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Global optimization using α-ordered proximal contractions in metric spaces with partial orders

Somayya Komal [a] and Poom Kumam [a,b,*]

[a] Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT),126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand (somayya.komal@mail.kmutt.ac.th, poom.kumam@mail.kmutt.ac.th)
[b] Theoretical and Computational Science (TaCS) Center, Science Laboratory Building, Faculty of Science King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand (poom.kumam@mail.kmutt.ac.th)

Abstract

The purpose of this article is to establish the global optimization with partial orders for the pair of non-self mappings, by introducing a new type of contractions like α-ordered contraction and α-ordered proximal contraction in the frame work of complete metric spaces. Also to calculate some fixed point theorems with the help of these generalized contractions. In addition, to establish an example which shows the validity of our main result. These results extend and unify many existing results in the literature.

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1. Introduction

It is obvious that best proximity point serves as an optimal approximate solution to the equation $Zx = x$, where Z is a non-self mapping from any two non-empty subsets of a metric space, a normed linear space or any other

*Corresponding author: poom.kumam@mail.kmutt.ac.th and poom.kum@kmutt.ac.th

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topological space. Also it is very interesting point that best proximity point theorems actually generalize the fixed point theorems in natural fashion by taking self mapping instead of non-self mapping in best proximity point theorem then we can get fixed point. Since $d(x, Zx) \geq d(A, B)$, for any $x \in A$, we obtain the global minimum of the mapping $x \mapsto d(x, Zx)$ as a best proximity point. For more details on this approach, we refer the reader to [2], [3], [4], [5], [6], [10], [7], [13], [11], [12], [14], [16], [1] and [15]. The basic purpose of this article is to establish some generalized notions and to derive new theorem of global optimization with partial orders in metric spaces. We have defined in this work an α-ordered contraction to find common best proximity points. The motivation of this paper is [9], we generalized that contraction of [9]. Also presented an example to verify the results.

2. Preliminaries

In this section let us take that A and B are non-void subsets of a metric space (X, d). we recall some definitions and notations in this section which will be used throughout this work.

Definition 2.1 ([8])**.** Let X be a metric space, A and B two nonempty subsets of X. Define

$$\begin{aligned} d(A,B) &= \inf\{d(a,b) : a \in A, b \in B\}, \\ A_0 &= \{a \in A : \textit{there exists some } b \in B \textit{ such that } d(a,b) = d(A,B)\}, \\ B_0 &= \{b \in B : \textit{there exists some } a \in A \textit{ such that } d(a,b) = d(A,B)\}. \end{aligned}$$

Definition 2.2 ([8])**.** Given non-self mappings $S : A \to B$ and $T : A \to B$, an element x^* is called common best proximity point of the mappings if this condition satisfied:

$$d(x^*, Sx^*) = d(x^*, Tx^*) = d(A, B).$$

We noticed here that common best proximity point is that element at which both functions S and T attain their global minimum, since $d(x, Sx) \geq d(A, B)$ and $d(x, Tx) \geq d(A, B)$ for all x.

Definition 2.3 ([9])**.** A mapping $S : A \to B$ is said to be an ordered contraction if there exists a non-negative real number $\gamma < 1$ such that

$$x_1 \preceq x_2 \;\Rightarrow\; d(Sx_1, Sx_2) \leq \gamma d(x_1, x_2),$$

for all $x_1, x_2 \in A$.

Definition 2.4 ([9])**.** A mapping $S : A \to B$ is said to be an ordered proximal contraction if there exists $\gamma < 1$ such that

$$x_1 \preceq x_2,$$
$$d(u_1, Sx_1) = d(A, B)$$

and

$$d(u_2, Sx_2) = d(A, B),$$

implies that $d(u_1, u_2) \leq \gamma d(x_1, x_2)$, for all $u_1, u_2, x_1, x_2 \in A$.

Definition 2.5 ([9])**.** Given non-self mappings $S : A \to B$ and $T : B \to A$, the pair (S, T) is said to form an ordered proximal cyclic contraction if there exists a non-negative real number $k < 1$ such that

$$x \preceq y,$$
$$d(u, Sx) = d(A, B)$$

and

$$d(v, Ty) = d(A, B),$$

implies that $d(u, v) \leq kd(x, y) + (1 - k)d(A, B)$, for all $u, x \in A$ and $v, y \in B$.

