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⋆-quasi-pseudometrics on algebraic structures

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Abstract

In this paper, we introduce some concepts of ⋆-(quasi)-pseudometric spaces, and give an example which shows that there is a ⋆-quasi-pseudometric space which is not a quasi-pseudometric space. We also study the conditions under which ⋆-quasi-pseudometric semitopological groups are paratopological groups or topological groups.

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1. Introduction

Finding a stronger topological structure is one of the central problems in topological algebra. In 1957, R. Ellis showed that Every locally compact Hausdorff semitopological group is a topological group [3]. In 1960, W. Zelazko established that each completely metrizable semitopological group is a topological group [19]. Later, in 1982, N. Brand proved that every Čech-complete paratopological group is a topological group [2].

In 1975, Kramosil and Michalek introduced a notion of metric fuzziness [10] which quickly became an important issue (for example, [4, 5, 6, 7, 8]).

Definition 1.1. A **fuzzy metric** (in the sense of Kramosil and Michalek) on a set X is a pair $(M, *)$ such that M is a fuzzy set in $X \times X \times [0, \infty)$ and $*$ is a continuous t-norm satisfying for all $x, y, z \in X$:

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1) $M(x, y, 0) = 0$;
2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
3) $M(x, y, t) = M(y, x, t)$;
4) $M(x, z, t + s) \geqslant M(x, y, t) * M(y, z, s)$ for all $t, s > 0$;
5) $M(x, y, \_) : [0, +\infty) \to [0, 1]$ is a left continuous function.

Recently, fuzzy metric topological groups have been widely studied in fuzzy topological algebra (see, among others, [15, 18]).

In particular, I. Sánchez and M. Sanchis found that some special fuzzy metrics (such as left invariant fuzzy quasi-pseudometrics and invariant fuzzy pseudometrics) can improve some topological algebraic structures into stronger topological structures. The main results are: (1) If $(G, M, *)$ is a fuzzy quasi-pseudometric right topological group such that $(M, *)$ is left-invariant, then $(G, M, *)$ is a fuzzy paratopological group (see [16, Theorem 3.2]). (2) If $(G, M, *)$ is a fuzzy pseudometric right topological group such that $(M, *)$ is left-invariant, then $(G, M, *)$ is a fuzzy topological group (see [16, Theorem 3.3]). (3) Let $(M, *)$ be a fuzzy quasi-pseudometric on a semigroup S. If $(M, *)$ is invariant, then $(S, M, *)$ is a fuzzy topological semigroup (see [16, Theorem 3.10]).

Given a function $d : X \times X \to \mathbb{R}^+$ on a set X, we consider the following conditions, for every $x, y, z \in X$:

(1) $d(x, x) = 0$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, y) \leqslant d(x, z) + d(z, y)$;
(4) if $d(x, y) = 0$, then $x = y$;
(4') if $d(x, y) = d(y, x) = 0$, then $x = y$,

for all $x, y, z \in X$.

The function d is called a *pseudometric* if it satisfies (1), (2) and (3). A pseudometric that also satisfies (4) is called a *metric*. A *quasi-pseudometric* on an arbitrary set X is a function $d : X \times X \to \mathbb{R}^+$ satisfying the conditions (1) and (3). If d satisfies further (4') then it is called a *quasi-metric*.

Recently, Khatami and Mirzavaziri (in [11]) generalized the concept of metric. They first gave a new operation called t-definer which is extended by t-conorm. It is defined as:

Definition 1.2 ([11, Definition 2.1]). A *t-definer* is a function $\star : [0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying the following conditions for each $a, b, c \in [0, \infty)$:

(T1) $a \star b = b \star a$;
(T2) $a \star (b \star c) = (a \star b) \star c$;
(T3) if $a \leqslant b$, then $a \star c \leqslant b \star c$;
(T4) $a \star 0 = a$;
(T5) $\star$ is continuous on its first component with respect to the Euclidean topology.

The residuum of a t-definer plays a role such as the role of minus operator for addition operator. Let $\star$ be a t-definer. The *residuum of* $\star$ is defined by

$$a \dot{\to} b = \inf\{c : c \star a \geqslant b\}.$$

Then, by the residuation property of $\star$ and $\dot{\to}$, we have

$$a \star (a \dot{\to} b) = \max\{a, b\}. \tag{1.1}$$

Khatami and Mirzavaziri changed the condition (3) in the metric axiom into the $\star$-triangle inequality. Then the following definition of $\star$-metrics can be obtained.

