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The dynamical look at the subsets of a group

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Abstract

We consider the action of a group G on the family $\mathcal{P}(G)$ of all subsets of G by the right shifts $A \mapsto Ag$ and give the dynamical characterizations of thin, n-thin, sparse and scattered subsets.
For $n \in \mathbb{N}$, a subset A of a group G is called n-thin if $g_0A \cap \cdots \cap g_nA$ is finite for all distinct $g_0, \ldots, g_n \in G$. Each n-thin subset of a group of cardinality $\aleph_0$ can be partitioned into n 1-thin subsets but there is a 2-thin subset in some Abelian group of cardinality $\aleph_2$ which cannot be partitioned into two 1-thin subsets. We eliminate the gap between $\aleph_0$ and $\aleph_2$ proving that each n-thin subset of an Abelian group of cardinality $\aleph_1$ can be partitioned into n 1-thin subsets.

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1. Introduction

Let G be a group with the identity e, $\mathcal{P}(G)$ denotes the family of all subsets of G, $[G]^{<\omega} = \{F \subseteq G : F \text{ is finite}\}$, $[G]^n = \{F \subseteq G : |F| = n\}$, $n \in \mathbb{N}$.

We say that a subset A of G is

- *thin* if $A \cap gA$ is finite for every $g \in G \setminus \{e\}$;
- *n-thin* if $g_0A \cap \cdots \cap g_nA$ is finite for any distinct $g_0, \ldots, g_n \in G$;
- *sparse* if, for every infinite subset $X \subseteq G$, there exists a finite subset $F \subset S$ such that $\bigcap_{g \in F} gA$ is finite;

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- *scattered* if, for any subset $B \subseteq A$, there exists $F \in [G]^{<\omega}$ such that, for each $H \in [G]^{<\omega}$, $F \cap H = \varnothing$, we can find $b \in B$ such that $Hb \cap B = \varnothing$;
- *thick* if, for any $F \in [G]^{<\omega}$, there exists $g \in G$ such that $Fg \subseteq A$.

In [3] C. Chou used thin subsets to prove that there are $2^{2^{|G|}}$ distinct left invariant Banach measures on each infinite amenable groups.

Clearly, thin subsets are precisely 1-thin subsets, n-thin subsets appeared in [10] in attempt to characterize the ideal in the Boolean algebra $\mathcal{P}(G)$ generated by thin subsets.

Sparse subsets appeared in [4] for characterization of strongly prime ultrafilters in the semigroup G^* of free ultrafilters on G and studied in [9].

Scattered subsets were introduced in [1] as asymptotic counterparts of scattered topological spaces.

Unexplicitely, thick subsets were used in [11] to partition of an infinite totally bounded group G into $|G|$ dense subsets. As to our knowledge, the name "thick subset" appeared in [2].

For every infinite group G, we have

$$\text{thin} \Rightarrow 2\text{-thin} \Rightarrow \cdots \Rightarrow n\text{-thin} \Rightarrow \cdots \Rightarrow \text{sparse} \Rightarrow \text{scattered}$$

and none of these arrows could be reversed. For "scattered $\not\Rightarrow$ sparse" see Remark 3.6.

More on these subsets and their applications one can find in the surveys [12], [16].

In this paper, we identify $\mathcal{P}(G)$ with $\{0,1\}^G$, endow $\mathcal{P}(G)$ with the product topology and consider the action of G on $\mathcal{P}(G)$ by the right shifts $A \mapsto Ag$. After short preliminary section 2, we give the dynamical characterizations to all above defined subsets in section 3. It should be mentioned that the dynamical approach is especially effective for finite partitions of groups [5].

By [10], every n-thin subset of a countable group G can be partitioned into n thin subsets (some cells of the partitions could be empty). Answering the question from [10], G. Bergman (see [15]) constructed an Abelian group G of cardinality $\aleph_2$ and a 2-thin subset of G which cannot be partitioned into two thin subsets. On the other hand [15], every n-thin subset of an Abelian group of cardinality $\aleph_m$ can be partitioned into n^{m+1} thin subsets but there is a 2-thin subset in some group of cardinality $\aleph_\omega$ which cannot be finitely partitioned. In section 4 we eliminate the gap between $\aleph_0$ and $\aleph_2$ proving that every n-thin subsets of an Abelian group of cardinality $\aleph_1$ is a union of n thin subsets.

