

Morse theory for C^* -algebras: a geometric interpretation of some noncommutative manifolds

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ABSTRACT

The approach we present is a modification of the Morse theory for unital C^ -algebras. We provide tools for the geometric interpretation of noncommutative CW complexes. Some examples are given to illustrate these geometric information. The main object of this work is a classification of unital C^* -algebras by noncommutative CW complexes and the modified Morse functions on them.*

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1. INTRODUCTION

Morse theory is an approach in the study of smooth manifolds by the tools from calculus. The classical Morse theory provides a connection between the topological structure of a manifold M and the homotopy type of critical points of a function $f : M \rightarrow \mathbb{R}$ (the Morse functions).

On a smooth manifold M , a point $a \in M$ is a *critical point* for a smooth function $f : M \rightarrow \mathbb{R}$, if the induced map $f_* : T_a(M) \rightarrow \mathbb{R}$ is zero. The real number $f(a)$ is then called a *critical value*. The function f is a *Morse function* if i) all the critical values are distinct and ii) its critical points are non degenerate, i.e. the Hessian matrix of second derivatives at the critical points has a

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non vanishing determinant. The number of negative eigenvalues of this Hessian matrix is *the index of f* at the critical point. The classical Morse theory states:

Theorem ([14]): *There exists a Morse function on any differentiable manifold and any differentiable manifold has the homotopy type of a CW complex with one λ -cell for each critical point of index λ .*

So once we have information around the critical points of a Morse function on M , we can reconstruct M by a sequence of surgeries.

A C*-algebraic approach which links operator theory and algebraic geometry, is obtained via a suitable set of equivalence classes of extensions of commutative C*-algebras. This provides a functor from locally compact spaces into abelian groups ([7], [12], [15]).

If J and B are two C*-algebras, an extension of B by J is a C*-algebra A together with morphisms $j : J \rightarrow A$ and $\eta : A \rightarrow B$ such that the following sequence is exact:

$$(1.1) \quad 0 \longrightarrow J \xrightarrow{j} A \xrightarrow{\eta} B$$

The aim of the extension problem is the characterization of those C*-algebras A satisfying the above exact sequence. This has something to do with algebraic topology techniques. In the construction of a CW complex, if X_{k-1} is a suitable subcomplex, I^k the unit ball and S^{k-1} its boundary, then the various solutions for the extension problem of $C(X_{k-1})$ by $C_0(I^k - S^{k-1})$ correspond to different ways of attaching I^k to X_{k-1} along S^{k-1} , via an attaching map $\varphi_k : S^{k-1} \rightarrow X_{k-1}$ which identify points $x \in S^{k-1}$ with their image $\varphi_k(x)$ in the disjoint union $X_{k-1} \cup I^k$.

After the construction of noncommutative geometry [1], there have been attempts to formulate the classical tools of differential geometry and topology in terms of C*-algebras (in some sense the dualization of the notions, [3], [4], [11]). The dual concept of CW complexes, with some regards, is the notion of noncommutative CW complexes ([7] and [15]). The approach of this paper is the geometric study of these structures. So many works have been done on the combinatorial structures of noncommutative simplicial complexes and their decompositions, for example [2], [6], [9], [10]. Following these works, together with some topological constructions, we show how a modification of the classical Morse theory to the level of C*-algebras will provide an innovative way to explain the geometry of noncommutative CW complexes through the critical ideals of the modified Morse function. This leads up to some classification theory.

This paper is prepared as follows. After introducing the notion of primitive spectrum of a C*-algebra, we will proceed the topological structure in detail and present some examples. Then we will study the noncommutative CW complexes and interpret their geometry by introducing the modified Morse function. All these provide tools for the modified Morse theory for C*-algebras. The last section is devoted to the proof of the following theorem:

Main Theorem: *Every unital C*-algebra with an acceptable Morse function on it is of pseudo-homotopy type as a noncommutative CW complex, having a k-th decomposition cell for each critical chain of order k.*

2. THE STRUCTURE OF THE PRIMITIVE SPECTRUM

The technique we follow to link the geometry, topology and algebra is the primitive spectrum point of view. In fact as we will see in our case it is a promising candidate for the noncommutative analogue of a topological manifold M . We review some preliminaries on the primitive spectrum. Details can be found in [5], [11], [13].