Definition 1.3 ([11, Definition 2.2])**.** Let X be a non-empty set and $\star$ a t-definer. If for every $x, y, z \in X$, a function $d^\star : X \times X \to [0, \infty)$ satisfies the following conditions:

(M1) $d^\star(x, y) = 0$ if and only if $x = y$;
(M2) $d^\star(x, y) = d^\star(y, x)$;
(M3) $d^\star(x, y) \leqslant d^\star(x, z) \star d^\star(z, y)$,

then $d^\star$ is called a $\star$-*metric* on X. The set X with a $\star$-metric is called $\star$-*metric space*, denoted by $(X, d^\star)$.

Assume that $(X, d^\star)$ is a $\star$-metric space. For any $a \in X$ and $r > 0$, denote by

$$B_{d^\star}(a, r) = \{x \in X : d^\star(a, x) < r\}$$

and

$$\mathscr{T}_{d^\star} = \{U \subseteq X : \text{ for each } a \in U \text{ there is } r > 0 \text{ such that } B_{d^\star}(a, r) \subseteq U\}.$$

Khatami and Mirzavaziri proved the following result:

Theorem 1.4 ([11, Theorems 3.2, 3.4, 3.5])**.** *For every $\star$-metric space $(X, d^\star)$, $\mathscr{T}_{d^\star}$ forms a Hausdorff topology on X and the topological space $(X, \mathscr{T}_{d^\star})$ is first countable and satisfied the normal separation axiom.*

Then, we have proved that

Theorem 1.5 ([9, Theorem 2.4])**.** *Every $\star$-metric space is metrizable.*

In this paper, we extend some concepts of $\star$-metric spaces (in [11]) to $\star$-quasi-pseudometric spaces, and give an example to show that $\star$-quasi-pseudometrics are not necessarily quasi-pseudometrics. Then, we will discuss the basic topological properties of $\star$-metric spaces. Further, we combine topological structure with algebraic structure. Our aim is to obtain conditions under which $\star$-quasi-pseudometric semitopological groups are paratopological groups or topological groups.

We show that: (1) if $(G, d^\star)$ is a $\star$-quasi-pseudometric right topological group such that $d^\star$ is left-invariant, then $(G, d^\star)$ is a paratopological group (see Theorem 3.5); (2) if $(G, d^\star)$ is a $\star$-quasi-pseudometric left topological group such that $d^\star$ is right-invariant, then $(G, d^\star)$ is a paratopological group. If in addition $(G, d^\star)$ is a $\star$-pseudometric left topological group, then $(G, d^\star)$ is a topological group (see Theorem 3.6); (3) let $d^\star$ be a left-invariant $\star$-quasi-pseudometric on a monoid G such that for each $x \in G$, λ_x is open and ρ_x is continuous at the identity e of $(G, d^\star)$. Then $(G, d^\star)$ is a topological semigroup (see Theorem 4.1).

2. Topology of $\star$-quasi-metric

In this section, we extend some concepts of $\star$-metric spaces to $\star$-quasi-metric spaces and $\star$-quasi-pseudometric spaces. Then we discussed the basic topological properties of $\star$-quasi-metric spaces and $\star$-quasi-pseudometric spaces.

Definition 2.1. Let X be a non-empty set and $\star$ a t-definer. A *$\star$-quasi-pseudometric* on X is a function $d^\star : X \times X \to [0, \infty)$ satisfying the following conditions:

(D1) $d^\star(x, x) = 0$;
(D2) $d^\star(x, y) \leqslant d^\star(x, z) \star d^\star(z, y)$.

In this case $(X, d^\star)$ is called a *$\star$-quasi-pseudometric space*.

In addition, if $d^\star$ is a $\star$-quasi-pseudometric and satisfies the condition:

(D3) for every $x, y \in X$, $if\ d^\star(x, y) = 0,\ then\ x = y$,

then $d^\star$ is called a *$\star$-quasi-metric* on X, and $(X, d^\star)$ is called a *$\star$-quasi-metric space*.

If $d^\star$ is a $\star$-quasi-pseudometric and satisfies the condition:

(D4) $d^\star(x, y) = d^\star(y, x)$,

then $d^\star$ is called a *$\star$-pseudometric* on X, and $(X, d^\star)$ is called a *$\star$-pseudometric space*.

The following example shows that there are $\star$-quasi-pseudometrics which are not quasi-pseudometrics.

Example 2.2. Let $X = [0, \infty)$. Clearly, $x \star y = (\sqrt{x} + \sqrt{y})^2$ is a t-definer, for every $x, y \in X$. The function

$$d^\star(x, y) = \begin{cases} (\sqrt{x} - \sqrt{y})^2, & x \geqslant y; \\ 0, & x < y. \end{cases}$$

forms an $\star$-quasi-pseudometric which is not a quasi-pseudometric.

Obviously, $d^\star(x, y)$ satisfies (D1) of Definition 2.1. Now, we show that also (D2) of Definition 2.1 holds.

