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On spaces with the property (wa)

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Abstract. A space X has the property (wa) (or is a space with the property (wa)) if for every open cover $\mathcal{U}$ of X and every dense subspace D of X, there exists a discrete subspace $F \subseteq D$ such that $\mathrm{St}(F,\mathcal{U}) = X$. In this paper, we give an example of a Tychonoff space without the property (wa), and also study topological properties of spaces with the property (wa) by using the example.

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1. Introduction

By a space, we mean a topological space. Matveev [2] defined a space X to have the *property* (a) if for every open cover $\mathcal{U}$ of X and every dense subspace D of X, there exists a discrete closed subspace $F \subseteq D$ such that $\mathrm{St}(F,\mathcal{U}) = X$, where

$$St(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \varnothing\}.$$

As a way to weaken the above definition, he also gave the following definition:

Definition 1.1 ([2]). *A space X has the property (wa) if for every open cover $\mathcal{U}$ of X and every dense subspace D of X, there exists a discrete subspace $F \subseteq D$ such that $St(F,\mathcal{U}) = X$.*

A space having the property (wa) is also called a space with the property (wa). From the above definitions, it is not difficult to see that every space with the property (a) is a space with the property (wa).

The purpose of this paper is to give an example of a Tychonoff space without the property (wa) and to study topological properties of spaces with the property (wa) by using the example.

As usual, $\mathbb{R}$, $\mathbb{P}$ and $\mathbb{Q}$ denote the set of all real numbers, all irrational numbers and all rational numbers, respectively. For a set A, $|A|$ denotes the cardinality

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of A. For a cardinal κ, κ^+ denotes the smallest cardinal greater than κ. In particular, let ω denote the first infinite cardinal, $\omega_1 = \omega^+$ and $\mathfrak{c}$ the cardinality of the continuum. As usual, a cardinal is the initial cardinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. For each ordinal α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$, $[\alpha, \beta) = \{\gamma : \alpha \leq \gamma < \beta\}$ and $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$. Other terms and symbols that we do not define will be used as in [1].

2. A Tychonoff space without the property (wa)

Matveev [2] gave an example of a T_1-space without the property (wa) and he asked if there exists a T_2 (T_3, Tychonoff) space without the property (wa). Yang [8] constructed a T_2 space without the property (wa). In this section, we give an example of a Tychonoff space without the property (wa). We omit the easy proof of the following lemma.

Lemma 2.1. *Let $\mathbb{R}$ be endowed with the usual topology and A a discrete subspace of $\mathbb{R}$. Then, $|A| \leq \omega$ and $\mathrm{cl}_{\mathbb{R}} A$ is nowhere dense in $\mathbb{R}$.*

Example 2.2. There exists a 0-dimensional, first countable, Tychonoff space without the property (wa).

Proof. Let $A = \bigcup_{n \in N} A_n$, where $A_n = \mathbb{Q} \times \{1/n\}$ and let $\mathcal{A} = \{S : S$ is a discrete subspace of $A\}$. Then, we have:

Claim 2.3. $|\mathcal{A}| = \mathfrak{c}$.

Proof. Since $|A| = \omega$, $|\mathcal{A}| \leq \mathfrak{c}$. Let $S = \{\langle n, 1\rangle : n \in N\} \subseteq A$. Since every subset of S is discrete, $\{F : F \subseteq S\} \subseteq \mathcal{A}$. Hence, $|\mathcal{A}| \geq |\{F : F \subseteq S\}| = \mathfrak{c}$. ☐

Since $|\mathcal{A}| = \mathfrak{c}$, we can enumerate the family $\mathcal{A}$ as $\{S_\alpha : \alpha < \mathfrak{c}\}$. For each $\alpha < \mathfrak{c}$ and each $n \in N$, put $S_{\alpha,n} = \{q \in \mathbb{Q} : \langle q, 1/n\rangle \in S_\alpha\}$.

