 1.1** ([28]). *Let  $(X, d)$  be a complete metric space and  $T$  an interpolative Kannan type operator. Then  $T$  is a Picard operator.*

For more results in this direction, we refer to [6, 8, 9, 19, 20, 21, 22, 29, 30, 31, 32, 33, 34, 35, 36, 38] and references mentioned therein.

Existence, uniqueness, stability, approximation and characterization of fixed points of certain operators are some of the main concerns of a metric fixed point theory. Contractive conditions on operators and distance structure of operator's domain play a vital role to ensure the convergence of iterative methods.

If for any  $x, y \in X$ , we have

$$d(Tx, Ty) \leq d(x, y).$$

Then  $T$  is called a nonexpansive operator. An operator  $T$  on  $X$  is called asymptotically regular on  $X$  if for all  $x$  in  $X$ ,

$$d(T^{n+1}x, T^n x) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The concept of asymptotic regularity plays an important role in approximating the fixed points of operators. Picard iteration method fails to converge to a fixed point of certain contractive mappings such as nonexpansive mappings on metric spaces. This led to the study of a variety of fixed point iteration procedures in the setup of Banach spaces.

In this paper, we shall approximate the fixed point of some nonlinear mappings through Krasnoselskii iteration method.

Let  $\lambda \in [0, 1]$ . A sequence  $\{x_n\}_{n=0}^\infty$  given by

$$(1.5) \quad x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \quad n = 0, 1, 2, \dots$$

is called the Krasnoselskii iteration.

Note that, Krasnoselskii iteration  $\{x_n\}_{n=0}^\infty$  sequence given by (1.5) is exactly the Picard iteration corresponding to an averaged operator

$$(1.6) \quad T_\lambda = (1 - \lambda)I + \lambda T.$$

Moreover, for  $\lambda = 1$  the Krasnoselskii iteration method reduces to Picard iteration method. Also,  $\text{Fix}(T) = \text{Fix}(T_\lambda)$ , for all  $\lambda \in (0, 1]$ .

On the other hand, Browder and Petryshyn [16] introduced the concept of asymptotic regularity in connection with the study of fixed points of nonexpansive mappings. As a matter of fact, the same property was used in 1955 by

Krasnoselskii [37] to prove that if  $K$  is a compact convex subset of a uniformly convex Banach space and  $T : K \rightarrow K$  is a nonexpansive mapping, then for any  $x_0 \in K$ , the sequence

$$(1.7) \quad x_{n+1} = \frac{1}{2}(x_n + Tx_n), \quad n \geq 0,$$

converges to fixed point of  $T$ .

In proving the above result, Krasnoselskii used the fact that if  $T$  is nonexpansive which, in general, is not asymptotically regular, then the averaged mapping  $T_{\frac{1}{2}}$  is asymptotically regular.

Therefore, an averaged operator  $T_\lambda$  enriches the class of nonexpansive mappings with respect to the asymptotic regularity. This observation suggested that one could enrich the classes of contractive mappings studied in the framework of metric spaces by imposing certain contractive condition on  $T_\lambda$  instead of  $T$  itself.

Employing this approach, the classes such as enriched contractions and enriched  $\phi$ -contractions [11], enriched Kannan contractions [12], enriched Chatterjea mappings [14], enriched nonexpansive mappings in Hilbert spaces [13], enriched multivalued contractions [3] and enriched Ćirić-Reich-Rus contraction [15], enriched cyclic contraction [5], enriched modified Kannan pair [2], enriched quasi contraction [4] were introduced and studied.

Abbas et al. [3] proved fixed point results by imposing the condition that  $T_\lambda$ -orbital subset is a complete subset of a normed space (see, Theorem 3 of [3]). Similarly, Górnicki and Bisht [24] considered the enriched Ćirić-Reich-Rus contraction operators and proved a fixed point theorem by imposing the condition that  $T_\lambda$  is asymptotically regular mapping (see, Theorem 3.1 of [24]).

Consistent with [12], let  $(X, \|\cdot\|)$  be a normed space. A operator  $T : X \rightarrow X$  is called an enriched Kannan contraction or  $(b, a)$ -enriched Kannan contraction if there exist  $b \in [0, \infty)$  and  $a \in [0, 0.5)$  such that for all  $x, y \in X$ , the following holds:

$$(1.8) \quad \|b(x - y) + Tx - Ty\| \leq a(\|x - Tx\| + \|y - Ty\|).$$

As shown in [12], several well-known contractive conditions in the existing literature imply the  $(b, a)$ -enriched Kannan contraction condition. It was proved in [12] that any enriched Kannan contraction operator defined on a Banach space has a unique fixed point which can be approximated by means of the Krasnoselskii iterative scheme.

