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La aritmética matricial como modelo del entorno cotidiano

The matrix arithmetic as a model of the everyday environment

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Abstract

This article is a proposition for the teaching / learning of some matrix calculation elements from mathematical modeling. As a matter of fact, some daily situations are established showing how we can create models to illustrate the matrix concept and also by introducing basic operations of difference and product of matrices. Firstly, a matrix is shown as a mathematical model of an image and then how the matrix difference becomes a model for image comparison is discussed. However, to do this task software such as Octave (or similar software) is necessary. This tool allows the research of a numerical model of a black and white image represented by a matrix. Furthermore, we see how the product of matrices is a model which can be naturally deduced from the grocery shopping routine. The main idea is to underline the matrix calculation epistemology in order to reinforce the students? cognitive character, bringing a contextual view of daily matters in real life at the same time, enriching the heuristic, thus allowing the visualization of the connection among the mathematical symbolism (introduced on the model) and the real situations.

Este artículo es una propuesta para la enseñanza/aprendizaje de algunos elementos de cálculo de matrices a partir del modelado matemático. De hecho, algunas situaciones cotidianas se establecen teniendo también las matrices y sus operaciones como modelo matemático, en particular mostrando cómo podemos crear modelos para ilustrar el concepto de matriz y también introduciendo operaciones básicas de diferencia y producto de matrices. En primer lugar, una matriz se muestra como un modelo matemático de una imagen y luego se discute cómo la diferencia de la matriz se convierte en un modelo para la comparación de imágenes.Sin embargo, para realizar esta tarea es necesario un software como Octave (o similar). Esta herramienta permite la búsqueda de un modelo numérico de una imagen en blanco y negro representada por una matriz. Además, vemos cómo el producto matriz es un modelo que puede deducirse naturalmente de la rutina de la compra de comestibles. La idea principal es subrayar la epistemología del cálculo matricial para reforzar el carácter cognitivo del alumno, aportando al mismo tiempo una visión contextual de lo cotidiano en la vida real, enriqueciendo lo heurístico, permitiendo así la visualización de la conexión entre el simbolismo matemático (introducido en el modelo) y las situaciones reales.

Keywords: teaching / learning, mathematical model, matrix concept
Palabras clave: enseñanza/aprendizaje, modelo matemático, matriz

1. Introduction

The concept of matrix is present in countless mathematical models of different situations ranging from Applied Sciences and Engineering to everyday life. However, its introduction into mathematical studies at the secondary level is often anecdotal and, at the tertiary level, it is almost always linked to the notion of linear mappings between vector spaces. This situation has two very negative effects. The first one is that students perceive matrices as abstract constructions that are alien to reality. The second one is that the understanding of the operations with matrices and their use in the various contexts of application where they appear becomes obscure for the students.

The attempts to contextualize mathematics in the field of tertiary education have been diverse, mainly in universities of the Catalan language area (see for example Sánchez Pérez, E.A., García-Raffi, L.M., Sánchez Pérez, J.V., 1999 and Joan Gómez Urgellés, 2007) and in the same manner the attempts to introduce matrices to students in applied contexts (see for example Jose M. Calabuig, Lluís M. García Raffi, Enrique A. Sánchez-Pérez 2013 and 2015). In this work different real situations are presented that provide frameworks where not only to apply the matrices as a mathematical model but to introduce in a natural way operations with them. Some of them have been applied to students of the first course of the Computer Science degree at EPSEVG University

2. Working with Images: The Matrices Difference as a Mathematical Model

2.1. A Matrix as a Mathematical Model of a Black and White image

When we talk about images, mathematics has an important role. Actually, technically each image can be seen as a table of numbers (formally known as a matrix). Then, defining an image as "composed by M per N pixels", it means that it can be represented by a matrix with M rows and N columns, generally with values between 0 and 255 (256 elements). The number of pixels is called "resolution". The procedure to obtain the matrix has been done through very sophisticated mathematical algorithms implemented by the software (as MatLab^{TM} with an expense of 50$ for students or 150$ for home users). When we say 15 per 13 pixels, we mean something similar to the figure below —see Fig. 1—.

