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Cellularity and density of balleans

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Abstract. A ballean is a set X endowed with some family $\mathcal{F}$ of balls in such a way that a ballean can be considered as an asymptotic counterpart of a uniform topological space. Then we define the asymptotic counterparts for dense and open subsets, introduce two cardinal invariants (density and cellularity) of balleans and prove some results concerning relationship between these invariants. We conclude the paper with applications of obtained partitions of left topological group in dense subsets.

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1. Introduction

Every infinite group G can be partitioned in $|G|$-many subsets dense in every totally bounded group topology on G. In [5] this statement was extracted from the following combinatorial claim. For every infinite group G there exists a disjoint family $\mathcal{F}$ of cardinality $|G|$ such that, for every $F \in \mathcal{F}$ and every finite subset K of G, there exists $g \in F$ such that $Kg \subseteq F$. Each subset $F \in \mathcal{F}$ looks like a set with non-empty interior in some structure dual to uniform topological space. To explain this duality we need some definitions and notations.

A *ball structure* is a triple $\mathcal{B} = (X, P, B)$ where X, P are non-empty sets and, for any $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the *set of radii*. Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}, B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

A ball structure is called

- *lower symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,
$$B^*(x, \alpha') \subseteq B(x, \alpha), B(x, \beta') \subseteq B^*(x, \beta);$$
- *upper symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,
$$B(x, \alpha) \subseteq B^*(x, \alpha'), B^*(x, \beta) \subseteq B(x, \beta');$$
- *lower multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,
$$B(B(x, \gamma), \gamma) \subseteq B(x, \alpha) \cap B(x, \beta);$$
- *upper multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,
$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

Let $\mathcal{B} = (X, P, B)$ be a lower symmetric and lower multiplicative ball structure. Then the family
$$\Big\{ \bigcup_{x \in X} B(x, \alpha) \times B(x, \alpha) : \alpha \in P \Big\}$$
is a base of entourages for some (uniquely determined) uniformity on X. On the other hand, if $\mathcal{U} \subseteq X \times X$ is a uniformity on X, then the ball structure $(X, \mathcal{U}, B)$ is lower symmetric and lower multiplicative, where $B(x, U) = \{y \in X : (x, y) \in U\}$. Thus, the lower symmetric and lower multiplicative ball structures can be identified with the uniform topological spaces.

A ball structure is said to be a *ballean* if it is upper symmetric and upper multiplicative. In entourage form the balleans arouse in coarse geometry [10] under name coarse structures and independently in combinatorics [6] under name uniform ball structures.

Now we define the mappings which play the parts of uniformly continuous and uniformly open mappings on the ballean stage.

Let $\mathcal{B}_1 = (X_1, P_1, B_1)$ and $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f : X_1 \to X_2$ is called a $\prec -$*mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$,
$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta).$$

A mapping $f : X_1 \to X_2$ is called $\succ$-*mapping* if, for every $\beta \in P_2$, there exists $\alpha \in P_1$ such that, for every $x \in X_1$
$$B_2(f(x), \beta) \subseteq f(B_1(x, \alpha)).$$

If $f : X_1 \to X_2$ is a bijection such that f is a $\prec$-mapping and f is a $\succ$-mapping, we say that f is an *asymorphism* and $\mathcal{B}_1$, $\mathcal{B}_2$ are *asymorphic.*

Given an arbitrary ballean $\mathcal{B} = B(X, P, B)$, we can replace every ball $B(x, \alpha)$ by $B^*(x, \alpha) \cap B(x, \alpha)$ and get an asymorphic ballean in which $B^*(x, \alpha) =$

$B(x, \alpha)$. *In what follows we shall assume that* $B^*(x, \alpha) = B(x, a)$ *for all* $x \in X$*,* $\alpha \in P$.

We need also some classification of subsets of X for a ballean $\mathcal{B} = (X, P, B)$. Given a subset $A \subseteq X$, we say that A is

- *large* if there exists $\alpha \in P$ such that $X = B(A, \alpha)$;
- *small* if $X \setminus B(A, \alpha)$ is large for every $\alpha \in P$;
- *thick* if, for every $\alpha \in P$ there exists $a \in A$ such that $B(a, \alpha) \subseteq A$.