Motivated by the work of Berinde and Păcurar [12], Abbas et al. [3] and Górnicki and Bisht [24], we propose a new class of enriched interpolative Kannan type operators. The purpose of this paper is to prove the existence of fixed point of such operators. Moreover, we study the well-posedness, Ulam-Hyers stability and periodic point property of the operators introduced herein. Finally, an application of our result to solve variational inequality problems is also given.

2. FIXED POINT APPROXIMATION OF ENRICHED INTERPOLATIVE KANNAN TYPE OPERATORS

In the sequel, the notations  $\mathbb{N}$  and  $\mathbb{R}$  will denote the set of all natural numbers and the set of all real numbers, respectively.

In this section, we present a new class of enriched interpolative Kannan type operators, which is first of its kind in the existing literature on metric fixed-point theory. Existence and convergence results of such class of operators are also obtained.

First, we introduce the following concept

**Definition 2.1.** Let  $(X, \|\cdot\|)$  be a normed space. A mapping  $T : X \rightarrow X$  is called enriched interpolative Kannan type operator if there exist  $b \in [0, \infty)$ ,  $a \in [0, 1)$  and  $\alpha \in (0, 1)$  such that for all  $x, y \in X$ , we have

$$(2.1) \quad \|b(x - y) + (Tx - Ty)\| \leq a(\|x - Tx\|)^\alpha (\|y - Ty\|)^{1-\alpha}.$$

To highlight an involvement of parameters  $a, b$  and  $\alpha$  in (2.1), we call  $T$  a  $(a, b, \alpha)$ -enriched interpolative Kannan type operator.

**Example 2.2.** Any interpolative Kannan type operator  $T$  satisfying (1.4) is a  $(a, 0, \alpha)$ -enriched interpolative Kannan contraction operator, that is,  $T$  satisfies (2.1) with  $b = 0$ .

We now give an example of an enriched interpolative Kannan type operator which is not a interpolative Kannan type contraction operator.

**Example 2.3.** Let  $(Y, \mu)$  be a finite measure space. The classical Lebesgue space  $X = L^2(Y, \mu)$  is defined as the collection of all Borel measurable functions  $f : Y \rightarrow \mathbb{R}$  such that  $\int_Y |f(y)|^2 d\mu(y) < \infty$ . We know that the space  $X$  equipped

with the norm  $\|f\|_X = \left(\int_Y |f|^2 d\mu\right)^{\frac{1}{2}}$  is a Banach space.

Define the operator  $T : L^2(Y, \mu) \rightarrow L^2(Y, \mu)$  by

$$Tf = g - 3f,$$

where  $g(y) = 1, \forall y \in Y$ . Clearly,  $g \in L^2(Y, \mu)$  as  $\mu(Y) < \infty$ .

Note that  $T$  is an  $(0.5, 3, 0.5)$ -enriched interpolative Kannan type operator but not an interpolative Kannan type contraction operator.

Indeed, if  $T$  would be a interpolative Kannan type operator then, by (1.4), there would exist  $a \in [0, 1)$  and  $\alpha \in (0, 1)$  such that

$$\|-3f + 3h\|_X \leq a(\|4f - g\|_X)^\alpha (\|4h - g\|_X)^{1-\alpha} \quad \forall f, h \in X,$$

which on taking  $f(y) = 0$  and  $h(y) = 1$ , for all  $y \in Y$  gives  $3^\alpha \leq a < 1$ , a contradiction.

We now present the following result.

**Theorem 2.4.** *Let  $(X, \|\cdot\|)$  be a normed space,  $T : X \rightarrow X$  a  $(a, b, \alpha)$ -enriched interpolative Kannan type mapping.*

*Then,*

- (1)  $Fix(T) = \{x^*\}$ ;
- (2) *There exist a  $T_\lambda$ -orbital sequence  $\{x_n\}_{n=0}^\infty$  around  $x_0$ , given by*

$$(2.2) \quad x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n; \quad n \geq 0,$$

*converges to  $x^*$  provided that, for  $x_0 \in X$ ,  $T_\lambda$ -orbital subset  $O(T_\lambda, x_0)$  is a complete subset of  $X$ , where  $\lambda = \frac{1}{b+1}$ .*

*Proof.* We divide the proof into the following two cases.

