# A Dependency Pair Framework for $A \vee C$-Termination

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**Abstract**

The development of powerful techniques for proving termination of rewriting modulo a set of equational axioms is essential when dealing with rewriting logic-based programming languages like CafeOBJ, Maude, ELAN, OBJ, etc. One of the most important techniques for proving termination over a wide range of variants of rewriting (strategies) is the *dependency pair approach*. Several works have tried to adapt it to rewriting modulo *associative and commutative* (AC) equational theories, and even to more general theories. However, as we discuss in this paper, no appropriate notion of minimality (and minimal chain of dependency pairs) which is well-suited to develop a *dependency pair framework* has been proposed to date. In this paper we carefully analyze the structure of infinite rewrite sequences for rewrite theories whose equational part is *any combination* of associativity and/or commutativity axioms, which we call $A \vee C$*-rewrite theories*. Our analysis leads to a more accurate and optimized notion of dependency pairs through the new notion of *stably minimal term*. We then develop a suitable dependency pair framework for proving termination of $A \vee C$-rewrite theories.

## 1 Introduction

Rewriting with rules $R$ modulo axioms $E$ is a widely used technique in both rule-based programming languages and in automated deduction. Consequently, termination of rewriting modulo specific equational axioms $E$ (e.g., associativity-commutativity, AC) has been studied. Methods for proving termination of rewriting systems modulo AC-axioms are known and even implemented. Several works have tried to adapt the *dependency pair approach* (DP-approach [1]) to rewriting modulo *associative and commutative* (AC) theories [20, 16, 17, 18, 21].

```
fmod LIST&SET is
  sorts Bool Nat List Set .
  subsorts Nat < List Set .
  ops true false : -> Bool .
  ops _and_ _or_ : Bool Bool -> Bool [assoc comm] .
  op 0 : -> Nat .
  op s_ : Nat -> Nat .
  op _;_ : List List -> List [assoc] .
  op null : -> Set .
  op __ : Set Set -> Set [assoc comm] .
  op _in_ : Nat Set -> Bool .
  op _==_ : List List -> Bool [comm] .
  op list2set : List -> Set .
  var  B : Bool .             vars N M : Nat .
  vars L L' : List .          var  S : Set .
  eq N N = N .
  eq true and B = B .         eq false and B = false .
  eq true or B = true .       eq false or B = B .
  eq 0 == s N = false .       eq s N == s M = N == M .
  eq N ; L == M = false .     eq N ; L == M ; L' = (N == M) and L == L' .
  eq L == L = true .
  eq list2set(N) = N .        eq list2set(N ; L) = N list2set(L) .
  eq N in null = false .      eq N in M S = (N == M) or N in S .
endfm
```

Figure 1: Example in Maude syntax [7]

The corresponding proof methods, though, cannot be applied to commonly occurring combinations of axioms that fall outside their scope.

**Example 1** *Consider the (order-sorted) TRS specified in Maude in Figure 1. It has four sorts:* `Bool`*,* `Nat`*,* `List`*, and* `Set`*, with* `Nat` *included in both* `List` *and* `Set` *as a subsort. That is, a natural number $n$ is simultaneously regarded as a list of length 1 and as a singleton set. The terms of each sort are, respectively, booleans, natural numbers (in Peano notation), lists of natural numbers, and finite sets of natural numbers. The rewrite rules in this module then define various functions such as* `_and_` *and* `_or_`*, a function* `list2set` *associating to each list its corresponding set, the set membership predicate* `_in_`*, and an equality predicate* `_==_` *on lists. Furthermore, the idempotency of set union is specified by the first equation. All these equations rewrite terms* modulo *the equational axioms declared in the module. Specifically,* `_and_` *and* `_or_` *have been declared associative and commutative with the* `assoc` *and* `comm` *keywords, the list concatenation operator* `_;_` *has been declared associative using the* `assoc` *keyword; the set union operator* `__` *has been declared associative, commutative using the* `assoc` *and* `comm`*, keywords; and the* `_==_` *equality predicate has been declared commutative using the* `comm` *keyword. The succinctness of this specification is precisely due to the power of rewriting modulo axioms, which typically uses considerably fewer rules that standard rewriting.*

Methods for proving termination of AC-theories could not be applied to prove termination of the TRS in Figure 1 (we would not care about sort information here), where we have an *arbitrary combination* of associative and/or commutative axioms which we call an *$A \vee C$-rewrite theory* in this paper. Furthermore, to the best of our knowledge, the *Dependency Pair Framework* (DP-framework [13]), which is the basis of state-of-the-art tools for proving termination of (different variants of) term rewriting has not yet been adapted to the AC case.