For some special balleans these types of subsets were introduced in [1] and [2]. We note also that large, small and thick subsets of a ballean may be considered as asymptotic duplicates of dense, nowhere dense and subsets with non-empty interior of uniform spaces.

Following this (non-formal) duality between uniform spaces and balleans, we define the *density* $d(\mathcal{B})$ and *cellularity* $c(\mathcal{B})$ as

$$d(\mathcal{B}) = min\{|L| : L \subseteq X, L \text{ is large}\},$$

$$c(\mathcal{B}) = sup\{|\mathcal{F}| : \mathcal{F} \text{ is a disjoint family of thick subsets of } X\}.$$

As in the case of uniform spaces, density of a ballean is much more easy to calculate or evaluate than its cellularity, so our main goal is to find some relationships between $d(\mathcal{B})$ and $c(\mathcal{B})$.

2. Observations

(1) Let $\mathcal{B} = (X, P, B)$ be a ballean, T be a thick subset of X and L be a large subset of X. Then there exists $\alpha \in P$ such that $X = B(L, \alpha)$ and $B(x, \alpha) \subseteq T$ for some $x \in T$, so $L \cap T \neq \varnothing$. Since every large subset meets every thick subset, we have $c(\mathcal{B}) \leqslant d(\mathcal{B})$.

(2) Given $\alpha \in P$ and $Y \subseteq X$, we say that Y is α-*discrete* if the family $\{B(y, \alpha) : y \in Y\}$ is pairwise disjoint. By Zorn Lemma, every α-discrete subset Y of X is contained in some maximal (by inclusion) α-discrete subset Z of X. If $y \in X$ then $B(y, \alpha) \cap B(Z, \alpha) \neq \varnothing$. We choose $\beta \in P$ such that $B(B(x, \alpha), \alpha) \subseteq B(x, \beta)$ for every $x \in X$. Then $y \in B(Z, \beta)$ and Z is large.

On the other hand, let L be a large subset of X, $X = B(L, \alpha)$ and let Z be a maximal α-disjoint subset of Y. Then $|Z| \leqslant |L|$ and Z is large.

Hence, $d(\mathcal{B})$ can be defined as the minimal cardinality of maximal α-disjoint subsets of X where α runs over P.

(3) Let (X, d) be a metric space. Given any $x \in X$, $n \in \omega$, we put $B_d(x, n) = \{y \in X : d(x, y) \leqslant n\}$ and say that $\mathcal{B}(X, d) = (X, \omega, B_d)$ is a *metric* ballean. A ballean $\mathcal{B}$ is called *metrizable* if $\mathcal{B}$ is asymorphic to some metric ballean. To characterize metrizable balleans we need two definitions.

A ballean $\mathcal{B} = (X, P, B)$ is called *connected* if, for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$.

We define the preordering $\leqslant$ on P by the rule: $\alpha \leqslant \beta$ if and only if $B(x,\alpha) \subseteq B(x,\beta)$ for every $x \in X$. A subset $P' \subseteq P$ is called *cofinal* if, for every $\alpha \in P$ there exists $\beta \in P'$ such that $\alpha \leqslant \beta$. The *cofinality* $cf(\mathcal{B})$ is the minimal cardinality of the cofinal subsets of P.

By [6, Theorem 9.1], a ballean $\mathcal{B}$ is metrizable if and only if $\mathcal{B}$ is connected and $cf(\mathcal{B}) \leqslant \aleph_0$. For approximation of arbitrary balleans via metrizable balleans see [7].

(4) A connected ballean $\mathcal{B} = (X, P, B)$ is called *ordinal* if there exists a well-ordered by $\leq$ cofinal subset of P. Replacing P to its minimal cofinal subset, we get the asymorphic ballean. Hence, we can write $\mathcal{B}$ as (X, β, B), where β is a regular cardinal (considered as a set of ordinals).

We note that every metrizable ballean is ordinal, and metric balleans are the main subject of asymptotic topology [3].