Case 1. If  $b > 0$ . Then  $\lambda = \frac{1}{b+1} \in (0, 1)$  and the enriched interpolative Kannan type operator (2.1) satisfies the following contraction condition:

$$\left\| \left( \frac{1}{\lambda} - 1 \right) (x - y) + Tx - Ty \right\| \leq a \|x - Tx\|^\alpha \|y - Ty\|^{1-\alpha}$$

and hence

$$\|(1 - \lambda)(x - y) + \lambda(Tx - Ty)\| \leq a\lambda \|x - Tx\|^\alpha \|y - Ty\|^{1-\alpha}$$

which can be written in an equivalent form as follows:

$$(2.3) \quad \|T_\lambda x - T_\lambda y\| \leq a \|x - T_\lambda x\|^\alpha \|y - T_\lambda y\|^{1-\alpha}, \quad \forall x, y \in X.$$

In view of (1.6), the Krasnoselskii iterative sequence defined by (2.2) is exactly the Picard's iteration associated with  $T_\lambda$ , that is,

$$(2.4) \quad x_{n+1} = T_\lambda x_n, \quad n \geq 0.$$

Take  $x := x_n$  and  $y := x_{n-1}$  in (2.3) to get

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|T_\lambda x_n - T_\lambda x_{n-1}\| \\ &\leq a \|x_n - T_\lambda x_n\|^\alpha \|x_{n-1} - T_\lambda x_{n-1}\|^{1-\alpha} \\ &\leq a \|x_n - x_{n+1}\|^\alpha \|x_{n-1} - x_n\|^{1-\alpha} \end{aligned}$$

which implies that

$$(2.5) \quad \|x_{n+1} - x_n\|^{1-\alpha} \leq a \|x_{n-1} - x_n\|^{1-\alpha}.$$

As  $\alpha \in (0, 1)$ ,

$$(2.6) \quad \|x_{n+1} - x_n\| \leq a \|x_{n-1} - x_n\|.$$

Inductively, we obtain that

$$(2.7) \quad \|x_{n+1} - x_n\| \leq a^n \|x_0 - x_1\|.$$

By (2.7) and triangular inequality, we have

$$(2.8) \quad \|x_n - x_{n+r}\| \leq \frac{a^n}{1 - a} \|x_0 - x_1\|, \quad r \in \mathbb{N}, \quad n \geq 1,$$

which, in view of  $0 < a < 1$  gives that  $\{x_n\}_{n=0}^\infty$  is a Cauchy sequence in the complete subset  $O(T_\lambda, x_0)$  of  $X$ .

Next, we assume that there exists an element  $x^*$  in  $O(T_\lambda, x_0)$  such that  $\lim_{n \rightarrow \infty} x_n = x^*$ . Note that

$$\begin{aligned} \|x^* - T_\lambda x^*\| &\leq \|x^* - x_{n+1}\| + \|x_{n+1} - T_\lambda x^*\| \\ &\leq \|x^* - x_{n+1}\| + \|T_\lambda x_n - T_\lambda x^*\| \\ &\leq \|x^* - x_{n+1}\| + a \|x_n - T_\lambda x_n\|^\alpha \|x^* - T_\lambda x^*\|^{1-\alpha} \\ &\leq \|x^* - x_{n+1}\| + a \|x_n - x_{n+1}\|^\alpha \|x^* - T_\lambda x^*\|^{1-\alpha}. \end{aligned}$$

On taking the limit as  $n \rightarrow \infty$  on both sides of the above inequality, we get that  $x^* = T_\lambda x^*$ .

Assume that  $x^*$  and  $y^*$  are two fixed points of  $T$ . Then from (2.3), we have

$$\begin{aligned} \|x^* - y^*\| &= \|T_\lambda x^* - T_\lambda y^*\| \leq a \|x^* - T_\lambda x^*\|^\alpha \|y^* - T_\lambda y^*\|^{1-\alpha} \\ &\leq a \|x^* - x^*\|^\alpha \|y^* - y^*\|^{1-\alpha}, \end{aligned}$$

which, gives  $x^* = y^*$ .