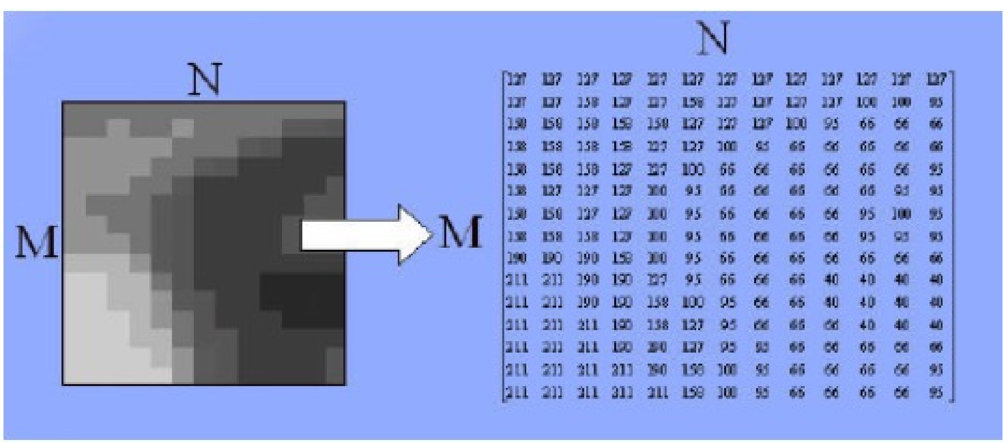

Figure 1 – The pixels and their correspondent array.

When we talk about "5 megapixels", we are really talking about 5 million pixels. However, if we read "640 x 480", that means a matrix of 640 columns per 480 rows Now lets go to analyze a real situation. Consider these violin pictures —see Fig. 2—:

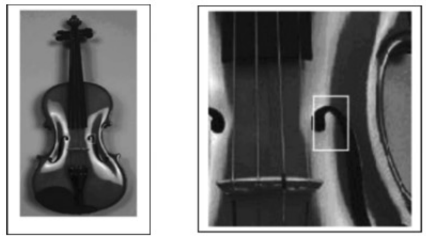

Figure 2 – Violin and Violin detail.

This violin picture image matches the Matrix below —see Fig. 3—:

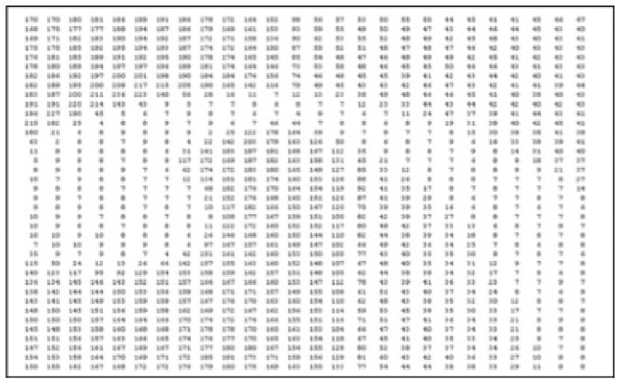

Figure 3 – Matrix of the Violin.

Take a glance at this unbelievable numerical table, even the density and the placement of the numbers drawing show the violin profile. Next, the procedure in order to get the model is fully explained.

Actually, we can choose any image in our computer, by selecting with the cursor over the image, doing right-clicking on it, and then open a *Properties Window* that shows all of the information about the image size. Depending on the resolution and the available space on the disk, it is possible to save the image in different formats such as BMP, TIFF, or JPEG as well —see Fig. 4—:

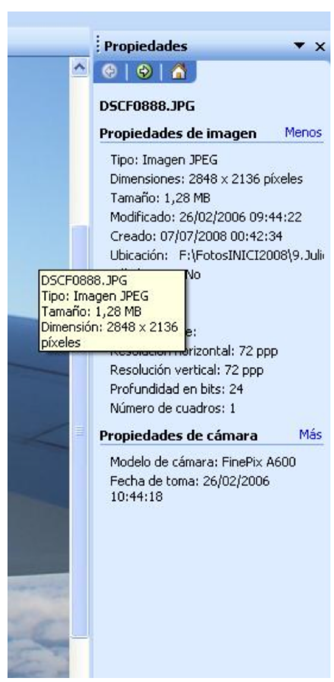

Figure 4 – Octave Properties Window.

In order to get a matrix model of an image, one needs software such as Octave. Formally Octave works with sophisticated numeric methods for obtaining the Matrix Model. The procedures are explained below.