Definition 2.6 ([9])**.** Given non-self mappings $S : A \to B$ and $T : B \to A$, the pair (S, T) is said to be proximally increasing if

$$x \preceq y,$$
$$d(u, Sx) = d(A, B)$$

and

$$d(v, Ty) = d(A, B),$$

implies that $u \leq v$, for all $u, x \in A$ and $v, y \in B$.

Definition 2.7 ([9])**.** Given non-self mapping $S : A \to B$ is said to be proximally increasing if it satisfies the condition:

$$x \preceq y,$$
$$d(u, Sx) = d(A, B)$$

and

$$d(v, Sy) = d(A, B),$$

implies that $u \leq v$, for all $u, v, x, y \in A$.

Definition 2.8 ([9])**.** Given non-self mapping $S : A \to B$ is said to be increasing if it satisfies the condition:

$$x \preceq y,$$

implies that $Sx \leq Sy$, for all $x, y \in A$.

Similarly, iteratively $S^n x \leq S^n y$, for $n \in \mathbb{N}$.

3. Main Results

Now, we are in position to define some notions and to prove some results.

Definition 3.1. A mapping $S : A \to B$ is said to be an α-ordered contraction if there exists $\beta \in \mathcal{F}$ and $\alpha : X \times X \to \mathbb{R}^+$ be a function such that

$$x_1 \preceq x_2 \;\Rightarrow\; \alpha(x_1, x_2)d(Sx_1, Sx_2) \leq \beta(d(x_1, x_2))d(x_1, x_2),$$

for all $x_1, x_2 \in A$.

We denote by $\mathcal{F}$ the class of all functions $\beta : [0, \infty) \to [0, 1)$ satisfying $\beta(t_n) \to 1$, implies $t_n \to 0$ as $n \to \infty$.

Definition 3.2. A mapping $S : A \to B$ is said to be an α-ordered proximal contraction if there exists $\beta \in \mathcal{F}$ and $\alpha : X \times X \to \mathbb{R}^+$ such that

$$x_1 \preceq x_2,$$

$$d(u_1, Sx_1) = d(A, B)$$

and

$$d(u_2, Sx_2) = d(A, B),$$

implies that $\alpha(x_1, x_2)d(u_1, u_2) \leq \beta d(x_1, x_2)$, for all $u_1, u_2, x_1, x_2 \in A$.

Definition 3.3. Given non-self mappings $S : A \to B$ and $T : B \to A$, the pair (S, T) is said to form an α-ordered proximal cyclic contraction if there exists a non-negative real number $k < 1$ such that

$$x \preceq y,$$

$$d(u, Sx) = d(A, B)$$

and

$$d(v, Ty) = d(A, B),$$

implies that $\alpha(x, y)d(u, v) \leq kd(x, y) + (1 - k)d(A, B)$, for all $u, x \in A$ and $v, y \in B$.

Theorem 3.4. *Let X be a non-empty set such that $(X, \preceq)$ is a partially ordered set and (X, d) is a complete metric space, $\alpha : X \times X \to \mathbb{R}^+$ be a function and let A, B be nonempty closed subsets of (X, d) such that A_0 and B_0 are non-void. Let $S : A \to B,\ T : B \to A$ and $g : A \cup B \to A \cup B$ satisfy the following conditions:*

(1) *S and T are α-ordered proximal contractions, proximally increasing;*
(2) *g is surjective isometry, its inverse is an increasing mapping;*
(3) *The pair (S, T) is proximally increasing, α-ordered proximal cyclic contraction;*
(4) *$S(A_0) \subseteq B_0,\ T(B_0) \subseteq A_0$;*
(5) *$A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$;*
(6) *S and T are α-proximal admissible maps;*
(7) *$\alpha(x_0, x_1) \geq 1$ for $x_0, x_1 \in X$;*
(8) *There exist elements x_0 and x_1 in A_0 and $y_0, y_1 \in B_0$ such that*

$$d(gx_1, Sx_0) = d(A, B),$$

and

$$d(gy_1, Ty_0) = d(A, B).$$

$$x_0 \preceq x_1, \quad y_0 \preceq y_1, \quad x_0 \preceq y_0.$$

(9) *If $\{x_n\}$ is an increasing sequence of elements in A converging to x, then $x_n \preceq x$, for all n. Also, if $\{y_n\}$ is an increasing sequence of elements in B converging to y, then $y_n \preceq y$ for all n.*