Let A be a unital C*-algebra. The *primitive spectrum* of A is the space of kernels of irreducible *-representations of A . It is denoted by $Prim(A)$. The topology on this space is given by the closure operation as follows:

For any subset $U \subseteq Prim(A)$, the closure of U is defined by

$$(2.1) \quad \bar{U} := \{I \in Prim(A) : \bigcap_{J \in U} J \subset I\}$$

Obviously $U \subseteq \bar{U}$. This operation defines a topology on $Prim(A)$ (the hull-kernel topology), making it into a T_0 -space ([8]).

Definition 2.1. A subset $U \subset Prim(A)$ is called *absorbing* if it satisfies the following condition:

$$(2.2) \quad I \in U, I \subseteq J \Rightarrow J \in U.$$

Lemma 2.2. *The closed subsets of $Prim(A)$ are exactly its absorbing subsets.*

Proof. It is clear from the definition of closed sets. □

In the special case, when M is a compact topological space, and $A = C(M)$ is the commutative unital C*-algebra of complex continuous functions on M , a homeomorphism between M and $prim(A)$ is obtained in the following way. For each $x \in M$ let

$$I_x := \{f \in A : f(x) = 0\};$$

I_x is a closed maximal ideal of A . It is in fact the kernel of the evaluation map

$$(ev)_x : A \longrightarrow \mathbb{C} \\ f \longmapsto f(x).$$

Now

$$(2.3) \quad I : M \rightarrow Prim(A)$$

defined by $I(x) := I_x$ is the desired homeomorphism.

Let A be an arbitrary unital C*-algebra. To each $I \in Prim(A)$, there corresponds an absorbing set

$$W_I := \{J \in Prim(A) : J \supseteq I\},$$

and an open set

$$O_I := \{J \in Prim(A) : J \subseteq I\},$$

containing I .

Being a T_0 -space, $Prim(A)$ can be made into a partially ordered set (poset) by setting,

$$I < J \Leftrightarrow I \subset J \quad \text{for } I, J \in Prim(A),$$

Remark 2.3. The following statements are equivalent:

- i) $I \subseteq J$.
- ii) $O_I \subseteq O_J$.
- iii) $W_I \supseteq W_J$.

The topology of $Prim(A)$ can be given equivalently by means of this partial order,

$$I < J \Leftrightarrow J \in \overline{\{I\}},$$

where $\overline{\{I\}}$ is the closure of the one point set $\{I\}$.

Since A is unital, $Prim(A)$ is compact([8]). Let $Prim(A) = \bigcup_{i=0}^n O_{I_i}$ be a finite open covering.

In general, let us suppose we have a topological space M together with an open covering $U = \{U_i\}$ which is also a topology for M . An equivalence relation on M is set by declaring that any two points $x, y \in M$ are equivalent if every open set U_i containing either x or y contains the other too. In this way the quotient space of M is made into a finite lattice.

In the same way an equivalence relation on $Prim(A)$ is given by

$$I \sim J \Leftrightarrow (J \in O_I \Leftrightarrow I \in O_J).$$

This is of course the trivial relation $I \sim J \Leftrightarrow I = J$.

In each O_{I_i} choose one I_i with respect to the above equivalence relation. Since $I \subseteq J$ implies $O_I \subseteq O_J$, O_{I_i} s can be chosen so that $I_i \neq I_j$ for $i \neq j$. Let I_0, I_1, \dots, I_n be chosen in this way so that $Prim(A)$ is made into a finite lattice for which the points are the equivalence classes of $[I_0], \dots, [I_n]$. For simplicity we show each class $[I_i]$ by its only representative I_i . Let

$$J_{i_0, \dots, i_k} := I_{i_0} \cap \dots \cap I_{i_k},$$

where $1 \leq i_0, \dots, i_k \leq n, 1 \leq k \leq n$.