Claim 2.4. *For each $\alpha < \mathfrak{c}$, $|\mathbb{R} \setminus \bigcup_{n \in N} \mathrm{cl}_{\mathbb{R}} S_{\alpha,n}| = \mathfrak{c}$.*

Proof. For each $\alpha < \mathfrak{c}$, let

$$X_\alpha = \mathbb{R} \setminus \bigcup_{n \in N} \mathrm{cl}_{\mathbb{R}}\, S_{\alpha,n}.$$

Since X_α is a G_δ-set in $\mathbb{R}$, X_α is a complete metric space. To show that X_α is dense in itself, suppose that X_α has an isolated point x. Then, there exists $\varepsilon > 0$ such that

$$(x - \varepsilon, x + \varepsilon) \cap X_\alpha = \{x\}.$$

Let $I = (x, x + \varepsilon)$, Then,

$$I \subset \mathbb{R} \setminus X_\alpha \subset \bigcup_{n \in N} \mathrm{cl}_{\mathbb{R}}\, S_{\alpha,n}.$$

Moreover, since I is open in $\mathbb{R}$, $\mathrm{cl}_{\mathbb{R}}\, S_{\alpha,n} \cap I \subseteq \mathrm{cl}_{\mathbb{R}}(S_{\alpha,n} \cap I)$. Hence,

$$I = (\bigcup_{n \in N} \mathrm{cl}_{\mathbb{R}}\, S_{\alpha,n}) \cap I = \bigcup_{n \in N} (\mathrm{cl}_{\mathbb{R}}\, S_{\alpha,n} \cap I) \subseteq \bigcup_{n \in N} \mathrm{cl}_{\mathbb{R}}(S_{\alpha,n} \cap I). \tag{6}$$

By Lemma 2.1, each $\mathrm{cl}_{\mathbb{R}}(S_{\alpha,n} \cap I)$ is nowhere dense in $\mathbb{R}$. Thus, (6) contradicts the Baire Category Theorem. Hence, X_α is dense in itself. It is known ([1, 4.5.5]) that every dense in itself complete metric space includes a Cantor set. Hence, $|X_\alpha| = \mathfrak{c}$. □

Claim 2.5. *There exists a sequence* $\{p_\alpha : \alpha < \mathfrak{c}\}$ *satisfying the following conditions*:

(1) *For each* $\alpha < \mathfrak{c}$, $p_\alpha \in \mathbb{P}$.
(2) *For any* $\alpha, \beta < \mathfrak{c}$, *if* $\alpha \neq \beta$, *then* $p_\alpha \neq p_\beta$.
(3) *For each* $\alpha < \mathfrak{c}$, $p_\alpha \notin \bigcup_{n\in N} \mathrm{cl}_{\mathbb{R}} S_{\alpha,n}$.

Proof. By transfinite induction, we define a sequence $\{p_\alpha : \alpha < \mathfrak{c}\}$ as follows: There is $p_0 \in \mathbb{P}$ such that $p_0 \notin \bigcup_{n\in N} \mathrm{cl}\, S_{0,n}$ by Claim 2.4. Let $0 < \alpha < \mathfrak{c}$ and assume that p_β has been defined for all $\beta < \alpha$. By Claim 2.4,

$$|\mathbb{R} \setminus \bigcup_{n\in N} \mathrm{cl}_{\mathbb{R}}\, S_{\alpha,n}| = \mathfrak{c}.$$

Hence, we can choose a point $p_\alpha \in (\mathbb{P} \setminus \bigcup_{n\in N} \mathrm{cl}_{\mathbb{R}}\, S_{\alpha,n}) \setminus \{p_\beta : \beta < \alpha\}$. Now, we have completed the induction. Then, the sequence $\{p_\alpha : \alpha < \mathfrak{c}\}$ satisfies the conditions (1) (2) and (3). □

Claim 2.6. *For each* $\alpha < \mathfrak{c}$, *there exists a sequence* $\{\varepsilon_{\alpha,n} : n \in N\}$ *in* $\mathbb{Q}$ *satisfying the following conditions:*

(1) *For each* $n \in N$, $(p_\alpha - \varepsilon_{\alpha,n}, p_\alpha + \varepsilon_{\alpha,n}) \cap S_{\alpha,n} = \varnothing$.
(2) *For each* $n \in N$, $\varepsilon_{\alpha,n} \geq \varepsilon_{\alpha,n+1}$.
(3) $\lim_{n\to\infty} \varepsilon_{\alpha,n} = 0$.