2. Some dynamics

Let G be a group. A topological space X is called a G-*space* if there is the action $X \times G \to X : (x, g) \mapsto xg$ such that, for each $g \in G$, the mapping $X \to X : x \mapsto xg$ is continuous.

Given any $x \in X$ and $U \subseteq X$, we set

$$[U]_x = \{g \in G : xg \in U\}$$

and denote

$$O(x) = \{xg : g \in G\}, \quad T(x) = clO(x),$$

$$W(x) = \{y \in T(X) : [U]_x \text{ is infinite for each neighbourhood } U \text{ of } y\}.$$

We recall also that $x \in X$ is a *recurrent* point if $x \in W(x)$.

Now we consider a group G, identify $\mathcal{P}(G)$ with the space $\{0,1\}^G$ and endow $\mathcal{P}(G)$ with the product topology. Thus, the subsets

$$U(F, H) = \{A \subseteq G : F \subseteq A, H \cap A = \varnothing\},$$

where $F \in [G]^{<\omega}$, $H \in [G]^{<\omega}$, form the base for the open sets on $\mathcal{P}(G)$.

In what follows, we consider $\mathcal{P}(G)$ as a G-space with the action defined by

$$A \mapsto Ag, \quad Ag = \{ag : a \in A\}.$$

We say that a subset A of G is *recurrent* if A is a recurrent point in $(\mathcal{P}(G), G)$.

3. Characterizations

All groups in this sections are supposed to be infinite.

Theorem 3.1. *For a subset A of a group G, the following statements hold*

(i) *A is finite if and only if $W(A) = \varnothing$;*

(ii) *A is thick if and only if $G \in W(A)$.*

Proof. (i) It suffices to note that A is finite if and only if, for every $x \in G$, the set $\{g \in G : x \in Ag\}$ is finite.

(ii) Suppose that $G \in W(A)$ and take an arbitrary finite subset F of G. Since $U(F, \varnothing)$ is a neighborhood of G in $\mathcal{P}(G)$, there exists $g \in G$ such that $Ag \in U(F, \varnothing)$, so $Fg^{-1} \subseteq A$ and A is thick.

Assume that A is thick and take an arbitrary finite subset F of G. Then we choose an injective sequence $(g_n)_{n\in\omega}$ in G such that $Fg_i \cap Fg_j = \varnothing$ for all distinct $i, j \in \omega$. For each in $n \in \omega$, we take $h_n \in G$ such that $(Fg_0 \cup \dots \cup Fg_n)h_n \subseteq A$, so $F \subseteq Ah_n^{-1}g_i^{-1}$, $i \in \{0, \dots, n\}$. It follows that $U(F, \varnothing)$ contains infinitely many points of the orbit $O(A)$, so $G \in W(A)$. □

Theorem 3.2. *For a subset A of a group G, the following statements hold*

(i) *A is n-thin if and only if $|Y| \leq n$ for every $Y \in W(A)$;*

(ii) *A is sparse if and only if each subset $Y \in W(A)$ is finite;*

(iii) *A is scattered if and only if, for every subset $B \subseteq A$ there exists $Y \in [G]^{<\omega}$ in the closure of $\{Bb^{-1} : b \in B\}$.*

Proof. Suppose that A is n-thin but $|Y| > n$ for some $Y \in W(A)$. Let $\{y_0, \dots, y_n\}$ be distinct elements from Y. Since $Y \in W(A)$ and the set $U(\{y_0, \dots, y_n\}, \varnothing)$ is a neighborhood of Y, the set $W = \{g \in G : \{y_0, \dots, y_n\} \subseteq Ag\}$ is infinite. We note that $W = \{g \in G : \{y_0g^{-1}, \dots, y_ng^{-1}\} \subseteq A\} = \{g \in G : g^{-1} \in y_0^{-1}A \cap \dots \cap y_n^{-1}A\}$. Hence, A is not n-thin.

