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Michael spaces and Dowker planks

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Abstract. We investigate the Lindelöf property of Dowker planks. In particular, we give necessary conditions such that the product of a Dowker plank with the irrationals is not Lindelöf. We also show that if there exists a Michael space, then, under some conditions involving singular cardinals, there is one that is a Dowker plank.

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1. Introduction

In 1963 E. Michael constructed, under the continuum hypothesis, a Lindelöf space whose product with the irrationals is not normal (see [7]). Such a space is known as a *Michael space*. An open problem is to construct a Michael space in ZFC without additional axioms.

The aim of this paper is to provide necessary conditions for the existence of a Michael space, and to give some examples of Michael spaces. Our work is associated to the results in [8].

In this note, $\mathbb{P}$ stands for the set of the irrational numbers, and the Cantor set $\mathbb{C}$ is viewed as a compactification of $\mathbb{P}$ obtained by adding a countable set $\mathbb{Q}_C$. Ordinal numbers are denoted by Greek letters; when viewed as topological spaces, they are given the order topology. Products of topological spaces are endowed with the standard product topology.

The symbol $[A]^\lambda$ denotes the family of subsets of A having size exactly λ. The symbols $[A]^{\leq\lambda}$ and $[A]^{<\lambda}$ have similar meaning.

Let $\leq_*$ be the quasi-order on a countable product of ordered sets that is associated to the coordinate-wise order on each set. Thus $f \leq_* g$ stands for $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. A subset of ${}^\omega\omega$ is *unbounded* if it is unbounded in $({}^\omega\omega, \leq_*)$. A *dominating family* is an unbounded set that is

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cofinal in $({}^{\omega}\omega, \leq_*)$. A subset of ${}^{\omega}\omega$ is a *scale* if it is a dominating family and is well-ordered by $\leq_*$.

Recall that $\mathbb{P}$ can be identified with ${}^{\omega}\omega$ with the product topology. For each $\xi \in {}^{<\omega}\omega = \{\eta \mid \eta : [0,n] \to \omega \text{ for some } n\}$, a basic open neighborhood of ξ in the product topology is $\{f \in {}^{\omega}\omega : \xi \subseteq f\}$. For every $g \in {}^{\omega}\omega$, the sets $\{f \in {}^{\omega}\omega : f \leq g\}$ and $\{f \in {}^{\omega}\omega : f \leq_* g\}$ are respectively compact and σ-compact (see [2]).

Let X and Y be topological spaces. A set $A \subseteq X$ is *Y-analytic* if it is a projection on X of a closed subset of $X \times Y$. In particular, $A \subseteq X$ is *analytic* if it is $\mathbb{P}$-analytic.

Given a function $f : X \to Y$, the small image of $A \subseteq X$ is defined by $f^\sharp(A) = \{y \in Y : f^{-1}(y) \subseteq A\}$. Sometimes we abuse of terminology and say that $f^\sharp$ is open, with the meaning that for each open subset A of X, $f^\sharp(A)$ is an open subset of Y.

In most cases we will employ the notation used in [4] and [6].

2. Michael sequences and Michael functions

We start the section with the definition of a Michael sequence. The first goal of this section is to show that Michael sequences may be assumed to be continuous.

Definition 2.1. Let $\{X_\xi\}_{\xi \leq \theta}$ be a decreasing sequence of sets. It is a *continuous sequence* if for any $\gamma \leq \theta$, with γ limit ordinal, $X_\gamma = \cap_{\xi<\gamma} X_\xi$.

Definition 2.2 (Moore [8])**.** *A decreasing sequence $\{X_\xi\}_{\xi \leq \theta}$ of subsets of a topological space Z is said to be a $\mathcal{K}$-*Michael sequence *if the following conditions hold:*

(i) *for each K compact subset of $Z \setminus X_\theta$ the ordinal $\delta_K = \min\{\xi \leq \theta : X_\xi \cap K = \varnothing\}$ does not have uncountable cofinality.*

*In particular an $\mathcal{F}$-*Michael sequence *is a $\mathcal{K}$-Michael sequence satisfying the following additional condition:*

$(\mathrm{ii})_{\mathcal{F}}$ *for each F closed subset of $Z \setminus X_\theta$ the ordinal $\delta_F = \min\{\xi \leq \theta : X_\xi \cap F = \varnothing\}$ is either θ or does not have uncountable cofinality.*

*Also given a topological space Y, an $\mathcal{A}(Y)$-*Michael sequence *is a $\mathcal{K}$-Michael sequence satisfying the following additional condition:*

$(\mathrm{ii})_{\mathcal{A}}$ *for each A which is Y-analytic in $Z \setminus X_\theta$ the ordinal $\delta_A = \min\{\xi \leq \theta : X_\xi \cap A = \varnothing\}$ is either θ or does not have uncountable cofinality.*

Remark 2.3. In the definition of a $\mathcal{K}$-Michael sequence, we observe that the property of being a continuous sequence is partially satisfied . In other words, for every limit ordinal $\gamma < \theta$ with $\mathrm{cf}\gamma > \omega$ it follows that $X_\gamma = \cap_{\xi<\gamma} X_\xi$. Indeed, let $x \in \cap_{\xi<\gamma} X_\xi \setminus X_\gamma$. Then $\{x\}$ is a compact subset of $Z \setminus X_\theta$, and $\delta_{\{x\}} = \gamma$, so that $\mathrm{cf}\delta_{\{\mathrm{x}\}} > \omega$ in contradiction with the definition of $\mathcal{K}$-Michael sequence.

Lemma 2.4. *Let θ be a cardinal and $\{X_\xi\}_{\xi\leq\theta}$ (strictly) decreasing sequence such that $X_\gamma = \cap_{\xi<\gamma} X_\xi$ for every limit ordinal $\gamma < \theta$ with $\mathrm{cf}\gamma > \omega$. Then there exists $\{Y_\xi\}_{\xi\leq\theta}$ continuous (strictly) decreasing sequence, such that $Y_\alpha = X_\alpha$ for every $\alpha < \theta$ with $\mathrm{cf}\alpha \neq \omega$.*

Proof. Let $\{X_\xi\}_{\xi\leq\theta}$ be decreasing sequence. Define $\{Y_\xi\}_{\xi\leq\theta}$ such that $Y_\alpha = X_\alpha$ for every $\alpha < \theta$ with $\mathrm{cf}\alpha > \omega$, otherwise $Y_\alpha = \cap_{\xi\leq\alpha} X_\xi$. Clearly $Y_\eta \supseteq Y_\xi$ for every $\eta < \xi \leq \theta$. Moreover for every $\alpha < \theta$ with $\mathrm{cf}\alpha = \omega$, $Y_\alpha \supseteq X_\alpha$. By construction, we have that $\{Y_\xi\}_{\xi\leq\theta}$ is a continuous sequence.

Assume that all the subsets $X_\xi \in \{X_\xi\}_{\xi\leq\theta}$ are distinct. Then $Y_\alpha \supseteq X_\alpha \supset X_{\alpha+1} = Y_{\alpha+1}$ implies that Y_α's are distinct. □

In case we have two or more sequences of subsets of Z of length $\theta+1$, having the same last element, and given H, we denote δ_H with respect the sequence $\{X_\xi\}_{\xi\leq\theta}$ with $\delta_H^{\tilde{X}}$.

Lemma 2.5. *Let θ be a cardinal with $\mathrm{cf}\theta > \omega$, $\{X_\xi\}_{\xi\leq\theta}$ and $\{Y_\xi\}_{\xi\leq\theta}$ two decreasing sequences of subsets of a topological space Z, such that $Y_\alpha = X_\alpha$ for every $\alpha < \theta$ with $\mathrm{cf}\alpha \neq \omega$. Then*

$$\delta_H^{\tilde{X}} < \delta_H^{\tilde{Y}} \;\Rightarrow\; (\,\delta_H^{\tilde{Y}} = \delta_H^{\tilde{X}} + 1\,) \wedge (\,\mathrm{cf}\delta_{\mathrm{H}}^{\tilde{X}} = \omega\,)$$

with $H \subseteq Z$.

Proof. From Remark 2.3 it follows that for every $\alpha < \theta$ with $\mathrm{cf}\alpha > \omega$, $X_\alpha = \cap_{\xi<\alpha} X_\xi$, and $X_\alpha = Y_\alpha \supseteq \cap_{\xi<\alpha} Y_\xi$. We have also that for every $\alpha < \theta$ with $\mathrm{cf}\alpha > \omega$ there exists a cofinal sequence $(\alpha_\eta)_{\eta<\mathrm{cf}\alpha}$ such that $Y_{\alpha_\eta} \subseteq X_{\alpha_\eta}$. Assume that $\mathrm{cf}\delta_{\mathrm{H}}^{\tilde{X}} = \omega$ and $\delta_H^{\tilde{Y}} \neq \delta_H^{\tilde{X}} + 1$, we want to show that $\delta_H^{\tilde{X}} = \delta_H^{\tilde{Y}}$. Two cases: (i) $\mathrm{cf}\delta_{\mathrm{H}}^{\tilde{Y}} > \omega$ and (ii) $\mathrm{cf}\delta_{\mathrm{H}}^{\tilde{Y}} = \omega$. If (i) holds, then $Y_{\delta_H^{\tilde{Y}}} \cap H = X_{\delta_H^{\tilde{Y}}} \cap H = \varnothing$, therefore $\delta_H^{\tilde{X}} \leq \delta_H^{\tilde{Y}}$. If $\delta_H^{\tilde{X}} < \delta_H^{\tilde{Y}}$ there exists α_η, such that $\delta_H^{\tilde{X}} < \alpha_\eta < \delta_H^{\tilde{Y}}$. Then by minimality of $\delta_H^{\tilde{Y}}$ we have $Y_{\alpha_\eta} \cap K \neq \varnothing$ and $Y_{\alpha_\eta} \cap K \subseteq X_{\alpha_\eta} \cap K$. Moreover $X_{\alpha_\eta} \subseteq X_{\delta_H^{\tilde{X}}}$, so $X_{\delta_H^{\tilde{X}}} \cap K \neq \varnothing$ which is in contradiction with the definition of δ_K. Thus $\delta_H^{\tilde{X}} = \delta_H^{\tilde{Y}}$. For (ii), assume by contradiction, that $\delta_H^{\tilde{X}} \neq \delta_H^{\tilde{Y}}$. Since $\mathrm{cf}\delta_{\mathrm{H}}^{\tilde{X}} = \mathrm{cf}\delta_{\mathrm{H}}^{\tilde{Y}} = \omega$, there exists α successor ordinal such that $\delta_H^{\tilde{Y}} < \alpha < \delta_H^{\tilde{X}}$. Then $X_\alpha = Y_\alpha$, and so $Y_{\delta_H^{\tilde{Y}}} \supseteq Y_\alpha = X_\alpha \supseteq X_{\delta_H^{\tilde{X}}}$. Therefore $X_\alpha \cap H = \varnothing$ which is in contradiction with the minimality of $\delta_H^{\tilde{X}}$. Thus $\delta_H^{\tilde{X}} = \delta_H^{\tilde{Y}}$. □

Corollary 2.6. *Let θ be a cardinal with $\mathrm{cf}\theta > \omega$, $\{X_\xi\}_{\xi\leq\theta}$ and $\{Y_\xi\}_{\xi\leq\theta}$ two decreasing sequences of subsets of Z, such that $Y_\alpha = X_\alpha$ for every $\alpha < \theta$ with $\mathrm{cf}\alpha \neq \omega$. Let $H \subset Z$, then $\delta_H^{\tilde{X}} = \delta_H^{\tilde{Y}}$ if either one has uncountable cofinality.*

Corollary 2.7. *Let θ be a cardinal with $\mathrm{cf}\theta > \omega$, $\{X_\xi\}_{\xi\leq\theta}$ and $\{Y_\xi\}_{\xi\leq\theta}$ two decreasing sequences of subsets of a topological space Z, such that $Y_\alpha = X_\alpha$ for every $\alpha < \theta$ with $\mathrm{cf}\alpha \neq \omega$.*

Then $\{X_\xi\}_{\xi\leq\theta}$ is a $\mathcal{K}$-Michael (resp., $\mathcal{F}$-Michael or $\mathcal{A}(Y)$-Michael) sequence if and only if $\{Y_\xi\}_{\xi\leq\theta}$ is a $\mathcal{K}$-Michael (resp., $\mathcal{F}$-Michael or $\mathcal{A}(Y)$-Michael) sequence.