Then there exists a point $x \in A$ and a point $y \in B$ such that

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

Moreover, the sequence $\{x_n\}$ in A_0, defined by

$$d(gx_{n+1}, Sx_n) = d(A, B) \quad (n \geq 1),$$

converges to the element x, and the sequence $\{y_n\}$ in B_0, defined by

$$d(gy_{n+1}, Ty_n) = d(A, B) \quad (n \geq 1),$$

converges to the element y.

Proof. Since $\alpha(x_0, x_1) \geq 1$ for $x_0, x_1 \in X$, and for $x_1 \in A_0$, $S(A_0) \subseteq B_0$ there exists $x_2 \in A_0$ such that

$$d(x_2, Sx_1) = d(A, B),$$

for $x_2 \in A_0$, $S(A_0) \subseteq B_0$ there exists $x_3 \in A_0$ such that

$$d(x_3, Sx_2) = d(A, B).$$

Since S is α-proximal admissible mapping, then from

$$d(x_2, Sx_1) = d(A, B)$$
$$d(x_3, Sx_2) = d(A, B),$$

implies that $\alpha(x_2, x_3) \geq 1$.
Proceeding in the same manner, we have

$$\alpha(x_n, x_{n+1}) \geq 1,$$

for $n \in \mathbb{N}$. The hypothesis (8) implies the existence of elements x_0 and x_1 in A_0 such that

$$d(gx_1, Sx_0) = d(A, B) \text{ and } x_0 \preceq x_1.$$

In view of the fact that $S(A_0) \subseteq B_0$, also it is given that $A_0 \subseteq g(A_0)$, there exists an element $x_2 \in A_0$ such that

$$d(gx_2, Sx_1) = d(A, B).$$

Since S is proximally increasing, $gx_1 \preceq gx_2$. As the inverse of mapping g is increasing, so $x_1 \preceq x_2$. Again, since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists an element $x_3 \in A_0$ such that

$$d(gx_3, Sx_2) = d(A, B).$$

Continuing in a similar fashion, one can find an element x_n in A_0 such that

$$d(gx_n, Sx_{n-1}) = d(A, B) \text{ and } x_{n-1} \preceq x_n.$$

In light of the fact that g is an isometry and that S is α-ordered proximal contraction, we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &= d(gx_n, gx_{n+1}) \\ &\leq \alpha(x_{n-1}, x_n) d(gx_n, gx_{n+1}) \\ &\leq \beta(d(x_{n-1}, x_n)) d(x_{n-1}, x_n). \end{aligned}$$

This shows that $\{d(x_{n+1}, x_n)\}$ is a decreasing sequence and bounded below. Hence there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$. Suppose that $r > 0$. Observed that

$$\frac{d(x_{n+1}, x_n)}{d(x_n, x_{n-1})} \leq \beta(d(x_{n-1}, x_n)).$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \beta(d(x_n, x_{n-1})) = 1.$$

Since $\beta \in \mathcal{F}$, so that $r = 0$, which is a contradiction to our supposition and hence

$$\lim_{n\to\infty} d(x_n, x_{n-1}) = 0. \tag{3.1}$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not Cauchy sequence. Then there exists $\epsilon > 0$ and subsequences $\{x_{m_k}\}, \{x_{n_k}\}$ of $\{x_n\}$ such that for any positive integers $n_k > m_k \geq k$

$$r_k := d(x_{m_k}, x_{n_k}) \geq \epsilon,$$

$d(x_{m_k}, x_{n_k-1}) < \epsilon$, for any $k \in \{1, 2, 3, ...\}$.
For each $n \geq 1$, let $\alpha_n := d(x_{n+1}, x_n)$. Then, we have

$$\begin{aligned} \epsilon \leq r_k &= d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \epsilon + \gamma_{n_k-1}. \end{aligned} \tag{3.2}$$