Now, we need to prove the following 6 cases.

(1) When $x \geqslant z \geqslant y$, we have

$$\begin{aligned} d^\star(x,y) &= (\sqrt{x}-\sqrt{y})^2 = (\sqrt{x}-\sqrt{z}+\sqrt{z}-\sqrt{y})^2 \\ &\leqslant \left[\sqrt{(\sqrt{x}-\sqrt{z})^2}+\sqrt{(\sqrt{z}-\sqrt{y})^2}\right]^2 \\ &= \left[\sqrt{d^\star(x,z)}+\sqrt{d^\star(z,y)}\right]^2 \\ &= d^\star(x,z)\star d^\star(z,y). \end{aligned}$$

(2) When $z \geqslant x \geqslant y$, we have $d^\star(x,y) = (\sqrt{x}-\sqrt{y})^2$, $d^\star(x,z) = 0$, $d^\star(z,y) = (\sqrt{z}-\sqrt{y})^2$. Therefore $d^\star(x,y) = (\sqrt{x}-\sqrt{y})^2 \leqslant (\sqrt{z}-\sqrt{y})^2 = 0\star(\sqrt{z}-\sqrt{y})^2 = d^\star(x,z)\star d^\star(z,y)$.

(3) When $x \geqslant y \geqslant z$, we have $d^\star(x,y) = (\sqrt{x}-\sqrt{y})^2$, $d^\star(x,z) = (\sqrt{x}-\sqrt{z})^2$, $d^\star(z,y) = 0$.Therefore $d^\star(x,y) = (\sqrt{x}-\sqrt{y})^2 \leqslant (\sqrt{x}-\sqrt{z})^2 = (\sqrt{x}-\sqrt{z})^2\star 0 = d^\star(x,z)\star d^\star(z,y)$.

(4) When $z \leqslant x < y$, we have $d^\star(x,z) = (\sqrt{x}-\sqrt{z})^2$, $d^\star(z,y) = 0$, $d^\star(x,y) = 0$. Therefore $d^\star(x,y) = 0 \leqslant (\sqrt{x}-\sqrt{z})^2 = (\sqrt{x}-\sqrt{z})^2\star 0 = d^\star(x,z)\star d^\star(z,y)$.

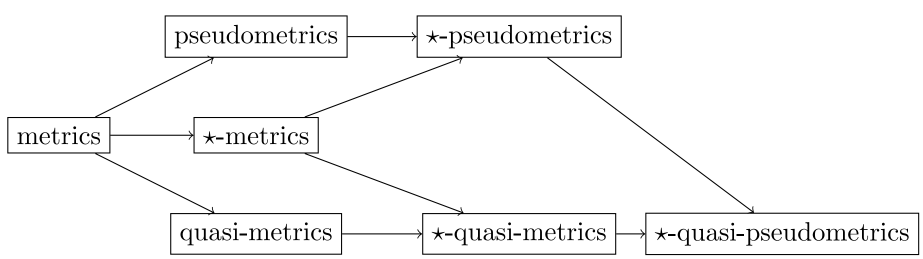
(5) When $x \leqslant z \leqslant y$, we have $d^\star(x,z) = 0$, $d^\star(z,y) = 0$, $d^\star(x,y) = 0$. Therefore $d^\star(x,y) = 0 = 0\star 0 = d^\star(x,z)\star d^\star(z,y)$.

(6) When $x < y \leqslant z$, $d^\star(x,z) = 0$, we have $d^\star(z,y) = (\sqrt{z}-\sqrt{y})^2$, $d^\star(x,y) = 0$. Therefore $d^\star(x,y) = 0 \leqslant (\sqrt{z}-\sqrt{y})^2 = 0\star(\sqrt{z}-\sqrt{y})^2 = d^\star(x,z)\star d^\star(z,y)$.

Thus, (D2) holds.

However, $d^\star(1,25) = 16 \nleqslant d^\star(1,16) + d^\star(16,25) = 10$, which means $d^\star(x,y)$ is not a quasi-pseudometric.

Khatami and Mirzavaziri gave a generalization of metrics, put forward ⋆-metrics, and give an example that ⋆-metrics are not metrics. Further, we extend the ⋆-metrics to obtain ⋆-quasi-metrics, ⋆-pseudometrics, and ⋆-quasi-pseudometrics. In Example 2.2, we find that there is a ⋆-quasi-pseudometric, which is not a quasi-pseudometric. This shows that our promotion is very meaningful. The following figure briefly describes the relationship between them.



Similar to metric spaces, we will give the definition of open balls in ⋆-quasi-pseudometric spaces below.