2.2. Octave: Generating the Pixels Matrix of an image

Octave is a free program available for Windows, Mac, and Linux, developed for Numeric Calculations. It is available on the web https://www.gnu.org/software/octave/ It was developed around 1988, created by Chemical Engineering students from Texas University and Wisconsin-Madison University to be applied to support Chemical Reactors drawing. Actually, Octave is a free option to the well-known MatLab^{TM}. Octave has a wide kit of tools to solve algebra, calculation, and statistics problems. It is also able to process digital images. A Smartphone version is available. We are working with Black and White images because the associated Matrix is bi-dimensional, which means that it is a Numbering Table with rows and columns. On the other hand, in the case of colored images, a "three-dimensional Matrix" would be generated, and each color would be obtained from the basic RGB (Red, Green, Blue). With Octave, we can generate the Pixel Matrix of any image. How can we do that? By following the steps below:

1. To install the program, go to the link: https://www.gnu.org/software/octave/ or ftp://ftp.gnu.org/gnu/octave/windows/octave-4.0.0_0-installer.exe

2. Once the Octave installation is completed, run the program by opening a similar window such as:

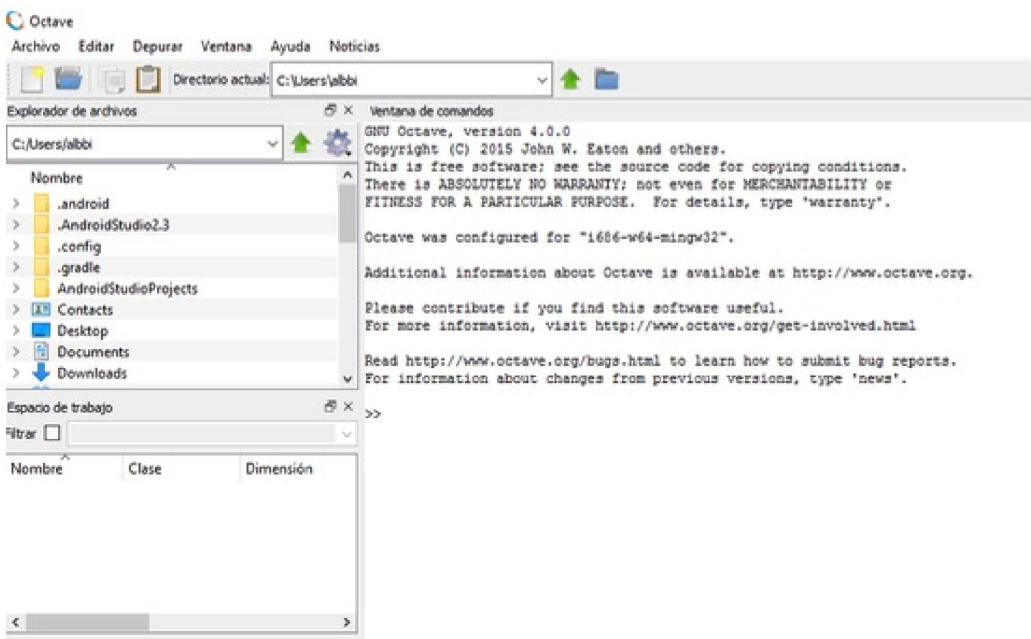

Figure 5 – Running Octave.

3. Choose a previously saved image in the directory.

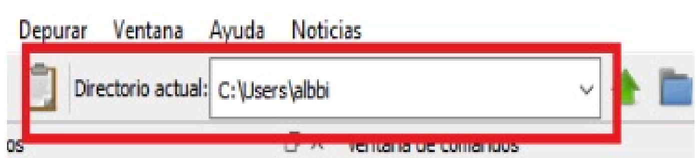

Figure 6 – Saving in the directory.

4. And then, select the image whose Pixel Matrix Associated we want to know.

Remember that it is necessary to save the image in the folder created by Octave, which in our case is C:\Users\albbi. Literally, in the Octave window we should go to the directory in which the images have been saved. With the image selected in the folder, introduce the following instruction in the command line **image = imread ('image name. extension')**

SUMMARIZING: We can introduce the concept of Matrix as a model of a Black and White image.