Set

$$W_{i_0, \dots, i_k} := \{J \in Prim(A) : J \supseteq J_{i_0, \dots, i_k}\}.$$

This is a closed subset of $Prim(A)$.

In what follows we see that the above construction makes it possible to obtain a cell complex decomposition for $Prim(C(X))$ when X has a CW complex structure. In fact the closed sets W_{i_0, \dots, i_k} corresponding to J_{i_0, \dots, i_k} play the role of chains in the construction.

Remark 2.4. If $J_{i_0, \dots, i_k} = 0$ for some $1 \leq i_0, \dots, i_k \leq n, 1 \leq k \leq n$, then $W_{i_0, \dots, i_k} = Prim(A)$. Also for each pair of indices $(i_0, \dots, i_t), \sigma(i_0, \dots, i_{t+m})$,

$$W_{i_0, \dots, i_t} \subseteq W_{\sigma(i_0, \dots, i_{t+m})}$$

where σ is a permutation on $t + m$ elements and $1 \leq i_0, \dots, i_{t+m} \leq n$.

Remark 2.5. A sequence

$$X_0 \subset X_1 \subset \dots \subset X_n = X$$

is an n-dimensional CW complex structure for a compact topological space X , where X_0 is a finite discrete space consisting of 0-cells, and for $k = 1, \dots, n$ each k -skeleton X_k is obtained by attaching λ_k number of k -disks to X_{k-1} via the attaching maps

$$\varphi_k : \bigcup_{\lambda_k} S^{k-1} \rightarrow X_{k-1}.$$

In other words

$$(2.4) \quad X_k = \frac{X_{k-1} \bigcup (\bigcup_{\lambda_k} I^k)}{x \sim \varphi_k(x)} := X_{k-1} \bigcup_{\varphi_k} (\bigcup_{\lambda_k} I^k)$$

where $I^k := [0, 1]^k$ and $S^{k-1} := \partial I^k$. The quotient map is denoted by

$$\rho : X_{k-1} \bigcup (\bigcup_{\lambda_k} I^k) \rightarrow X_k.$$

For more details see [12].

Now let

$$X_0 \subset X_1 \subset \dots \subset X_n = X$$

be an n-dimensional CW complex structure for the compact space X . A cell complex structure is induced on $\text{Prim}(C(X))$ by the following procedure:

Let $A_k = C(X_k)$, $k = 0, 1, \dots, n$. Set $A = C(X) = C(X_n) = A_n$. Let $I : X \rightarrow \text{Prim}(C(X))$ be the homeomorphism of relation (4). For each k -cell C_k in the k -skeleton X_k , let

$$I_{C_k} = \bigcap_{x \in C_k} I_x = \{f \in A : f(x) = 0; x \in C_k\},$$

for $0 \leq k \leq n$. By considering the restriction of functions on X to X_k , I_{C_k} will be an ideal in A_k .

Definition 2.6. In the above notations, I_{C_k} is called a k -ideal in A (or A_k) and the restriction of its corresponding closed set W_{i_0, \dots, i_k} in $\text{Prim}(A_k)$, i.e.

$$W_{i_0, \dots, i_k} = \{J \in \text{Prim}(A_k) : J \supseteq I_{C_k}\}$$

is called a k -chain.

In the following two examples we identify the k -ideals and the k -chains for the CW complex structures of the closed interval $[0, 1]$ and the 2-torus $S^1 \times S^1$.