(5) A subset $Y \subseteq X$ is called *bounded* if there exist $y \in Y$ and $\alpha \in P$ such that $Y \subseteq B(y,\alpha)$. A ballean is called bounded if its support is bounded. Clearly, $d(\mathcal{B}) = c(\mathcal{B}) = 1$ for every bounded ballean $\mathcal{B}$.

3. Results

Theorem 3.1. *For every ordinal ballean $\mathcal{B}$, $c(\mathcal{B}) = d(\mathcal{B})$ and there exists a disjoint family $\mathcal{F}$ of cardinality $d(\mathcal{B})$ consisting of thick subsets of X.*

Proof. Let $\mathcal{B} = (X, \rho, B)$, $\kappa = d(\mathcal{B})$ and $cf(\kappa)$ be the cofinality of κ. If $\mathcal{B}$ is bounded, we use observation 5, so we assume that $\mathcal{B}$ is unbounded. We fix some element $x_0 \in X$ and consider four cases.

Case $\rho < cf\kappa$. We prove the following auxiliary statement. For every $\alpha < \rho$, there exist β, $\alpha < \beta < \rho$ and an α-discrete subset Y_α of X such that $x_0 \in Y_\alpha$, $B(Y_\alpha, \alpha) \subseteq B(x_0, \beta)$ and $|Y_\alpha| = \kappa$.

Let Z be a maximal α-discrete subset of X such that $x_0 \in Z$. By observation 2, $|Z| \geqslant \kappa$. For every $\lambda < \rho$, we put $Z_\lambda = Z \cap B(x_0, \lambda)$. Since $Z = \bigcup_{\lambda<\rho} Z_\lambda$, $|Z| \geqslant \kappa$ and $\rho < cf(\kappa)$, there exists $\mu < \rho$ such that $|Z_\mu| \geqslant \kappa$. We choose $\beta < \rho$ such that $\alpha < \beta$ and $B(B(x_0,\mu),\alpha) \subseteq B(x_0,\beta)$. Then $B(Z_\mu, \alpha) \subseteq B(x_0, \beta)$ and we can choose a subset $Y_\alpha \subseteq Z_\mu$ such that $x_0 \in Y$ and $|Y_\alpha| = \kappa$.

Using the auxiliary statement and regularity of ρ, we can define inductively a mapping $f : \rho \to \rho$ and a family $\{Y_{f(\alpha)} : \alpha < \rho\}$ of subsets of X such that, for every $\alpha < \rho$, $f(\alpha) > \alpha$, $Y_{f(\alpha)}$ is $f(\alpha)$-discrete, $|Y_{f(\alpha)}| = \kappa$ and

$$B(Y_{f(\alpha)}, f(\alpha)) \subseteq B(x_0, f(\alpha+1)) \setminus B(x_0, f(\alpha)).$$

For every $\alpha < \rho$, we enumerate $Y_{f(\alpha)} = \{y(f(\alpha),\lambda) : \lambda < \kappa\}$ and, for every $\lambda < \kappa$, put

$$T_\lambda = \bigcup_{\alpha<\rho} B(y(f(\alpha),\lambda), f(\alpha)).$$

Clearly, every subset T_λ is thick and the family $\{T_\lambda : \lambda < \kappa\}$ is disjoint.

Case $cf(\kappa) \leqslant \rho < \kappa$. We prove the following auxiliary statement. For any $\alpha < \rho$ and $k' < \kappa$, there exist β, $\alpha < \beta < \rho$ and an α-discrete subset Y_α of X such that $x_0 \in Y_\alpha$, $B(Y_\alpha, \alpha) \subseteq B(x_0, \beta)$ and $|Y_\alpha| > \kappa'$.

Let Z be a maximal α-discrete subset of X such that $x_0 \in Z$. By observation 2, $|Z| \geqslant \kappa$. Let I be a cofinal subset of κ such that $|I| = \rho$. For every $\lambda \in I$, we put $Z_\lambda = Z \cap B(x_0, \lambda)$. Clearly, $Z = \bigcup_{\lambda \in I} Z_\lambda$. If $|Z_\lambda| \leqslant \kappa'$ for every $\lambda \in I$, then $|Z| \leqslant \kappa'|I| = \kappa'\rho < \kappa$. Hence, there exists $\mu \in I$ such that $|Z_\mu| > \kappa'$. We choose $\beta < \rho$ such that $\alpha < \beta$ and $B(B(x_0, \mu), \alpha) \subseteq B(x_0, \beta)$. Then $B(Z_\mu, \alpha) \subseteq B(x_0, \beta)$ and we put $Y_\alpha = Z_\mu$.