3. Modeling Experience

Now, we are doing an experience which has been explained during the first course of the computer Science degree at the Universitat Politècnica de Catalunya (UPC) by showing the Matrix difference as a Mathematical Model of an image. In our example, we are considering two different black and white images, previously saved in our computer, and comparing both. In our computer, these images have been saved as "imatge1.jpg" and "imatge2.jpg" —see Fig. 7—.

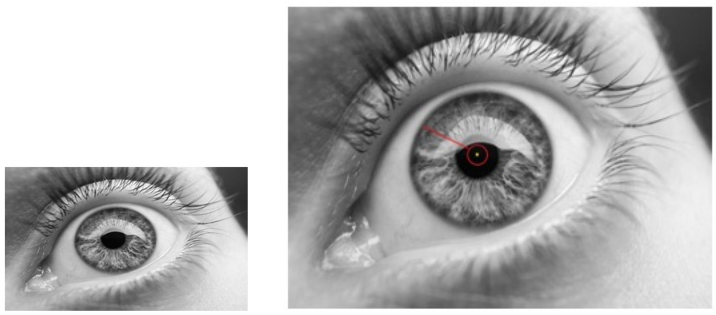

Figure 7 – "imatge1.jpg" and "imatge2.jpg".

Apparently, these pictures seem identical, but this is not true; the second picture has a small different dot in yellow in order to be easily seen. Now we are introducing the following lines in octave command Variable Name :

```
>>I = imread (''fotografia1.jpg'')
```

where"I" has been chosen as a Variable Name. In this line, we are saving the Variable "I", the "imatge 1". However, we are really saving the Matrix of picture 1. Next, we are introducing the command Variable Name

```
>> I2= imread (''fotografia2.jpg'')
```

saving the variable "I2" ("picture 2").

By pressing "Enter" after each instruction, the window shows respectively the Matrix as a Mathematical Model of each image respectively obtaining the result shown in Fig. 8.

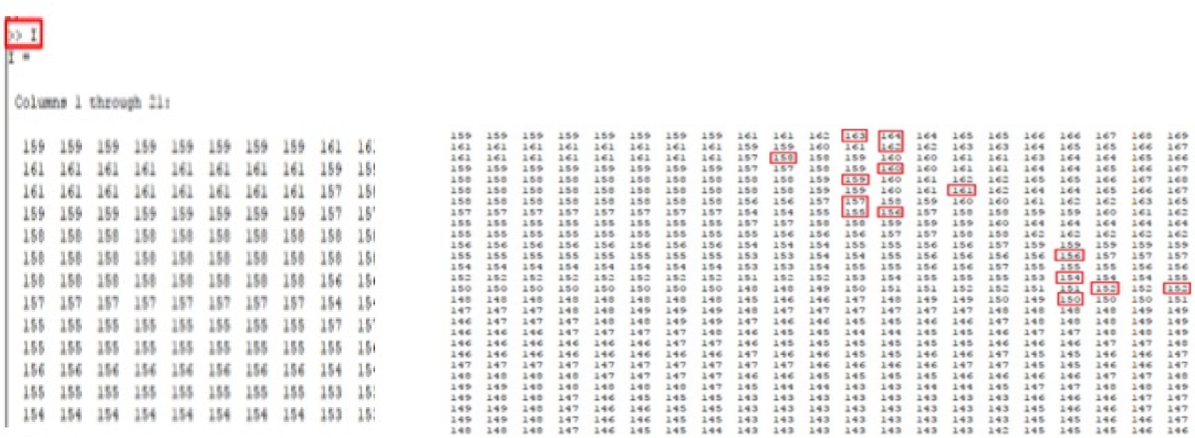

Figure 8 – Left: Image Partial view (imatge 1). Right: Full Capture of the"imatge 1" Matrix.