Example 2.7. Let $X_0 = \{0, 1\}$ and $X_1 = [0, 1]$ be the zero and the one skeleton for a CW complex structure of $[0, 1]$. Then we have $A_0 = C(X_0) \simeq \mathbb{C} \oplus \mathbb{C}$ and $A = A_1 = C(X_1)$. The 0-ideals I_0 and I_1 and their corresponding 0-chains W_0 and W_1 are as follow:

$$I_0 = \{f \in A_0 : f(0) = 0\} \simeq \mathbb{C}, I_1 = \{f \in A_0 : f(1) = 0\} \simeq \mathbb{C},$$

$$W_0 = \{J \in \text{Prim}(A_0) : J \supseteq I_0\} \simeq \{0\}, W_1 = \{J \in \text{Prim}(A_0) : J \supseteq I_1\} \simeq \{1\}.$$

Corresponding to the 1-chain $C_1 = [0, 1]$, the only 1-ideal is

$$I = \bigcap_{x \in C_1} I_x = \{f \in A : f(x) = 0; x \in [0, 1]\} = 0,$$

with the corresponding 1-chain

$$W_I = \{J \in \text{Prim}(A) : J \supseteq I\} = \text{Prim}(A) \simeq [0, 1].$$

Example 2.8. Let

$$X_0 = \{0\}, X_1 = \{\alpha, \beta\}, X_2 = T^2 = S^1 \times S^1$$

be the skeletons of a CW complex structure for the 2-torus T^2 , where α, β are homeomorphic images of S^1 (closed curves with the origin 0). Let $A_0 = C(X_0) = \mathbb{C}$, $A_1 = C(X_1)$ and $A = A_2 = C(T^2)$. The 0-ideal and its corresponding 0-chain are as follow:

$$I_0 = \{f \in A_0 : f(0) = 0\},$$

$$W_0 = \{J \in \text{Prim}(A_0) : J \supseteq I_0\} \simeq \text{Prim}(A_0) = \{0\}.$$

Also the 1-ideals I_1, I_2 and 1-chains W_{I_1}, W_{I_2} are

$$I_1 = \{f \in A_1 : f(\alpha) = 0\} = \bigcap_{x \in \alpha} I_x \simeq \mathbb{C},$$

$$I_2 = \{f \in A_1 : f(\beta) = 0\} = \bigcap_{x \in \beta} I_x \simeq \mathbb{C},$$

$$W_{I_1} = \{J \in \text{Prim}(A_1) : J \supseteq I_1\} \simeq \{\alpha\},$$

$$W_{I_2} = \{J \in \text{Prim}(A_1) : J \supseteq I_2\} \simeq \{\beta\}.$$

Finally the only 2-ideal and its corresponding 2-chain are

$$I = \{f \in A : f(T^2) = 0\} \simeq 0,$$

$$W_I = \{J \in \text{Prim}(A) : J \supseteq I\} \simeq T^2.$$

3. THE NONCOMMUTATIVE CW COMPLEXES

In this section we see how the construction of the primitive spectrum of the previous section helps us to study the noncommutative CW complexes.

For a continuous map $\phi : X \rightarrow Y$ between compact topological spaces X and Y , the C^* -morphism induced on their associated C^* -algebras is denoted by $C(\phi) : C(Y) \rightarrow C(X)$ which is defined by $C(\phi)(g) := g \circ \phi$ for $g \in C(Y)$.

Definition 3.1. Let A_1, A_2 and C be C^* -algebras. A pull back for C via morphisms $\alpha_1 : A_1 \rightarrow C$ and $\alpha_2 : A_2 \rightarrow C$ is the C^* -subalgebra of $A_1 \oplus A_2$ denoted by $PB(C, \alpha_1, \alpha_2)$ defined by

$$PB(C, \alpha_1, \alpha_2) := \{a_1 \oplus a_2 \in A_1 \oplus A_2 : \alpha_1(a_1) = \alpha_2(a_2)\}.$$

For any C^* -algebra A , let

$$S^n A := C(S^n \rightarrow A), I^n A := C([0, 1]^n \rightarrow A), I_0^n A := C_0((0, 1)^n \rightarrow A),$$

where S^n is the n -dimensional unit sphere.

We review the definition of noncommutative CW complexes from [7], [15].