Proof. Let $\alpha < \mathfrak{c}$. For $n = 1$, since $p_\alpha \notin \mathrm{cl}_{\mathbb{R}}\, S_{\alpha,1}$, there exists a rational $\varepsilon_{\alpha,1} > 0$ such that

$$(p_\alpha - \varepsilon_{\alpha,1}, p_\alpha + \varepsilon_{\alpha,1}) \cap S_{\alpha,1} = \varnothing.$$

Let $n > 1$ and assume that we have defined $\{\varepsilon_{\alpha,m} : m < n\}$ satisfying that

$$\varepsilon_{\alpha,1} > \varepsilon_{\alpha,2} > \cdots > \varepsilon_{\alpha,n-1}.$$

Since $p_\alpha \notin \mathrm{cl}_{\mathbb{R}}\, S_{\alpha,n}$, there exists a rational $\varepsilon'_{\alpha,n}$ such that

$$(p_\alpha - \varepsilon'_{\alpha,n}, p_\alpha + \varepsilon'_{\alpha,n}) \cap S_{\alpha,n} = \varnothing.$$

Put

$$\varepsilon_{\alpha,n} = n^{-1} \min\{\varepsilon_{\alpha,n-1}, \varepsilon'_{\alpha,n}\}.$$

Now, we have completed the induction. Then, the sequence $\{\varepsilon_{\alpha,n} : n \in N\}$ satisfies (1) (2) and (3). □

Define $X = A \cup B$, where $B = \{\langle p_\alpha, 0\rangle : \alpha < \mathfrak{c}\}$. Topologize X as follows: A basic neighborhood of a point in A is a neighborhood induced from the usual topology on the plane. For each $\alpha < \mathfrak{c}$, a neighborhood base $\{U_n\langle p_\alpha, 0\rangle : n \in \omega\}$ of $\langle p_\alpha, 0\rangle \in B$ is defined by

$$U_n\langle p_\alpha, 0\rangle = \{\langle p_\alpha, 0\rangle\} \cup (\bigcup_{i\geq n} \{((p_\alpha - \varepsilon_{\alpha,i}, p_\alpha + \varepsilon_{\alpha,i}) \cap \mathbb{Q}) \times \{1/i\}\}).$$

for each $n \in \omega$. Then, X is a first countable T_2-space. For each $\alpha < \mathfrak{c}$ and each $n \in \omega$. $U_n\langle p_\alpha, 0\rangle$ is open and closed in X, because $p_\alpha \pm \varepsilon_{\alpha,i} \notin \mathbb{Q}$ for each $i \in \omega$. It follows that X is 0-dimensional, and hence, a Tychonoff space.

Claim 2.7. *The space X has not the property* (wa).

Proof. Let

$$\mathcal{U} = \{A\} \cup \{U_1\langle p_\alpha, 0\rangle : \alpha < \mathfrak{c}\}.$$

Then, $\mathcal{U}$ is an open cover of X and A is a dense subspace of X. For each discrete subset F of A, there exists $\alpha < \mathfrak{c}$ such that $F = S_\alpha$. Since $U_1\langle p_\alpha, 0\rangle \cap S_\alpha = \varnothing$, $\langle p_\alpha, 0\rangle \notin \mathrm{St}(F, \mathcal{U})$. This shows that X does not have the property (wa). □

□

Remark 2.8. The above example was announced in [6]. The author does not know if there exists a normal space without the property (*wa*).

Remark 2.9. Just, Matveev and Szeptycki [5] constructed an example that has similar properties as example 2.2, but the construction of our example seems to be simpler than their example.

3. Some topological properties of spaces with the property (wa)

First, we give an example showing that a continuous image of a space with the property (wa) need not be a space with the property (wa).

Example 3.1. There exists a continuous bijection $f : X \to Y$ from a Tychonoff space X with the property (wa) to a Tychonoff space Y without the property (wa).

Proof. We define the space X by changing the topology of the space of Example 2.2 by the discrete space. Then, the space X is a space with property (wa). Let Y be the space of Example 2.2 as in the proof of Example 2.2. Then, the space Y is a Tychonoff space without property (wa). Let $f : X \to Y$ be the identity map. Clearly f is continuous, which completes the proof. □

Let us recall that a mapping $f : X \to Y$ is *varpseudocompact* if $Int(f(U)) \neq \varnothing$ for every non-empty set U of X.

Theorem 3.2. *Let X be a space with the property* (*wa*) *and $f : X \to Y$ be a varpseudocompact continuous closed mapping. Then, Y is a space with the property* (*wa*).