Proof. Let $\{X_\xi\}_{\xi\le\theta}$ be a K-Michael (resp., $\mathcal{F}$-Michael or $\mathcal{A}(Y)$-Michael) sequence. By hypothesis $Y_\theta = X_\theta$. Let $H \subseteq (Z \setminus X_\theta)$ compact (resp., closed or analytic). Then $\mathrm{cf}\delta_{\mathrm{H}}^{\tilde{\mathrm{X}}} \le \omega$ (resp., either $\delta_H^{\tilde{\mathrm{X}}} \le \omega$ or $\delta_H^{\tilde{\mathrm{X}}} = \theta$). We want to check that $\mathrm{cf}\delta_{\mathrm{K}}^{\tilde{\mathrm{Y}}} \le \omega$ (resp., either $\delta_H^{\tilde{\mathrm{Y}}} \le \omega$ or $\delta_H^{\tilde{\mathrm{Y}}} = \theta$). Assume not, i.e., $\mathrm{cf}\delta_{\mathrm{K}}^{\tilde{\mathrm{Y}}} > \omega$, (resp., $\omega < \mathrm{cf}\delta_{\mathrm{K}}^{\tilde{\mathrm{Y}}} < \theta$) Corollary 2.6 implies that $\delta_K^{\tilde{\mathrm{X}}} = \delta_K^{\tilde{\mathrm{Y}}}$, which is a contradiction. □

Corollary 2.8. *Let θ be a cardinal with* $\mathrm{cf}\theta > \omega$. *The following are equivalent:*

(i) *there exists $\{X_\xi\}_{\xi\le\theta}$ which is $\mathcal{K}$-Michael (resp., $\mathcal{F}$-Michael or $\mathcal{A}(Y)$-Michael strictly decreasing) sequence;*

(ii) *there exists $\{X_\xi\}_{\xi\le\theta}$ continuous $\mathcal{K}$-Michael (resp., $\mathcal{F}$-Michael or $\mathcal{A}(Y)$-Michael strictly decreasing) sequence.*

Next we introduce the definition of Michael function and we analyze the relationship between Michael functions and Michael sequences.

Definition 2.9. Let Z be a topological space and $f : Z \to \theta + 1$ an arbitrary function. Then f is said to be a *$\mathcal{K}$-Michael function* if the following condition holds:

(i) for each K compact subset of $Z \setminus f^{-1}(\{\theta\})$, $\sup_{\mathrm{x}\in\mathrm{K}}\mathrm{f(x)} + 1$ does not have uncountable cofinality.

In particular an *$\mathcal{F}$-Michael function* is a $\mathcal{K}$-Michael function satisfying the following additional condition:

(ii) for every F closed subset of $Z \setminus f^{-1}(\{\theta\})$, $\sup_{\mathrm{x}\in\mathrm{F}}\mathrm{f(x)} + 1$ is either θ or does not have uncountable cofinality.

Also given a topological space Y, an *$\mathcal{A}(Y)$-Michael function* is a $\mathcal{K}$-Michael function satisfying the following additional condition:

(ii) for every A which is Y-analytic in $Z \setminus f^{-1}(\{\theta\})$, $\sup_{\mathrm{x}\in\mathrm{A}}\mathrm{f(x)}+1$ is either θ or does not have uncountable cofinality.

In the next proposition we will show the equivalence of continuous $\mathcal{K}$-Michael sequences with $\mathcal{K}$-Michael functions $f : Z \to \theta + 1$.

Lemma 2.10. *Let Z be a topological space, $f : Z \to \theta + 1$ be an arbitrary function with θ cardinal. If $X_\xi = \{x \in Z : f(x) \ge \xi\}$ for every $\xi \in \theta$, then $\delta_H = \sup_{\mathrm{x}\in\mathrm{H}}\mathrm{f(x)} + 1$ for every $H \subseteq Z \setminus X_\theta$.*

Proof. By definition we have that $\delta_H = \min\{\xi \le \theta : K \cap X_\xi = \varnothing\} = \min\{\xi \le \theta : \forall x \in K\ (x \notin X_\xi)\} = \min\{\xi \le \theta : \forall x \in K\ (f(x) < \xi)\} = \sup\{\mathrm{f(x)} + 1 : \mathrm{x} \in \mathrm{K}\}$. □

Lemma 2.11. *Let θ be a cardinal, $\{X_\xi\}_{\xi\le\theta}$ a continuous sequence of subsets of topological space Z, and $f : Z \to \theta+1$ a function defined by $f(x) = \sup\{\gamma \in \theta + 1 : \mathrm{x} \in \mathrm{X}_\gamma\}$. Then we have:*

(i) *$X_\xi = \{x \in Z : f(x) \ge \xi\}$ for every $\xi \in \theta$;*

(ii) *f is surjective if and only if $\{X_\xi\}_{\xi\le\theta}$ is strictly decreasing.*

Proof. To show (i), we have that for every $\xi \in \theta$, $\{x \in Z : f(x) \geq \xi\} = \{x \in Z : \sup\{\gamma \in \theta : \mathrm{x} \in \mathrm{X}_\gamma\} \geq \xi\}$. From the continuity follow $\{x \in Z : \sup\{\gamma \in \theta : \mathrm{x} \in \mathrm{X}_\gamma\} \geq \xi\} = \{\mathrm{x} \in \mathrm{Z} : \mathrm{x} \in \mathrm{X}_\xi\} = \mathrm{X}_\xi$.

For (ii), first assume that f is surjective. By (i) we have that $X_\xi = \{x \in Z : f(x) \geq \xi\}$ for every $\xi \in \theta$, and so $\{X_\xi\}_{\xi \leq \theta}$ is a decreasing sequence. Assume that there exist α, $\beta \in \theta$ with $\alpha < \beta$ such that $X_\alpha = X_\beta$. Thus there exist $\xi \in \theta$ with $\alpha < \xi \leq \beta$ and $z \in Z$ such that $f(x) = \xi$. Hence $x \in X_\beta$ but $x \notin X_\alpha$, a contradiction.

On the other hand, assume that $\{X_\xi\}_{\xi \leq \theta}$ is strictly decreasing, and f is not surjective. Then there exists $\alpha < \theta$ such that $f(x) \neq \alpha$ for any $x \in Z \setminus X_\theta$, with $X_\theta = f^{-1}(\{\theta\})$. Let $f(x) > \alpha$. From (i) it follows that there exists $\alpha < \theta$ such that $(Z \setminus X_\theta) \subseteq X_\alpha$. Thus $X_\beta = X_\alpha$ for any $\beta \leq \alpha$, which contradicts the fact that the sequence is strictly decreasing. If $f(x) < \alpha$, follow that $(Z \setminus X_\theta) \cap X_\alpha = \varnothing$, which is a contradiction. □

Proposition 2.12. *Let Z and Y be two topological spaces, θ a cardinal with* $\mathrm{cf}\theta > \omega$. *For every $Q \subseteq Z$, the following statements are equivalent:*

(i) *there exists a continuous $\mathcal{K}$-Michael (resp., $\mathcal{F}$-Michael or $\mathcal{A}$- Michael) sequence $\{X_\xi\}_{\xi \leq \theta}$ with $Q = X_\theta$;*
(ii) *there exists $\mathcal{K}$-Michael (resp., $\mathcal{F}$-Michael or $\mathcal{A}(Y)$-Michael) function $f : Z \to \theta + 1$, with $Q = f^{-1}(\{\theta\})$;*
(iii) *there exists $\mathcal{K}$-Michael (resp., $\mathcal{F}$-Michael or $\mathcal{A}(Y)$-Michael) sequence $\{X_\xi\}_{\xi \leq \theta}$ with $Q = X_\theta$.*

Proof. (i) ⇒ (ii). Let $\{X_\xi\}_{\xi \leq \theta}$ be a continuous $\mathcal{K}$-Michael sequence. Define the following map $f : Z \to \theta + 1$ such that $f(x) = \sup\{\gamma \in \theta + 1 : x \in X_\gamma\}$. Clearly $f^{-1}(\{\theta\}) = X_\theta$. Now let $H \subset (Z \setminus Q)$ be a compact (resp., closed or analytic) subset. By Lemma 2.11, for any $\alpha \leq \theta$, $X_\alpha = \{x \in Z : f(x) \geq \alpha\}$. By Lemma 2.10, $\sup_{\mathrm{x}\in\mathrm{H}}\mathrm{f(x)} + 1) = \delta_\mathrm{H}$. Since $\mathrm{cf}\delta_\mathrm{K} \leq \omega$ (resp., either $\mathrm{cf}\delta_\mathrm{H} \leq \omega$ or $\mathrm{cf}\delta_\mathrm{H} = \theta$), then cf $(\sup_{\mathrm{x}\in\mathrm{K}}\mathrm{f(x)} + 1)) \leq \omega$ (resp., either cf $(\sup_{\mathrm{x}\in\mathrm{H}}\mathrm{f(x)} + 1) \leq \omega$ or cf $(\sup_{\mathrm{x}\in\mathrm{H}}\mathrm{f(x)} + 1) = \theta$).

(ii) ⇒ (iii). Let $f : Z \to \theta + 1$ be a $\mathcal{K}$-Michael function with $Q = f^{-1}(\{\theta\})$. For any $\alpha \leq \theta$ define $X_\alpha = \{x \in Z : f(x) \geq \alpha\}$. Clearly $X_\theta = f^{-1}(\{\theta\})$ and $X_\xi \supseteq X_\eta$ for any $\xi < \eta \leq \theta$. Let now $H \subset (Z \setminus Q)$ be a compact (resp., closed or analytic) subset, we want to show that $\mathrm{cf}\delta_\mathrm{H} \leq \omega$. By Lemma 2.10, $\sup_{\mathrm{x}\in\mathrm{K}}\mathrm{f(x)} + 1 = \delta_\mathrm{H}$. Since cf $(\sup_{\mathrm{x}\in\mathrm{H}}\mathrm{f(x)} + 1) \leq \omega$ (resp., either cf $(\sup_{\mathrm{x}\in\mathrm{H}}\mathrm{f(x)} + 1) \leq \omega$ or cf $(\sup_{\mathrm{x}\in\mathrm{H}}\mathrm{f(x)} + 1) = \theta$), then $\mathrm{cf}\delta_\mathrm{H} \leq \omega$ (resp., either $\mathrm{cf}\delta_\mathrm{H} \leq \omega$ or $\mathrm{cf}\delta_\mathrm{H} = \theta$).

(iii) ⇒ (i). Follow from Corollary 2.8.

□

Corollary 2.13. *Let θ be a cardinal of uncountable cofinality, Z and Y two topological spaces. There exists a continuous $\mathcal{K}$-Michael (resp., $\mathcal{F}$-Michael or $\mathcal{A}(Y)$-Michael) strictly decreasing sequence $\{X_\xi\}_{\xi \leq \theta}$ of subsets of Z, if and only if there exists $\mathcal{K}$-Michael (resp., $\mathcal{F}$-Michael or $\mathcal{A}(Y)$-Michael) function $f : Z \to \theta + 1$ that is surjective.*

3. Local properties of Michael functions

In this section we want to analyze and characterize the properties of being a Michael function. First we need the following definition.

Definition 3.1. Let Z be a topological space, $h : Z \to \theta + 1$ an arbitrary function with θ cardinal. For every $\alpha \leq \theta$ we say that h is *Michael at* α if

$$\mathrm{cf}\alpha > \omega \ \Rightarrow \ (\forall \mathrm{F} \subseteq \mathrm{Z}\ \mathrm{closed}\,(\forall \mathrm{z} \in \mathrm{F}\ \mathrm{h(z)} < \alpha) \Rightarrow (\mathrm{sup}_{\mathrm{z}\in\mathrm{F}}\mathrm{h(z)} < \alpha)).$$

Moreover h is *σ-Michael at* α if

$$\mathrm{cf}\alpha > \omega \ \Rightarrow \ (\forall \mathrm{C} \subseteq \mathrm{Z}\ \mathrm{F}_\sigma\text{-set}\ \ (\forall \mathrm{z} \in \mathrm{C}\ \mathrm{h(z)} < \alpha) \Rightarrow (\mathrm{sup}_{\mathrm{z}\in\mathrm{C}}\mathrm{h(z)} < \alpha)).$$

Directly from the definition follow:

Lemma 3.2. *Let Z be a topological space, $h : Z \to \theta + 1$ an arbitrary function with θ cardinal. The following statements are equivalent:*

(i) *h is Michael at α,*

(ii) $\mathrm{cf}\alpha > \omega \Rightarrow (\forall \mathrm{U} \subseteq \mathrm{Z}\,\mathrm{open}\,(\mathrm{h}^{-1}[\alpha, \theta] \subseteq \mathrm{U}) \Rightarrow (\mathrm{sup}_{\mathrm{z}\in\mathrm{Z}\setminus\mathrm{U}}\mathrm{h(z)} < \alpha))$.