Suppose that A is not n-thin. We take $g_0, \dots, g_n \in G$ such that the subset $B = g_0A \cap \dots \cap g_nA$ is infinite. If $b \in B$ then $\{g_0^{-1}b, \dots, g_n^{-1}b\} \subseteq A$ so $g_0^{-1}, \dots, g_n^{-1} \subseteq Ab^{-1}$. We take an arbitrary limit point L of the set $\{Ab^{-1} : b \in B\}$. Then $L \in W(A)$ but $\{g_0^{-1}, \dots, g_n^{-1}\} \subseteq L$, so $|L| > n$.

(ii) Suppose that A is sparse but some subset $Y \in W(A)$ is infinite. We take a countable subset $\{y_n : n \in \omega\}$ of Y and put $U_n = U(\{y_0, \dots, y_n\}, \varnothing)$. Then we choose an injective sequence $(g_n)_{n\in\omega}$ in G such that $g_n \in [U_n]_A$ for each $n \in \omega$. We note that $\{y_0, \dots, y_n\} \subseteq Ag_n$ so $g_n^{-1} \in y_0A \cap \dots \cap y_nA$. We put $X = \{y_n^{-1} : n \in \omega\}$. Then $\bigcap_{g\in F} gA$ is infinite for each finite subset F of X. Hence, A is not sparse.

Assume that each subset $Y \in W(A)$ is finite but A is not sparse. Then there exists an injective sequence $(g_n)_{n\in\omega}$ in G such that $g_0A \cap \dots \cap g_nA$ is infinite for each $n \in \omega$. We choose an injective sequence $(y_n)_{n\in\omega}$ in G such that $y_n \in g_0A \cap \dots \cap g_nA$, so $\{g_0^{-1}, \dots, g_n^{-1}\} \subseteq Ay_n^{-1}$ for each $n \in \omega$. Let L be an arbitrary limit point of $\{Ay_n^{-1} : n \in \omega\}$. Then $\{g_n^{-1} : n \in \omega\} \subseteq L$ and $L \in W(A)$.

(iii) Suppose that A is scattered and B is a subset of G. We choose corresponding $F \in [G]^{<\omega}$and take an arbitrary $H \in [G]^{<w}$ such that $F \cap H = \varnothing$. By the definition, there exists $b_H \in B$ such that $Hb_H \cap B = \varnothing$ so $H \cap Bb_H^{-1} = \varnothing$. It follows that the closure of $\{Bb_H^{-1} : H \in [G]^{<\omega}, H \cap F = \varnothing\}$ contains some point Y such that $Y \subseteq F$.

To prove the converse statement, given $B \subseteq A$, we choose $Y \in [G]^{<\omega}$ in the closure of $\{Bb^{-1} : b \in B\}$. Then, for every $H \in [G]^{<\omega}$, $H \cap Y = \varnothing$, there exists $b_H \in B$ such that $H \cap Bb_H^{-1} = \varnothing$. Hence, A is scattered. □

Let $(g_n)_{n\in\omega}$ be an injective sequence in G. The set

$$FP(g_n)_{n\in\omega} = \{g_{i_1}g_{i_2}\dots g_{i_n} : 0 \le i_1 < i_2 < \dots < i_n < \omega\}$$

is called an *FP-set*. By [8, Theorem 5.12], a subset A of G contains an FP-set if and only if A is a member of some idempotent of the semigroup G^* of all free ultrafilters on G.

Given a sequence $(b_n)_{n\in\omega}$ in G, the set

$$\{g_{i_1}g_{i_2}\dots g_{i_n}b_{i_n} : 0 \le i_1 < i_2 < \dots < i_n < \omega\}$$

is called a *(right) piecewise shifted FP-set* [1].

