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Topological groups with dense compactly generated subgroups

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Abstract. A topological group G is: (i) *compactly generated* if it contains a compact subset algebraically generating G, (ii) σ*-compact* if G is a union of countably many compact subsets, (iii) $\aleph_0$*-bounded* if arbitrary neighborhood U of the identity element of G has countably many translates xU that cover G, and (iv) *finitely generated modulo open sets* if for every non-empty open subset U of G there exists a finite set F such that $F \cup U$ algebraically generates G. We prove that: (1) a topological group containing a dense compactly generated subgroup is both $\aleph_0$-bounded and finitely generated modulo open sets, (2) an almost metrizable topological group has a dense compactly generated subgroup if and only if it is both $\aleph_0$-bounded and finitely generated modulo open sets, and (3) an almost metrizable topological group is compactly generated if and only if it is σ-compact and finitely generated modulo open sets.

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1. Preliminaries

All topological groups in this article are assumed to be T_1 (and thus Tychonoff). For subsets A and B of a group G let $AB = \{ab : a \in A \text{ and } b \in B\}$ and $A^{-1} = \{a^{-1} : a \in A\}$. For $a \in A$ and $b \in B$ we write aB or Ab rather than $\{a\}B$ or $A\{b\}$. If A is a subset of a group G, then the smallest subgroup of G that contains A is denoted by $\langle A \rangle$.

Recall that a topological group G is said to be:

(i) *compactly generated* if $G = \langle K \rangle$ for some compact subspace K of G,
(ii) *sigma-compact* provided that there exists a sequence $\{K_n : n \in \omega\}$ of compact subsets of G such that $G = \bigcup\{K_n : n \in \omega\}$,

(iii) $\aleph_0$-*bounded* if for every neighborhood U of the unit element there exists a countable set $S \subset G$ such that $US = G$ ([2]),

(iv) *totally bounded* if for every neighborhood U of the unit element there exists a finite set $S \subset G$ such that $US = G$,

(v) *finitely generated modulo open sets* if for every non-empty open set $U \subseteq G$, there exists a finite set $F \subseteq G$ such that $\langle F \cup U \rangle = G$ ([1]).

Clearly, compactly generated groups are σ-compact. It is well-known that σ-compact groups, separable groups and their dense subgroups are $\aleph_0$-bounded ([2]).

2. The results

The main purpose of this note is to study the following question: When does a topological group contain a dense compactly generated subgroup? Our first result provides two necessary conditions:

Theorem 2.1. *If a topological group G contains a dense compactly generated subgroup, then G is both $\aleph_0$-bounded and finitely generated modulo open sets.*

Proof. Let G be a topological group and K be its compact subset such that $\langle K \rangle$ is dense in G. Then G is $\aleph_0$-bounded ([2]), so it remains only to show that G is finitely generated modulo open sets. Given a non-empty open set U, the group G is divided into pairwise disjoint left-congruence classes modulo its subgroup $\langle U \rangle$. Let X be a complete set of representatives of these congruence classes: $G = \bigcup_{x \in X} x\langle U \rangle$. Since each congruence class is an open set, finite number of those classes must cover the compact set K. Therefore there is a finite set $F \subset X$ such that $F\langle U \rangle \supseteq K$. Since $\langle K \rangle$ is dense in G, it follows that $G = U\langle K \rangle \subseteq U\langle F\langle U \rangle \rangle \subseteq \langle F \cup U \rangle \subseteq G$. □

In our future arguments we will make use of the following easy lemma:

Lemma 2.2. *Let X be a topological space. Let $K \subset X$ be a compact set with a neighborhood base $\{U_n\}_{n \in \omega}$. Suppose that we have compact sets $C_n \subset \bigcap_{k \leq n} U_k$ for all $n \in \omega$. Then the set $C = K \cup \bigcup_{n \in \omega} C_n$ is also compact.*

A topological group G is *almost metrizable* if there exist a non-empty compact set $K \subset G$ and a sequence $\{U_n\}_{n \in \omega}$ of open subsets of G such that (1) $K \subset U_n$ for all $n \in \omega$ and (2) if O is an open set containing K, then there is an $n \in \omega$ such that $K \subset U_n \subset O$. (Such a sequence $\{U_n\}_{n \in \omega}$ is called a *neighborhood base* of K in G.) Both metric groups and locally compact groups are almost metrizable ([3]).

Our next theorem demonstrates that the necessary conditions for a topological group G to have a dense compactly generated subgroup found in Theorem 2.1 are also sufficient in case G is almost metrizable.