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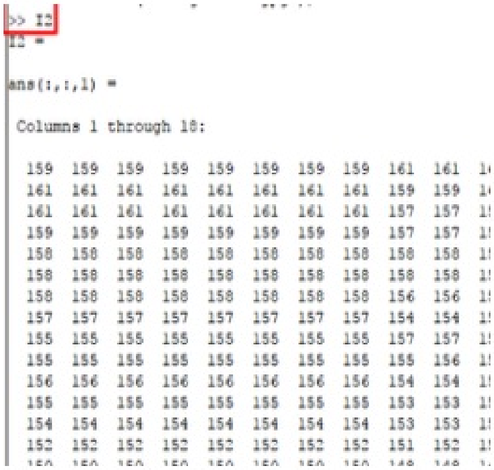
Similarly for the second image:



```
>> I2
I2 =

ans(:,:,1) =

 Columns 1 through 18:

  159  159  159  159  159  159  159  159  161  161
  161  161  161  161  161  161  161  161  159  159
  161  161  161  161  161  161  161  161  157  157
  159  159  159  159  159  159  159  159  157  157
  158  158  158  158  158  158  158  158  158  158
  158  158  158  158  158  158  158  158  158  158
  158  158  158  158  158  158  158  158  156  156
  157  157  157  157  157  157  157  157  154  154
  155  155  155  155  155  155  155  155  157  157
  155  155  155  155  155  155  155  155  155  156
  156  156  156  156  156  156  156  156  154  154
  155  155  155  155  155  155  155  155  153  153
  154  154  154  154  154  154  154  154  153  153
  152  152  152  152  152  152  152  152  151  152
```

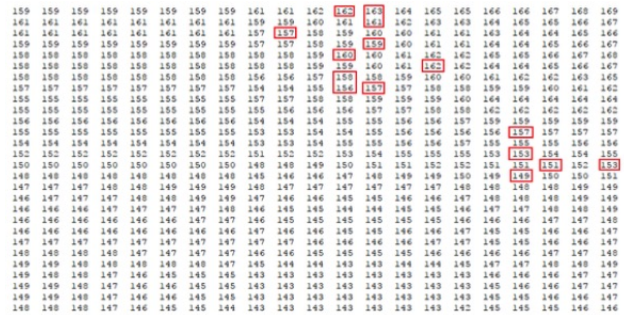


Figure 9 – Left: Image Partial view (imatge 2). Right: Full Capture of the 'imatge 2" Matrix.

As you can see the differences between the images are clear because the values of the matrix models are diferent. Strictly Speaking: we can get the differences between images by subtracting these matrices and concluding that the regions with zeros do not have changes. On the other hand, the regions with values different from zero mean that they do have changes.

SUMMARIZING: The difference between Matrices would be a Mathematical Modeling to compare images.

The Model has been applied in the class to the First Course of Computer Science EPSEVG University, as a group work developed by students, despite the fact that they had never worked with Matrices before. However, they were able to explain the work in the classroom to their other classmates, as shown in Fig 10.

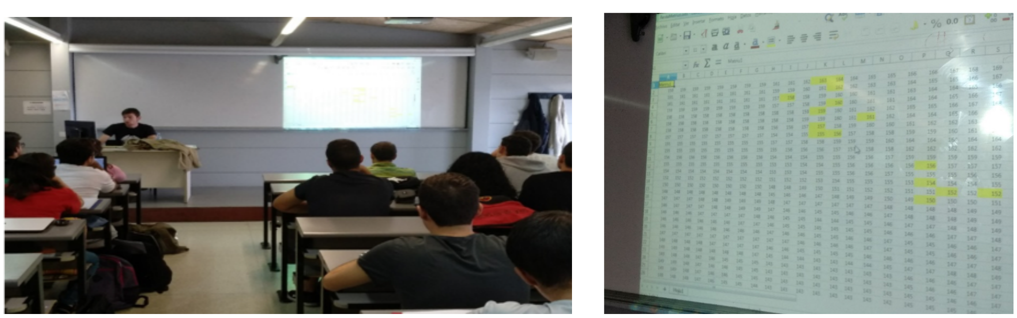


Figure 10 – Left: Computer Science University class. Right: The Matrix obtained by the Computer Science students.

Octave also allows subtracting the obtained matrices. It's even possible to select the Matrices by attaching them to the Excel database and then subtracting them. The results of our experience can be seen in Fig 11.