Proof. Let $f : X \to Y$ be a varpseudocompact continuous closed mapping. Let $\mathcal{U}$ be an open cover of Y and D a dense subspace of Y. Then, $\mathcal{U}_0 = \{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of X and $D_0 = f^{-1}(D)$ is dense in X since f is varpseudocompact. Then, there is a discrete subset $B \subseteq D_0$ such that $St(B, \mathcal{U}_0) = X$, since X is a space with property (wa). Let $F = f(B)$. Then, F is a discrete subset of D since f is closed, and $St(F, \mathcal{U}) = Y$, which completes the proof. □

In the following, we give an example to show that a regular-closed subset of a space with the property (a) (hence, (wa)) need not be a space with the property (wa). Recall [3] that a space X is *absolutely countably compact* if for every open cover $\mathcal{U}$ of X and every dense subspace D of X, there exists a finite subset $F \subseteq D$ such that $St(F,\mathcal{U}) = X$. It is known that every absolutely countably compact T_2 space is countably compact and has the property (a) (see [2, 3]). Moreover, Vaughan [7] proved that every countably compact GO-space is absolutely countably compact. Thus, every cardinality with uncountable cofinality is absolutely countably compact.

Example 3.3. There exists a Tychonoff space X with the property (a) (hence,(wa)) having a regular-closed subspace without the property (wa).

Proof. Let $X = A \cup B$ be as in the proof of Example 2.2. Let

$$S_1 = (\mathfrak{c}^+ \times A) \cup B.$$

We topologize S_1 as follows: $\mathfrak{c}^+ \times B$ has the usual product topology and is an open subspace of S_1. For each $\alpha < \mathfrak{c}$, a basic neighbourhood of $\langle p_\alpha, 0\rangle$ takes the form

$$G_{\beta,n}(\langle p_\alpha, 0\rangle) = \{\langle p_\alpha, 0\rangle\} \cup (\{\alpha : \beta < \alpha < \mathfrak{c}^+\} \times (U_n\langle p_\alpha, 0\rangle \setminus \{\langle p_\alpha, 0\rangle\})).$$

for $\beta < \mathfrak{c}^+$ and $n \in N$, where $U_n\langle p_\alpha, 0\rangle$ is defined in Example 2.2. Then, the space S_1 is Tychonoff. Now, we show that S_1 has the property (a). For this end, let $\mathcal{U}$ be an open cover of S_1. Let D_0 be the set of all isolated points of $\mathfrak{c}^+$ and let $D = D_0 \times A$. Then, D is dense in S_1 and every dense subspace of S_1 contains D. Thus, it suffices to show that there exists a subset $F \subseteq D$ such that F is discrete closed in S_1 and $St(F,\mathcal{U}) = S_1$. For each $q \in \mathbb{Q}$ and each $n \in N$, since $\mathfrak{c}^+ \times \{\langle q, 1/n\rangle\}$ is absolutely countably compact, there exists a finite subset $F_{q,n} \subseteq D_0 \times \{\langle q, 1/n\rangle\}$ such that

$$\mathfrak{c}^+ \times \{\langle q, 1/n\rangle\} \subseteq St(F_{q,n},\mathcal{U}).$$

Let

$$F' = \bigcup\{F_{q,n} : q \in \mathbb{Q} \text{ and } n \in \omega\}.$$

Then,

$$\mathfrak{c}^+ \times A \subseteq St(F',\mathcal{U}).$$

For each $\alpha < \mathfrak{c}$, take $U_\alpha \in \mathcal{U}$ with $\langle p_\alpha, 0\rangle \in U_\alpha$, and fix $\beta_\alpha < \mathfrak{c}^+$ and $n_\alpha \in N$ such that

$$\{\langle \alpha, \langle p_\alpha, 0\rangle\rangle : \beta\alpha < \alpha < \mathfrak{c}^+\} \subseteq U_\alpha.$$

For each $n \in N$, let $B_n = \{\alpha < \mathfrak{c} : n_\alpha = n\}$ and choose $\beta_n \in S$ with $\beta_n > \sup\{\beta_\alpha : \alpha \in B_n\}$. Then,

$$B_n \subseteq St(\langle \beta_n, n\rangle,\mathcal{U}).$$

Thus, if we put

$$F'' = \{\langle \beta_n, n\rangle : n \in N\}.$$

Then $B \subseteq St(F'',\mathcal{U})$. Let $F = F' \cup F''$. Then, F is a countable subset of D such that $S_1 = St(F,\mathcal{U})$. Since $F \cap (\mathfrak{c}^+ \times \{\langle q, n\rangle\})$ is finite for each $q \in \mathbb{Q}$

and each $n < \omega$, F is discrete and closed in S_1, which shows that S_1 has the property (a).

Let S_2 be the same space X as in Example 2.2. Then, the space S_2 is a Tychonoff space without the property (wa).