Lemma 3.3. *Let Z be a topological space, $h : Z \to \theta + 1$ an arbitrary function with θ cardinal. The following statements are equivalent:*

(i) *h is σ-Michael at α,*

(ii) $\mathrm{cf}\alpha > \omega \Rightarrow (\forall \mathrm{G} \subseteq \mathrm{Z}\,\mathrm{G}_\delta\text{-set}\,(\mathrm{h}^{-1}([\alpha, \theta]) \subseteq \mathrm{G}) \Rightarrow (\mathrm{sup}_{\mathrm{z}\in\mathrm{Z}\setminus\mathrm{G}}\mathrm{h(z)} < \alpha))$.

Lemma 3.4. *Let Z be a topological space, $h : Z \to \theta + 1$ an arbitrary function with θ cardinal. Then h is σ-Michael at α if and only if h is Michael at α.*

Proof. Let $\mathrm{cf}\alpha > \omega$, and $C = \bigcup_{n\in\omega} F_n$ with F_n closed subset of Z such that for every $z \in C\, h(z) < \alpha$. Then for every $n \in \omega$ and for every $z \in F_n$, $h(z) < \alpha$. Let $\alpha_n = \mathrm{sup}_{\mathrm{z}\in\mathrm{F_n}}\mathrm{h(z)}$. Since h is Michael at α, follow $\alpha_n < \alpha$ for every $n \in \omega$. Then $\mathrm{sup}_{\mathrm{z}\in\mathrm{C}}\mathrm{h(z)} = \mathrm{sup}_{\mathrm{n}\in\omega}\alpha_{\mathrm{n}}$. From $\mathrm{cf}\alpha > \omega$ if follows that $\mathrm{sup}_{\mathrm{n}\in\omega}\alpha_{\mathrm{n}} < \alpha$. □

An arbitrary function $h : Z \to \theta+1$, induces a new function $\hat{h} : Z\times Y \to \theta+1$, defined by $\hat{h}(x) = h(\pi_1(x))$ for every $x \in Z \times Y$, where Y is an arbitrary topological space, and π_1 the projection of $Z \times Y$ onto its first coordinate space. Clearly this raises the question whether $\hat{h}$ is Michael at some ordinal $\alpha \leq \theta$.

Lemma 3.5. *Let Z, Y be two topological spaces, $h : Z \to \theta + 1$ is an arbitrary function, $\hat{h} : Z \times Y \to \theta + 1$ with θ cardinal. Then the following statements are equivalent:*

(i) *$\hat{h}$ is Michael at α,*

(ii) $\mathrm{cf}\alpha > \omega \Rightarrow (\forall \mathrm{A} \subseteq \mathrm{Z}\,\mathrm{Y}\text{-analytic}\,(\forall \mathrm{z}\in\mathrm{A}\ \mathrm{h(z)} < \alpha) \Rightarrow \mathrm{sup}_{\mathrm{z}\in\mathrm{A}}\mathrm{h(z)} < \alpha)$.

Moreover, if $\hat{h}$ is Michael at α for some $\alpha \leq \theta$, then h is Michael at the same ordinal. But the converse does not hold.

Now, given a function $h : Z \to \theta + 1$, we want to characterize the property of being Michael at some ordinal for h, in term of a Michael function.

Proposition 3.6. *Let Z be a topological space, $h : Z \to \theta + 1$ a function with θ cardinal. Then the following statements are equivalent:*

(i) *h is a $\mathcal{F}$-Michael function;*
(ii) *h is Michael at α for every $\alpha \leq \theta$.*

Proof. Assume that h is not a $\mathcal{F}$-Michael function. Then there is a closed set $F \subseteq (Z \setminus h^{-1}(\{\theta\}))$ such that $\mathrm{cf}(\mathrm{sup}_{z\in F}\mathrm{h(z)} + 1) > \omega$. Let $\alpha = \mathrm{sup}_{z\in K}\mathrm{h(z)} + 1$. Note that $h(z) < \alpha$ for every $z \in F$. Since h is Michael at α, from Lemma 3.2 follow that $\mathrm{sup}_{z\in F}\mathrm{h(z)} < \alpha$, which is a contradiction.

Vice versa, Assume that h is a $\mathcal{F}$-Michael function. Let $\alpha \in \theta$ such that $\mathrm{cf}\alpha > \omega$. We want to show that h is Michael at α. Let U be an open set of Z such that $h^{-1}([\alpha, \theta]) \subset U$. Then $Z \setminus U$ is such that $h(z) < \alpha$ for every $z \in Z$. Therefore $\mathrm{cf}(\mathrm{sup}_{Z\setminus U}\mathrm{h(z)} + 1) \leq \omega$. Since $\mathrm{cf}\alpha > \omega$ there exists $\beta < \alpha$ such that $\mathrm{sup}_{z\in Z\setminus U}\mathrm{h(z)} + 1 \leq \beta < \alpha$. □

From the previous proof we can argue that if h is Michael at α for every $\alpha \in \theta$, then h is $\mathcal{F}$-Michael function, and so $\mathcal{K}$-Michael function, but the vice versa does not hold. Clearly it is true in case Z is a compact space.

Moreover we have shown that if h is a $\mathcal{F}$-Michael function, then there exists an ordinal α such that h is Michael at α. The vice versa does not hold, we needed the property of being Michael to be satisfied at each ordinal into the codomain of h.

Proposition 3.7. *Let Z, Y be two topological spaces, $h : Z \to \theta + 1$ and $\hat{h} : Z \times Y \to \theta + 1$ functions with θ cardinal. Then the following statements are equivalent:*

(i) *h is a $\mathcal{A}(Y)$-Michael function,*
(ii) *$\hat{h}$ is Michael at α for each $\alpha \leq \theta$.*

Proof. Assume that h is not a $\mathcal{A}(Y)$-Michael function. Then there is a set $A \subseteq (Z \setminus h^{-1}(\{\theta\}))$ which is the projection onto Z of a closed subset F of $Z \times Y$, such that $\omega < \mathrm{cf}(\mathrm{sup}_{z\in A}\mathrm{h(z)} + 1) < \theta$. Let $\alpha = \sup_{z\in A} h(z) + 1$. Since A is a Y-analytic subset of Z and $\hat{h}$ is Michael at α it follows that $\mathrm{sup}_{z\in A}\mathrm{h(z)} < \alpha$ which is in contradiction with $\mathrm{sup}_{z\in A}\mathrm{h(z)} = \{\beta \leq \alpha : \mathrm{h(z)} \geq \beta \text{ for same } \mathrm{z} \in \mathrm{A}\} = \alpha$.

Assume that h is a $\mathcal{A}(Y)$-Michael function. Let $\alpha < \theta$ with $\mathrm{cf}\alpha > \omega$. Let A be a Y-analytic subset of Z, i.e., $A = \pi(F)$ where F is a closed subset of $Z \times Y$ such that $h(z) < \alpha$ for every $z \in A$. Then $\mathrm{cf}(\mathrm{sup}_{z\in A}\mathrm{h(z)} + 1) \leq \omega$. Since $\mathrm{cf}\alpha > \omega$, it follows that $\mathrm{sup}_{z\in A}\mathrm{h(z)} < \alpha$. □

Remark 3.8. Note that by Proposition 2.12 and Proposition 3.6, it follows that if $h : \mathbb{C} \to \theta + 1$ is such that $\mathbb{Q}_C = h^{-1}(\{\theta\})$, the property of being Michael at α for every $\alpha \leq \theta$ is equivalent to the notion of $\mathcal{K}$-Michael sequence $\{X_\xi\}_{\xi\leq\theta}$ [M [8]], where for every $\xi \in \theta$, $X_\xi \subseteq \mathbb{C}$ and $X_\theta = \mathbb{Q}_C$.

The next Proposition give us conditions on the function $h : Z \to \theta + 1$, so that the function $\hat{h}$ is not Michael at θ.

Proposition 3.9. *Let θ be a cardinal with* $\mathrm{cf}\theta > \omega$, *$Z$ a topological space. Let $h : Z \to \theta + 1$ be a function such that $h(Z) \cap (\alpha, \theta) \neq \varnothing$ for every $\alpha < \theta$. Then $\hat{h}$ is not Michael at θ, where $\hat{h} : Z \times (Z \setminus h^{-1}(\{\theta\})) \to \theta + 1$.*

Proof. Set $\Delta = \{(z, z) : z \in Z \setminus h^{-1}(\{\theta\})\}$. Then Δ is a closed subset of $Z \times (Z \setminus h^{-1}(\{\theta\}))$ such that for every $z \in Z \setminus h^{-1}(\{\theta\})$ we have $h(z) < \theta$. But $\sup_{z \in Z \setminus h^{-1}(\{\theta\})} h(z) = \theta$. If not, there exists $\alpha < \theta$ such that $\sup_{z \in Z \setminus h^{-1}(\{\theta\})} h(z) = \alpha$. Since $h(Z) \cap (\alpha, \theta) \neq \varnothing$, there exists β with $\alpha < \beta < \theta$, $z \in Z \setminus h^{-1}(\{\theta\})$ such that $h(z) = \beta$ which is a contradiction. □

4. NL Property

In this section we introduce the new definition of NL Property at some ordinal, and we give examples of functions which have this property.

Definition 4.1. Let X be a topological space, θ a cardinal and $\jmath : X \to \theta$ an arbitrary function. For each $\alpha \leq \theta$ with $\mathrm{cf}\alpha > \omega$, we say that $\jmath$ *has the property NL at* α if for every $A \subseteq X$ such that $\jmath(A)$ is cofinal in α, A is not Lindelöf

Remark 4.2. A banal case for the function $\jmath : X \to \theta + 1$ to have the property NL at each $\alpha \leq \theta$ with $\mathrm{cf}\alpha > \omega$, is for $\jmath = id_{\theta+1}$. Indeed every subset of α which is cofinal in α cannot be Lindelöf

Another simple case in which $\jmath$ has the property NL at each $\alpha \leq \theta$ with $\mathrm{cf}\alpha > \omega$ is when $\jmath^{-1}(\beta)$ is open in X for every $\beta < \alpha$. Indeed, assume that $A \subseteq X$ such that $\jmath(A)$ is cofinal in α, and by contradiction A is Lindelöf. Then $\{\jmath^{-1}(\beta)\}_{\beta \in \alpha}$ is an open cover for A, therefore there exist $\beta_0 \in \alpha$ countable such that $A \subseteq \bigcup_{\beta \in \beta_0} \jmath^{-1}(\beta)$. Thus $A \subseteq \jmath^{-1}(\beta_0)$ which is a contradiction.

Other examples of function with the property NL are given. Before we need the following definitions.

Definition 4.3. Let θ be a cardinal and X a topological space. The family $\{A_\alpha\}_{\alpha \in \theta}$ is a *special G_δ family* of X, if for every $\alpha \in \theta$, $A_\alpha = \bigcap_{n \in \omega} A^n_\alpha$ where each A^n_α is open in X and for every $n \in \omega$, $\{A^n_\alpha\}_{\alpha \in \theta}$ is an increasing family.

Definition 4.4. Let θ be a cardinal and X a topological space. The function $\jmath : X \to \theta + 1$ is a *special at* α with $\alpha \leq \theta$, if there exists a sequence of continuous functions $(\jmath_n)_{n \in \omega}$ with $\jmath_n : X \to \theta + 1$ such that for every $n \in \omega$,

(i) $\jmath^{-1}(\alpha) \subseteq \jmath_n^{-1}(\alpha)$,
(ii) $\jmath(x) \leq \jmath_n(x)$ for every $x \in X$,
(iii) $\{\jmath_n^{-1}(\alpha)\}_{\alpha \leq \theta}$ is an increasing family.