Theorem 3.3. *For a subset A of a group G, the following statements hold*

- (i) *A is not n-thin if and only if there exist $F \in [G]^{n+1}$ and an injective sequence $(x_n)_{n<\omega}$ in G such that $Fx_n \subseteq A$ for each $n \in \omega$;*
- (ii) *A is not sparse if and only if there exists two injective sequences $(x_n)_{n<\omega}$ and $(y_n)_{n\in\omega}$ such that $x_ny_m \in A$ for each $0 \le n \le m < \omega$;*
- (iii) *A is not scattered if and only if A contains a piecewise shifted FP-set;*
- (iv) *A contains a recurrent subset if and only if there exists $x \in A$ and an FP-set Y such that $xY \subseteq A$.*

Proof. The statements (i) and (ii) follow easily from the definitions of n-thin and sparse subsets, (iii) was proved in [1, Theorem 1].

To prove (iv), we suppose that A contains a recurrent subset B. We take an arbitrary $x \in B$ and choose inductively an injective sequence $(g_n)_{n\in\omega}$ in G such that, for each $n \in \omega$,

$$xFP(g_i)_{i=0}^{n-1} \subseteq Bg_n^{-1} \cap B,$$

where $FP(g_i)_{i=0}^{n-1} = \{g_{i_0} g_{i_1} \dots g_{i_k} : 0 \le i_0 < i_1 < \dots < i_k < n\}$. After ω steps, we have $xFP(g_n)_{n\in\omega} \subseteq B$.

If $xFP(g_n)_{n\in\omega} \subseteq A$ then, passing to some subsequence $(h_n)_{n\in\omega}$ of $(g_n)_{n\in\omega}$, we make $xFP(h_n)_{n\in\omega}$ to be recurrent. $\square$

Corollary 3.4. *Every scattered subset of a group G has no recurrent points.*

Proof. We observe that, by corresponding definitions, if a subset A is scattered or has a recurrent subset then the same is valid for each subset $x^{-1}Ax$, $x \in G$.

Suppose that a subset A of G has some recurrent subset. By Theorem 3.3(iv), there exist $x \in A$ and an FP-set Y such that $xY \subseteq A$. Then $x^{-1}Ax$ contains a piecewise shifted FP-set Yx so, by Theorem 3.3(iii), $x^{-1}Ax$ is not scattered and A is not scattered. $\square$

Remark 3.5. By [1, Theorem 2], every scattered subset A of an amenable group G is absolute null, i.e. $\mu(A) = 0$ for every left invariant Banach measure μ on G. But this statement could not be generalized to subsets with no recurrent points. By [7, Theorem 11.6], there is a subset A of $\mathbb{Z}$ of positive Banach measure such that $(a + B) \setminus A \neq \varnothing$ for any FP-set B. By Theorem 3.3(iv), A has no recurrent subsets.

Remark 3.6. Let G be an arbitrary infinite group. We construct two injective sequences $(x_n)_{n\in\omega}$, $(y_n)_{n\in\omega}$ in G such the set $\{x_n y_m : 0 \le n \le m < \omega\}$ is scattered. By Theorem 3.3(ii), this subset is not sparse.

We choose a countable subgroup H of G and write H as the union $H = \bigcup_{n\in\omega} H_n$ of finite subsets such that $H_0 = \{e\}$, $H_n = H_n^{-1}$, $H_n \subset H_{n+1}$ for each $n \in \omega$. We take arbitrary $x_0, y_0 \in H$ and suppose that we have chosen $x_0, \dots, x_n$ and $y_0, \dots .y_n$ in H. At the next step, we choose y_{n+1} and then x_{n+1} to satisfy

(1) $H_{n+1} x_i y_{n+1} \cap \{x_k y_j : 0 \le k \le j \le n\} = \varnothing,\ i \in \{0, \dots, n\}$;

(2) $H_{n+1} x_{n+1} y_{n+1} \cap (\{x_k y_j : 0 \le k \le j \le n\} \cup \{x_i y_{n+1} : 0 \le i \le n\}) = \varnothing$.

After ω steps, we put $A = \{x_n y_m : 0 \le n \le m < \omega\}$. To show that A is scattered, we take an arbitrary infinite subset Y of A and consider two cases.

Case 1. For each $k \in \omega$, there exist $n, m \in \omega$ such that $n > k$ and $x_n y_m \in Y$. Let K be a finite subset of G such that $e \notin K$. We take $n \in \omega$ such that $K \cap H \subseteq H_n$. and $x_n y_m \in Y$ for some $m \in \omega$. By (2), $Kx_n y_m \cap A = \varnothing$, in particular, $Kx_n y_m \cap Y = \varnothing$.

