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Few remarks on maximal pseudocompactness

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Abstract

A pseudocompact space is maximal pseudocompact if every strictly finer topology is no longer pseudocompact. The main result here is a counterexample which answers a question raised by Alas, Sanchis and Wilson.

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1. Introduction

For undefined notions we refer to [5] and [3]. Given a space X, we denote by $\tau(X)$ its topology.

A Tychonoff space is pseudocompact if every real valued continuous function defined on it is bounded. Equivalently, a Tychonoff space X is pseudocompact if and only if every locally finite family of open sets is finite [5]. In a serie of papers Ofelia Alas, Richard Wilson and their co- authors have investigated the notion of maximal pseudocompactness (see [1], [2] and [8]). This notion is justified because a pseudocompact space can have a strictly finer Tychonoff topology which is still pseudocompact: consider for instance the compact space ω_1+1 with the order topology. Indeed, let X be the space obtained by isolating the point ω_1, i.e. $X=\omega_1\oplus\{\omega_1\}$. As X is the topological sum of two countably compact Tychonoff spaces, it is pseudocompact and it clearly has a topology strictly finer than ω_1+1. A Tychonoff space (X,τ) is maximal pseudocompact

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if (X,τ) is pseudocompact but (X,σ) is not pseudocompact for any Tychonoff topology σ strictly finer than τ.

2. Results

An easy but useful fact is in the following:

Lemma 2.1. *Let (X,τ) be a T_1 space and $p \in X$ be a point of countable character. If σ is a Tychonoff topology on X such that $\sigma \supseteq \tau$ and (X,σ) is pseudocompact, then σ coincides with τ at p (i.e. p has the same system of neighbourhoods in both topologies).*

Proof. Fix a decreasing local base of open sets $\{U_n : n < \omega\}$ at p in τ. If σ differs from τ at p, then there exists a closed neigbourhood V of p in σ which is not a neighbourhood of p in τ. But then, $\{U_n \setminus V : n < \omega\}$ would be a locally finite family of non-empty open sets in (X,σ). This family is infinite because (X,τ) is T_1 and we reach a contradiction. □

Therefore, a first countable pseudocompact space is always maximal pseudocompact.

We begin by formulating a better sufficient condition. Let us say that a collection $\mathcal{S}$ hits a set A if $S \cap A \neq \varnothing$ for each $S \in \mathcal{S}$.

A set $S \subseteq X$ is co-pseudocompact [resp. co-singleton] if $X \setminus S$ is pseudocompact [resp. $|X \setminus S| = 1$].

Proposition 2.2. *Let (X,τ) be a pseudocompact space and assume that for every co-pseudocompact set $A \subseteq X$ and every point $p \in \overline{A}$ there exists a sequence of open sets in X which hits A and converges to p. Then X is maximal pseudocompact.*

Proof. Assume by contradiction that there is a topology σ strictly finer than τ such that (X,σ) is still pseudocompact. If $\sigma \neq \tau$ at a point p, we may fix a regular closed neighbourhood V of p in σ which is not a neighbourhood of $p$in τ. The set $X \setminus V$ is co-pseudocompact and $p \in \overline{X \setminus V}^{\tau}$. Therefore, there exists a sequence $\{U_n : n < \omega\} \subseteq \tau$ converging to p and satisfying $U_n \cap (X \setminus V) \neq \varnothing$ for every n. But then, $\{U_n \setminus V : n < \omega\}$ would be a locally finite infinite family of open sets in (X,σ), in contrast with the pseudocompactness of σ. □

The next observation shows that maximal pseudocompact spaces are "very close to" first countable.

Proposition 2.3. *If X is maximal pseudocompact, then each $p \in X$ is the limit of a convergent sequence of non-empty open sets.*

Proof. Proposition 3.1 in [2] states that a maximal pseudocompact space has countable π-character, but the proof of this result actually establishes the stronger statement that every point is the limit of a convergent sequence of non-empty open sets. Indeed, let p be a non-isolated point of X. The maximal pseudocompactness of X implies that $X \setminus \{p\}$ is not pseudocompact and so there is an infinite family of disjoint open sets $\{U_n : n < \omega\} \subseteq X \setminus \{p\}$ which is discrete in $X \setminus \{p\}$. We claim that the sequence $\{U_n : n < \omega\}$ converges to p. If not, there would be an infinite set $S \subseteq \omega$ and a closed neighbourood V of p such that $U_n \setminus V \neq \varnothing$ for each $n \in S$. But then, the infinite family of open sets $\{U_n \setminus V : n \in S\}$ would be discrete in X, in contrast with the pseudocompactness of X. □

The above proposition shows that maximal pseudocompactness imposes strong conditions to the topology. Another non-trivial consequence is described in the following:

Corollary 2.4. *If X is maximal pseudocompat, then $|X| \leq 2^{c(X)}$.*

Proof. By Proposition 3.1 in [2] and Šapirovskiĭ's formula $w(X) \leq \pi\chi(X)^{c(X)}$, there exists a dense set D such that $|D| \leq 2^{c(X)}$. But, by Proposition 2.3 each point of X is the limit of a sequence contained in D and so $|X| \leq |D|^{\omega} \leq 2^{c(X)}$. □

Thus, there are plenty of compact spaces which are not maximal pseudocompact. In addition, by Corollary 3.5 in [2] every compactification of a non-compact pseudocompact space is not maximal pseudocompact. Propositions 2.2 and 2.3 seem to suggest that in the class of pseudocompact spaces maximal pseudocompactness (briefly MP) is a *convergent-like* property. Indeed, if $\mathcal{P}^+$:= "for every co-pseudocompact set $A \subseteq X$ and every point $p \in \overline{A}$ there exists a sequence of open sets in X which hits A and converges to p" and $\mathcal{P}^-$:= "every point is the limit of a converging sequence of non-empty open sets", then

$$\text{pseudocompact} + \mathcal{P}^+ \Longrightarrow \text{MP} \Longrightarrow \mathcal{P}^-$$

The one-point compactification of an uncountable discrete space is a compact space satisfying $\mathcal{P}^+$ which is not first countable. We believe there should exist a maximal pseudocompact space which does not satisfy $\mathcal{P}^+$, but at moment we do not have such a space. On the other direction, let $X = \mathcal{A} \cup \omega$ be a Ψ-space over a MAD family $\mathcal{A}$ on ω and let $X \cup \{\infty\}$ be its one-point compactification. Fix $A_0 \in \mathcal{A}$ and let Z be the quotient space of $X \cup \{\infty\}$ obtained by identifying A_0 and ∞ to a point p. Z is a compact space which satisfies $\mathcal{P}^-$. This is evident for each point of $Z \setminus \{p\}$. For p observe that $\{\{n\} : n \in A_0\}$ is a sequence of open sets in Z converging to p. But, Z is not maximal pseudocompact, because the function $f : X \to Z$, defined by letting $f(A_0) = p$ and $f(x) = x$ for every $x \in X \setminus \{A_0\}$, is a continuous bijection which is not open.

We conclude that the unknown property $\mathcal{P}$ which characterizes maximal pseudocompactness within the class of pseudocompact space lies in between $\mathcal{P}^+$ and $\mathcal{P}^-$ and differs from the latter.

Since $\mathcal{P}^-$ is just $\mathcal{P}^+$ restricted to co-singleton sets (a subclass of co-pseudocompact sets), property $\mathcal{P}$ should involve an appropriate subclass of co-pseudocompact sets.

Question 2.5. *What is the convergent property $\mathcal{P}$ such that a pseudocompact space X is maximal pseudocompact if and only if X satisfies $\mathcal{P}$?*

As pointed out in [1], a relevant role in studying maximal pseudocompactness is played by the notion of accessibility from a dense subset. Given a space X and a dense set $D \subseteq X$, we say that X is strongly accessible from D if for any $x \in X \setminus D$ and any $A \subseteq D$ such that $x \in \overline{A}$, there exists a countable sequence $S \subseteq A$ converging to x.

Proposition 2.6 ([1, Theorem 2.4])**.** *Let X be a pseudocompact space and D a dense set of isolated points. If X is strongly accessible from D, then X is maximal pseudocompact.*

Therefore, any compactification $\gamma(\mathbb{N})$ of the set of integers $\mathbb{N}$ with the discrete topology such that $\gamma(\mathbb{N})$ is strongly accessible from $\mathbb{N}$ is maximal pseudocompact [1]. In some case, $\gamma(\mathbb{N}) \setminus \mathbb{N}$ can be homeomorphic to $\omega_1 + 1$, thus showing for instance that a compact maximal pseudocompact space need not be Fréchet.

The first construction of this kind, discovered in the attempt to find a compact radial separable non Fréchet space, is the space $\delta(\mathbb{N})$ given in [7] by assuming the Continuum Hypothesis. A similar example, obtained under the weaker assumption $\mathfrak{d} = \omega_1$, is given in [6]. But perhaps, the easiest way to obtain it is by using a tower.

Recall that the cardinal $\mathfrak{t}$ is the smallest size of a tower, i.e. a well-ordered by reverse almost inclusion family of subsets of $\mathbb{N}$ without any infinite pseudointersection (see [3] for more).