Definition 3.2. A *0-dimensional noncommutative CW complex* is any finite dimensional C*-algebra A_0 . Recursively an *n-dimensional noncommutative CW complex* is any C*-algebra appearing in the following diagram

$$(3.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & I_0^n F_n & \longrightarrow & A_n & \xrightarrow{\pi} & A_{n-1} & \longrightarrow & 0 \\ & & \parallel & & \downarrow f_n & & \downarrow \varphi_n & & \\ 0 & \longrightarrow & I_0^n F_n & \longrightarrow & I^n F_n & \xrightarrow{\delta} & S^{n-1} F_n & \longrightarrow & 0 \end{array}$$

Where the rows are extensions, A_{n-1} an $(n-1)$ -dimensional noncommutative CW complex, F_n some finite (linear) dimensional C*-algebra of dimension λ_n , δ the boundary restriction map, φ_n an arbitrary morphism (called the connecting morphism), for which

$$(3.2) \quad A_n = PB(S^{n-1} F_n, \delta, \varphi_n) := \{(\alpha, \beta) \in I^n F_n \oplus A_{n-1} : \delta(\alpha) = \varphi_n(\beta)\},$$

and f_n and π are respectively projections onto the first and second coordinates.

With these notations $\{A_0, \dots, A_n\}$ is called *the noncommutative CW complex decomposition of dimension n for $A = A_n$* .

For each $k = 0, 1, \dots, n$, A_k is called *the k-th decomposition cell*.

Proposition 3.3. *Let X be an n-dimensional CW complex containing cells of each dimension $k = 0, \dots, n$. Then there exists a noncommutative CW complex decomposition of dimension n for $A = C(X)$.*

Conversely if $\{A_0, \dots, A_n\}$ be a noncommutative CW complex decomposition for the C-algebra A such that A_i s ($i = 0, \dots, n$) are unital, Then there exists an n-dimensional CW complex structure on $\text{Prim}(A)$.*

Proof. Let

$$X_0 \subset X_1 \subset \dots \subset X_n = X$$

be a CW complex structure for X where X_k (for each $k \leq n$) is the k -skeleton defined in relation (2.4). Let $A_k = C(X_k)$ and $i : \bigcup_{\lambda_k} S^{k-1} \rightarrow \bigcup_{\lambda_k} I^k$, $\varphi_k : \bigcup_{\lambda_k} S^{k-1} \rightarrow X_{k-1}$ be the injection and attaching maps respectively, and $C(i)$ and $C(\varphi_k)$ be their induced maps. Let

$$PB := PB\left(C\left(\bigcup_{\lambda_k} S^{k-1}\right), C(\varphi_k), C(i)\right),$$

and define

$$\begin{aligned} \theta : C(X_k) &\longrightarrow PB \\ f &\longmapsto (f \circ \rho)_1 \oplus (f \circ \rho)_2, \end{aligned}$$

Where $(f \circ \rho)_1$ and $(f \circ \rho)_2$ are the restrictions of $(f \circ \rho)$ to $\bigcup_{\lambda_k} I^k$ and X_{k-1} respectively.

θ is well defined since $C(\varphi_k)((f \circ \rho)_1) = C(i)((f \circ \rho)_2)$. Also $C(\varphi_k)(h) = C(i)(g)$ for $(h, g) \in PB$, and so if $f \in C(X_k)$ be defined by

$$f(y) = \begin{cases} g(y) & y \in \bigcup_{\lambda_k} I^k \\ h(y) & y \in X_{k-1} \end{cases}$$

then $\theta(f) = (h, g)$. Now the noncommutative CW complex decomposition of dimension n for $A = C(X)$ is given by $\{A_0, \dots, A_n\}$.