Case 2.  $b = 0$ . In this case, the enriched interpolative Kannan type operator (2.1) becomes

$$(2.9) \quad \|Tx - Ty\| \leq a \|x - Tx\|^\alpha \|y - Ty\|^{1-\alpha} \quad \forall x, y \in X,$$

where  $a \in (0, 1)$ . That is,  $T$  is an interpolative Kannan type contraction operator and hence by Theorem 2.2 of [28],  $T$  has a unique fixed point.  $\square$

The following example illustrate the above theorem.

**Example 2.5.** Let  $X = \mathbb{R} \setminus \{\frac{1}{5}, \frac{4}{5}\}$  be endowed with the usual norm and  $T : X \rightarrow X$  be defined by  $Tx = 1 - x, \forall x \in X$ . Clearly  $T$  is a  $(1, 1)$ -enriched interpolative Kannan type operator. Now  $\lambda = \frac{1}{1+b}$  gives that  $\lambda = \frac{1}{2}$ . Let  $x_0 = 1/2$  be the fixed in  $X$ . Then, we have

$$T_{\frac{1}{2}}(x_0) = \frac{1}{2}x_0 + \frac{1}{2}Tx_0 = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{2}.$$

Pick  $x_1 = T_{\frac{1}{2}}x_0 = 1/2$ . Continuing this way, we obtained  $x_{n+1} = T_{\frac{1}{2}}x_n$ , where  $x_{n+1} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$ . Note that  $\{x_n\}_{n=0}^\infty$  is a Cauchy sequence which converges to  $1/2$  and  $1/2$  is the fixed point of  $T$ .

We now present the following fixed point theorem for  $(a, b, \alpha)$ -enriched interpolative Kannan type operator on a Banach space.

**Corollary 2.6.** *Let  $(X, \|\cdot\|)$  be a Banach space and  $T : X \rightarrow X$  a  $(a, b, \alpha)$ -enriched interpolative Kannan type operator. Then  $T$  has a unique fixed point.*

*Proof.* Following arguments similar to those in proof Theorem 2.4, the result follows.  $\square$

If we take  $b = 0$  in the Corollary 2.6, we obtain Theorem 2.2 of [28] in the setting of Banach spaces.

**Corollary 2.7** ([28]). *Let  $T$  be an interpolative Kannan type operator on a Banach space  $(X, \|\cdot\|)$ . Then  $T$  is a Picard operator.*

By Corollary 2.6, we obtain the following result.

**Corollary 2.8** ([12]). *Let  $(X, \|\cdot\|)$  be a Banach space and  $T : X \rightarrow X$  an  $(b, a)$ -enriched Kannan contraction, that is, for all  $x, y \in X$ , it satisfies the following inequality;*

$$(2.10) \quad \|b(x - y) + Tx - Ty\| \leq a\{ \|x - Tx\| + \|y - Ty\| \}$$

with  $b \in [0, \infty)$  and  $a \in [0, 0.5)$ . Then  $T$  has a unique fixed point.

*Proof.* Take  $\lambda = \frac{1}{b+1}$ . Obviously,  $0 < \lambda < 1$  and the  $(b, a)$ -enriched Kannan contraction condition (2.10) becomes

$$\begin{aligned} & \left\| \left( \frac{1}{\lambda} - 1 \right) (x - y) + Tx - Ty \right\| \\ & \leq a\{ \|x - Tx\| + \|y - Ty\| \}, \quad \forall x, y \in X, \end{aligned}$$

which can be written in an equivalent form as follows;

$$(2.11) \quad \|T_\lambda x - T_\lambda y\| \leq a\{ \|x - T_\lambda x\| + \|y - T_\lambda y\| \}, \quad \forall x, y \in X.$$

By (2.11),  $T_\lambda$  is a Kannan contraction. It follows from [28] that  $T_\lambda$  satisfies condition (2.11) and condition (2.3). Since, for  $\lambda = \frac{1}{b+1}$ , the inequality (2.3) is same as the condition (2.1). This suggests that  $T$  is an enriched interpolative Kannan type operator and then the Corollary 2.6 leads to the conclusion.  $\square$

### 3. WELL-POSEDNESS, PERIODIC POINT AND ULAM-HYERS STABILITY RESULTS

We now present the well-Posedness, perodic point and Ulam-Hyers stability results for  $(a, b, \alpha)$ -enriched interpolative Kannan type operators.