Definition 2.3. Let $(X, d^\star)$ be a $\star$-quasi-pseudometric space. We define open ball $B_{d^\star}(x, r)$ with $x \in X$ and radius $r > 0$ as

$$B_{d^\star}(x, r) = \{y \in X : d^\star(x, y) < r\}.$$

Theorem 2.4. *Let $(X, d^\star)$ be a $\star$-quasi-pseudometric space. Define*

$$\mathscr{T}_{d^\star} = \{U \subseteq X : \text{ for each } x \in U \text{ there is } r > 0 \text{ such that } B_{d^\star}(x, r) \subseteq U\}.$$

Then $\mathscr{T}_{d^\star}$ is a topology on X.

Lemma 2.5. *In $\star$-quasi-pseudometric space $(X, d^\star)$ every open ball is an open set.*

Proof. Let $\star$ be a t-definer, $\dot{\to}$ be the residuum of $\star$. For every $x \in X$ and $r > 0$, we claim that there exist $\epsilon > 0$, such that for every $y \in B_{d^\star}(x, r)$, we have

$$B_{d^\star}(y, \epsilon) \subseteq B_{d^\star}(x, r).$$

In fact, take $\epsilon = d^\star(x, y) \dot{\to} r$ and for every $z \in B_{d^\star}(y, \epsilon)$, then $d^\star(y, z) < d^\star(x, y) \dot{\to} r$. By formula (1.1), we have

$$d^\star(x, y) \star d^\star(y, z) < d^\star(x, y) \star (d^\star(x, y) \dot{\to} r) = r.$$

Therefore, we have $d^\star(x, z) \leq d^\star(x, y) \star d^\star(y, z) < r$ which shows that $z \in B_{d^\star}(x, r)$. □

Now, by Definition 2.3 and Lemma 2.5, for a $\star$-quasi-(pseudo)metric space $(X, d^\star)$, the set $\mathscr{B} = \{B_{d^\star}(x, \epsilon) \mid x \in X, \epsilon > 0\}$ is a base for the topology induced by $d^\star$ on X.

Definition 2.6. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of a $\star$-quasi-pseudometric space $(X, d^\star)$, and $x \in X$. If for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $d^\star(x, x_n) < \epsilon$ whenever $n \geqslant k$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ *converges x under $d^\star$.*

The following propositions are easy to prove.

Proposition 2.7. *Let $(X, d^\star)$ be a $\star$-quasi-pseudometric space. Then the following statements are equivalent:*

(1) $\{x_n\}_{n\in\mathbb{N}}$ *converges to x_0 under $\mathscr{T}_{d^\star}$;*
(2) $\{x_n\}_{n\in\mathbb{N}}$ *converges to x_0 under $d^\star$.*

Remark 2.8. The Proposition 2.7 illustrates that for a $\star$-quasi-pseudometric space, $x_n \to x$ if and only if $d^\star(x, x_n) \to 0$.

Proposition 2.9. *Let $(X, d^\star)$ be a $\star$-quasi-pseudometric space. Then the set X with the topology induced by $d^\star$ is first countable.*

In Proposition 2.9, we get that, for every $x \in X$, $\mathscr{B}_x = \{B_{d^\star}(x, \frac{1}{n}) : n \in \mathbb{N}\}$ is a neighborhood base at x in the $\star$-quasi-pseudometric space $(X, d^\star)$.

Proposition 2.10. *Every $\star$-quasi-metric space $(X, d^\star)$ is a Hausdorff space.*

Proof. Choose two distinct points $x, y \in X$. We shall show that, there exists $r > 0$ such that $B_{d^\star}(x,r) \cap B_{d^\star}(y,r) = \varnothing$. Since the $\star$ is continuous and $d^\star(x,y) > 0$, we have $d^\star(x,y) > r \star r$. Now, we assume that there exists $z \in B_{d^\star}(x,r) \cap B_{d^\star}(y,r)$ then we get the following contradiction:

$$d^\star(x,y) \leqslant d^\star(x,z) \star d^\star(z,y) < r \star r < d^\star(x,y).$$

Hence, $B_{d^\star}(x,r) \cap B_{d^\star}(y,r) = \varnothing$. □

The notions and concepts of topological spaces are defined as usual (e.g. see [1] or [13]). Unless otherwise stated, $\star$-quasi-metric spaces and $\star$-quasi-pseudometric spaces do not satisfy any separation axiom.

3. $\star$-quasi-pseudometric topological groups

We now move on to notions from topological algebra. Let G be an algebraic group. For a fixed element $x \in G$. The function $\lambda_x\colon G \to G$ defined by $\lambda_x(g) = xg$ is called the *left translation* of x on G. Similarly, $\rho_x\colon G \to G$ defined as $\rho_x(g) = gx$ is known as the *right translation* of x on G.