Case 2. There exists $k \in \omega$ such that $Y \subseteq \{x_n y_m : n \le k, m \ge n\}$. We take an arbitrary $K \in [G]^{<\omega}$ such that $K \cap H_k = \varnothing$. Then we choose $n, m \in \omega$, such that $K \cap H \subseteq H_n$, $n > k$ and $x_n y_m \in Y$. By (1), $Kx_n y_m \cap \{x_n y_m : n \le k, m \ge n\} = \varnothing$, in particular, $Kx_n y_m \cap Y = \varnothing$.

4. Partitions into thin subsets

We fix a subset T of a group G and, for every $g \in G \setminus \{e\}$, consider the *orbital graph* Γ_g with the set of vertices T and the set of edges

$$E_g = \{\{s, t\} \in [T]^2 : st^{-1} \in \{g, g^{-1}\}\}.$$

We say that a graph (T, E) is an *orbital companion* of T if, for each $g \in G \setminus \{e\}$, the set E contains all but finitely many edges from E_g so $E \setminus E_g$ if finite.

For $n \in \mathbb{N}$, a graph Γ is called *n-discrete* if each connected component of Γ has no more than n vertices.

Given a graph (V, E), we take any subsets E', E'', from E such that $E = E' \cup E''$ and say that (V, E) is a union of the graphs (V, E') and (V, E'').

In this section we use the following equivalent definition of an n-thin subset T of a group (see Theorem 3.3): for every $F \in [G]^{<\omega}$, there exists $H \in [G]^{<\omega}$ such that, for every $g \in G \setminus H$, we have $|Fg \cap T| \leq n$.

Lemma 4.1. *Every n-thin subset T of a countable group G has an n discrete orbital companion Γ.*

Proof. We write G as the union of an increasing chain $\{F_i : i < \omega\}$ of finite subsets such that $e \in F_0$, $F_i \subset F_{i+1}$ and $F_i = F_i^{-1}$. Then we choose a chain $\{V_i : i < \omega\}$ of finite subsets of G such that, for any $i < \omega$ and $g \in G \setminus V_i$, we have $V_i \subset V_{i+1}$ and $|F_i^m g \cap T| \leq n$. We define the set E of edges of Γ by the rule:

$$\{s, t\} \in E \Leftrightarrow \exists k : s \notin V_K \text{ and } st^{-1} \in F_k.$$

Given any $g \in G \setminus \{e\}$, we pick $k < \omega$ such that $g \in F_k$. If $t \in T$ and $gt \in T$ then either $\{t, gt\} \subset V_k$ or $(t, gt) \in E$. Since V_k is finite, we conclude that Γ is an orbital companion of T.

Suppose that Γ is not n-discrete and choose a subset $S \in [T]^{n+1}$ such that the induced graph Γ_S is connected. We find the minimal number k such that, for any $\{s_1, s_2\} \in [S]^2 \cap E$, $s_1 s_2^{-1} \in F_k$ and so there are $\{s, s'\} \in [S]^2 \cap E$ such that $s' s^{-1} \in F_k \setminus F_{k-1}$. It follows that $s \in G \setminus V_k$ and $S \subset F_k^n s$ but $|S| > n$ and we get a contradiction with the choice of V_k. $\square$

Lemma 4.2. *Let G be an Abelian group of cardinality $\aleph_1$ and let T be an n-thin subset of G. Then some orbital companion of T is a union of two n-discrete graphs.*

Proof. Applying Lemma 2 from [15], we represent G as the union of increasing chain $\{G_i : i < \omega_1\}$ of countable subgroups such that

$$(*) \textit{ for any } i < \omega_1 \textit{ and } g \in G_{i+1} \setminus G_i,\ |G_i g \cap T| \leq n.$$

For every $i < \omega_1$, we consider the n-thin subset $T_i = T \cap (G_{i+1} \setminus G_i)$ of G_{i+1} and choose n-discrete companion (T_i, E_i) of T_i given by Lemma 4.1. We denote by E_i' the union of E_i and the set of all $\{t, t'\} \in [T_i]^2$ such that $t^{-1}t' \in G_i$. Since G is Abelian, we have $G_i g = gG_i$ and by $(*)$, (T_i, E_i') is a union of two n-discrete graphs (T_i, E_i) and $(T_i, E_i' \setminus E_i)$.