#### 3.1. Well-Posedness.

Let us start with the following definition.

**Definition 3.1** ([39]). Let  $(X, d)$  be a metric space and  $T : X \rightarrow X$ . The fixed point problem  $FPP(T)$  is said to be well-posed if  $T$  has unique fixed point  $x^*$  (say) and for any sequence  $\{x_n\}$  in  $X$  satisfying  $\lim_{n \rightarrow \infty} d(Tx_n, x_n) = 0$ , we have  $\lim_{n \rightarrow \infty} x_n = x^*$ .

Since  $Fix(T) = Fix(T_\lambda)$ , we conclude that the fixed point problem of  $T$  is well-posed if and only if the fixed point problem of  $T_\lambda$  is well-posed.

Well-posedness of certain fixed point problems has been studied by several mathematicians, see for example, [10], [39] and references mentioned therein.

We now study the well-posedness of a fixed point problem of mappings in Theorem 2.4 and Corollary 2.6.

**Theorem 3.2.** *Let  $(X, \|\cdot\|)$  be a Banach space. Suppose that  $T$  is an operator on  $X$  as in the Theorem 2.4. Then,  $FPP(T)$  is well-posed.*



*Proof.* It follows from Theorem 2.4 that  $x^*$  is the unique fixed point of  $T$ . Suppose that  $\lim_{n \rightarrow \infty} \|T_\lambda x_n - x_n\| = 0$ . Using (2.3) we have,

$$\begin{aligned} \|x_n - x^*\| &\leq \|x_n - T_\lambda x_n\| + \|T_\lambda x_n - x^*\| \\ &= \|x_n - T_\lambda x_n\| + \|T_\lambda x_n - T_\lambda x^*\| \\ &\leq \|x_n - T_\lambda x_n\| + a \|x_n - T_\lambda x_n\|^\alpha \|x^* - T_\lambda x^*\|^{1-\alpha} \end{aligned}$$

that is,

$$(3.1) \quad \|x_n - x^*\| \leq \|x_n - T_\lambda x_n\|.$$

It follows from (3.1) that  $\lim_{n \rightarrow \infty} x_n = x^*$  provided that  $\lim_{n \rightarrow \infty} \|T_\lambda x_n - x_n\| = 0$ . This complete the proof.  $\square$

**Corollary 3.3.** *Let  $(X, \|\cdot\|)$  be Banach space. Suppose that  $T$  is an operator on  $X$  as in the Corollary 2.6. Then the fixed point problem is well-posed.*

*Proof.* Following arguments similar to those in the proof Theorem 3.2, the result follows.  $\square$

**3.2. Periodic Point Result.** Clearly, a fixed point  $x^*$  of  $T$  is also a fixed point of  $T^n$  for every  $n \in \mathbb{N}$ . However, the converse is false. For example, if we take,  $X = [0, 1]$  and define an operator  $T$  on  $X$  by  $Tx = 1 - x$ . Then  $T$  has a unique fixed point  $1/2$ , and for each even integer  $n$ ,  $n^{th}$ -iterate of  $T$  is an identity map and hence every point of  $[0, 1]$  is a fixed point of  $T^n$ . Also, if  $X = [0, \pi]$ ,  $Tx = \cos x$ , then every iterate of  $T$  has the same fixed point as  $T$ . If a map  $T$  satisfies  $Fix(T) = Fix(T^n)$  for each  $n \in \mathbb{N}$ , then it is said to have a periodic point property  $P$  [23].

Since  $Fix(T) = Fix(T_\lambda)$ , we conclude that the mapping  $T$  has property  $P$  if and only the mapping  $T_\lambda$  has property  $P$ .