A *topological semigroup* (G, τ) is an algebraic semigroup G with a topology τ that makes the multiplication in G jointly continuous. A *paratopological group* G is a topological semigroup such that G is an algebraic group. A *topological group* G is a paratopological group G such that the inverse mapping is continuous.

(G, τ) is said to be a *left (respectively, right) topological group* if the translations λ_x (respectively, ρ_x) are continuous in G for all $x \in G$, and a *semitopological group* is a left topological group which is also a right topological group.

Next, we will give the definitions related to $\star$-quasi-pseudometric topological groups.

Definition 3.1. By a $\star$*-(quasi)-pseudometric semigroup* we mean a pair $(G, d^\star)$ such that $(G, d^\star)$ is a $\star$-(quasi)-pseudometric space and $(G, \mathscr{T}_{d^\star})$ is a topological semigroup.

A $\star$*-(quasi)-pseudometric paratopological group* is a $\star$-(quasi)-pseudometric semigroup $(G, d^\star)$ such that G is an algebraic group.

Definition 3.2. By a $\star$*-(quasi)-pseudometric right (left) topological group* we mean a pair $(G, d^\star)$ such that $(G, d^\star)$ is a $\star$-(quasi)-pseudometric space and $(G, \mathscr{T}_{d^\star})$ is a right (left) topological group.

We give the definition of left (right) invariance in $\star$-(quasi)-pseudometric topological groups. This notion plays an important role in our results.

Definition 3.3. A $\star$-(quasi)-pseudometric $d^\star$ on a group G is *left-invariant* (respectively, *right-invariant*) if $d^\star(x,y) = d^\star(ax, ay)$ (respectively, $d^\star(x,y) = d^\star(xa, ya)$) whenever $a, x, y \in G$. We say that $d^\star$ is *invariant* if it is both left-invariant and right-invariant.

Now, we give a well known result which is an internal characterization of a (para)topological group.

Proposition 3.4 ([1, Theorem 1.2.12]). *Let G be a group with identity e and $\mathscr{U}$ a family of subsets of G containing e. If $\mathscr{U}$ satisfies the following conditions:*

(i) *for every $U, V \in \mathscr{U}$, there exists an $W \in \mathscr{U}$ such that $W \subseteq U \cap V$;*
(ii) *for every $U \in \mathscr{U}$ and $x \in U$, there exists an $V \in \mathscr{U}$ such that $Vx \subseteq U$;*
(iii) *for every $U \in \mathscr{U}$ and $x \in G$, there exists an $V \in \mathscr{U}$ such that $xVx^{-1} \subseteq U$;*
(iv) *for every $U \in \mathscr{U}$, there exists an $V \in \mathscr{U}$ such that $V^2 \subseteq U$;*

then the family $\{Ux : x \in G, U \in \mathscr{U}\}$ is a base for a topology $\tau_{\mathscr{U}}$ on G. With this topology, G is a paratopological group, and the family $\{xU : x \in G, U \in \mathscr{U}\}$ is a base for the same topology on G. In addition, if $\mathscr{U}$ satisfies

(v) *for every $U \in \mathscr{U}$, there exists an $V \in \mathscr{U}$ such that $V^{-1} \subseteq U$.*

Then $(G, \tau_{\mathscr{U}})$ is a topological group.

Theorem 3.5. *If $(G, d^\star)$ is a $\star$-quasi-pseudometric right topological group such that $d^\star$ is left-invariant, then $(G, d^\star)$ is a paratopological group.*

Proof. Let e be the identity of G. According to Proposition 2.9, $\mathscr{B}_e = \{B_{d^\star}(e, \frac{1}{n}) : n \in \mathbb{N}\}$ is a local base at e. Let us show that $\mathscr{B}_e = \{B_{d^\star}(e, \frac{1}{n}) : n \in \mathbb{N}\}$ satisfies conditions $(i) - (iv)$ in Theorem 3.4, that is, the topology $\mathscr{T}_{\mathscr{B}_e}$ associated to the family $\mathscr{B}_e$ makes G into a paratopological group.

(i). It follows from the fact that $\mathscr{B}_e$ is a local base at e in $(G, \mathscr{T}_{d^\star})$. So, $\mathscr{B}_e$ satisfies (i).

(ii). Take $n \in \mathbb{N}$ and $x \in B_{d^\star}(e, \frac{1}{n})$. Since ρ_x is continuous at e and $\rho_x(e) = ex = x \in B_{d^\star}(e, \frac{1}{n})$, there exists $m \in \mathbb{N}$ such that

$$\rho_x(B_{d^\star}(e, \frac{1}{m})) = B_{d^\star}(e, \frac{1}{m})x \subseteq B_{d^\star}(e, \frac{1}{n}).$$

Thus, (ii) holds.