We put $E = \bigcup_{i<\omega_1} E_i'$ and note that (T, E) is a union of n-discrete graphs because the subsets $\{T_i : i < \omega_1\}$ are pairwise disjoint.

Now it remains to prove that (T, E) is an orbital companion of T. Given $g \in G \setminus \{e\}$, we choose $i < \omega_1$ such that $g \in G_{i+1} \setminus G_i$. Since G is Abelian, $T \cap Tg^{-1} = T \cap g^{-1}T$ and then the set $\{t \in T \cap g^{-1}T : (t, gt) \notin E\}$ is the union of the following sets

$$\bigcup_{i<j<\omega_1} \{t \in T_j \cap g^{-1}T_j : (t, gt) \notin E\}, \quad \{t \in T_i \cap g^{-1}T_i : (t, gt) \notin E\},$$

$$\{t \in T_i \cap g^{-1}T : tg \in G_i, (t, gt) \in E\}.$$

The first subset is empty by the choice of E'_j, the second is finite by the choice of Γ_i, the third subset is finite by $(*)$. □

Lemma 4.3. *If a graph Γ is a union of two n-discrete graphs then the chromatic number $\chi(\Gamma)$ does not exceed n.*

Proof. We use the induction by n. For $n = 1$, the statement is evident. Let $\Gamma = (V, E)$ be the union of two n-discrete graphs $\Gamma_1 = (V, E_1)$ and $\Gamma_2 = (V, E_2)$. Extending if necessary the sets V, E_1, E_2, we may suppose that every connected component of Γ_1 and Γ_2 is a complete graph on n vertices. Applying the Hall's Theorem [6], we get a subset X of V such that $|X \cap K| = 1$ for every $K \in U_1 \cup U_2$. We fix some color to color all vertices from X, delete X from V and apply the inductive assumption to the obtained graph. □

Theorem 4.4. *Every n-thin subset T of an Abelian group G of cardinality $\aleph_1$ can be partitioned into n thin subsets.*

Proof. We take an orbital companion Γ of T given by Lemma 4.2. By Lemma 4.3, $\chi(\Gamma) \le n$ so any coloring of T witnessing $\chi(\Gamma)$ gives the desired partition of T. □

Remark 4.5. We do not know how to extend Theorem 4.4 to any group of cardinality $\aleph_1$.

Theorem 4.6. *Let A be an infinite subset of a group G such that the set $\Delta(A) = \{g \in G : Ag \cap A$ is infinite$\}$ is finite. Then A can be partitioned into $\le |\Delta(A)|$ thin subsets.*

Proof. We consider a graph Γ with the set of vertices A and the set of edges $E = \{\{x, y\} \in [A]^2 : xy^{-1}\}$. Since the local degree of each vertex of Γ does not exceed $|\Delta(A)| - 1$, the chromatic number $\chi(F) \le |\Delta(A)|$ and the partition of A witnessing $\chi(F)$ is desired. □

Clearly, an infinite subset A of G is thin if and only if $\Delta(A) = \{e\}$. On the other hand [9, Theorem 3.2], every infinite group G contains a 2-thin subset such that $\Delta(A)$ is infinite.

Remark 4.7. By [1, Theorem 3], every infinite group G can be partitioned into $\aleph_0$ scattered subsets. For a group G we denote by $\mu(G)$ and $\eta(G)$ the minimal cardinalitis of partitions of G into thin and sparse subsets respectivly. By [13], for an infinite group G, $\mu(G) = |G|$ if $|G|$ is a limit cardinal and $\mu(G) = \gamma$ if $|G| = \gamma^+$. It is an open question how to detect $\eta(G)$. Some very preliminary results are in [14], but we do not know even if $\eta(G) = \aleph_0$ for any group G of cardinality $2^{\aleph_0}$.

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