Fix a family $\mathcal{A} = \{A_\alpha : \alpha \in \omega_1\}$ of subsets of $\mathbb{N}$, well-ordered by $\subset^*$. Furthermore, put $A_{-1} = \varnothing$ and $A_{\omega_1} = \mathbb{N}$.

We define a topology on the set $\gamma(\mathbb{N}) = \mathbb{N} \cup \omega_1 + 1$ by declaring each point of $\mathbb{N}$ isolated and by taking as a local base at each $\alpha \in \omega_1 + 1$ the sets $]\beta, \alpha] \cup A_\alpha \setminus (A_\beta \cup F)$, where F is a finite subset of $\mathbb{N}$ and $-1 \leq \beta < \alpha$. To be more formally correct, we should replace in the previous definition $\omega_1 + 1$ for instance with $\{x_\alpha : \alpha \in \omega_1 + 1\}$. However, we believe our semplified notation does not cause any trouble to the reader.

The space $\gamma(\mathbb{N})$ is compact Hausdorff and first countable at each $\alpha < \omega_1$.

Proposition 2.7. *The space $\gamma(\mathbb{N})$ may fail to be maximal pseudocompact if and only if $\mathfrak{t} = \omega_1$.*

Proof. We begin by showing that $\mathfrak{t} > \omega_1$ implies the maximal pseudocompactness of $\gamma(\mathbb{N})$. By Proposition 2.6, it suffices to check that $\gamma(\mathbb{N})$ is strongly accessible from $\mathbb{N}$. As the only point of uncountable character in $\gamma(\mathbb{N})$ is ω_1, we only need to consider the case of a set $A \subseteq \mathbb{N}$ such that $\omega_1 \in \overline{A}$. This clearly implies $|(\mathbb{N} \setminus A_\alpha) \cap A| = \omega$ for each α. Since $\mathfrak{t} > \omega_1$, the family $\{(\mathbb{N} \setminus A_\alpha) \cap A : \alpha < \omega_1\}$ is not a tower and hence we may take an infinite set $S \subseteq A$ satisfying $S \subseteq^* \mathbb{N} \setminus A_\alpha$ for every $\alpha < \omega_1$. This actually means that S converges to ω_1 and we are done.

To complete the proof, we now verify that $\mathfrak{t} = \omega_1$ implies that $\gamma(\mathbb{N})$ may fail to be maximal pseudocompact. The point is that, by assuming $\mathfrak{t} = \omega_1$, we may choose the family $\mathcal{A}$ in such a way that $\{\mathbb{N} \setminus A_\alpha : \alpha < \omega_1\}$ is a tower.

Consequently, if we take an infinite set $S \subseteq \mathbb{N}$, then there exists some $\alpha < \omega_1$ such that $|S \cap A_\alpha| = \omega$. If α is the least ordinal with this property, then the set $S \cap A_\alpha$ is actually a sequence converging to α. Therefore, no subsequence of $\mathbb{N}$ can converge to ω_1. In particular, the point ω_1 cannot be the limit of a sequence of non-empty open subsets of $\gamma(\mathbb{N})$ and so by Proposition 2.3 the space $\gamma(\mathbb{N})$ is not maximal pseudocompact. □

The above discussion suggests the following:

Question 2.8 (ZFC)**.** *Is there a compactification $\gamma(\mathbb{N})$ of $\mathbb{N}$ which is maximal pseudocompact but not Fréchet?*

Any radial compactification of $\mathbb{N}$ is obviously maximal pseudocompact, but Dow [4] has shown that there are models where every compact separable radial space is Fréchet.

In [1], Question 2.17 asks whether Proposition 2.6 is reversible, namely:

"Suppose that X is a maximal pseudocompact space with a dense set of isolated points D. Is X strongly accessible from D?"

Mimiking the construction given in [1], Example 2.14, we can give a consistent negative answer to the above question.

Example 2.9. $[\mathfrak{t} > \omega_1]$ or $[\mathfrak{d} = \omega_1]$ A maximal pseudocompact space which is not accessible from a dense set of isolated points.

Proof. Take the space $\gamma(\mathbb{N}) = \mathbb{N} \cup \omega_1 + 1$ described above under the assumption $\mathfrak{t} > \omega_1$ or $\mathfrak{d} = \omega_1$and let $Y = (\omega + 1) \times \omega_1$. Let X be the quotient space of $Y \oplus \gamma(\mathbb{N})$, obtained by identifying the set $\{\omega\} \times \omega_1 \subseteq Y$ with the copy of ω_1 in $\gamma(\mathbb{N})$ (i.e. $(\omega, \alpha) \equiv \alpha$ for each $\alpha \in \omega_1$). Recall that if $q : Y \oplus \gamma(\mathbb{N}) \to X$ is the quotient map, then $V \in \tau(X)$ if and only if $q^{-1}(V) \in \tau(Y \oplus \gamma(\mathbb{N}))$ if and only if $q^{-1}(V) \cap Y \in \tau(Y)$ and $q^{-1}(V) \cap \gamma(\mathbb{N}) \in \tau(\gamma(\mathbb{N}))$.