We assume that $S_1 \cap S_2 = \varnothing$. Let $\varphi : B \to B$ be the identify map. Let X be the quotient space obtained from the discrete sum $S_1 \oplus S_2$ by identifying $\langle p_\alpha, 0\rangle$ with $\varphi(\langle p_\alpha, 0\rangle)$ for each $\alpha < \mathfrak{c}$. Let $\pi : S_1 \oplus S_2 \to X$ be the quotient map. It is easy to check that $\pi(S_2)$ is a regular-closed subset of X, however, it is not a subspace of X with the property (wa), since it is homeomorphic to S_2.

Next, we show that X has the property (a). For this end, let $\mathcal{U}$ be an open cover of X. Let $S = \pi(A \cup D)$ Then, S is dense in X and every dense subspace of X contains S, since each point of S is a isolated point of X. Thus, it suffices to show that there exists a subset C of S such that C is discrete closed in X and $X = St(C, \mathcal{U})$. Since $\pi(S_1)$ is homeomorphic to the space S_1, then there exists a discrete closed subset $C_0 \subseteq \pi(D)$ such that

$$\pi(S_1) \subseteq St(C_0, \mathcal{U}).$$

Since $\pi(S_1)$ is closed in X, then C_0 is closed in X. Let $C_1 = X \setminus St(\pi(C_0, \mathcal{U})$. Then, $C_1 \subseteq S$. If we put $C = C_0 \cup C_1$, Then $X = St(C, \mathcal{U})$. Since $C \subseteq S$ and C is a discrete closed subset of X, Then X has the property (a), which completes the proof. □

Considering other types of subspaces, we arrive to the following result, which is rather unexpected even thought the Lindelöf property is preserved by arbitrary F_σ-subspaces, and which is a minor improvement of Theorem 84 from [4]. Recall that a space is a P-space if every G_δ-set is open.

Theorem 3.4. *An open F_σ-subset of a P-space with the property (wa) has the property (wa).*

Proof. Let X be a P-space with the property (wa) and let $Y = \bigcup\{H_n : n \in \omega\}$ be an open F_σ-subset in X (each H_n is closed in X). Let $\mathcal{U}$ be an open cover of Y and let D be a dense subset of Y. We have to find a discrete set $F \subseteq D$ such that $St(F, \mathcal{U}) = Y$. For each $n \in \omega$, let us consider the open cover

$$\mathcal{U}_n = \mathcal{U} \cup \{X \setminus H_n\}$$

of X and the dense subset $D \cup (X \setminus Y)$ of X. Since X has the property (wa), there is a discrete subset $B_n \subseteq D \cup (X \setminus Y)$ such that $St(B_n, \mathcal{U}_n) = X$. Put $A_n = B_n \cap D$. It is clear that $H_n \subseteq St(A_n, \mathcal{U})$. Put $F = \bigcup\{A_n : n \in \omega\}$. Then F is a discrete subset of D, since X is a P-space and $St(F, \mathcal{U}) = Y$, which completes the proof. □

Since a cozero-set is open F_σ-set, thus we have the following corollary.

Corollary 3.5. *A cozero-set of a P-space with the property (wa) has the property (wa).*

Recall that the Alexandorff duplicate $A(X)$ of a space X is constructed as follows. The underlying set of $A(X)$ is $X \times \{0,1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is the set of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X.

Theorem 3.6. *Let X be any space. Then, $A(X)$ is a space with the property (wa).*

Proof. Let D_0 be the set of all isolated points of X. If we put $D = D_0 \cup (X \times \{1\})$, then D is a dense subset of $A(X)$. Since each point of $X \times \{1\}$ is a isolated point of $A(X)$, then every dense subset of $A(X)$ contains D. We show that $A(X)$ is a space with the property (wa). For this end, let $\mathcal{U}$ be an open cover of $A(X)$. It suffices to show that there exists a discrete subspace F of D such that $St(F,\mathcal{U}) = A(X)$. Since D is dense in $A(X)$ and each point of D is isolated. Taking $F = D$, then D is discrete and $St(D,\mathcal{U}) = A(X)$, since D is dense in $A(X)$,which completes the proof. □

The following corollary follows directly from Theorem 3.6:

Corollary 3.7. *Every space can be embedded as a closed subset into a space with the property (wa).*

Just, Matveev and Szeptycki proved in Theorem 16 of [5] that the product of a countably paracompact (a)-space and a compact metrizable space is a (a)-space. In a similar way, we may prove the following:

Theorem 3.8. *Let X be a countably paracompact space with the property (wa) and Y a compact metric space.Then, $X \times Y$ is a space with the property (wa).*

Remark 3.9. The author does not know if the assumption that X is countably paracompact can be removed.

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