Taking limit as $k \to \infty$, we get

$$\begin{aligned} \epsilon &\leq \lim_{k\to\infty} r_k \\ &< \epsilon + \lim_{k\to\infty} \gamma_{n_k-1}. \end{aligned}$$

(3.3)

It follows that

$$\epsilon \leq \lim_{k\to\infty} r_k < \epsilon + 0$$

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \tag{3.4}$$

Notice also that

$$\begin{aligned} \epsilon \leq r_k &= d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k+1}) + d(x_{n_k+1}, x_{n_k}) + d(x_{m_k+1}, x_{n_k+1}) \\ &= \gamma_{m_k} + \gamma_{n_k} + d(x_{m_k+1}, x_{n_k+1}) \\ &= \gamma_{m_k} + \gamma_{n_k} + d(gx_{m_k+1}, gx_{n_k+1}) \\ &\leq \gamma_{m_k} + \gamma_{n_k} + \alpha(x_{m_k}, x_{n_k}) d(gx_{m_k+1}, gx_{n_k+1}) \\ &\leq \gamma_{m_k} + \gamma_{n_k} + \beta(d(x_{m_k}, x_{n_k})) d(x_{m_k}, x_{n_k}), \end{aligned}$$

implies that

$$\frac{d(x_{m_k}, x_{n_k}) - \gamma_{m_k} - \gamma_{n_k}}{d(x_{m_k}, x_{n_k})} \leq \beta(d(x_{m_k}, x_{n_k})).$$

Taking limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} \beta(d(x_{m_k}, x_{n_k})) = 1,$$

since $\beta \in \mathcal{F}$, so

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = 0.$$

Hence $\epsilon = 0$, which is a contradiction. So $\{x_n\}$ is a Cauchy sequence and converges to some element $x \in A$. So, we have $x_n \preceq x$ for any n.
Similarly, in view of the fact that $T(B_0) \subseteq A_0$ and $B_0 \subseteq g(B_0)$, it is ascertained that there is a sequence $\{y_n\}$ of elements in B_0 such that

$$(gy_{n+1}, Ty_n) = d(A, B).$$

Since T is proximally increasing and the inverse of g is an increasing mapping, $y_n \preceq y_{n+1}$. Since g is an isometry and T is an α-ordered proximal contraction, it follows that

$$\begin{aligned} d(y_n, y_{n+1}) &= d(gy_n, gy_{n+1}) \\ &\leq \alpha(y_{n-1}, y_n) d(gy_n, gy_{n+1}) \\ &\leq \beta(d(y_{n-1}, y_n)) d(y_{n-1}, y_n). \end{aligned}$$

Similarly, there exists a Cauchy sequence $\{y_n\}$ such that it converges to some element $y \in B$. Therefore, it follows that $y_n \preceq y$ for all n. Further, since the pair (S, T) is proximally increasing and the inverse of g is an increasing mapping, we have $x_n \preceq y_n$, for all n. Since the pair (S, T) forms an α-ordered proximal cyclic contraction and g is an isometry, it follows that

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &= d(gx_{n+1}, gy_{n+1}) \\ &\leq \alpha(x_n, y_n) d(gx_{n+1}, gy_{n+1}) \\ &\leq kd(x_n, y_n) + (1-k)d(A, B). \end{aligned}$$

Letting $n \to \infty$, it follows that

$$d(x, y) = kd(x, y) + (1-k)d(A, B)$$

$$\Rightarrow d(x, y) = d(A, B). \tag{3.5}$$

Thus $x \in A_0$ and $y \in B_0$. Since $S(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$, there exists $u \in A$ and $v \in B$ such that

$$\left. \begin{aligned} d(u, Sx) = d(A, B) \\ d(v, Ty) = d(A, B). \end{aligned} \right\} \tag{3.6}$$

Since S is α-ordered proximal contraction, we get from $d(u, Sx) = d(A, B)$ and $d(gx_{n+1}, Sx_n) = d(A, B)$ as

$$d(u, gx_{n+1}) \leq \alpha(x_n, x) d(u, gx_{n+1}) \leq \beta(d(x, x_n)) d(x, x_n). \tag{3.7}$$