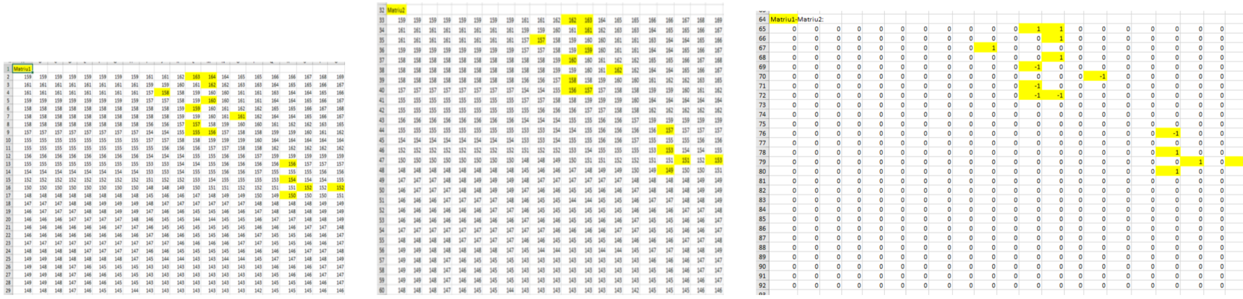


Figure 11 – Result of the difference between Matrix 1 and Matrix 2.

The resultant matrix clearly shows the regions of the image in which all differences have been observed.

Another example has been gathered from the Written Press. It refers to finding / spotting the differences —see Fig. 12—. In detail, we realized that as a model, they have respectively their Numerical Matrix —see Fig. 13—.

Figure 12 – The Game of Differences.

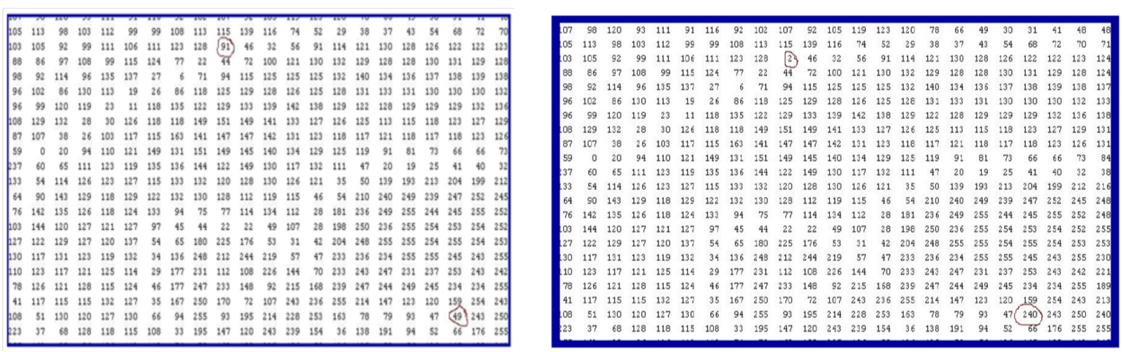


Figure 13 – Searching the differences.

As previously shown, the differences between images have been found in row 3, column 10. Over there, number 91 has been converted to 2. Also in row 21, in column 20 number 49 has been converted to 240. This means that by subtracting both matrices, zero is obtained in all operations except in row 3, column 10, and in row 21, column 20 as well.

To our students, we can comment on another daily example when the difference between matrices reaches a remarkable role in the security field.

3.1. The difference between matrices as a Model of Security System.

According to our previous results, we can compare Images. Think about a hypothetical frame series (in black and white) captured by surveillance camera inside a bank. Now, considering the models of two consecutive images, the “intelligent camera” processes the difference between two matrices. If a “zero” matrix (all the elements null) is obtained, we realize that no movement has been made, in this case it is not necessary to record all associated images to those matrices. On the other hand, all the matrices with a difference not null will

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be recorded registering the movement inside the bank. This simplified example explains how the "intelligent surveillance cameras" do a night surveillance.

4. Shopping at a Supermarket: A Model of the Product between Matrices

In the following situation, the students, naturally discovering how to make the product of matrices into a Model, link the quantities of buying products at a Supermarket to their prices with total expenses. Now we will do an easy study in order to clarify which one between two supermarkets is the least expensive merchant. We are shopping twice a week, buying the articles and quantities according to the Table 1.

	Pork Loin(kg)	Oranges (kg)	Lettuce (3 units \ tray)
1^{st} Day	1	3	1
2^{nd} Day	3	2	2

Table 1 – A Daily Consumption.

This table can be written in a different way:

$$\begin{pmatrix} 1 & 3 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

And then for each supermarket we can calculate:

- What are the expenses for the first day?
- What are the expenses for the second day?

We can also do a global calculation:

- What is the least expensive option for each day?

These questions could be proposed to our students so that they can calculate and achieve their own conclusions.