Lemma 4.5. *Let θ be a cardinal, X a topological space and $\jmath : X \to \theta + 1$ a function. The following statements are equivalent:*

(i) *$\jmath$ is special at each $\alpha \leq \theta$*
(ii) *$\{\jmath^{-1}(\alpha)\}_{\alpha \leq \theta}$ is a special G_δ family of X.*

Proof. Let $\alpha \leq \theta$ and $\jmath$ be special at α. Let $(\jmath_{n,\alpha})_{n \in \omega}$ be a sequence of continuous functions $\jmath_{n,\alpha} : X \to \theta + 1$ satisfying properties in Definition 4.4. By continuity of each $\jmath_{n,\alpha}$, the set $\jmath_{n,\alpha}^{-1}(\alpha)$ is open in X for each $n \in \omega$. Since $\jmath^{-1}(\alpha) \subseteq \jmath_{n,\alpha}^{-1}(\alpha)$, it follows that $\jmath^{-1}(\alpha) \subseteq \bigcap_{n\in\omega} \jmath_{n,\alpha}^{-1}(\alpha)$ for each $\alpha \in \omega$. We show that $\bigcap_{n\in\omega} \jmath_{n,\alpha}^{-1}(\alpha) \subseteq \jmath^{-1}(\alpha)$. Let $x \in \bigcap_{n\in\omega} \jmath_{n,\alpha}^{-1}(\alpha)$, hence $x \in \jmath_{n,\alpha}^{-1}(\alpha)$ for each $n \in \omega$, i.e., for each n, $\jmath_{n,\alpha}(x) \in \alpha$. Since $\jmath(x) \leq \jmath_{n,\alpha}(x)$ for all $n \in \omega$ and $x \in X$, we have that $\jmath(x) \leq \alpha$. Thus $x \in \jmath^{-1}(\alpha)$ and $\jmath^{-1}(\alpha) = \bigcap_{n\in\omega} \jmath_{n,\alpha)}^{-1}(\alpha)$ for each $\alpha \leq \theta$. Moreover for every $n \in \omega$, we have that $\{\jmath_{n,\alpha}^{-1}(\alpha)\}_{\alpha\leq\theta}$ is an increasing family.

Vice versa, assume that $\{\jmath^{-1}(\alpha)\}_{\alpha\leq\theta}$ is a special G_δ family of X. Let $\alpha \leq \theta$. By hypothesis, $\jmath^{-1}(\alpha) = \bigcap_{n\in\omega} A_\alpha^n$ with the property that A_α^n is an open set and for every n the family $\{A_\alpha^n\}_{\alpha\leq\theta}$ is increasing. Define for each $n \in \omega$, the function $\jmath_n : X \to \theta + 1$ by $\jmath_n(x) = \min\{\xi \in \theta + 1 : x \in A_\xi^n\}$. We have that for each $\alpha \leq \theta$, $\jmath_n^{-1}(\alpha) = A_\alpha^n$. Indeed, $A_\alpha^n \subseteq \jmath_n^{-1}(\alpha)$ and for each $\gamma > \alpha$ there is not $y \in A_\gamma^n \setminus A_\alpha^n$ such that $y \in \jmath_n^{-1}(\alpha)$. Otherwise from $y \in \jmath_n^{-1}(\alpha)$, it follows that $y \in A_\alpha^n$ which is a contradiction. Thus $\jmath_n$ is continuous for each n and the family $\{\jmath_n^{-1}(\alpha)\}_{\alpha\leq\theta}$ is an increasing. Since $\jmath^{-1}(\alpha) = \bigcap_{n\in\omega} \jmath_n^{-1}(\alpha)$, we have that for each $n \in \omega$, $\jmath^{-1}(\alpha) \subseteq \jmath_n^{-1}(\alpha)$. Let $x \in X$. It remains to prove that $\jmath(x) \leq \jmath_n(x)$ for every $n \in \omega$. Let $\jmath(x) = \alpha$. Hence $x \in \jmath^{-1}(\alpha)$ and $x \in \jmath_n^{-1}(\alpha)$ for every $n \in \omega$, i.e., the point x is such that $\min\{\xi \in \theta + 1 : x \in A_\xi^n\} = \alpha$ for each n. Therefore $\jmath_n(x) \geq \alpha$ for each $n \in \omega$. □

Proposition 4.6. *Let X be a topological space, θ a cardinal and $\jmath : X \to \theta + 1$ a function. If $\{\jmath^{-1}(\alpha)\}_{\alpha\in\theta}$ is a special G_δ family, then $\jmath$ has the property NL for every $\alpha \leq \theta$.*

Proof. Let $A \subseteq X$, $\alpha \leq \theta$ with $\mathrm{cf}\alpha > \omega$ and $\jmath(A)$ is cofinal in α. For every $\beta \in \theta$, we have $\jmath^{-1}(\beta) = \bigcap_{n\in\omega} G_\beta^n$ such that for every $n \in \omega$, $\{G_\beta^n\}_{\beta\in\theta}$ is an increasing family of open sets. Since $\jmath(A)$ is cofinal in α, for all $\beta \in \alpha$ $A \setminus \bigcap_{n\in\omega} G_\beta^n \neq \varnothing$, i.e., for all $\beta \in \alpha$ there exists $n \in \omega$ such that $A \setminus G_\beta^n \neq \varnothing$. There exist $n \in \omega$ and $(\beta_\xi)_{\xi\in\mathrm{cf}\alpha}$ increasing sequence with $\beta_\xi < \alpha$, such that $A \setminus G_{\beta_\xi}^n \neq \varnothing$. Now, fixed $n \in \omega$, we have that $A \subseteq \bigcup_{\xi\in\mathrm{cf}\alpha} G_{\beta_\xi}^n$. Therefore the family $\{G_{\beta_\xi}^n\}_{\xi\in\mathrm{cf}\alpha}$ is an open cover of A. If A was Lindelöf, there should be β_0 countable such that $G_{\beta_0}^n$ would cover A, which is a contradiction. □

Proposition 4.7. *Let $X = \prod_{n\in\omega} \theta + 1$, $\jmath : X \to \theta + 1$ defined by $\jmath(f) = \min\{\xi \in \theta + 1 : f \leq f_\xi\}$, where $\{f_\alpha\}_{\alpha\in\theta} \subseteq \prod_{n\in\omega} \theta + 1$ such that for every $\alpha < \alpha'$ $f_\alpha \leq f_{\alpha'}$. Then $\jmath$ has the property NL at every $\alpha \leq \theta$.*

Proof. By definition $\jmath^{-1}(\alpha) = \{f \in X : \forall n \in \omega\ f(n) < f_\alpha(n)\} = \bigcap_{n\in\omega}\{f \in X : f(n) < f_\alpha(n)\}$. Set $G_\alpha^n = \{f \in X : f(n) < f_\alpha(n)\}$, then for every $\alpha \in \theta+1$, G_α^n is an open set in X, and moreover for every $n \in \omega$, $\{G_\alpha^n\}_{\alpha\in\theta}$ is an increasing family. Hence $\{\jmath^{-1}(\alpha)\}_{\alpha\in\theta}$is a special G_δ family of X. Proposition 4.6 ends the proof. □

Corollary 4.8. *Let $X = \prod_{n\in\omega} \theta + 1$, $\jmath : X \to \theta + 1$ defined by $\jmath(f) = \min\{\xi \in \theta + 1 : f \leq f_\xi\}$, where f_ξ is a constant function with value ξ for every $\xi \leq \theta$. Then $\jmath$ has the property NL at every $\alpha \leq \theta$.*

Remark 4.9. Given $X = \prod_{n\in\omega} \theta + 1$, a family $\{f_\alpha\}_{\alpha\in\theta} \subseteq \prod_{n\in\omega} \theta + 1$, a sequence of function $\jmath_n(f) = \min\{\xi \in \theta + 1 : f(n) \leq f_\xi(n)\}$ and a function $\jmath(f) = \min\{\xi \in \theta + 1 : f \leq f_\xi\}$, all of them defined in X with value in $\theta + 1$. Then $\jmath > \sup \jmath_n$, and the equality does not hold. Indeed let $f : \omega \to \theta + 1$ defined by $f(n) = 0$ for every $n \neq 0$ and $f(0) = 2$, and $\{f_\xi\}_{\xi\in\theta}$ defined by $f_\xi = \vec{\xi}$ for every $\xi \in \theta$ with $\xi \neq 2$ and $f_2(n) = 0$ for every $n \in \omega \setminus \{0, 2\}$ and $f(0) = 2$, $f(2) = 0$. Then $\jmath(f) = 3$ and $\sup_n \jmath_n(f) = 2$.

There are examples of chain for countable product of ordered spaces, not considering the constant value function, which is a banal example. For example $X = \prod_{n\in\omega\setminus\{0\}} \aleph_{\omega\cdot n}$. In $(X, \leq)$ there exists a chain C such that $\mathrm{ot}(C) = \aleph_{\omega\cdot\omega}$ but not $\mathrm{ot}(C) = \aleph_{\omega\cdot\omega+1}$.

Given $\{\alpha_n\}_{n\in\omega}$ ordinals, what is the set of β such that there exists a function $f : \beta \hookrightarrow \pi\alpha_n$?

Remark 4.10. Let κ be a cardinal with $\mathrm{cf}\kappa > \omega$, $X = \prod_{n\in\omega} \kappa + 1$ and $\{f_\xi\}_{\xi\in\kappa} \subseteq X$ such that $f_\alpha \leq f_\beta$ for every $\alpha < \beta < \kappa$. Let $\jmath : X \to \kappa + 1$, defined by $\jmath(f) = \min\{\xi \in \kappa : f \leq f_\xi\}$. We have that the function $\jmath$ has the property NL at κ.
Let $A \subset X$ such that $\jmath(A)$ is cofinal in κ. Then for every $\alpha \in \kappa$ $A \nsubseteq \jmath^{-1}(\alpha)$, i.e., for every $\alpha \in \kappa$ and for every $n \in \omega$ $A \nsubseteq \{g \in X : g(n) \leq f_\alpha(n)\}$. Let $Vn, \alpha = \{g \in X : g(n) \leq f_\alpha(n)\}$. Then $\{V_{n,\alpha}\}_{n,\alpha}$ is an uncountable open cover of A. If A was Lindelöf, there should exist $\alpha_0 \in \kappa$ countable such that $\{V_{n,\alpha}\}^{n\in\omega}_{\alpha\in\alpha_0}$ is a cover for A which is a contradiction with $\mathrm{cf}\kappa > \omega$.

We give an example of function which has the property NL only at some ordinal.

Proposition 4.11. *Let $X = \prod_{n\in\omega} \theta_n + 1$ with every θ_n cardinal with $\mathrm{cf}\theta_n > \omega$, and $\jmath : X \to \kappa + 1$ defined by $\jmath(f) = min\{\xi \in \kappa : f \leq_* f_\xi\}$ where κ is a cardinal with $\mathrm{cf}\kappa > \omega$ and $\kappa > \theta_n$ for every $n \in \omega$, $\{f_\xi\}_{\xi\in\kappa}$ a dominating family in $(\prod_{n\in\omega} \theta_n, \leq_*)$. Then $\jmath$ has the property NL at κ.*

Proof. Let $A \subset X$ such that $\jmath(A)$ is cofinal in κ. Then for every $\alpha \in \kappa$ $A \nsubseteq \jmath^{-1}(\alpha)$, i.e., for every $\alpha \in \kappa$ $A \nsubseteq \{g \in X : g \leq_* f_\alpha\}$. Then there exists $n \in \omega$ such that $\{g(n) : g \in A\}$ is unbounded in θ_n. If not, for every $n \in \omega$ $\{g(n) : g \in A\}$ is bounded in θ_n, and since the family $\{f_\xi\}_{\xi\in\kappa}$ is an $\leq_*$-dominating in $(\prod_{n\in\omega} \theta_n, \leq_*)$, there should exists $\xi \in \kappa$ such that for every $g \in A$ $g \leq_* f_\xi$, which is a contradiction. Thus there exist $n \in \omega$ such that for every $\alpha \in \theta_n$ $A \nsubseteq \{g \in A : g(n) < \alpha\}$. Let $V_{n,\alpha} = \{g \in A : g(n) < \alpha\}$. Then $\{V_{n,\alpha}\}_{n,\alpha}$ is an uncountable open cover of A. If A was Lindelöf, there should exist $\alpha_0 \in \theta_n$ countable such that $\{V_{n,\alpha}\}^{n\in\omega}_{\alpha\in\alpha_0}$ is a cover for A which is a contradiction with $\mathrm{cf}\theta_n > \omega$. □

5. Closed mapping properties

In this section we investigate different properties of the projection map, introducing two new definitions.

Let us recall that if $f : X \to Y$ is a function and $A \subseteq X$, then the restriction of f to A, $f{\restriction}A$, is closed if the image of a closed subset of A is a closed subset of Y.

Definition 5.1. Given two arbitrary topological spaces X and Y, we say that the function $f : X \to Y$ is *σ-closed* if the image of a closed subset of X is an F_σ subset of Y.

Definition 5.2. Let X, Y be two topological spaces. $f : X \to Y$ is *strongly σ-closed* if there exists $(K_n)_{n\in\omega}$ with K_n's closed subsets of X such that $X = \bigcup_{n\in\omega} K_n$ and $f{\restriction}K_n$ is closed for every $n \in \omega$.