(iii). First we show that, for each $n \in \mathbb{N}$ and $x \in G$, we have

$$xB_{d^\star}(e, \frac{1}{n}) = B_{d^\star}(x, \frac{1}{n}). \tag{1}$$

In fact, take $y \in B_{d^\star}(e, \frac{1}{n})$, namely $xy \in xB_{d^\star}(e, \frac{1}{n})$. Since $d^\star$ is left-invariant, we have

$$d^\star(x, xy) = d^\star(e, y) < \frac{1}{n}.$$

By the foregoing, $xB_{d^\star}(e, \frac{1}{n}) \subseteq B_{d^\star}(x, \frac{1}{n})$.

On the other hand, take $z \in B_{d^\star}(x, \frac{1}{n})$. Because $d^\star$ is left-invariant, we have

$$d^\star(e, x^{-1}z) = d^\star(x, z) < \frac{1}{n}.$$

This proves that $x^{-1}z \in B_{d^\star}(e, \frac{1}{n})$, and from this it follows further that $z \in xB_{d^\star}(e, \frac{1}{n})$ which shows (1).

Now, we shall show (iii). Take $n \in \mathbb{N}$ and $x \in G$. Note that every right translation is a homeomorphism and $x \in B_{d^\star}(e, \frac{1}{n})$. So $B_{d^\star}(e, \frac{1}{n})x$ is an open neighborhood of x. Hence there is $m \in \mathbb{N}$ such that $B_{d^\star}(x, \frac{1}{m}) \subseteq B_{d^\star}(e, \frac{1}{n})x$. From this and (1) it follows that

$$xB_{d^\star}(e, \frac{1}{m})x^{-1} = B_{d^\star}(x, \frac{1}{m})x^{-1} \subseteq B_{d^\star}(e, \frac{1}{n}).$$

So, $\mathscr{B}_e$ satisfies (iii).

(iv). For every $n \in \mathbb{N}$, since the $\star$ is continuous, there is $m \in \mathbb{N}$ such that $\frac{1}{m} \star \frac{1}{m} < \frac{1}{n}$. Then for each $y, z \in B_{d^\star}(e, \frac{1}{m})$, the following inequalities hold

$$d^\star(e, yz) \leqslant d^\star(e, y) \star d^\star(y, yz) = d^\star(e, y) \star d^\star(e, z) < \frac{1}{m} \star \frac{1}{m} < \frac{1}{n}.$$

Therefore, $B_{d^\star}(e, \frac{1}{m})B_{d^\star}(e, \frac{1}{m}) \subseteq B_{d^\star}(e, \frac{1}{n})$, $\mathscr{B}_e$ satisfies (iv).

By Proposition 3.4, $(G, \mathscr{T}_{\mathscr{B}_e})$ is a paratopological group and $\{xB_{d^\star}(e, \frac{1}{n}) : x \in G, n \in \mathbb{N}\}$ is a base for $\mathscr{T}_{\mathscr{B}_e}$. Notice that equation (1) implies that $\{xB_{d^\star}(e, \frac{1}{n}) : x \in G, n \in \mathbb{N}\}$ also is a base for $\mathscr{T}_{d^\star}$ so that $\mathscr{T}_{\mathscr{B}_e} = \mathscr{T}_{d^\star}$. This shows that $(G, d^\star)$ is a paratopological group. □

Theorem 3.6. *If $(G, d^\star)$ is a $\star$-pseudometric right topological group such that $d^\star$ is left-invariant, then $(G, d^\star)$ is a topological group.*

Proof. Since, $\star$-quasi-pseudometrics are $\star$-pseudometrics, according to Theorem 3.5, $(G, d^\star)$ is a paratopological group. To complete the proof, it is enough to show that the family $\mathscr{B} = \{B_{d^\star}(e, \frac{1}{n}) : n \in \mathbb{N}\}$ satisfies (v) of Proposition 3.4. For every $n \in \mathbb{N}$. Take $x \in B_{d^\star}(e, \frac{1}{n})$. As a consequence of left-invariance of $d^\star$, we have

$$d^\star(e, x^{-1}) = d^\star(x, e) = d^\star(e, x) < \frac{1}{n}.$$

We conclude that $x^{-1} \in B_{d^\star}(e, \frac{1}{n})$. So, $(G, d^\star)$ is a topological group. □

Similar to the proof of Theorems 3.5 and 3.6, we can obtain the following Theorem.

Theorem 3.7. *If $(G, d^\star)$ is a $\star$-quasi-pseudometric left topological group such that $d^\star$ is right-invariant, then $(G, d^\star)$ is a paratopological group. If furthermore $(G, d^\star)$ is a $\star$-pseudometric left topological group, then $(G, d^\star)$ is a topological group.*

Since a semitopological group is both a left and right topological group. According to the result of Theorems 3.5, 3.6 and 3.7 we can get the following corollary.