Conversely let A_n be as in (3.2), and

$$\varphi_n^* : S^{n-1} \rightarrow \text{Prim}(A_{n-1})$$

be the attaching map induced by the connecting morphism

$$\varphi_n : A_{n-1} \rightarrow S^{n-1}F_n$$

of diagram (3.1). Then using the notation in relation (2.4),

$$\text{Prim}(A_n) = \text{Prim}(A_{n-1}) \bigcup_{\varphi_n^*} I^n.$$

We note that $\varphi_n^* = C(\varphi_n)$. Furthermore for $k \leq n$, $\varphi_k^*(S^{k-1})$ is a closed subset of $\text{Prim}(A_{k-1})$. It is of the form

$$\varphi_k^*(S^{k-1}) = \{J \in \text{Prim}(A_{k-1}) : I_{k-1} \subset J\}$$

for some ideal I_{k-1} in A_{k-1} . In fact $I_{k-1} = \bigcap_{J \in \varphi_k^*(S^{k-1})} J$. So $\text{Prim}(A)$ has an n -dimensional CW-structure with $X_k = \text{Prim}(A_k)$ as its k -skeleton for $k = 0, \dots, n$. \square

Example 3.4. *Following the notations of diagram (3.1), a 1-dimensional non-commutative CW complex decomposition for $A = C([0, 1]) = C(I)$ is given by*

$$A_0 = \mathbb{C} \oplus \mathbb{C}, A_1 = C([0, 1]).$$

Let $F_1 = \mathbb{C}$, then

$$I_0^1 F_1 = C_0((0, 1)), I^1 F_1 = C([0, 1]), S^0 F_1 = \mathbb{C} \oplus \mathbb{C}$$

and $\varphi_1 = id$. Also

$$C(I) = PB(S^0 F_1, \delta, \varphi_1) = \{f \oplus (\lambda \oplus \mu) \in C([0, 1]) \oplus (\mathbb{C} \oplus \mathbb{C}) : f(0) = \lambda, f(1) = \mu\}$$

together with the maps

$$\begin{aligned} \pi : A_1 &\longrightarrow A_0 \\ f \oplus (\lambda \oplus \mu) &\longmapsto \lambda \oplus \mu, \end{aligned}$$

$$\begin{aligned} f_1 : A_1 &\longrightarrow I^1 F_1 = A_1 \\ f \oplus (\lambda \oplus \mu) &\longmapsto f, \end{aligned}$$

and

$$\begin{aligned} \delta : I^1 F_1 = A_1 &\longrightarrow S^0 F_1 = \mathbb{C} \oplus \mathbb{C} \\ f &\longmapsto f(0) \oplus f(1). \end{aligned}$$

4. MODIFIED MORSE THEORY ON C*-ALGEBRAS

In this section, following the study of the Morse theory for the cell complexes in [2], [6], [9], [10], with some modification, we define the Morse function for the C*-algebras and state and prove the modified Morse theory for the non-commutative CW complexes. This is a classification theory in the category of C*-algebras and noncommutative CW complexes.

Definition 4.1. If A, B are two C*-algebras, two morphisms $\alpha, \beta : A \rightarrow B$ are *homotopic*, written $\alpha \sim \beta$, if there exists a family $\{H_t\}_{t \in [0,1]}$ of morphisms $H_t : A \rightarrow B$ such that for each $a \in A$ the map $t \mapsto H_t(a)$ is a norm continuous path in B with $H_0 = \alpha$ and $H_1 = \beta$. The C*-algebras A and B are said to *have the same homotopy type*, if there exists morphisms $\varphi : A \rightarrow B$ and $\psi : B \rightarrow A$ such that $\varphi \circ \psi \sim id_B$ and $\psi \circ \varphi \sim id_A$. In this case the morphisms φ and ψ are called *homotopy equivalence*.

Definition 4.2. Let A and B be unital C*-algebras. We say A is of *pseudo-homotopy type* as B if $C(Prim(A))$ and B have the same homotopy type.

Remark 4.3. In the case of unital commutative C*-algebras, by the GNS construction [11], $C(Prim(A)) = A$. So the notions of pseudo-homotopy type and the same homotopy type are equivalent.