In this paper, we address these two problems. Giesl and Kapur generalized the previous works on AC-termination with dependency pairs to deal with more general kinds of equational theories $E$ satisfying some restrictions [10]. In principle, the $A \vee C$-theories that we investigate here fit the main outlines of Giesl and Kapur's approach. However, as we discuss below, the paper [10] did not provide any definition of *minimal chain*, which is needed for further developments in the DP-framework. In the DP-framework, the central notion regarding termination proofs is that of *DP problem*: the goal is checking the absence (or presence) of the so-called *infinite minimal chains*, where the notion of minimal chain can be thought of as an abstraction of the infinite rewrite sequences starting from *minimal non-terminating terms*. The most important notion regarding mechanization of the proofs is that of *processor*. A (correct) processor basically transforms DP problems into (hopefully) *simpler* ones, in such a way that the existence of an infinite chain in the original DP problem implies the existence of an infinite chain in the transformed one. Here 'simpler' usually means that fewer pairs are involved. Processors are used in a pipe (more precisely, a *tree*) to incrementally simplify the original DP problem as much as possible, possibly decomposing it into smaller pieces which are then independently treated in the very same way. This is the crucial new feature of the DP-framework w.r.t. the DP-approach of [1]. This makes it very powerful as a basis for implementing termination provers.

Before being able to extend the DP-framework to deal with $A \vee C$-theories, we start by giving a more refined notion of minimality. In fact, the notion of minimality which is used in [10] is the straightforward extension of the one which is used to prove termination of standard rewriting but without dealing with *E-equivalence preservation* which, as we show below, is essential to provide an appropriate notion of minimal $E$-nonterminating term for $A \vee C$-theories $E$ which can be used to define a suitable $A \vee C$-DP-framework. We carefully analyze the structure of infinite rewrite sequences for $A \vee C$-rewrite theories. This leads to appropriate definitions of $A \vee C$-dependency pair and of minimal chain.

## 1.1 Plan of the paper

The results, techniques, and tools that derive from our work can be of interest to a fairly wide audience. The material in this paper will be more familiar, however, to specialists interested in termination and in proving termination of rewriting modulo equational theories. Throughout the paper we made a serious effort to provide sufficient intuition and informal descriptions for our main definitions and results. After some technical preliminaries, in Sections 2 and 3, the paper is structured in three main parts:

1. In Section 4 we investigate the drawbacks of previous notions of minimal term when modeling infinite $A\vee C$-rewrite sequences. Then, we introduce the notion of *stably minimal* $E$-nonterminating term, which is the basis of our development. Section 5 investigates the structure of infinite sequences starting from such stably minimal terms. This analysis is essential in order to provide an appropriate definition of $A\vee C$-dependency pair and the related notions of chain, graph, etc.

2. Section 6 uses these results to formalize our notions of $A\vee C$-dependency pairs and of minimal chains and shows how to use them to *characterize* termination of $A\vee C$-rewrite theories.

3. We describe a suitable framework for dealing with proofs of $A\vee C$-termination by using these results. Section 7 extends the *dependency pair framework* [12, 13] to $A\vee C$-termination by defining appropriate notions of $A\vee C$ *problem* and $A\vee C$ *processor* that rely on the results obtained in the second part of the paper. In particular, we introduce the notion of $A\vee C$-dependency graph and the associated $A\vee C$ processor. We also show how to use orderings for defining a second processor and other auxiliary processors. Section 8 formalizes the use of usable rules and usable equations with orderings. Section 9 shows the performance of our techniques in practice, after implementing them in the termination tool MU-TERM. Section 10 compares our approach with related work and concludes the paper.

## 2 Preliminaries

This section collects a number of definitions and notations about term rewriting. More details and missing notions can be found in [4, 22, 27].

Let $\mathsf{A}$ be a set and $\mathsf{R} \subseteq \mathsf{A} \times \mathsf{A}$ be a binary relation on $\mathsf{A}$. We denote the transitive closure of $\mathsf{R}$ by $\mathsf{R}^+$ and its reflexive and transitive closure by $\mathsf{R}^*$. We say that $\mathsf{R}$ is *terminating* (*strongly normalizing*) if there is no infinite sequence $a_1 \;\mathsf{R}\; a_2 \;\mathsf{R}\; a_3 \cdots$. A reflexive and transitive relation $\mathsf{R}$ is a quasi-ordering.