Remark 5.3. We are dealing with three different properties of the function $f : X \to Y$. The following implications hold

$$f \text{ closed } \Rightarrow\ f \text{ strongly } \sigma\text{-closed } \Rightarrow\ f\ \sigma\text{-closed}$$

Example 5.4. First note that for every countable topological space X which is T_1, the map $f : X \to Y$ is strongly σ-closed for every topological space Y which is T_1. Therefore the map $f : \mathbb{Q} \to \mathbb{R}$ with $f = id_{\mathbb{Q}}$ is strongly σ-closed, but it is not a closed map.

Example 5.5. [*AC*] Under the Axiom of choice, the set ω_1 can be partitioned in ω stationary sets S_n such that $\omega_1 = \bigcup_{n\in\omega} S_n$. In other words,there exists a function $f : \omega_1 \to \omega + 1$ defined by $f^{-1}(n) = S_n$ for every $n \in \omega$. By definition of stationary set, it follows that for every $n \in \omega$ and for every club C in ω_1 we have $C \cap f^{-1}(n) \neq \varnothing$. Clearly f is σ-closed. We claim that f is not strongly σ-closed, which is equivalent to show that for every $(K_n)_{n\in\omega}$ with K_n's closed subsets of X such that $X = \bigcup_{n\in\omega} K_n$ there exists $n_0 \in \omega$ such that the map $f{\restriction}K_{n_0}$ is not closed. Indeed let $(K_n)_{n\in\omega}$ be any countable family of closed subsets of ω_1 such that $\omega_1 = \bigcup_{n\in\omega} K_n$. Then there exist $n_0 \in \omega$ such that $|K_{n_0}| > \aleph_0$. Then K_{n_0} is a club in ω_1, therefore $K_{n_0} \cap f^{-1}(n) \neq \varnothing$ for every $n \in \omega$. Thus $f(K_{n_0}) = \omega$, and so $f{\restriction}K_{n_0}$ is not closed, because the set ω is not closed in its compactification $\omega + 1$.

Lemma 5.6. *Let X, Z be topological spaces, such that $X = \bigcup_{n\in\omega} K_n$. Let F be a subset of $X \times Z$ and $F_n = F \cap (K_n \times Z)$. Let $\pi : X \times Z \to Z$ be the projection map. Then $\pi(F) = \bigcup_{n\in\omega} \pi{\restriction}(K_n \times Z)(F_n)$*

Proof. Note that $X \times Z = \bigcup_{n\in\omega}(K_n \times Z)$, and for every $n \in \omega$, F_n is a subset of $K_n \times Z$ such that $F = \bigcup_{n\in\omega} F_n$. Let $p_n = \pi{\restriction}(K_n \times Z)$. For every $n \in \omega$, $p_n(F_n) \subseteq \pi(F)$. Indeed if $z \in p_n(F_n)$, there exists $(x, z) \in F_n$ such that $p_n(x, z) = z$, therefore there exists $(x, z) \in F$ such that $\pi(x, z) = z$. Thus $z \in \pi(F)$. On the other side, if $z \in \pi(F)$, there exists $(x, z) \in \bigcup_{n\in\omega} F_n$ such that $\pi(x, z) = z$. Therefore there exists $n \in \omega$ such that $(x, z) \in F_n$ such that $p_n(x, z) = z$. □

The Kuratowski Theorem is useful:

Theorem 5.7. *Given a compact Hausdorff space X, the projection map $\pi : X \times Z \to Z$ is a closed map, for every topological space Z.*

An application is given by:

Proposition 5.8. *Given an Hausdorff space X and the projection map $\pi : X \times Z \to Z$, the following implications hold*

$$X\ \sigma\text{-compact} \Rightarrow \pi\ \text{strongly}\ \sigma\text{-closed} \Rightarrow \pi\ \sigma\text{-closed}$$

Proof. First we show that π is a strongly σ-closed map. From X $\sigma-$compact,let $X = \bigcup_{n\in\omega} K_n$ where K_n's are compact in X. Therefore $K_n \times Z$ is closed in $X \times Z$. The Kuratowski Theorem assures that the projection map $\pi\restriction K_n \times Z$ is a closed map, for every topological space Z. For the second implication, let $X \times Z = \bigcup_{n\in\omega} K_n$ where K_n's are closed. Let F be a closed subset of $X \times Z$, and $F_n = F \cap K_n$. Then for every $n \in \omega$ F_n is a closed subset of K_n such that $F = \bigcup_{n\in\omega} F_n$. From Lemma 5.6, $\pi(F) = \bigcup_{n\in\omega} \pi\restriction K_n(F_n)$; moreover for every $n \in \omega$ $\pi\restriction K_n(F_n)$ is closed. It follows that $\pi(F)$ is F_σ in Z. □

The use of the small image of the projection map will recur often. So let us state an useful basic property:

Lemma 5.9. *Let X and Y be two topological spaces and $\pi : X \times Y \to Y$ a projection map. Then for every $A \subseteq Y$ and $B, K \subseteq X \times Y$,*

(i) $A \subseteq (\pi\restriction K)^\sharp(B \cap K) \Leftrightarrow (X \times A) \cap K \subseteq B$;
(ii) $A \subseteq \pi^\sharp(B) \Leftrightarrow X \times A \subseteq B$.

Proof. $A \subseteq (\pi\restriction K)^\sharp(B \cap K) = \{y \in Y : \pi^{-1}(y) \cap K \subseteq B \cap K\} \Leftrightarrow \forall y \in A\ \pi^{-1}(y) \cap K \subseteq B \cap K \Leftrightarrow \forall y \in A\, \{(x, y) \in K : x \in X \wedge \pi(x, y) = y\} \subseteq B \Leftrightarrow \{(x, y) \in K \cap (X \times A) : \pi(x, y) = y\} \subseteq B \Leftrightarrow (X \times A) \cap K \subseteq B$. □

Lemma 5.10. *Let X, Y be two topological spaces, $f : X \to Y$ an arbitrary function. If f is a closed map, then for every U open in X, $f^\sharp(U)$ is an open subset of Y.*
Moreover if f is σ-closed map, then $f^\sharp(U)$ is a G_δ subset of Y.

Proposition 5.11. *Let the projection $\pi : K \times Z \to Z$ be σ-closed, and $X \subseteq Z$. Let U be an open subset in $K \times Z$ which cover $K \times X$, then there exists $H \supseteq X$ which is a G_δ in Z such that U cover $K \times H$.*

Proof. Set $H = \pi^\sharp(U)$. Then, since π is σ-closed, H is a G_δ in Z. By Lemma 5.9 follow that $K \times H \subseteq U$, and $X \subseteq H$. □

Proposition 5.12. *Let X be a subset of a topological space Z, and $K \times Z \subseteq \bigcup_{n\in\omega} K_n$ with every K_n Lindelöf, and for every $n \in \omega$ $\pi\restriction K_n$ is closed, where $\pi : K \times Z \to Z$ is the projection map. If X is Lindelöf, then $K \times X$ is Lindelöf.*

Proof. Let $\mathcal{U}$ be a cover of $K \times X$ made by open sets of $K \times Z$. Without loss of generality we can assume that $\mathcal{U}$ is closed under countable unions. Fix $n \in \omega$, for each $z \in X$, $K_n \cap (K \times \{z\})$ is Lindelöf. For every $z \in X$, since $\mathcal{U}$ is closed under countable unions there exists $U_z \in \mathcal{U}$ such that $K_n \cap (K \times \{z\}) \subset U_z$. Set $A_{z,n} = (\pi\restriction K_n)^\sharp(U_z \cap K_n)$. Then $A_{z,n}$ is an open subset of Z containing z. From Lemma 5.9 follow that $(K \times A_{z,n}) \cap K_n \subseteq U_z$. For a fixed $n \in \omega$, $\{A_{z,n}\}_{z\in X}$ is a family of open sets in Z which covers X. Since X is Lindelöf there exists countably many z_i^n's such that $\{A_{z_i^n,n}\}_{i\in\omega}$ cover X. Moreover we have that for every $n \in \omega$ $(K \times A_{z_i^n,n}) \cap K_n \subseteq U_{z_i^n}$. We claim that $\{U_{z_i^n}\}_{i,n\in\omega}$ covers $K \times X$. Indeed, let $(k, z) \in K \times X$, then there exists $n \in \omega$ such that $(k, z) \in (K \times X) \cap K_n$. Fixed such n, there exists $i \in \omega$ such that $z \in A_{z_i^n,n}$. Thus $(k, z) \in (K \times A_{z_i^n,n}) \cap K_n \subseteq U_{z_i^n}$. Therefore we have that $\{U_{z_i^n}\}_{i,n\in\omega}$ is a countable family of open sets of $\mathcal{U}$ which covers $K \times X$. □

Corollary 5.13. *Let X be a subset of a topological space Z, and $K = \bigcup_{n\in\omega} K_n$ with every K_n Lindelöf, and for every $n \in \omega$ $\pi\restriction K_n \times Z$ is closed, where $\pi : K \times Z \to Z$ is the projection map. If X is Lindelöf, then $K \times X$ is Lindelöf.*

Remark 5.14. From the proof of Lemma 5.12 we can also get that if K, X and π satisfy the assumptions, there exists $U \in \mathcal{U}$ such that it covers $K \times X$, where $\mathcal{U}$ is an open cover of $K \times X$ made by open set in $K \times Z$ closed under countable union.

Corollary 5.15. *Let K, Z two topological spaces, $\pi : K\times Z \to Z$ the projection map, and $X \subset Z$. If*

(i) *X is Lindelöf,*
(ii) *K is Lindelöf,*
(iii) *π is strongly σ-closed,*

then $K \times X$ is Lindelöf.

Corollary 5.16. *Let K, Z two topological spaces, $\pi : K\times Z \to Z$ the projection map and $X \subset Z$. Let $\mathcal{U}$ be a family of open sets of $K \times Z$ which covers $K \times X$. If*

(i) *X is Lindelöf,*
(ii) *K is Lindelöf,*
(iii) *π is strongly σ-closed,*

then there exists $H \supset X$ which is a G_δ in Z and a countable subfamily of $\mathcal{U}$ which covers $K \times H$.

Proof. Without loss of generality we can assume that $\mathcal{U}$ is closed under countable unions. By Corollary 5.15, $K\times X$ is Lindelöf, therefore there exists $\mathcal{U}_0 \subseteq \mathcal{U}$ countable such that cover $K \times X$. Set $U_0 = \bigcup \mathcal{U}_0$. Then $U_0 \in \mathcal{U}$. By Proposition 5.11, there exists $H \supseteq X$ which is a G_δ in Z, such that $U \supseteq K \times H$. □

Lemma 5.17. *Let $\mathcal{U}$ be a family of open sets in $K \times Z$ which covers $K \times X$, with $X \subset Z$. Let $\pi : K \times Z \to Z$ be the projection map. If*

(i) *X is Lindelöf,*
(ii) *K is Lindelöf,*
(iii) *$K = \bigcup_{n\in\omega} K_n$ with K_n closed and $\pi\restriction K_n \times Z$ is closed,*

then there exists a countable subfamily of $\mathcal{U}$ that covers $K \times X$.

Corollary 5.18. *Let K be a σ- compact space, X a Lindelöf subset of a topological space Z. Let $\mathcal{U}$ be a family of open subsets in $K\times Z$ which cover $K\times X$, then there exists $H \supseteq H$ which is a G_δ in Z, and $\mathcal{U}_0 \subseteq \mathcal{U}$ countable which cover $K \times H$.*

6. Lindelof Haydon planks

In this section we construct a Dowker-Style plank, i.e., a variation of Dowker's idea of 1955 in which we take the subspace of all points in the product lying below the graph of a function (see [3]). Planks have been extensively studied by Watson in [9].