For a unital C*-algebra A let

$$\Sigma = \{W_{i_1, \dots, i_k}\}_{1 \leq i_1, \dots, i_k \leq n, 1 \leq k \leq n}$$

be the set of all k -chains ($k = 1, \dots, n$) in $Prim(A)$, and

$$\Gamma = \{I_{i_1, \dots, i_k}\}_{1 \leq i_1, \dots, i_k \leq n, 1 \leq k \leq n}$$

be the set of all k -ideals corresponding to the k -chains of Σ for $k = 1, \dots, n$.

Lemma 4.4. Γ is an absorbing set.

Proof. This follows from the fact that for each $I_{i_1, \dots, i_k} \in \Gamma$, $J \in Prim(A)$ the relation $I_{i_1, \dots, i_k} \subset J$ is equivalent to $J = I_{i_1, \dots, i_t}$ for some $t \leq k$, meaning $J \in \Gamma$. \square

Definition 4.5. Let $f : \Sigma \rightarrow \mathbb{R}$ be a function. The k -chain $W_k = W_{i_1, \dots, i_k}$ is called a *critical chain of order k* for f , if for each $(k+1)$ -chain W_{k+1} containing W_k and for each $(k-1)$ -chain W_{k-1} contained in W_k , we have

$$f(W_{k-1}) \leq f(W_k) \leq f(W_{k+1}).$$

The corresponding ideal I_k to W_k is called *the critical ideal of order k* .

Definition 4.6. Let f has a critical chain of order k . We say f is an *acceptable Morse function*, if it has a critical chain of order i , for all $i \leq k$.

Definition 4.7. A function $f : \Sigma \rightarrow \mathbb{R}$ is called a *modified Morse function* on the C*-algebra A , if for each k -chain W_k in Σ , there is at most one $(k+1)$ -chain W_{k+1} containing W_k and at most one $(k-1)$ -chain W_{k-1} contained in W_k , such that

$$f(W_{k+1}) \leq f(W_k) \leq f(W_{k-1}).$$

Here we state the discrete Morse theory of Forman from [9], and state and prove our modification of it.

Theorem (Discrete Morse Theory): Suppose Δ is a simplicial complex with a discrete Morse function. Then Δ is homotopy equivalent to a CW complex with one cell of dimension p for each critical p -simplex.

Lemma 4.8. *If f is an acceptable modified Morse function on A , then $Prim(A)$ is homotopy equivalent to a CW complex with exactly one cell of dimension p for each critical chain of order p .*

Proof. In the discrete Morse theory it suffices to substitute Γ for the simplicial complex Δ . Since Γ is absorbing, it satisfies the properties of the simplicial complex Δ in the discrete Morse theorem. It follows that $Prim(A)$ is homotopy equivalent to a CW complex with exactly one cell of dimension p for each critical chain of order p . \square

Now we state our main theorem which provides a condition for a unital C^* -algebra to admit a noncommutative CW-complex decomposition. This is what we call the geometric condition.

Theorem 4.9. *Every unital C^* -algebra A with an acceptable modified Morse function f on it, is of pseudo-homotopy type as a noncommutative CW complex having a k -th decomposition cell for each critical chain of order k .*

Proof. If A is a unital C^* -algebra, then the acceptable modified Morse function on A is in fact a function on the simplicial complex of all k -ideals in $Prim(A)$ (a function on Γ). From lemma 4.8 we conclude that $Prim(A)$ is homotopy equivalent to a finite dimensional CW complex Ω . From the proposition 3.3 there is a noncommutative CW-complex decomposition for $C(\Omega)$ making it into a noncommutative CW complex. Now $C(Prim(A))$ and $C(\Omega)$ have the same homotopy type, which means A is of pseudo-homotopy type of the noncommutative CW complex $C(\Omega)$. Furthermore since f is acceptable, from the proof of proposition 3.3 it follows that if there exists a critical k -chain for f , then there exists C^* -algebras A_i for each $i \leq k$ so that $\{A_0, \dots, A_k\}$ is a noncommutative CW complex decomposition for $C(Prim(A))$ yielding a noncommutative CW complex decomposition for $C(\Omega)$. \square

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