Letting $n \to \infty$ in the above inequality, we have

$$d(u, gx) = 0$$

and so $u = gx$. It follows that $\{gx_n\}$ converges to u. Further, as g is an isometry, the sequence $\{gx_n\}$ converges to gx as well. Thus, we write as

$$d(gx, Sx) = d(u, Sx) = d(A, B).$$

In the same manner, we have $v = gy$ and so it can be prove that

$$d(gy, Ty) = d(v, Ty) = d(A, B).$$ □

Example 3.5. Consider $X = \mathbb{R}^2$ be an Euclidean metric space with partially ordered set X. Let us define the sets $A = \{1\} \times [0, \infty)$ and $B = \{2\} \times [0, \infty)$. Take $A_0 = A$ and $B_0 = B$. Obviously, $d(A, B) = 1$. Let $g : A \cup B \to A \cup B$ be an identity mapping, the mapping g is surjective isometry, its inverse is an increasing mapping, $A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$.
Let us define $S : A \to B$ and $T : B \to A$ as:

$$S(1, x) = (2, \frac{x}{x+1}),$$

and

$$T(2, x) = (1, \frac{x}{x+1}).$$

where $(1, x) \in A,\ (2, x) \in B$ and $x \in [0, \infty)$.
Let $\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ defined as:

$$\alpha(x, y) = \begin{cases} 1 & if \text{ x=1 } or \text{ x=2 } and\ y \in [0, \infty), \\ 0 & elsewhere. \end{cases}$$

Clearly, S and T are proximally increasing and α-ordered proximal contractions with these assumptions such that $S(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$. The pair (S, T) is proximally increasing, α-ordered proximal cyclic contraction. Thus, all other assumptions of the Theorem (3.1) are also satisfied. Finally, very easily one can say that the element $(1, 0)$ in A and the element $(2, 0)$ in B satisfy the conclusion of the preceding result.

If g is the identity mapping in the Theorem 3.4, then we obtain the following:

Corollary 3.6. *Let X be a non-empty set such that $(X, \preceq)$ is a partially ordered set and (X, d) is a complete metric space, $\alpha : X \times X \to \mathbb{R}^+$ be a function and let A, B be nonempty closed subsets of (X, d) such that A_0 and B_0 are non-void. Let $S : A \to B,\ T : B \to A$ satisfy the following conditions:*

1. *S and T are α-ordered proximal contractions, proximally increasing;*
2. *The pair (S, T) is proximally increasing, α-ordered proximal cyclic contraction;*
3. *$S(A_0) \subseteq B_0,\ T(B_0) \subseteq A_0$;*
4. *S and T are α-proximal admissible maps;*
5. *$\alpha(x_0, x_1) \geq 1$ for $x_0, x_1 \in X$;*

(6) *There exist elements x_0 and x_1 in A_0 and $y_0, y_1 \in B_0$ such that*

$$d(gx_1, Sx_0) = d(A, B),$$

and

$$d(gy_1, Ty_0) = d(A, B).$$

$$x_0 \preceq x_1, \quad y_0 \preceq y_1, \quad x_0 \preceq y_0.$$

(7) *If $\{x_n\}$ is an increasing sequence of elements in A converging to x, then $x_n \preceq x$, for all n. Also, if $\{y_n\}$ is an increasing sequence of elements in B converging to y, then $y_n \preceq y$ for all n.*

Then there exists a point $x \in A$ and a point $y \in B$ such that

$$d(x, Sx) = d(y, Ty) = d(x, y) = d(A, B).$$

Moreover, the sequence $\{x_n\}$ in A_0, defined by

$$d(x_{n+1}, Sx_n) = d(A, B) \quad (n \geq 1),$$

converges to the element x, and the sequence $\{y_n\}$ in B_0, defined by

$$d(y_{n+1}, Ty_n) = d(A, B) \quad (n \geq 1),$$

converges to the element y.

If $\alpha(x_0, x_1) = 1$ and $\beta(t) = k$, where $k \in [0, 1)$ in the Corollary (3.1), then we obtain the following corollary of [9].