Let $\varphi : \rho \to \kappa$ be an injective mapping such that $\varphi(\rho)$ is cofinal in κ and $\alpha < \beta < \rho$ implies $\varphi(\alpha) < \varphi(\beta) < \kappa$. Using the auxiliary statement and regularity of κ, we can define inductively a mapping $f : \rho \to \rho$ and a family $\{Y_{f(\alpha)} : \alpha < \rho\}$ of subsets of X such that

- (i) $f(\rho)$ is cofinal in ρ and $\alpha < \beta < \rho$ implies $f(\alpha) < f(\beta) < \rho$;
- (ii) $Y_{f(\alpha)}$ is $f(\alpha)$-discrete;
- (iii) $B(Y_{f(\alpha)}, f(\alpha)) \subseteq B(x_0, f(\alpha + 1)) \setminus B(x_0, f(\alpha))$;
- (iv) $|Y_{f(\alpha)}| = \varphi(\alpha)$.

For every $\alpha < \rho$, we enumerate $Y_{f(\alpha)} = \{y(f(\alpha), \lambda) : \lambda < \varphi(\alpha)\}$. Then for any α and γ such that $\varphi(\alpha) \leqslant \gamma < \varphi(\alpha + 1)$, we put

$$T_\gamma = \bigcup \big\{B(f(\beta), \gamma) : \alpha + 1 < \beta < \rho\big\}.$$

By (i), T_γ is thick. By (ii) and (iii), the family

$$\mathcal{F} = \big\{T_\gamma : \varphi(\alpha) \leqslant \gamma < \varphi(\alpha + 1), \alpha < \rho\big\}$$

is disjoint. Since $\varphi(\rho)$ is cofinal in κ, by (iv), we have $|\mathcal{F}| = \kappa$.

Case $\rho = \kappa$. Using the assumption, we can construct inductively the subset $\{y_\alpha : \alpha < \kappa\}$ of X such that the family $\{B(y_\alpha, \alpha) : \alpha < \kappa\}$ is disjoint. Then we partition $\kappa = \bigcup_{\lambda < \kappa} I_\lambda$ into κ cofinal subsets and, for every $\lambda < \kappa$, put

$$T_\lambda = \bigcup \big\{B(y_\alpha, \alpha) : \alpha \in I_\lambda\big\}.$$

Clearly, every subset T_λ is thick and the family $\{T_\lambda : \lambda < \kappa\}$ is disjoint.

Case $\rho > \kappa$. We show that this variant is impossible. Suppose the contrary. Let Z be a large subset of X such that $|Z| = \kappa$ and $X = B(Z, \alpha)$. For every $z \in Z$, we pick $\alpha(z) < \rho$ such that $B(z, \alpha) \subseteq B(x_0, \alpha(z))$. Since ρ is regular and $\kappa < \rho$, there exists $\beta < \rho$ such that $\beta > \alpha(z)$ for every $z \in Z$. Then $B(z, \alpha) \subseteq B(x_0, \beta)$ for every $z \in Z$, so $X = B(x_0, \beta)$ and $\mathcal{B}$ is bounded. $\square$

Corollary 3.2. *For every metrizable ballean $\mathcal{B}$, $c(\mathcal{B}) = d(\mathcal{B})$ and there exists a disjoint family $\mathcal{F}$ of cardinality $d(\mathcal{B})$ consisting of thick subsets of X.*

Theorem 3.3. *Let $\mathcal{B} = (X, P, B)$ be a ballean, $|X| = \kappa$ and let $|P| \leqslant \kappa$. Then $c(\mathcal{B}) = d(\mathcal{B}) = \kappa$ and there exists a disjoint family $\mathcal{F}$ of cardinality κ consisting of thick subsets of X provided that one of the following conditions is satisfied:*

- *(i) there exists $\kappa' < \kappa$ such that $B(x, \alpha) \leqslant \kappa'$ for all $x \in X$ and $\alpha \in P$;*
- *(ii) $|B(x, \alpha)| < \kappa$ for all $x \in X$, $\alpha \in P$ and κ is regular.*

Proof. Let L be a large subset of X, $\alpha \in P$ and $X = B(L, \alpha)$. Then each of the assumptions (i) and (ii) gives $|L| = \kappa$ so $d(\mathcal{B}) = \kappa$.