Corollary 3.8. *Suppose that $(G, \mathscr{T}_{d^\star})$ is a semitopological group whose topology $\mathscr{T}_{d^\star}$ is induced by a right-(or left-)invariant $\star$-quasi-pseudometric $d^\star$. Then $(G, \mathscr{T}_{d^\star})$ is a paratopological group.*

Corollary 3.9. *Suppose that $(G, \mathscr{T}_{d^\star})$ is a semitopological group whose topology $\mathscr{T}_{d^\star}$ is induced by a right-(or left-)invariant $\star$-pseudometric $d^\star$. Then $(G, \mathscr{T}_{d^\star})$ is a topological group.*

It is known that a (quasi-)pseudometric is a $\star$-(quasi-)pseudometric. So, it is easy to draw the following conclusions:

Corollary 3.10 ([16, Corollary 3.4]). *Suppose that (G, τ) is a left (right) topological group whose topology τ is induced by a right-(left-)invariant quasi-pseudometric. Then (G, τ) is a paratopological group.*

Corollary 3.11 ([16, Corollary 3.8]). *Suppose that G is a left (right) topological group whose topology is induced by a right-(left-)invariant pseudometric. Then G is a topological group.*

We said that a topological space X is said to be $\star$-*(quasi-)metrizable* if there exists a $\star$-(quasi-)metric $d^\star$ on the set X that induces the topology of X. A $\star$-quasi-metric $d^\star(x, y)$ is called *left-continuous* if $d^\star(x, \_)$ is continuous.

Recall that a topological space X is called a *sequential space* if a set $A \subset X$ is closed if and only if together with any sequence it contains all its limits.

Theorem 3.12. *Suppose that G is a $\star$-quasi-metrizable paratopological group with respect to a left continuous, left-invariant $\star$-quasi-metric. Then G is a $\star$-metrizable topological group.*

Proof. First we prove that the $\star$-quasi-metrizable paratopological group G is a topological group. It is sufficient to prove that the inverse operation is continuous.

Let G be a paratopological group with respect to a left continuous, left-invariant $\star$-quasi-metric $d^\star$ and e be the neutral element. First we prove that if $x_n \to x$, then $x_n^{-1} \to x^{-1}$. Since $x_n \to x$ and $d^\star$ is left continuous, then $d^\star(x_n, x) \to d^\star(x, x) = 0$. As a consequence of the left invariance of $d^\star$, we have

$$d^\star(e, x_n^{-1}x) = d^\star(x_n e, x_n x_n^{-1} x) = d^\star(x_n, x) \to 0.$$

Then $x_n^{-1}x \to e$ by Proposition 2.7. By the foregoing, $x_n^{-1} \to x^{-1}$. Let U be open. We shall prove that U^{-1} is open. Since G is a sequential space, it is sufficient to prove U^{-1} is sequential open. Let $y_n \to y \in U^{-1}$, then $y_n^{-1} \to y^{-1} \in U$. Since U is open, $\{y_n^{-1} : n \in \mathbb{N}\}$ is eventually in U. Hence $\{y_n : n \in \mathbb{N}\}$ is eventually in U^{-1}. Therefore, U^{-1} is open.

The inverse operation on G is continuous, hence G is a topological group. According to [1, Theorem 3.3.12], A Hausdorff topological group satisfying the first-countable axiom is metrizable. By Propositions 2.9 and 2.10, G is a Hausdorff topological group satisfying the first-countable axiom and from this it follows by the foregoing that G is metrizable. Therefore G is $\star$-metrizable by Theorem 1.5. $\square$

From Theorem 3.12, we can easily get Liu's conclusion in [12]

Corollary 3.13 ([12, Theorem 2.1]). *Suppose that G is a quasi-metrizable paratopological group with respect to a left continuous, left-invariant quasi-metric. Then G is a metrizable topological group.*

4. ⋆-quasi-pseudometric topological semigroups

We now move on to ⋆-quasi-pseudometric semigroups.