Definition 6.1. Let X, Z be topological spaces, θ a cardinal, $h : Z \to \theta + 1$ an arbitrary function, and $\jmath : X \to \theta + 1$ surjective. Define the plank

$$Y_{\jmath,h} = \{(x, z) \in X \times Z : h(z) \geq \jmath(x)\}$$

For every $\xi \leq \gamma \leq \theta$ denote

$$Y_{\jmath,h}\restriction(\xi, \cdot) = \{(x, z) \in Y_{\jmath,h} : \jmath(x) < \xi\}$$

and

$$Y_{\jmath,h}\restriction(\xi, \gamma) = \{(x, z) \in Y_{\jmath,h} : \jmath(x) < \xi \wedge h(z) < \gamma\}.$$

We investigate more in detail the relation between the plank and the functions. In the following, unless we state otherwise, we assume that the X, Z and the function h and $\jmath$ are defined as in the Definition 6.1

Proposition 6.2. *Let $\alpha \leq \theta$, if $\jmath$ has the property NL at α, then*

$$(\ \exists B \ Lindelöf : Y_{\jmath,h}\restriction(\alpha, \alpha) \subseteq B \subseteq Y_{\jmath,h} \Rightarrow h \, is \, Michael \, at \, \alpha).$$

Proof. Let $\alpha \leq \theta$ with $\mathrm{cf}\alpha > \omega$, and B Lindelöf subset of $Y_{\jmath,h}$ such that $Y_{\jmath,h}\restriction(\alpha, \alpha) \subseteq B$. Let $F \subset Z$ be closed such that for every $z \in F$, $h(z) < \alpha$. Then $B \cap (X \times F)$ is Lindelöf. Let $A = \pi_X(B \cap (X \times F))$. Thus A is a Lindelöf subset of X, such that for every $x \in A$ $\jmath(x) < \alpha$. From $\jmath$ NL at α we have $\jmath(A)$ is not cofinal in α, i.e, there exist $\beta < \alpha$ such that for every $x \in A$ $\jmath(x) \leq \beta$. Since $\jmath$ is surjective, for every $z \in F$ we can choose $x \in X$ with $\jmath(x) = h(z)$. Then $(x, z) \in Y_{\jmath,h}\restriction(\alpha, \alpha) \cap (X \times F)$, therefore $(x, z) \in B \cap (X \times F)$. It follows that $x \in A$. Hence for every $z \in F$ there exists $x \in A$ with $\jmath(x) = h(z) \leq \beta$. Thus $\mathrm{sup_{z\in F}h(z)} \leq \beta < \alpha$, i.e., h is Michael at α. □

Corollary 6.3. *Let θ be a cardinal. Assume that $\jmath$ has the property NL at each $\alpha \leq \theta$. If $Y_{\jmath,h}$ is Lindelöf then for each $\alpha \in \theta$, h is Michael at α.*

Proof. Let $\alpha \in \theta$ with $\mathrm{cf}\alpha > \omega$. Assume by contradiction that h is not Michael at α. Then $Y_{\jmath,h}$ is Lindelöf and $Y_{\jmath,h}\restriction(\alpha, \alpha) \subset Y_{\jmath,h}$. From Proposition 6.2 follows that h is Michael at α. □

From Proposition 3.9 and Proposition 6.2 follow:

Corollary 6.4. *Let θ be a cardinal with* $\mathrm{cf}\theta > \omega$*. If*

(i) *$\jmath$ has the property NL at θ,*

(ii) *for each $\alpha < \theta$, $h(Z) \cap (\alpha, \theta) \neq \varnothing$,*

then $Y_{\jmath,h} \times (Z \setminus h^{-1}(\{\theta\})$ is not Lindelöf.

Now we want to investigate when the plank $Y_{\jmath,h}$ is Lindelöf, and we give an inductive proof. First we need the following lemma.

Lemma 6.5. *Let $\mathcal{U}$ be a family of open sets in $\jmath^{-1}([0,\alpha]) \times Z$ which covers $Y_{\jmath,h}\restriction(\alpha+1,\cdot)$. Let $\pi : \jmath^{-1}([0,\alpha]) \times Z \to Z$ be the projection map. If*

(i) *there exists $U_0 \in \mathcal{U}$ which covers $\jmath^{-1}([0,\alpha]) \times h^{-1}([\alpha,\theta])$,*
(ii) *h is Michael at α,*
(iii) *foe each $\xi < \alpha$, $Y_{\jmath,h}\restriction(\xi+1,\cdot)$ is Lindelöf;,*
(iv) *π is σ-closed,*

then there exists a countable subfamily of $\mathcal{U}$ that covers $Y_{\jmath,h}\restriction(\alpha+1,\cdot)$.

Proof. Note that $Y_{\jmath,h}\restriction(\alpha+1,\cdot) = (\jmath^{-1}([0,\alpha]) \times h^{-1}([\alpha,\theta])) \cup (\bigcup_{\xi<\alpha} Y_{\jmath,h}\restriction(\xi+1,\cdot)$. Let $U_0 \in \mathcal{U}$ that covers $\jmath^{-1}([0,\alpha]) \times h^{-1}([\alpha,\theta])$.
If $\mathrm{cf}\alpha = \omega$, there exists an increasing sequence of ordinal $(\alpha_n)_{n\in\omega}$ such that $\bigcup_{\xi<\alpha} Y_{\jmath,h}\restriction(\xi+1,\cdot) = \bigcup_{n\in\omega} Y_{\jmath,h}\restriction(\alpha+1,\cdot)$, therefore there exists $\mathcal{U}_1 \subset \mathcal{U}$ which cover $\bigcup_{\xi<\alpha} Y_{\jmath,h}\restriction(\xi+1,\cdot)$. Then $\mathcal{U}_1 \cup \{U_0\}$ is a countable subcover of $\mathcal{U}$ that covers $Y_{\jmath,h}\restriction(\alpha+1,\cdot)$.

Assume that $\mathrm{cf}\alpha > \omega$. Let $U_0^c = (\jmath^{-1}([0,\alpha]) \times Z) \setminus U_0$. Then U_0^c is closed in $\jmath^{-1}([0,\alpha]) \times Z$ and for every $(x,z) \in U_0^c \cap Y_{\jmath,h}\restriction(\alpha+1,\cdot)$ we have that $\jmath(x) < \alpha$ and $h(z) < \alpha$. From (iv) follow that $C = \pi(U_0^c)$ is an F_σ subset of Z, and for every $z \in C$ $h(z) < \alpha$. From (ii) follow that $\delta = \sup_{z\in C} h(z) < \alpha$. We claim that $Y_{\jmath,h}\restriction(\alpha+1,\cdot) \setminus U_0 \subseteq Y_{\jmath,h}\restriction(\delta+1,\cdot)$. Let $(x,z) \in Y_{\jmath,h}\restriction(\alpha+1,\cdot) \setminus U_0$. From $(x,z) \in U_0^c$, follow that $z \in C$; from $h(z) < \delta$ and $\jmath(x) \le h(z)$, follow that $(x,z) \in Y_{\jmath,h}\restriction(\delta+1,\cdot)$. By hypothesi, $Y_{\jmath,h}\restriction(\delta+1,\cdot)$ is Lindelöf, therefore there exists a countable subfamily $\mathcal{U}_1 \subset \mathcal{U}$ which is a cover for a $Y_{\jmath,h}\restriction(\delta+1,\cdot)$. Thus $\{U_0\} \cup \mathcal{U}_1$ is a countable subcover for $\mathcal{U}$ that covers $Y_{\jmath,h}\restriction(\alpha+1,\cdot)$. □

Proposition 6.6. *Let Z be a Lindelöf space, θ a cardinal and $\alpha \le \theta$. If*

(i) *h is Michael at α,*
(ii) *$h^{-1}([\alpha,\theta])$ is Lindelöf,*
(iii) *$\jmath^{-1}([0,\alpha])$ is Lindelöf,*
(iv) *for each $\xi < \alpha$, $Y_{\jmath,h}\restriction(\xi+1,\cdot)$ is Lindelöf,*
(v) *$\pi : \jmath^{-1}([0,\alpha]) \times Z \to Z$ is strongly σ-closed,*

then $Y_{\jmath,h}\restriction(\alpha+1,\cdot)$ is Lindelöf.

Proof. Let $\mathcal{U}$ be a cover of $Y_{\jmath,h}\restriction(\alpha+1,\cdot)$ made by open sets of $\jmath^{-1}([0,\alpha]) \times Z$, and without loss of generality we can assume that it is closed under countable union. Note that $Y_{\jmath,h}\restriction(\alpha+1,\cdot) = (\jmath^{-1}([0,\alpha]) \times h^{-1}([\alpha,\theta])) \cup (\bigcup_{\xi<\alpha} Y_{\jmath,h}\restriction(\xi+1,\cdot)$. From Corollary 5.15, there exists $\mathcal{U}_0 \subset \mathcal{U}$ countable such that it covers $\jmath^{-1}([0,\alpha]) \times h^{-1}([\alpha,\theta])$. Let $U_0 = \bigcup \mathcal{U}_0$, then $U_0 \in \mathcal{U}$. Lemma 6.5 ends the proof. □

Proposition 6.7. *Let Z be a Lindelöf space, θ a cardinal, and $\alpha \le \theta$. If for each $\beta \le \alpha$*

(i) *h is Michael at β,*
(ii) *$h^{-1}([\beta,\theta])$ is Lindelöf,*
(iii) *$\jmath^{-1}([0,\beta])$ is Lindelöf,*
(iv) *$\pi : \jmath^{-1}([0,\beta]) \times Z \to Z$ is strongly σ-closed,*

then $Y_{\jmath,h}\restriction(\alpha+1,\cdot)$ is Lindelöf.

Proof. Assume that h is Michael at β for every $\beta \leq \alpha$. From Proposition 6.6, it remains to show that $Y_{\jmath,h}\restriction(\beta+1,\cdot)$ is Lindelöf for every $\beta < \alpha$. Suppose not, there exists $\beta < \alpha$ such that $Y_{\jmath,h}\restriction(\beta+1,\cdot)$ is not Lindelöf, and assume that β is the minimum ordinal with this property. Then, for every $\gamma < \beta$, $Y_{\jmath,h}\restriction(\gamma+1,\cdot)$ is Lindelöf, and for every $\gamma < \beta$, h is Michael at γ. From Proposition 6.6 follow that $Y_{\jmath,h}\restriction(\beta+1,\cdot)$ is Lindelöf, a contradiction. □

Theorem 6.8. *Let Z be a Lindelöf space, θ a cardinal. If for each $\alpha \leq \theta$*

(i) *h is Michael at α;,*
(ii) *$h^{-1}([\alpha,\theta])$ is Lindelöf;,*
(iii) *$\jmath^{-1}([0,\alpha])$ is Lindelöf,*
(iv) *$\pi : \jmath^{-1}([0,\alpha]) \times Z \to Z$ is strongly σ-closed,*

then $Y_{\jmath,h}$ is Lindelöf.

Note that the problem to determinate when given an arbitrary topological space Y, the product $Y_{\jmath,h} \times Y$ is Lindelöf becomes a problem to find condition on $\hat{h}$ and $\jmath$ so that $Y_{\jmath,\hat{h}}$ is a Lindelöf space where $\hat{h} : Z \times Y \to \theta + 1$.

Simply applying Proposition 6.2 and Corollary 6.8 to $\hat{h}$ the following corollary give us conditions to determinate when $Y_{\jmath,h} \times Y$ is Lindelöf.

Corollary 6.9. *Let Z be a Lindelöf space, X, Y a topological spaces, θ a cardinal, $\hat{h} : Z \times Y \to \theta + 1$. If for each $\alpha \leq \theta$*

(i) *$\hat{h}$ is Michael at α,*
(ii) *$\hat{h}^{-1}([\alpha,\theta]) \times Y$ is Lindelöf,*
(iii) *$\jmath^{-1}([0,\alpha])$ is Lindelöf,*
(iv) *$\pi : \hat{\jmath}^{-1}([0,\alpha]) \times Z \times Y \to Z \times Y$ is strongly σ-closed,*

then $Y_{\jmath,h} \times Y$ is Lindelöf.

The next Theorem give us a necessary condition to find a Michael space:

Theorem 6.10. *Let X be a topological space, θ a cardinal with uncountable cofinality and $h : \mathbb{C} \to \theta + 1$. If*

(i) *$\mathbb{Q}_C = h^{-1}(\{\theta\})$,*
(ii) *for each $\alpha \leq \theta$, h is Michael at α,*
(iii) *for each $\alpha < \theta$, $h(\mathbb{C}) \cap (\alpha, \theta) \neq \varnothing$,*
(iv) *$\jmath$ has the property NL at θ,*
(v) *for each $\alpha \leq \theta$, $\jmath^{-1}([0,\alpha])$ is Lindelöf,*
(vi) *for each $\alpha \leq \theta$, $\pi : \jmath^{-1}([0,\alpha]) \times Z \to Z$ is strongly σ-closed,*

then $Y_{\jmath,h}$ is a Michael space.

7. Special cases

One special case is obtained choosing $X = \theta+1$ and the map $\jmath$ as the identity map on $\theta + 1$. The plank $Y_{\jmath,h}$ becomes

$$Y_h = \{(\alpha, z) \in (\theta + 1) \times Z : h(z) \geq \alpha\}$$

subset of $(\theta+1)\times Z$, and it is an Haydon Plank [(see [5]). For every $\alpha\in\theta$ denote $Y_h\restriction\alpha=\{(\delta,z)\in Y_h:\delta<\alpha\}$.

In this case, the plank is characterized as Lindelöf, and it is also an example of Michael space.