(i) Let Z be a subset of X such that $|Z| < \kappa$, $\alpha \in P$. Let Y be a maximal (by inclusion) α-discrete subset of X such that $B(Z, \alpha) \cap B(Y, \alpha) = \varnothing$. If $x \in X$ then $B(x, \alpha) \cap B(Z \cup Y, \alpha) \neq \varnothing$. Hence, $Z \cup Y$ is large and $|Y| = \kappa$.

We fix a bijection $f : P \times \kappa \to \kappa$ and note that, for every $\alpha \in P$, the set $f(\alpha, \kappa)$ is cofinal in κ (as a set of ordinals). We define also two mappings $\varphi : \kappa \to P$ and $\psi : \kappa \to \kappa$ be the rule: if $f(\alpha, \lambda) = \gamma$ then $\varphi(\gamma) = \alpha$, $\psi(\gamma) = \lambda$.

We take an arbitrary $\varphi(0)$-discrete subset Y_0 of X such that $|Y_0| = \psi(0)$. Assume that, for some $\gamma < \kappa$, we have defined the family $\{Y_\lambda : \lambda < \gamma\}$ of subsets of X such that each subset Y_λ is $\varphi(\lambda)$-discrete, $|Y_\lambda| = \psi(\lambda)$ and the family $\{B(Y_\lambda, \varphi(\lambda)) : \lambda < \gamma\}$ is disjoint. Put $Z = \bigcup_{\lambda<\gamma} B(Y_\lambda, \psi(\lambda))$. In view of above paragraph there exists a $\varphi(\gamma)$-discrete subset Y_γ such that $|Y_\gamma| = \psi(\gamma)$ and $Z \cap B(Y_\gamma, \varphi(\gamma)) = \varnothing$. After κ steps we get the family $\{Y_\gamma : \gamma < \kappa\}$.

For every $\gamma < \kappa$, we enumerate $Y_\gamma = \{y(\lambda, \gamma) : \lambda < |Y_\gamma|\}$ and put $T_0 = \bigcup_{\gamma<\kappa} B(y(0, \gamma), \varphi(\gamma))$. Since φ is surjective, T_0 is thick. Assume that, for some $\delta < \kappa$, we have defined disjoint family $\{T_\mu : \mu < \delta\}$ of thick subsets of X. Put $T = \bigcup_{\mu<\delta} T_\mu$. To define T_δ we denote $I = \{\gamma : \gamma < \kappa, Y_\gamma \setminus T \neq \varnothing\}$ and put

$$T_\delta = \bigcup_{\gamma \in I} B(y(\delta, \gamma), \varphi(\gamma)).$$

After κ steps we put $\mathcal{F} = \{T_\delta : \delta < \gamma\}$. Since $f(\alpha, \kappa)$ is cofinal in κ for every $\alpha \in P$, every subset T_δ is thick.

(ii) Let $\gamma < \kappa$ and $\{Y_\lambda : \lambda < \gamma\}$ be a family of subsets of X such that $|Y_\lambda| < \kappa$ for every $\lambda < \gamma$. Let $\{p_\lambda : \lambda < \gamma\}$ be a subset of P. We put $Z = \bigcup_{\lambda<\gamma} B(Y_\lambda, \lambda)$. By (ii), $|Z| < \kappa$. Hence, for any $\alpha \in P$ and $\kappa' < \kappa$, we can take an α-disjoint subset $Y_(\gamma, \alpha)$ of X such that $B(Y(\gamma, \alpha), \alpha) \cap Z = \varnothing$ and $|Y(\gamma, \alpha)| \geqslant \kappa'$. Using this remark, we can construct the family $\mathcal{F}$ as in (i). $\square$

4. Examples

We show that, for every infinite cardinal κ, there exists a metric space X such that $d(\mathcal{B}(X)) = \kappa$.