Corollary 3.7. *Let X be a non-empty set such that $(X, \preceq)$ is a partially ordered set and (X, d) is a complete metric space, let A, B be nonempty closed subsets of (X, d) such that A_0 and B_0 are non-void. Let $S : A \to B$, $T : B \to A$ satisfy the following conditions:*

(1) *S and T are ordered proximal contractions, proximally increasing;*
(2) *The pair (S, T) is proximally increasing, ordered proximal cyclic contraction;*
(3) *$S(A_0) \subseteq B_0$, $T(B_0) \subseteq A_0$;*
(4) *There exist elements x_0 and x_1 in A_0 and $y_0, y_1 \in B_0$ such that*

$$d(gx_1, Sx_0) = d(A, B),$$

and

$$d(gy_1, Ty_0) = d(A, B).$$

$$x_0 \preceq x_1, \quad y_0 \preceq y_1, \quad x_0 \preceq y_0.$$

(5) *If $\{x_n\}$ is an increasing sequence of elements in A converging to x, then $x_n \preceq x$, for all n. Also, if $\{y_n\}$ is an increasing sequence of elements in B converging to y, then $y_n \preceq y$ for all n.*

Then there exists a point $x \in A$ and a point $y \in B$ such that

$$d(x, Sx) = d(y, Ty) = d(x, y) = d(A, B).$$

Moreover, the sequence $\{x_n\}$ in A_0, defined by

$$d(x_{n+1}, Sx_n) = d(A, B) \quad (n \geq 1),$$

converges to the element x*, and the sequence* $\{y_n\}$ *in* B_0*, defined by*

$$d(y_{n+1}, Ty_n) = d(A, B) \quad (n \geq 1),$$

converges to the element y*.*

By taking $\alpha(x_0, x_1) = 1$ and $\beta(t) = k$, where $k \in [0, 1)$ in the Theorem (3.1), we get the main result of [9] as:

Corollary 3.8. *Let* X *be a non-empty set such that* $(X, \preceq)$ *is a partially ordered set and* (X, d) *is a complete metric space and let* A, B *be nonempty closed subsets of* (X, d) *such that* A_0 *and* B_0 *are non-void. Let* $S : A \to B,\ T : B \to A$ *and* $g : A \cup B \to A \cup B$ *satisfy the following conditions:*

1. *S and T are ordered proximal contractions, proximally increasing;*
2. *g is surjective isometry, its inverse is an increasing mapping;*
3. *The pair (S, T) is proximally increasing, ordered proximal cyclic contraction;*
4. *$S(A_0) \subseteq B_0,\ T(B_0) \subseteq A_0$;*
5. *$A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$;*
6. *There exist elements x_0 and x_1 in A_0 and $y_0, y_1 \in B_0$ such that*

$$d(gx_1, Sx_0) = d(A, B),$$

and

$$d(gy_1, Ty_0) = d(A, B).$$

$$x_0 \preceq x_1, \quad y_0 \preceq y_1, \quad x_0 \preceq y_0.$$

7. *If $\{x_n\}$ is an increasing sequence of elements in A converging to x, then $x_n \preceq x$, for all n. Also, if $\{y_n\}$ is an increasing sequence of elements in B converging to y, then $y_n \preceq y$ for all n.*

Then there exists a point $x \in A$ *and a point* $y \in B$ *such that*

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

Moreover, the sequence $\{x_n\}$ *in* A_0*, defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B) \quad (n \geq 1),$$

converges to the element x*, and the sequence* $\{y_n\}$ *in* B_0*, defined by*

$$d(gy_{n+1}, Ty_n) = d(A, B) \quad (n \geq 1),$$

converges to the element y*.*

If we take $A = B = X$, and $\alpha(x_0, x_1) = 1$ in our main result (3.3), we get the following fixed point corollary, which is also the result of [9].

Corollary 3.9. *Let* X *be a non-empty set such that* $(X, \preceq)$ *is a partially ordered set and* (X, d) *is a complete metric space. Let* $S : X \to X$ *satisfy the following conditions:*

1. *S is increasing, ordered contraction;*
2. *There exist elements x_0 in A such that $x_0 \preceq Sx_0$;*
3. *If $\{x_n\}$ is an increasing sequence of elements in A converging to x, then $x_n \preceq x$, for all n.*

Then S *has a fixed point.*

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