**Theorem 3.4.** *Let  $(X, \|\cdot\|)$  be a Banach space. Suppose that  $T$  is an operator on  $X$  as in the Theorem 2.4. Then  $T$  has property  $P$ .*

*Proof.* From Theorem 2.4,  $T$  has a fixed point. Let  $y^* \in Fix(T^n)$ . Now from (2.3), we have

$$\begin{aligned} \|y^* - T_\lambda y^*\| &= \|T_\lambda^n y^* - T_\lambda(T_\lambda^n y^*)\| \\ &= \|T_\lambda(T_\lambda^{n-1} y^*) - T_\lambda(T_\lambda^n y^*)\| \\ &\leq a \|T_\lambda^{n-1} y^* - T_\lambda^n y^*\|^\alpha \|T_\lambda^n y^* - T_\lambda^{n+1} y^*\|^{1-\alpha}, \end{aligned}$$

that is,

$$(3.2) \quad \|T_\lambda^n y^* - T_\lambda^{n+1} y^*\|^\alpha \leq a \|T_\lambda^{n-1} y^* - T_\lambda^n y^*\|^\alpha.$$

Since  $\alpha \in (0, 1)$ , then (3.2) becomes

$$\begin{aligned} \|y^* - T_\lambda y^*\| &= \|T_\lambda^n y^* - T_\lambda^{n+1} y^*\| \leq a \|T_\lambda^{n-1} y^* - T_\lambda^n y^*\| \leq \\ &\leq a^2 \|T_\lambda^{n-2} y^* - T_\lambda^{n-1} y^*\| \leq \dots \leq a^n \|y^* - T_\lambda y^*\|. \end{aligned}$$

Now,  $0 \leq a < 1$  implies that  $\|y^* - T_\lambda y^*\| = 0$  and hence  $y^* = Ty^*$ .  $\square$

**3.3. Ulam-Hyers Stability.** Let  $(X, d)$  be a metric space,  $T : X \rightarrow X$  and  $\epsilon > 0$ . A point  $w^* \in X$  called an  $\epsilon$ -solution of the fixed point problem  $FPP(T)$  if  $w^*$  satisfies the following inequality

$$d(w^*, Tw^*) \leq \epsilon.$$

Let us recall the notion of Ulam-Hyers stability.

**Definition 3.5** ([42]). Let  $(X, d)$  be a metric space,  $T : X \rightarrow X$  and  $\epsilon > 0$ . The fixed point problem  $FPP(T)$  is called generalized Ulam-Hyers stable if and only if there exists an increasing and continuous function  $\phi : [0, \infty) \rightarrow [0, \infty)$  with  $\phi(0) = 0$  such that for each  $\epsilon$ -solution  $w^* \in X$  of the fixed point equation  $Tx = x$ , there exists a solution  $x^*$  of  $Tx = x$  in  $X$  such that

$$d(x^*, w^*) \leq \phi(\epsilon).$$

*Remark 3.6.* If the function  $\phi$  in the above definition is given by  $\phi(t) = mt$  for all  $t \geq 0$ , where  $m > 0$ , then the fixed point equation  $Tx = x$  is said to be Ulam-Hyers stable.

**Theorem 3.7.** Let  $(X, \|\cdot\|)$  be a Banach space. Suppose that  $T$  is an operator on  $X$  as in the Theorem 2.4. Then the fixed point problem is Ulam-Hyers stable.

*Proof.* Since  $Fix(T) = Fix(T_\lambda)$ , it follows that the fixed point problem  $Tx = x$  is equivalent to the fixed point problem

$$(3.3) \quad x = T_\lambda x.$$

Let  $w^*$  be  $\epsilon$ -solution of the fixed point equation (3.3), that is,

$$(3.4) \quad d(w^*, T_\lambda w^*) \leq \epsilon.$$

Using (2.3) and (3.4), we get

$$\begin{aligned} \|x^* - w^*\| &= \|T_\lambda x^* - w^*\| \leq \|T_\lambda x^* - T_\lambda w^*\| + \|T_\lambda w^* - w^*\| \\ &\leq a \|x^* - T_\lambda x^*\|^\alpha \|w^* - T_\lambda w^*\|^{1-\alpha} + \epsilon \\ &= \epsilon. \end{aligned}$$

□

#### 4. APPLICATION TO VARIATIONAL INEQUALITY PROBLEM

Variational inequality theory provides some important tools to handle the problems arising in economic, engineering, mechanics, mathematical programming, transportation and others. Many numerical methods have been constructed for solving variational inequalities and optimization problems. The aim of this section is to present generic convergence theorems for Krasnoselskii type algorithms that solve variational inequality problems.

Let  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ ,  $C \subset H$  a closed and convex set and  $S : H \rightarrow H$ .