Example 4.1. Let $\mathbb{I}$ be a (non-directed) graph with the set of vertices ω and the set of edges $\{(i, i+1) : i \in \omega\}$. We consider the set $\{\mathbb{I}_\gamma : \gamma < \kappa\}$ of copies of $\mathbb{I}$, identify the terminal vertices of these copies and denote by Γ the resulting graph. Let $X = V(\Gamma)$ be the set of vertices of Γ. We endow X with path metric: the distance between two vertices $u, v \in X$ is the length of the shortest path between u and v. If L is a large subset of X, then $L \cap V(\mathbb{I}_\gamma)$ is infinite for every $\gamma < \kappa$, so $|L| = \kappa$ and $d(\mathcal{B}(X)) = \kappa$.

The next two examples show that cellularity of a ballean could be much more smaller than its density.

Example 4.2. Let X be a set and φ be a filter on X. For any $x \in X$ and $F \in \varphi$, we put

$$B_\varphi(x, F) = \begin{cases} \{x\}, \text{ if } x \in F; \\ X \setminus F, \text{ if } x \notin F \end{cases}$$

and consider the ballean $\mathcal{B}(X, \varphi) = (X, \varphi, B_\varphi)$. A ballean $\mathcal{B} = (X, P, B)$ is called *pseudodiscrete* if, for every $\alpha \in P$, there exists a bounded subset V of X such that $B(x, \alpha) = \{x\}$ for every $x \in X \setminus V$. By [8], a ballean $\mathcal{B}$ is pseudodiscrete if and only if there exists a filter φ on X such that $\mathcal{B}$ is asymorphic to $\mathcal{B}(X, \varphi)$.

Now let X be infinite and $\cap\varphi = \varnothing$. Then $\mathcal{B}(X, \varphi)$ is an unbounded connected ballean. A subset $L \subseteq X$ is large if and only if $L \in \varphi$, so $d(\mathcal{B}) = min\{|F| : F \in \varphi\}$. On the other hand, let T be a thick subset of X, $F \in \varphi$. We take $x \in X$ such that $B_\varphi(x, F) \subseteq X$. Then either $x \in F$ or $X \setminus F \subseteq T$. It follows that T is cofinal with respect to φ, i.e.: $F \cap T \neq \varnothing$ for every $F \in \varphi$. Hence, if φ is an ultrafilter then any two thick subsets of X have non-empty intersection and $c(\mathcal{B}(X, \varphi)) = 1$.

Example 4.3. Let X be an infinite set of regular cardinality κ. Denote by $\mathcal{F}$ the family of all subsets of X of cardinality $< \kappa$. Let P be a set of all mappings $f : X \to \mathcal{F}$ such that, for every $x \in X$, we have $x \in f(x)$ and

$$\big|\{y \in X : x \in f(y)\}\big| < \kappa.$$

Given any $x \in X$ and $f \in P$, we put $B(x, f) = f(x)$ and consider the ball structure $\mathcal{B} = (X, P, B)$. Since $B^*(x, f) = \{y \in X : x \in f(y)\}$, $\mathcal{B}$ is upper symmetric. Since κ is regular, $\mathcal{B}$ is upper multiplicative. Hence, $\mathcal{B}$ is a ballean. Clearly, $\mathcal{B}$ is connected and unbounded.

If L is a large subset of X, by regularity of κ, we have $|L| = \kappa$. If A is a subset of X and $|A| = \kappa$, we fix an arbitrary bijection $h : A \to X$ and put

$$f(x) = \begin{cases} \{x\}, \text{ if } x \notin A; \\ \{x, h(x)\}, \text{ if } x \in A \end{cases}$$

Then $f \in P$ and $B(A, f) = X$. Hence, a subset L of X is large if and only if $|L| = \kappa$, so $d(\mathcal{B}) = \kappa$.

If $A \subseteq X$ and $|X \setminus A| = \kappa$, by observation 1 and above paragraph, A is not thick. It means that any two thick subsets of X are not disjoint and $c(\mathcal{B}) = 1$.