Theorem 7.1. *Let Z be a Lindelöf space, $h:Z\to\theta+1$ a function, θ a cardinal with* $\mathrm{cf}\theta>\omega$. *Then Y_h is Lindelöf if and only if for every $\alpha\le\theta$*

(i) *h is Michael at α;*
(ii) *$h^{-1}([\alpha,\theta])$ is Lindelöf.*

Proof. Let Y_h be Lindelöf. Then, for every $\alpha\in\theta$, $Y_h\restriction\alpha+1$ is Lindelöf. By Proposition 6.2, h is Michael at α for every $\alpha\in\theta$. Moreover $Y_h\cap(\{\alpha\}\times Z)\cong h^{-1}([\alpha,\theta])$. Theorem 6.8 ends the proof. □

Corollary 7.2. *Let θ be a cardinal with* $\mathrm{cf}\theta>\omega$, *$h:\mathbb{C}\to\theta+1$ an arbitrary function. Then Y_h is Lindelöf if and only if for each $\alpha\le\theta$, h is Michael at α.*

Lemma 7.3. *Let Y be a topological space, θ cardinal with* $\mathrm{cf}\theta>\omega$. *If $Y_h\times Y$ is Lindelöf, then for every $\alpha\le\theta$, $\hat{h}$ is Michael at α, where $\hat{h}:Z\times Y\to\theta+1$.*

Moreover, if $h^{-1}([\alpha,\theta])\times Y$ is Lindelöf for every $\alpha\le\theta$, then the converse holds.

Proof. Follows from Corollary 6.8 (applied to $\hat{h}$), Proposition 6.2 and Remark 4.2. □

Corollary 7.4. *Let Y be a topological space, θ cardinal, $h:Z\to\theta+1$ a function such that for every $\alpha\le\theta$ $h^{-1}([\alpha,\theta])\times Y$ is Lindelöf. Then the following statements are equivalent:*

(i) *h is $\mathcal{A}(Y)$-Michael function,*
(ii) *$Y_h\times Y$ is Lindelöf.*

Proof. Follows from Theorem 7.3 and Proposition 3.7. □

Corollary 7.5. *Let θ be a cardinal with uncountable cofinality, Z a Lindelöf space and $h:Z\to\theta+1$ a function such that for every $\alpha<\theta$ $h(Z)\cap(\alpha,\theta)\neq\varnothing$. Then $Y_h\times(Z\setminus h^{-1}(\{\theta\}))$ is not Lindelöf*

Proof. Follows from Corollary 6.4. □

Corollary 7.6. *Let θ be a cardinal with uncountable cofinality, Z a Lindelöf space and $h:Z\to\theta+1$ a function. If*

(i) *for each $\alpha\le\theta$, h is Michael at α,*
(ii) *for each $\alpha<\theta$, $h(Z)\cap(\alpha,\theta)\neq\varnothing$,*
(iii) *for each $\alpha\le\theta$, $h^{-1}([\alpha,\theta])$ is Lindelöf,*

then $Y_h\subseteq(\theta+1)\times Z$ is a Lindelöf space such that $Y_h\times(Z\setminus h^{-1}(\{\theta\}))$ is a non-Lindelöf space.

Proof. Follows from Corollary 6.8 and Corollary 7.5. □

Theorem 7.7. *Let θ be a cardinal with uncountable cofinality and h a $\mathcal{K}$-Michael function defined on $\mathbb{C}$ such that*

(i) $\mathbb{Q}_C = h^{-1}(\{\theta\})$,

(ii) $h(\mathbb{C}) \cap (\alpha, \theta) \neq \varnothing$ *for every* $\alpha < \theta$.

Then Y_h, subspace of $(\theta + 1) \times \mathbb{C}$, is a Michael space.

We give some other examples of planks which are Michael spaces.

Definition 7.8. A special plank is given by choosing $X = \prod_{n\in\omega} \theta + 1$ with θ of uncountable cofinality, and the map $\jmath : X \to \theta+1$ defined by $\jmath(f) = \min\{\xi \in \theta + 1 : f \leq f_\xi\}$, where f_α is a constant function with value α for every $\alpha \leq \theta$. We denote this plank $Y^P_{\jmath,h}$.

Theorem 7.9. *Let θ be a cardinal with* $\mathrm{cf}\theta > \omega$ *and h a $\mathcal{K}$- Michael function defined on $\mathbb{C}$ such that*

(i) $\mathbb{Q}_C = h^{-1}(\{\theta\})$,

(ii) *for every* $\alpha < \theta$, $h(\mathbb{C}) \cap (\alpha, \theta) \neq \varnothing$.

Then $Y^P_{\jmath,h}$ is a Michael space.

Proof. By Corollary 4.8, follow that the map $\jmath$ has the property NL at α for every $\alpha \leq \theta$. Moreover, for every $\alpha \leq \theta$, $\jmath^{-1}([0, \alpha]) = \{f \in \prod_{n\in\omega} \theta + 1 : \forall n \in \omega\ f(n) \leq \alpha\}$ is a compact subset of $\prod_{n\in\omega} \theta+1$. By Lemma 5.8, the projection $\pi : \jmath^{-1}([0, \alpha]) \times Z \to Z$ is strongly σ-closed. Theorem 6.10 ends the proof. □

Another special plank is obtained for a particular choice of the map $\jmath$.

Definition 7.10. Let $X = \prod_{n\in\omega} \theta_n + 1$ with every θ_n cardinal with uncountable cofinality, and $\jmath : X \to \kappa + 1$ defined by $\jmath(f) = \min\{\xi \in \kappa : f \leq_* f_\xi\}$ where κ is a cardinal with $\mathrm{cf}\kappa > \omega$ and $\{f_\xi\}_{\xi\in\kappa}$ is a dominating family in $(\prod_{n\in\omega} \theta_n, \leq_*)$. We denote this plank $Y^{\prod}_{\jmath,h}$.

Remark 7.11. The definition of a dominating family and the definition of the map $\jmath$ in the plank $Y^{\prod}_{\jmath,h}$, imply that $\kappa > \theta_n$ for every $n \in \omega$. Indeed considering the special case in which the family $\{f_\xi\}_{\xi\in\kappa}$ is a family of constant functions, we need to have the function which assumes constant value θ_n. Therefore $\kappa > \theta_n + 1$ for every $n \in \omega$.

Remark 7.12. Let $(X, \leq)$ be a partial order, $\mathcal{F} \subseteq X$ with $\mathcal{F} = \{f_\xi\}_{\xi\in\kappa}$ a dominating family in X, (i.e. for all $x \in X$, there exists $f_\xi \in \mathcal{F}$ such that $x \leq f_\xi$). Define $\jmath : X \to \mathcal{F}$ by $\jmath(x) = \min\{f_\xi \in \mathcal{F} : x \leq f_\xi\}$. We have that $\jmath$ is surjective if and only if $f_\alpha \not\geq f_\beta$ for every $\alpha < \beta$.

Remark 7.13. Let $X = \prod_{n\in\omega} \theta_n + 1$ with every θ_n cardinal with uncountable cofinality, κ a cardinal with $\mathrm{cf}\kappa > \omega$ and $\{f_\xi\}_{\xi\in\kappa}$ a dominating family in $(\prod_{n\in\omega} \theta_n, \leq_*)$. The map $\jmath : X \to \kappa+1$ defined by $\jmath(f) = \min\{\xi \in \kappa : f \leq_* f_\xi\}$ might not be surjective. Since $\mathcal{F}' = \{f_\xi \in \mathcal{F} : f_\xi \in \jmath(X)\}$ is still a dominating family of X, when $\jmath(X)$ has order type κ, we can assume without loss of generality that $\jmath$ is surjective. Further,if the dominating family is a scale of X, we can consider $\mathcal{F}^{'}$, the dominating family of minimum cardinality which is a scale, i.e., $|\mathcal{F}^{'}| = \mathbf{d}$. Such a family is a dominating family with order type $\mathbf{d}$ and the map $\jmath^{'} : X \to \mathcal{F}^{'}$ defined by $\jmath^{'}(x) = \min\{f_\xi \in \mathcal{F}^{'} : x \leq f_\xi\}$ is surjective.

An example of $Y^{\prod}_{\jmath,h}$-plank is given by cardinal of countable cofinality. Indeed, from the Theorem of Shelah [B.M. [1]], given θ with $\mathrm{cf}\theta = \omega$, there exists an increasing sequence of regular cardinals $\{\theta_n\}_{n\in\omega}$ cofinal in θ, and a scale $\{f_\xi\}_{\xi\in\theta^+}$ on $(\prod_{n\in\omega}\theta_n, \leq_*)$. In this case choose $X = \prod_{n\in\omega}\theta_n + 1$ and the map $\jmath : X \to \theta^+ + 1$ defined by $\jmath(f) = \min\{\xi \in \theta^+ : f \leq_* f_\xi\}$.

Then we have:

Theorem 7.14. *Let θ be a cardinal with $\mathrm{cf}\theta > \omega$ and h a $\mathcal{K}$- Michael function defined on $\mathbb{C}$ such that*

(i) $\mathbb{Q}_C = h^{-1}(\{\theta\})$,
(ii) *for every $\alpha < \theta$, $h(\mathbb{C}) \cap (\alpha, \theta) \neq \varnothing$.*

Then $Y^{\prod}_{\jmath,h}$ is a Michael space.

Proof. For every $\alpha \in \kappa$, we have $\jmath^{-1}([0,\alpha]) = \{f \in X : f \leq_* f_\alpha\} = \bigcup_{F\in[\omega]^{<\omega}}\{f \in X : \forall n \notin F\ f(n) \leq f_\alpha(n)\}$. Therefore for every $\alpha \in \kappa$, the set $\jmath^{-1}([0,\alpha]) \subset X$ is σ-compact, hence by Lemma 5.8, the projection map $\pi : \jmath^{-1}([0,\alpha]) \times Z \to Z$ is strongly σ-closed. Moreover from Proposition 4.11, the map $\jmath$ has the property NL at κ. Theorem 6.10 ends the proof. □

8. The cardinal L

If X is a non-Lindelöf space, $L(X)$ denote the minimum cardinality of an uncountable open cover of X with no countable subcover, and if X is Lindelöf, define $L(X) = \infty$. Note that for a non-Lindelöf space, $L(X) \leq w(X)$, where $w(X)$ denote the weight of the topological space X, and $L(X)$ is either a regular cardinal or has countable cofinality.

The following lemma give us some relations between the L cardinals of related spaces.

Lemma 8.1. *Let X, Y be topological spaces. The following properties hold:*

(i) *If X is Lindelöf and $X \times Y$ is not Lindelöf, then $L(X \times Y) \leq |Y|$.*
(ii) *If $F \subseteq X$ is closed and not Lindelöf space, then $L(X) \leq L(F)$.*
(iii) *If f is a continuous open map, such that $f(X)$ is not Lindelöf space, then $L(f(X)) = L(X)$.*

Proof. (i) Let $\mathcal{U}$ be an open cover of $X \times Y$ witnessing $L(X \times Y)$. For every $y \in Y$, let $\mathcal{U}(y) = \{U_n(y) : n \in \omega\} \subset \mathcal{U}$ be a countable open subcover of $X \times \{y\}$. Thus $\mathcal{V} = \{U_n(y) : n \in \omega \wedge y \in Y\} \subset \mathcal{U}$ is an open cover of $X \times Y$, such that $|\mathcal{V}| \leq |Y|$ with no countable subcover. Therefore $L(X \times Y) \leq |Y|$.

(ii) Let $\mathcal{U}$ be an open cover of F with $|\mathcal{U}| = L(F)$ with no countable subcover. Then $\mathcal{U} \cup \{F^c\}$ is an open cover for X of the same kind.