Figure 14 – Expenses at Supermarket 1

In the case of Supermarket 1 we have the price list in the Table 2, and for Supermarket 2, in Table 3:

Supermarket 1	Pork Loin(kg)	Oranges (kg)	Lettuce (3 units \ tray)
Price	6.49 €	1.25 €	0.75 €

Table 2 – Articles and Prices in Supermarket 1

Figure 15 – Expenses at Supermarket 2

Supermarket 1	Pork Loin(kg)	Oranges (kg)	Lettuce (3 units \ tray)
Price	5.90 €	0.85 €	1.25 €

Table 3 – Articles and Prices in Supermarket 2

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Then, we can perform the daily calculation of our expenses in each supermarket:

- Supermarket 1
  - 1^{st} Day $\longrightarrow$ $1 \cdot 6.49 + 3 \cdot 1.25 + 1 \cdot 0.75 = 10.99$ €
  - 2^{nd} Day $\longrightarrow$ $3 \cdot 6.49 + 2 \cdot 1.25 + 2 \cdot 0.75 = 23.47$ €
- Supermarket 2
  - 1^{st} Day $\longrightarrow$ $1 \cdot 5.90 + 3 \cdot 0.85 + 1 \cdot 1.25 = 9.70$ €
  - 2^{nd} Day $\longrightarrow$ $3 \cdot 5.90 + 2 \cdot 0.85 + 2 \cdot 0.75 = 21.90$ €

Mathematically, we can write the expenses in each supermarket as:

- Supermarket 1
  - 1^{st} Day $\longrightarrow$ $(1, 3, 1) \cdot (6.49, 1.25, 0.75) = 10.99$ €
  - 2^{nd} Day $\longrightarrow$ $(3, 2, 2) \cdot (6.49, 1.25, 0.75) = 23.47$ €
- Supermarket 2
  - 1^{st} Day $\longrightarrow$ $(1, 3, 1) \cdot (5.90, 0.85, 1.25) = 9.70$ €
  - 2^{nd} Day $\longrightarrow$ $(3, 2, 2) \cdot (5.90, 0.85, 1.25) = 21.90$ €

You can see that what we apply is the so called Scalar Euclidian Product. It is remarkable tha the students had been building the scalar product in an intuitive manner.

SUMMARIZING: A daily matter such as shopping at a Supermarket has a Mathematical Model, the "Scalar Euclidean Product".

Globally, we can write the expenses in each supermarket as:

$$\begin{pmatrix} 1 & 3 & 1 \\ 3 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 6.49 \\ 1.25 \\ 0.75 \end{pmatrix} = \begin{pmatrix} 10.99 \\ 23.47 \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} 1 & 3 & 1 \\ 3 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 5.90 \\ 0.85 \\ 1.25 \end{pmatrix} = \begin{pmatrix} 9.70 \\ 21.90 \end{pmatrix} \tag{2}$$

Here we are naturally building the product of a matrix by a vector column. Then the Matrix Model can be represented as the next matrices:

- Q: Quantity of Products

$$Q = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

- P: Price in each supermarket

$$P = \begin{pmatrix} 6.49 & 5.90 \\ 1.25 & 0.85 \\ 0.75 & 1.25 \end{pmatrix}$$

- D: Expenses

$$\begin{pmatrix} 10.90 & 9.70 \\ 23.47 & 21.90 \end{pmatrix}$$

By doing it as previously mentioned, it is possible to introduce the matrix product. The following Mathematical Model links Quantities (Q), Price (P), and Expenses (D) by Eq. 3.

$$Q \cdot P = D \tag{3}$$

SUMMARIZING: We realize that by linking purchased quantities and prices, it is possible to naturally obtain the algorithm to multiply matrices.

Going back to the previous supermarket comparison, the Supermarket 2 has the best deals. The total expenses in Supermarket 2 mean significant money saving when compared with those in Supermarket1. The students who did this exercise discovered how to multiply Matrices naturally. Now, the professor feels free to propose situations by introducing the inverse matrix concept and other elements of Matrix calculation.

5. Conclusions

These examples are useful to show how the use of real life situations makes it feasible to find patterns (models) giving information about the proposed situations.

Obviously, the reader can translate the involvement of the Mathematical Modeling in the Academic Curricula, at the same time realizing that competent teaching is taking place.

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