We claim that X is maximal pseudocompact. Indeed, let σ be a pseudocompact topology finer than the topology τ on X. Since $\gamma(\mathbb{N})$ is compact, $q \restriction \gamma(\mathbb{N})$ is an embedding and so we may identify $\gamma(\mathbb{N})$ with $q(\gamma(\mathbb{N})) \subseteq X$. We must have $\omega_1 \in \overline{\mathbb{N}}^\sigma$, otherwise a sequence $S \subseteq \mathbb{N}$, converging in τ to ω_1, would provide a discrete infinite family of open singletons in σ. Since by Lemma 2.1 σ coincides with τ at each $\alpha \in \omega_1$, we see that $\overline{\mathbb{N}}^\sigma = \gamma(\mathbb{N})$. Thus $\gamma(\mathbb{N})$ is regular closed in σ and hence pseudocompact in σ. Since $\gamma(\mathbb{N})$ is maximal pseudocompact, we conclude that σ coincides with τ on $\gamma(\mathbb{N})$, i.e. $\sigma \restriction \gamma(\mathbb{N}) = \tau \restriction \gamma(\mathbb{N})$. Since X is first countable at each point of $q(Y)$, again by Lemma 2.1, σ coincides with τ on $q(Y)$, so we also have $\sigma \restriction q(Y) = \tau \restriction q(Y)$.

Now, take any $V \in \sigma$. From $V \cap q(Y) \in \sigma \restriction q(Y) = \tau \restriction q(Y)$, it follows $q^{-1}(V) \cap Y \in \tau(Y)$. In a similar manner, from $V \cap \gamma(\mathbb{N}) \in \sigma \restriction \gamma(\mathbb{N}) = \tau \restriction \gamma(\mathbb{N}) = \tau \restriction q(\gamma(\mathbb{N}))$, it follows $q^{-1}(V) \cap \gamma(\mathbb{N}) \in \tau(\gamma(\mathbb{N}))$. This suffices to conclude that $V \in \tau$ and hence $\sigma = \tau(X)$.

X has a dense set of isolated points, namely $D = (\omega \times \{0, \alpha + 1 : \alpha < \omega_1\}) \cup \mathbb{N}$. But, X is not strongly accessible from D, because ω_1 is in the closure of $\omega \times \{\alpha + 1 : \alpha < \omega_1\}$, but no subsequence of it can converge to ω_1. This last thing depends on the countable compactness of Y. □

In [1] a space X was called hereditarily maximal pseudocompact (briefly HMP) if each closed subspace of X is maximal pseudocompact.

Since pseudocompactness is preserved by passing to regular closed subspaces, the following way to define the hereditary version of maximal pseudocompactness certainly makes sense.

A space is weakly hereditarely maximal pseudocompact (briefly wHMP) if every regular closed subspace is maximal pseudocompact.

Example 2.14 in [1] as well as the space X in Example 2.9 above provide maximal pseudocompact spaces which are not wHMP. Regarding Example 2.9, observe that the set $q(\omega \times \omega_1)$ is open in X and $\overline{q(\omega \times \omega_1)}$ is homeomorphic to $Y \cup \{\omega_1\}$. But the latter is a regular closed subspace of X which is not maximal pseudocompact.

A non-trivial difference between HMP and wHMP emerges from the following:

Proposition 2.10. *If the pseudocompact space X is strongly accessible from a dense set D of isolated points, then X is wHMP.*

Proof. Let Y be a regular closed subset of X. Since $Y = \overline{U}$ for some open set U, Y is strongly accessible from the dense subset of isolated points $U \cap D$ and we are done. □

Any version of the space $\gamma(\mathbb{N})$, mentioned above, is therefore wHMP but not HMP.

In connection with Question 2.17 in [1], consider the following:

Proposition 2.11. *Let X be a wHMP space. If D is a dense set of isolated points, then X is strongly accessible from D.*

Proof. Take a point $x \in X \setminus D$ and a set $A \subseteq D$ such that $x \in \overline{A}$. Since A is open, we see that the subspace $\overline{A}$ is maximal pseudocompact. Therefore, by Proposition 2.3 (or Lemma 2.8 in [1]), there is a sequence in A converging to x. □

Propositions 2.10 and 2.11 show that Proposition 2.6 is reversible precisely for wHMP spaces in the class of pseudocompact spaces.

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