Theorem 2.3. *An almost metrizable topological group G contains a dense compactly generated subgroup if and only if it is $\aleph_0$-bounded and finitely generated modulo open sets.*

Proof. The "only if" part of our theorem follows from Theorem 2.1, so it remains only to prove the "if" part. Let K be a compact subset of G with a neighborhood base $\{U_n\}_{n\in\omega}$. Since G is $\aleph_0$-bounded, for each $n \in \omega$ there is a countable set $S_n \subset G$ such that $G = S_n U_n$. The set $S = \bigcup_{n\in\omega} S_n$ is countable, so we can fix its enumeration $S = \{s_n\}_{n\in\omega}$. Let $g \in G$. Let V be any neighborhood of the unit element of G. Then KV^{-1} is an open set containing K, and so there is an $n \in \omega$ such that $U_n \subseteq KV^{-1}$. Since $S_n U_n = G$, there is an $s \in S_n$ such that $g \in sU_n \subseteq sKV^{-1}$. Let $g = skv^{-1}$ with $k \in K$ and $v \in V$. Then $gv = sk \in gV \cap SK \neq \varnothing$. Since V and g are arbitrary, it follows that SK is dense in G. Since G is finitely generated modulo open sets, there are finite sets F_n such that $G = \langle F_n \cup U_n \rangle$ for each $n \in \omega$. Set $E_0 = F_0 \cup \{s_0\}$. It follows that $G = \langle E_0 \cup U_0 \rangle$. So there is a finite set $E_1 \subseteq U_0$ such that $F_1 \cup \{s_1\} \subset \langle E_0 \cup E_1 \rangle$. From this it follows that $\langle E_0 \cup E_1 \cup U_1 \rangle = G$. So there is a finite set $E_2 \subseteq U_1$ such that $F_2 \cup \{s_2\} \subset \langle E_0 \cup E_1 \cup E_2 \rangle$. In this way we obtain finite sets $E_{n+1} \subset U_n$ (for $n \in \omega$) such that $F_{n+1} \cup \{s_{n+1}\} \subseteq \langle E_0 \cup E_1 \cup \cdots \cup E_{n+1} \rangle$. By Lemma 2.2, the set $C = K \cup \bigcup_{n\in\omega} E_n$ is compact. The subgroup $\langle C \rangle$ is dense, since it contains SK. Thus G contains a compactly generated dense subgroup. □

Since every metrizable group is almost metrizable ([3]), and $\aleph_0$-boundedness is equivalent to separability for metrizable groups, from Theorem 2.3 we obtain:

Corollary 2.4. *A metrizable group contains a dense compactly generated subgroup if and only if it is separable and finitely generated modulo open sets.*

Our next result generalizes Theorem 4 from [1].

Lemma 2.5. *If a σ-compact almost metrizable group G contains a dense compactly generated subgroup, then G itself is compactly generated.*

Proof. Suppose $G = \bigcup_{n\in\omega} L_n$, with L_n compact. Suppose also that $H = \langle L_0 \rangle$ is dense in G. Let $K \subseteq G$ be a compact set with a neighborhood base $\{U_n\}_{n\in\omega}$. By regularity of the topology of G and compactness of K, we may assume without loss of generality that each U_n contains the closure of U_{n+1}. By compactness of L_n and denseness of H, there is a finite subset F_n of H such that $L_n \subset U_{n+1} F_n$. Let $C_n = \overline{L_n F_n^{-1} \cap U_{n+1}}$. Then C_n is compact, because it is a closed subset of the union of finitely many copies of L_n. We also have $C_n \subset U_n$ and $L_n \subset C_n F_n \subset \langle C_n \cup L_0 \rangle$. Therefore, setting $C = L_0 \cup K \cup \bigcup_{n\in\omega} C_n$, we obtain $\langle C \rangle = G$. It follows from Lemma 2.2 that C is compact. □

Combining Theorem 2.3 and Lemma 2.5, we obtain our next theorem which extends the main result of [1]:

Theorem 2.6. *An almost metrizable topological group is compactly generated if and only if it is σ-compact and finitely generated modulo open sets.*

Theorems 2.3 and 2.6 become especially simple for locally compact groups:

Theorem 2.7. *For a locally compact group G the following conditions are equivalent:*

(i) *G has a dense compactly generated subgroup,*
(ii) *G is compactly generated,*
(iii) *G is finitely generated modulo open sets.*

Proof. Let U be an open neighbourhood of the identity element having compact closure $\overline{U}$.

(i)→(ii). Let K be a compact subset of G such that $\langle K\rangle$ is dense in G. Then $\overline{U} \cup K$ is also compact and $\langle \overline{U} \cup K\rangle \supseteq U\langle K\rangle = G$ because $\langle K\rangle$ is dense in G and U is an open neighbourhood of the identity.

(ii)→(iii) follows from Theorem 2.1.

(iii)→(i). Assume that G is finitely generated modulo open sets. Then there exists a finite set $F \subseteq G$ with $\langle F \cup U\rangle = G$. Now note that $G = \langle F \cup U\rangle \subseteq \langle F \cup \overline{U}\rangle \subseteq G$. Since $\langle F \cup \overline{U}\rangle$ is compact, G is compactly generated. □

Since for every non-empty open subset U of a topological group G the set $\langle U\rangle$ is an open subgroup of G, it follows that a topological group without proper open subgroups is finitely generated modulo open sets ([1]). Therefore, from Theorem 2.6 we obtain

Corollary 2.8. *An almost metrizable, σ-compact group without proper open subgroups is compactly generated.*

Corollary 2.9. *A metric σ-compact group without proper open subgroups is compactly generated.*

Totally bounded groups are finitely generated modulo open sets, and so we get

Corollary 2.10. *Every σ-compact totally bounded almost metrizable group is compactly generated.*

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