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F-points in countably compact spaces

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Abstract. Answering a question of A. V. Arhangel′skiĭ, we show that any extremally disconnected subspace of a compact space with countable tightness is discrete.

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1. The results

In [1] Arhangel′skiĭ noticed that under PFA any extremally disconnected subspace of a compact space with countable tightness is discrete. Then, he posed the natural question whether this result holds in ZFC.

The aim of this short note is just to provide the full answer to such question. We actually manage to generalize the result by weakening the compactness assumption.

Henceforth, all space are assumed to be Tychonoff spaces, unless otherwise specified.

Recall that, for a given topological space X,the tightness at the point $x \in X$, denoted by $t(x, X)$, is the smallest cardinal κ such that, whenever $x \in \overline{A} \subseteq X$, there exists a set $B \subseteq A$ satisfying $|B| \leq \kappa$ and $x \in \overline{B}$.

We say that $x \in X$ is a F-point if there are no disjoint open F_σ-sets U and V such that $x \in \overline{U} \cap \overline{V}$.

It is evident that X is a F-space if and only if each $x \in X$ is a F-point in X.

If Y is a subspace of the space X, we say that Y is countably compact in X provided that every infinite subset of Y has an accumulation point in X. Of course, a space is countably compact if and only if it is countably compact in itself. If Y is countably compact in X, then $\overline{Y}$ is pseudocompact but not necessarily countably compact.

The next Lemma is essentially a combination of Theorem 1 in [2] and Proposition 3.1 in [3].

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Lemma 1.1. *Let X be a space, $Y \subseteq X$ and y be a non-isolated point of Y. If Y is countably compact in X and $t(y,X) \leq \aleph_0$ then y is in the closure of some countable discrete subspace of Y.*

Proof. Let A be a countable subset of $Y \setminus \{y\}$ such that $y \in \overline{A}$ and fix a family $\mathcal{U} = \{U_n : n \in \omega\}$ of closed neighbourhoods of y in X chosen so that $A \cap \bigcap \mathcal{U} = \varnothing$. Let S be the subset of X consisting of all points which are in the closure of some countable set $\{z_n : n \in \omega\}$ where $z_n \in U_n \cap A$ for each n. Since Y is countably compact in X and $y \in \overline{U_n \cap A}$ for each n, it easily follows that $y \in \overline{S}$. As $t(y,X) \leq \aleph_0$, we may select a set $\{s_n : n \in \omega\} \subseteq S$ so that $y \in \overline{\{s_n : n \in \omega\}}$. Let $\{z_n^i : n \in \omega\}$ be a sequence witnessing that $s_i \in S$ and put $B = \{z_j^h : 0 \leq h \leq j < \omega\}$. B is a subset of A which contains each s_i in its closure and so $y \in \overline{B}$. Furthermore, since $B \cap \bigcap \mathcal{U} = \varnothing$ and $B \setminus U_n$ is finite for every n, it follows that B is discrete. $\square$

Lemma 1.2. *Let x be a F-point of the space X. If x is in the closure of some countable discrete set $N \subseteq X$, then $N \cup \{x\}$ is homeomorphic to a subspace of βN.*

Proof. By contradiction, let us assume that there are two disjoint subset $A, B \subset N$ such that $x \in \overline{A} \cap \overline{B}$. As N is discrete, for each $y \in N$ there is an open neighbourhood $V(y)$ of y such that $\overline{V(y)} \cap N = \{y\}$. Let $\{a_i : i \in \omega\}$ and $\{b_i : i \in \omega\}$ be enumerations of A and B. Let P_i be an open F_σ-set satisfying $a_i \in P_i \subseteq V(a_i) \setminus \bigcup\{\overline{V(b_j)} : j \leq i\}$ and let Q_i be an open F_σ-set satisfying $b_i \in Q_i \subseteq V(b_i) \setminus \bigcup\{\overline{V(a_j)} : j \leq i\}$. But now, letting $P = \bigcup\{P_i : i \in \omega\}$ and $Q = \bigcup\{Q_i : i \in \omega\}$, we get disjoint open F_σ-sets of X satisfying $x \in \overline{P} \cap \overline{Q}$ – in contrast with the fact that x is a F-point. $\square$

A crucial role is played here by the following result, which is a bit more general version than the one proved for countably compact spaces in [5], Corollary 4.

Lemma 1.3. *Let $p \in \beta N \setminus N$. If the subspace $N \cup \{p\}$ is countably compact in the space X, then $t(p,X) > \aleph_0$.*

Proof. Let us assume that $N \cup \{p\}$ is countably compact in the space X. Of course, we may identify p with the trace in N of the family of all neighbourhoods of p in X.

Suppose first that p is a P-point. For any $P \in p$, choose a partition of P consisting of two infinite sets P' and P'' so that $P' \in p$ and let $x(P)$ be an accumulation point of the set P'' in X. The regularity of X guarantees that p is in the closure of the set $A = \{x(P) : P \in p\} \subseteq X$. If X has countable tightness at p, then we could find $\{P_n : n < \omega\} \subseteq p$ such that $p \in \overline{\{x(P_n) : n < \omega\}}$. But, since p is a P-point, there exists a neighbourhood U of p in X such that $U \cap P_n''$ is finite for each $n < \omega$ and so $p \notin \overline{\{x(P_n) : n < \omega\}}$ – a contradiction.