Now we compare cellularity and density with another cardinal invariant of balleans, namely resolvability, defined in [9]. Given a ballean $\mathcal{B} = (X, P, B)$ and a cardinal κ, we say that $\mathcal{B}$ is κ- resolvable if X can be partitioned in κ-many large subsets. The resolvability of $\mathcal{B}$ is the cardinal

$$r(\mathcal{B}) = sup\{\kappa : \mathcal{B} \text{ is } \kappa\text{-resolvable}\}.$$

If $\mathcal{B}$ is a ballean from Example 4.1, by [9, Theorem 2.3], $r(\mathcal{B}) = \aleph_0$. By Corollary 3.2, $d(\mathcal{B}) = c(\mathcal{B}) = \kappa$.

If $\mathcal{B}$ is a ballean from Example 4.2, defined by free ultrafilter, then $c(\mathcal{B}) = r(\mathcal{B}) = 1$, but $d(\mathcal{B}) = min\{|F| : F \in \varphi\}$.

If $\mathcal{B}$ is a ballean from Example 4.3, then $r(\mathcal{B}) = d(\mathcal{B}) = \kappa$, $c(\mathcal{B}) = 1$.

The above remarks show that there are no direct correlations between resolvability on one hand and density or cellularity on the other hand.

5. Applications

Let G be a group with the identity e endowed with some topology. Then G is called *left topological* if all the left shifts $x \mapsto gx$, $g \in G$ are continuous.

Let G be an infinite left topological group, $|G| = \kappa$, γ be an infinite cardinal such that $\gamma \leqslant \kappa$. We say that G is

- *totally bounded* if, for every nieghbourhood U of e, there exists a finite subset F of G such that $G = FU$;
- γ-*bounded* if, for every neighbourhood U of e, there exists a subset F of G such that $|F| < \gamma$ and $G = FU$;
- *weakly bounded* if, for every neighbourhood U of e, there exists a subset F of G such that $|F| < \kappa$ and $G = FU$.

In this terminology, totally bounded groups are $\aleph_0$-bounded and weakly bounded groups are κ-bounded.

We denote by $\mathcal{F}_\gamma$ the family of all subsets F of X such that $e \in F$, $F = F^{-1}$ and $|F| < \gamma$. Given any $g \in G$ and $F \in \mathcal{F}_\gamma$, we put

$$B(g, F) = Fg$$

and denote by $\mathcal{B}(G, \gamma)$ the ballean $(G, \mathcal{F}_\gamma, B)$.

If $\gamma = \aleph_0$ then a subset L is large (with respect to $\mathcal{B}(G, \aleph_0)$) if and only if $G = FL$ for some finite subset F of G. By Theorem 3.2 (case (ii) for $\kappa = \aleph_0$ and case (i) for $\kappa > \aleph_0$), there exists a disjoint family $\mathcal{F}$ of cardinality κ consisting of thick subsets. By observation 1, every subset $F \in \mathcal{F}$ meets every large subset L. It follows that $F \cap gU \neq \varnothing$ for every $g \in G$ and every neighbourhood U of e in every totally bounded topology τ on G, so F is dense in τ. Hence, G can be partitioned to κ subsets dense in each totally bounded topology.

If $\gamma < \kappa$, the same arguments applying to $\mathcal{B}(G, \gamma)$ and Theorem 3.2 (i) prove that G can be partitioned to κ subsets dense in every γ-bounded topology on G.

if $\gamma = \kappa$ and κ is regular, we apply either Theorem 3.1 or Theorem 3.2 (ii) and conclude that G can be partitioned in κ-many subsets dense in every weakly bounded topology on G.

What about $\gamma = \kappa$ and κ is singular? This is old (and unsolved) problem posed by the author [4, Problem 13.45] in the following weak form.

Problem 5.1. *Every infinite group G of regular cardinality κ can be partitioned $G = A_1 \cup A_2$ so that $FA_1 \neq G$ and $FA_2 \neq G$ for every subset F of G such that $|F| < \kappa$. Is the same true for groups of singular cardinalities?*

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