The variational inequality problem with respect to  $S$  and  $C$ , denoted by  $VIP(S, C)$ , is to find  $x^* \in C$  such that

$$\langle Sx^*, x - x^* \rangle \geq 0, \forall x \in H.$$

It is well known [17] that if  $\gamma > 0$ , then  $x^* \in C$  is a solution of  $VIP(S, C)$  if and only if  $x^*$  is a solution of fixed point problem of  $P_C \circ (I - \gamma S)$ , where  $P_C$  is the nearest point projection onto  $C$ .

Amongst many others results in [17], it was proved that if  $I - \gamma S$  and  $P_C \circ (I - \gamma S)$  are averaged nonexpansive operators, then, under some additional assumptions, the iterative algorithm  $\{x_n\}_{n=0}^\infty$  defined by

$$x_{n+1} = P_C(I - \gamma S)x_n, n \geq 0,$$

converges weakly to a solution of the  $VIP(S, C)$ , if such solutions exist.

In the case of averaged nonexpansive mappings, the problem of replacing the weak convergence in the above result with strong convergence has received a much attention of researchers.

Our alternative is to consider  $VIP(S, C)$  for enriched interpolative Kannan type contraction operators, which are in general discontinuous operators in contrast of nonexpansive operators which are always continuous.

In this case, we shall have  $VIP(S, C)$  with a unique solution, as shown in the next theorem. Moreover, the considered algorithm (4.1) will converge strongly to the solution of the  $VIP(S, C)$ .

**Theorem 4.1.** *Assume that for  $\gamma > 0$ ,  $P_C(I - \gamma S)$  is enriched interpolative Kannan type contraction operator. Then there exists  $\lambda \in (0, 1]$  such that the iterative algorithm  $\{x_n\}_{n=0}^\infty$  defined by*

$$(4.1) \quad x_{n+1} = (1 - \lambda)x_n + \lambda P_C(I - \gamma S)x_n, n \geq 0,$$

*converges strongly to the unique solution  $x^*$  of the  $VIP(S, C)$  for any  $x_0 \in C$ .*

*Proof.* Since  $C$  is closed, we take  $X := C$  and  $T := P_C(I - \gamma S)$  and apply Corollary 2.6. □

**Example 4.2.** Let  $X = \mathbb{R}^2$  and for any  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$  in  $X$ , the inner product is defined by

$$\langle x, y \rangle = x_1y_1 + x_2y_2.$$

Then  $X$  equipped with the above inner product is a Hilbert space. The above inner product gives the norm given by

$$\|x\| = (\langle x, x \rangle)^{1/2}.$$

Define  $S : X \rightarrow X$  by

$$S(x) = \frac{(1, 0) + x}{\gamma} \quad \forall x \in X,$$

where  $\gamma > 0$  be fixed real number.  
 For a mapping  $P_C : X \rightarrow C$  defined by

$$P_C(x) = \begin{cases} \frac{x}{\|x\|} & ; x \notin C \\ x & ; x \in C, \end{cases}$$

where  $C = \{x \in X : \|x\| \leq 1\}$ , it is easy to check that  $P_C(I - \gamma S)$  is a  $(a, 0, \alpha)$  enriched interpolative Kannan type contraction.

By Corollary 2.6,  $P_C(I - \gamma S)$  has a unique solution, which in turns a solution for  $VIP(S, C)$ .

## 5. CONCLUSIONS

- (1) We introduced a large class of contractive operators, called enriched interpolative Kannan type operators, that includes usual interpolative Kannan type operators and enriched Kannan operators.
- (2) We presented examples to show that the class of enriched interpolative Kannan type operators strictly includes the interpolative Kannan type operator in the sense that there exist operators which are not interpolative Kannan type and belong to the class of enriched interpolative Kannan type operators.
- (3) We studied the set of fixed point (Theorem 2.4) and constructed an algorithm of Krasnoselskii type in order to approximate fixed point of enriched interpolative Kannan type operators and we proved a strong convergence theorem.
- (4) We also obtained Theorems 3.2, 3.4 and 3.7 for a well-posedness, periodic point and Ulam-Hyers stability problem of the fixed point problem for enriched interpolative Kannan type operators, respectively.
- (5) As application of our main results (Corollary 2.6), we presented two Krasnoselskii projection type algorithms for solving variational inequality problems for the class of enriched interpolative Kannan type operators, thus improving the existence and weak convergence results for variational inequality problems in [17] to existence and uniqueness as well as to strong convergence theorems.

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