Theorem 4.1. *Suppose that $d^\star$ be a ⋆-quasi-pseudometric on a semigroup S. If $d^\star$ is invariant, then $(S, d^\star)$ is a topological semigroup.*

Proof. Take $y, z \in S$. Since the $\star$ is continuous, for every $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that $\frac{1}{m} \star \frac{1}{m} < \frac{1}{n}$. We can claim that $B_{d^\star}(y, \frac{1}{m}) B_{d^\star}(z, \frac{1}{m}) \subseteq B_{d^\star}(yz, \frac{1}{n})$. Choose $a \in B_{d^\star}(y, \frac{1}{m})$ and $b \in B_{d^\star}(z, \frac{1}{m})$, then $ab \in B_{d^\star}(y, \frac{1}{m}) B_{d^\star}(z, \frac{1}{m})$. Since $d^\star$ is invariant, we have

$$d^\star(yz, ab) \leqslant d^\star(yz, yb) \star d^\star(yb, ab) = d^\star(z, b) \star d^\star(y, a) < \frac{1}{m} \star \frac{1}{m} < \frac{1}{n}.$$

We have proved that multiplication is continuous in $(S, \mathscr{T}_{d^\star})$. As a consequence, $(S, d^\star)$ is a topological semigroup. □

Let us recall that a *monoid* is a semigroup with a neutral element.

Theorem 4.2. *Let $d^\star$ be a left-invariant ⋆-quasi-pseudometric on a monoid G such that for each $x \in G$, λ_x is open and ρ_x is continuous at the identity e of $(G, d^\star)$. Then $(G, d^\star)$ is a topological semigroup.*

Proof. Let e be the identity of G. We claim that for each $n \in \mathbb{N}$ and $x \in G$ we have

$$xB_{d^\star}(e, \frac{1}{n}) \subseteq B_{d^\star}(x, \frac{1}{n}). \tag{2}$$

Indeed, take $y \in B_{d^\star}(e, \frac{1}{n})$. Since $d^\star$ is left-invariant, we have

$$d^\star(x, xy) = d^\star(e, y) < \frac{1}{n}.$$

This proves (2). As a consequence of (2), we have that left translations are continuous at e.

Now, we shall show that for every $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ satisfying

$$B_{d^\star}(e, \frac{1}{m}) B_{d^\star}(e, \frac{1}{m}) \subseteq B_{d^\star}(e, \frac{1}{n}). \tag{3}$$

Since the $\star$ is continuous, for every $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that $\frac{1}{m} \star \frac{1}{m} < \frac{1}{n}$. Then, for each $y, z \in B_{d^\star}(e, \frac{1}{m})$, the following inequalities hold:

$$d^\star(e, yz) \leqslant d^\star(e, y) \star d^\star(y, yz) = d^\star(e, y) \star d^\star(e, z) < \frac{1}{m} \star \frac{1}{m} < \frac{1}{n}.$$

Now, we will prove that the multiplication is continuous in $(G, \mathscr{T}_{d^\star})$. Take $x, y \in G$ and $n \in \mathbb{N}$. By (2), we have $xyB_{d^\star}(e, \frac{1}{n}) \subseteq B_{d^\star}(xy, \frac{1}{n})$. By (3), $B_{d^\star}(e, \frac{1}{m}) B_{d^\star}(e, \frac{1}{m}) \subseteq B_{d^\star}(e, \frac{1}{n})$ for some $m \in \mathbb{N}$. Therefore

$$xyB_{d^\star}(e, \frac{1}{m}) B_{d^\star}(e, \frac{1}{m}) \subseteq xyB_{d^\star}(e, \frac{1}{n}) \subseteq B_{d^\star}(xy, \frac{1}{n}). \tag{4}$$

It follows from the hypothesis that left translations are open. Hence $yB_{d^\star}(e, \frac{1}{m})$ is an open set in $(G, d^\star)$ which contains y. According to assumptions ρ_x is continuous at e. Hence there is $k \in \mathbb{N}$ satisfying

$$\rho_y(B_{d^\star}(e, \frac{1}{k})) = B_{d^\star}(e, \frac{1}{k})y \subseteq yB_{d^\star}(e, \frac{1}{m}). \tag{5}$$

According to (4)-(5), we have

$$xB_{d^\star}(e, \frac{1}{k})yB_{d^\star}(e, \frac{1}{m}) \subseteq xyB_{d^\star}(e, \frac{1}{m})B_{d^\star}(e, \frac{1}{m}) \subseteq B_{d^\star}(xy, \frac{1}{n}).$$

Since left translations are open, $xB_{d^\star}(e, \frac{1}{k})$ and $yB_{d^\star}(e, \frac{1}{m})$ are open neighborhoods of x and y, respectively. Hence multiplication in $(G, d^\star)$ is continuous. □

Applying the previous results, we get the following results in semigroups and topological monoids.

Corollary 4.3 ([16, Corollary 3.12])**.** *Suppose that d is a invariant quasi-pseudometric on a semigroup S. Then (S, d) is a topological semigroup.*

Corollary 4.4 ([16, Corollary 3.14])**.** *Let d be a left-invariant quasi-pseudometric on a monoid G such that for each $x \in G$, λ_x is open and ρ_x is continuous at the identity e of (G, d). Then (G, d) is a topological semigroup.*

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