If p is not a P-point, then there is a partition $\{A_n : n < \omega\}$ of N into infinite sets such that for every n, $A_n \notin p$, and if $P \in p$ then $|P \cap A_n| = \omega$ for infinitely many n. Now put

$$\mathcal{T} = \{\bigcup_{n<\omega} F_n : F_n \subseteq A_n \text{ finite}, n < \omega\}.$$

For any $T \in \mathcal{T}$, select an accumulation point $x(T)$ of T in X and let $A = \{x(T) : T \in \mathcal{T}\}$. Again, the regularity of X guarantees that $p \in \overline{A}$. If X has countable tightness at p, then there exists a family $\{T_n : n < \omega\} \subseteq \mathcal{T}$ such that $p \in \overline{\{x(T_n) : n < \omega\}}$. For every $n < \omega$, let $T_n = \bigcup_{m<\omega} F_m^n$, with F_m^n a finite subset of A_m for every m. Put

$$K = F_0^0 \cup (F_1^0 \cup F_1^1) \cup (F_2^0 \cup F_2^1 \cup F_2^2) \cup \ldots.$$

Then $K \cap A_n$ is finite for every n, and as a consequence, $P = N \setminus K \in p$. Thus, taking a neighbourhood U of p in X so that $U \cap K = \varnothing$, we have that $U \cap T_n$ is finite for every n. But this is a contradiction with $p \in \overline{\{x(T_n) : n < \omega\}}$. □

Theorem 1.4. *A non-isolated F-point in a countably compact space has not countable tightness.*

Proof. Let x be a non-isolated F-point in the countably compact space X and assume that the tightness at x is countable. Thanks to Lemma 1.1, there is a countable discrete $N \subset X$ such that $x \in \overline{N}$. By Lemma 1.2, we have that $N \cup \{x\}$ is homeomorphic to a subspace of βN. But, according to Lemma 1.3, X has not countable tightness at x – a contradiction. □

Corollary 1.5. *A space containing a non-isolated F-point cannot be densely embedded into a countably compact space with countable tightness.*

Proof. It is enough to observe that if Y is a dense estention of X, then a F-point of X is still a F-point in the space Y. □

Now, we easily get the answer to Arhangel′skiĭ's question, in the following more general form:

Corollary 1.6. *An extremally disconnected subspace which is countably compact in a space with countable tightness is discrete.*

Observe that most of the results stated here are no longer true outside the class of Tychonoff spaces. Indeed, for any countable Hausdorff space X fix a maximal almost disjoint family $\mathcal{A}$ of infinite closed discrete subsets of X and denote by $\psi(X)$ the set $X \cup \mathcal{A}$, topologized in such a way that X is an open subspace of $\psi(X)$ and a local base at $A \in \mathcal{A}$ consists of the sets $\{A\} \cup (A \setminus F)$, where F is a finite subset of A. It is easy to see that $\psi(X)$ is a Hausdorff space with countable tightness and X is countably compact in $\psi(X)$. Now, if we take as X a countable maximal space [4], then Lemma 1.1 fails for $\psi(X)$, because no point of X is in the closure of a discrete subset of $\psi(X)$. If we take as X the space $N \cup \{p\}$, for some $p \in \beta N \setminus N$, then Lemma 1.3 fails for such $\psi(X)$. Moreover, in both cases, X is a non-discrete extremally disconnected space which is countably compact in a Hausdorff space with countable tightness, that is a failure of Corollary 1.6.

Coming back to Tychonoff spaces, Corollary 1.6 may naturally suggest the following:

Question 1.7. *Is it true that an extremally disconnected (dense) subspace of a pseudocompact space with countable tightness is necessarely discrete?*

It is possible to show that in a pseudocompact space with character at most $\aleph_1$ every non-isolated point is in the closure of a discrete set. However, it is not clear whether Lemma 1.1 holds for pseudocompact spaces.

Question 1.8. *Does every non-isolated point of a pseudocompact space with countable tightness belong to the closure of a discrete set?*

We finish by observing that, at first glance, Corollary 1.6 could have been better formulated for F-spaces, but this is only an impression.

Indeed, we could consider the notion of *point of extremal disconnectedness* (ED-point), as a point $x \in X$ such that there are no disjoint open sets $U, V \subseteq X$ for which $x \in \overline{U} \cap \overline{V}$. It is obvious that any ED-point is a F-point and the two notions are in general different, look, for instance, at the point ω_1 in the space $\omega_1 + 1$. However, this is not the case for spaces of countable tightness:

Proposition 1.9. *Let x be a point of countable tightness in a Tychonoff space X. If $x \in X$ is a F-point then x is also a ED-point.*

Proof. It is enough to observe that if U is an open set and $x \in \overline{U}$ then there exists an open F_σ set $V \subseteq U$ such that $x \in \overline{V} \subseteq \overline{U}$. For this, let $A \subseteq U$ be a countable set such that $x \in \overline{A}$ and for any $a \in A$ fix an open F_σ set V_a satisfying $a \in V_a \subseteq U$. It is clear that the set $V = \bigcup\{V_a : a \in A\}$ is what we are looking for. $\square$

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