(iii) Let $\mathcal{U}$ be an open cover of $f(X)$ with $|\mathcal{U}| = L(f(X))$ with no countable subcover. Then $f^{-1}(\mathcal{U})$ is an open cover for X. Thus $L(X) \leq L(f(X))$. If $\mathcal{V}$ is an open cover of X with $|\mathcal{V}| = L(X)$ with no countable subcover. Then $f(\mathcal{V})$ is an open cover for $f(X)$ of the same kind. □

Lemma 8.2. *Let X, Y be topological with X Lindelöf. For every $F \subseteq Y$ closed such that $L(X \times Y) > |F|$, $X \times F$ is Lindelöf.*

Proof. If $X \times Y$ is Lindelöf, then $L(X \times Y) = \infty$, and $X \times F$ is Lindelöf. Now, assume that $X \times Y$ and $X \times F$ are not Lindelöf. Since $X \times F$ is closed in $X \times Y$, from Lemma 8.1 we have that $|F| < L(X \times Y) \leq L(X \times F)$. Lemma 8.1 ends the proof. □

Corollary 8.3. *Let X, Y be topological spaces with X Lindelöf and $L(X \times Y) = |Y|$. Then for every closed $F \subseteq Y$ with $|F| < |Y|$ follow that $X \times F$ is Lindelöf.*

Lemma 8.4. *Let X, Y be topological spaces with X Lindelöf and $L(X \times Y) = |Y|$, then Y is not union of less than $|Y|$ many closed subsets of Y with cardinality less than $|Y|$.*

Proof. Let $\mathcal{U} = \{U_\xi\}_{\xi < |Y|}$ be an open cover of $X \times Y$ witnessing $L(X \times Y)$. Let κ cardinal with $\kappa < |Y|$. Assume by contradiction that $Y = \bigcup_{\xi \in \kappa} Y_\xi$ where for every $\xi \in \kappa$ Y_ξ are closed in Y and $|Y_\xi| < |Y|$. Therefore from Corollary 8.3 follow that $X \times Y_\xi$ is Lindelöf for every $\xi \in \kappa$, and so there exists $\mathcal{U}_\xi \subset \mathcal{U}$ countable subcover of $X \times Y_\xi$. Set $\mathcal{V} = \{\mathcal{U}_\xi : \xi \in \kappa\} \subset \mathcal{U}$. Then $\mathcal{V} \subseteq \mathcal{U}$ is an open cover of $X \times Y$ of size κ. From $L(X \times Y) = |Y|$ follow that there exist a countable subcover $\mathcal{V}' \subset \mathcal{V}$ of $X \times Y$. Then $\mathcal{V}'$ is also a countable subcover from $\mathcal{U}$ which is a contradiction. □

Let X, Y be topological spaces, θ a cardinal, and $P(X, Y, \theta)$ states that X is a Lindelöf space such that $X \times Y$ is not Lindelöf space and $L(X \times Y) = \theta$.

Theorem 8.5. *Let X be a topological space and θ a cardinal. If Y satisfies $P(X, Y, \theta)$ and $|Y| < \kappa$, with κ infinite cardinal, then there exists Y' which satisfies $P(X, Y', \theta)$ and $|Y'| = \kappa$.*

Proof. Let $Y' = Y \oplus \alpha D(\kappa)$, where $\alpha D(\kappa)$ is the one-point compactification of a discrete set of cardinality κ. Clearly $|Y'| = \kappa$. Since the space $X \times Y$ is a closed subset of $X \times Y'$, it follows that $X \times Y'$ is not Lindelöf. It remains to show that $L(X \times Y') = \theta$, assuming that $L(X \times Y) = \theta$. Since the space $X \times Y$ is a closed subset of $X \times Y'$, from Lemma 8.1 follow that $L(X \times Y) \leq L(X \times Y')$, and so $L(X \times Y') \geq \theta$. Now, let $\mathcal{U}$ be an open cover for $X \times Y$ of size θ with no countable subcover. We have that $X \times Y'$ is homeomorphic to $(X \times Y) \oplus (X \times \alpha D(\kappa))$ hence it follows that $\mathcal{U}$ is an open family in $X \times Y'$ such that $U \cap (X \times \alpha D(\kappa)) = \varnothing$ for every $U \in \mathcal{U}$. Let $\mathcal{V} = \mathcal{U} \cup \{X \times \alpha D(\kappa)\}$. Then $\mathcal{V}$ is an open cover of Y' of size θ with no countable subcover. □

Remark 8.6. In other words we have that for a fixed topological space X and a cardinal θ, if there exists Y such that $P(X,Y,\theta)$, then the set $A_{X,\theta} = \{\kappa : \kappa \text{ is cardinal} \wedge \quad \exists Y\ P(X,Y,\theta) \wedge |Y| = \kappa\}$ is non empty and $A_{X,\theta} = [\min A_{X,\theta}, +\infty)$.

We conclude this work showing that if there is a Michael space, then under some conditions involving singular cardinals, there must be one which is a Haydon plank.

Theorem 8.7. *Let X be a Lindelöf space, Y a topological space such that $X \times Y$ is not Lindelöf, θ a cardinal with $\mathrm{cf}\theta = \omega$ and $L(X \times Y) = |Y| = \theta$. Let cY be any compactification of Y. Then there exists a function $f : cY \to \theta + 1$ such that*

(i) $f^{-1}(\{\theta\}) = cY \setminus Y$,
(ii) *for every $\alpha \leq \theta$, f is Michael at α,*
(iii) *for every $\alpha < \theta$, $f(cY) \cap (\alpha, \theta) \neq \varnothing$.*

Proof. Let $\theta = L(X \times Y)$, and $\mathcal{U}$ be an open cover of $X \times Y$ witnessing $L(X \times Y)$. Fix an enumeration $\{y_\xi\}_{\xi<\theta}$ of Y of order type θ. Given $y \in Y$, let $\mathcal{U}(y) = \{U_n(y) : n \in \omega\} \subset \mathcal{U}$ a countable open subcover of $X \times \{y\}$. Thus $\mathcal{V} = \{U_n(y) : n \in \omega \wedge y \in Y\} \subset \mathcal{U}$ is an open cover of $X \times Y$, such that $|\mathcal{V}| = \theta$.

Let cY be a compactification of Y. Define the function $f : cY \to \theta + 1$ as follows: for every $y \in Y$, $f(y) = \sup\{\gamma \in \theta : \mathrm{X} \times \{\mathrm{y}\} \nsubseteq \cup_{\xi<\gamma}(\cup_{\mathrm{n}\in\omega} \mathrm{U_n}(\mathrm{y}_\xi))\}$ and for every $y \in cY \setminus Y$ $f(y) = \theta$. Then, by definition of $\mathcal{V}$, there is not $y \in Y$ such that $X \times \{y\} \nsubseteq \cup_{\xi<\alpha}(\cup_{n\in\omega} U_n(y_\xi))$ for every $\alpha \leq \theta$. Thus $f^{-1}(\{\theta\}) = cY \setminus Y$.

Let $\alpha \in \theta$ with $\mathrm{cf}\alpha > \omega$, and $F \subset cY$ closed such that $f(y) < \alpha$ for every $y \in F$. Assume by contradiction that $\mathrm{sup_{y\in F} f(y)} = \alpha$. By definition of α we have that $X \times F \subseteq \cup_{\xi<\alpha}(\cup_{n\in\omega} U_n(y_\xi))$. Then $\{U_n(y_\xi) : n \in \omega \wedge \xi < \alpha\}$ is an uncountable cover of $X \times F$ with F compact. We want to show that it has no countable subcover which contradict $X \times F$ to be Lindelöf. Indeed if $\{U_m(y_{\xi_n})\}_{n,m\in\omega}$ was a countable subcover of $X \times F$. Let $\nu = \mathrm{sup_{n\in\omega}\xi_n}$. Since $\mathrm{cf}\alpha > \omega$, $\nu < \alpha$. By definition of ν there exists $y \in F$ such that $X \times \{y\} \nsubseteq \cup_{n\in\omega} U_m(y_{\xi_n})$ which is a contradiction.

Now, by contradiction, there exists $\alpha \in \theta$ such that $f(cY) \cap (\alpha, \theta) = \varnothing$, i.e., there exists $\alpha \in \theta$ such that for every $y \in Y$, $f(y) < \alpha$. Therefore for every $y \in Y$, $X \times \{y\} \subseteq \cup_{\xi<\alpha}(\cup_{n\in\omega} U_n(y_\xi))$. Thus $\{U_n(y_\xi) : n \in \omega \wedge \xi < \alpha\}$ is an open cover of $X \times Y$ with $\alpha < \theta$. By definition of $L(X \times Y) = \theta$, there exists a countable subcover for $X \times Y$ from $\{U_n(y_\xi) : n \in \omega \wedge \xi < \alpha\}$, and therefore from $\mathcal{U}$, which is a contradiction. □

Theorem 8.8. *Let X be a Lindelöf space such that $X \times Y$ is not Lindelöf, θ a regular cardinal such that $L(X \times Y) = \theta$. Let cY be any compactification of Y. Then there exists a function $f : cY \to \theta + 1$ such that*

(i) $f^{-1}(\{\theta\}) = cY \setminus Y$,
(ii) *for every $\alpha \leq \theta$, f is Michael at α,*
(iii) *for every $\alpha < \theta$, $f(cY) \cap (\alpha, \theta) \neq \varnothing$.*

Proof. Let $\theta = L(X \times Y)$. Fix an enumeration $\{U_\xi\}_{\xi<\theta}$ of an open cover of $X \times Y$ witnessing $L(X \times Y)$. Let cY be a compactification of Y. Define the function $f : cY \to \theta + 1$ as follows: for every $y \in Y$, $f(y) = \sup\{\gamma \in \theta : \mathrm{X} \times \{\mathrm{y}\} \nsubseteq \cup_{\xi<\gamma}\mathrm{U}_\xi\}$ and for every $y \in cY \setminus Y$ $f(y) = \theta$. Since θ is regular and $\{U_\xi\}_{\xi<\theta}$ is an open cover of $X \times Y$, there is not $y \in Y$ such that for every $\alpha \leq \theta$ $X \times \{y\} \nsubseteq \cup_{\xi<\alpha}U_\xi$. Thus $f^{-1}(\{\theta\}) = cY \setminus Y$.

Let now $\alpha \in \theta$ with $\mathrm{cf}\alpha > \omega$, and $F \subset cY$ closed such that $f(y) < \alpha$ for every $y \in F$. Assume by contradiction that $\sup_{\mathrm{y}\in\mathrm{F}}\mathrm{f(y)} = \alpha$. By definition of α we have that for every $\beta \geq \alpha$, $X \times \{y\} \subseteq \cup_{\xi<\beta}U_\xi$ for every $y \in F$, therefore $X \times F \subseteq \cup_{\xi<\alpha}U_\xi$. Then $\{U_\xi\}_{\xi<\theta}$ is an uncountable cover of $X \times F$ with F compact. We want to show that it has no countable subcover which contradict $X \times F$ to be Lindelöf. Indeed if $\{U_{\xi_n}\}_{n\in\omega}$ was a countable subcover of $X \times F$. Let $\nu = \sup_{\mathrm{n}\in\omega}\xi_{\mathrm{n}}$. Since $\mathrm{cf}\alpha > \omega$, $\nu < \alpha$. By definition of ν there exists $y \in F$ such that $X \times \{y\} \nsubseteq \cup_{n\in\omega}U_{\xi_n}$, contradiction.

Now, by contradiction, there exists $\alpha \in \theta$ such that $f(cY) \cap (\alpha, \theta) = \varnothing$, i.e., there exists $\alpha \in \theta$ such that for every $y \in Y$, $f(y) < \alpha$. Therefore for every $y \in Y$, $X \times \{y\} \subseteq \cup_{\xi<\alpha}U_\xi$. Thus $\{U_\xi\}_{\xi<\alpha}$ is an open cover of $X \times Y$ with $\alpha < \theta$. By definition of $L(X \times Y) = \theta$, there exists a countable subcover for $X \times Y$ from $\{U_\xi\}_{\xi<\alpha}$, and therefore from $\{U_\xi\}_{\xi<\theta}$, which is a contradiction. □

In the Theorem 8.8, θ is a regular cardinal, we do not need any assumption about the cardinality of Y (as in Theorem 8.7) because the open cover$\{U_\xi\}_{\xi<\theta}$ witnessing $L(X \times Y) = \theta$ is never cofinal. This guarantee that $f^{-1}(\{\theta\}) = cY \setminus Y$.

From Theorem 7.7 it follows:

Theorem 8.9. *Let M be a Michael space, θ a regula cardinal such that $L(M \times \mathbb{P}) = \theta$. Then there exists a function f and $Y_f \subseteq (\theta + 1) \times \mathbb{C}$ which is a Michael space.*

The aim of Proposition 8.7 and Proposition 8.8 is to produce the following statement: given a Lindelöf space X such that $L(X \times Y) = \theta$, there exists f and $Y_f \subseteq (\theta + 1) \times cY$ such that Y_f is Lindelöf and $Y_f \times Y$ is not Lindelöf, where cY is any compactification of Y.

We require the property that for all $\alpha \leq \theta$, $f^{-1}([\alpha, \theta])$ is Lindelöf. Clearly this is always true when Y admits an hereditarily Lindelöf compactification. When does the property hold?

Is there a function $f : cY \to \theta + 1$ such that satisfy the property of Proposition 8.7 when X is a Lindelöf space, Y a topological space such that $X \times Y$ is not Lindelöf, θ a cardinal of countable cofinality such that $L(X \times Y) = \theta$, and $|Y| > \theta$?

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