Given relations $\mathsf{R}$ and $\mathsf{R}'$ over the same set $\mathsf{A}$, we define its *composition* $\mathsf{R}\circ\mathsf{R}'$ as follows: for all $a, b \in \mathsf{A}$, $a\,(\mathsf{R}\circ\mathsf{R}')\,b$ if there is $c \in \mathsf{A}$ such that $a\,\mathsf{R}\,c$ and $c\,\mathsf{R}'\,b$.

Throughout the paper, $\mathcal{X}$ denotes a countable set of variables and $\Sigma$ (equivalently $\mathcal{F}$ and $\Gamma$) denotes a signature, i.e., a set of function symbols $\{f, g, \ldots\}$, each having a fixed arity given by a mapping $ar : \Sigma \to \mathbb{N}$. The set of terms built from $\Sigma$ and $\mathcal{X}$ is $\mathcal{T}(\Sigma, \mathcal{X})$. $\mathcal{V}ar(t)$ is the set of variables occurring in a term $t$.

Terms are viewed as labelled trees in the usual way. Positions $p, q, \ldots$ are represented by chains of positive natural numbers used to address subterms of $t$. We denote the empty sequence by $\Lambda$. Given positions $p, q$, we denote their concatenation as $p.q$. Positions are ordered by the standard prefix ordering: $p \leq q$ if $\exists q'$ such that $q = p.q'$. If $p$ is a position, and $Q$ is a set of positions, then $p.Q = \{p.q \mid q \in Q\}$. The set of positions of a term $t$ is $\mathcal{P}os(t)$. Positions of nonvariable symbols in $t$ are denoted as $\mathcal{P}os_\Sigma(t)$, and $\mathcal{P}os_\mathcal{X}(t)$ are the positions of variables. The subterm at position $p$ of $t$ is denoted as $t|_p$, and $t[s]_p$ is the term $t$ with the subterm at position $p$ replaced by $s$.

We write $s \trianglerighteq t$, read $t$ *is a subterm of* $s$, if $t = s|_p$ for some $p \in \mathcal{P}os(s)$ and $s \vartriangleright t$ if $s \trianglerighteq t$ and $s \neq t$. We write $s \ntrianglerighteq t$ and $s \ntriangleright t$ for the negation of the corresponding properties. The symbol labeling the root of $t$ is denoted as $root(t)$. A *context* is a term $C \in \mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{X})$ with a 'hole' $\Box$ (a fresh constant symbol). We write $C[\;]_p$ to denote that there is a (usually single) hole $\Box$ at position $p$ of $C$. Generally, we write $C[\;]$ to denote an arbitrary context and make the position of the hole explicit only if necessary. $C[\;] = \Box$ is called the *empty* context.

A substitution is a mapping $\sigma : \mathcal{X} \to \mathcal{T}(\Sigma, \mathcal{X})$. Denote as $\varepsilon$ the 'identity' substitution: $\varepsilon(x) = x$ for all $x \in \mathcal{X}$. The set $\mathcal{D}om(\sigma) = \{x \in \mathcal{X} \mid \sigma(x) \neq x\}$ is called the *domain* of $\sigma$. A *renaming* is an injective substitution $\rho$ such that $\rho(x) \in \mathcal{X}$ for all $x \in \mathcal{X}$. A substitution $\sigma$ such that $\sigma(s) = \sigma(t)$ for two terms $s, t \in \mathcal{T}(\Sigma, \mathcal{X})$ is called a *unifier* of $s$ and $t$; we also say that $s$ and $t$ unify (with substitution $\sigma$). If two terms $s$ and $t$ unify, then there is a unique *most general unifier* $\sigma$ (up to renaming of variables) such that for every other unifier $\tau$, there is a substitution $\theta$ such that $\theta \circ \sigma = \tau$.

A binary relation $\mathsf{R} \subseteq \mathcal{T}(\Sigma, \mathcal{X}) \times \mathcal{T}(\Sigma, \mathcal{X})$ on terms is *stable* if, for all terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and substitutions $\sigma$, we have $\sigma(s) \mathrel{\mathsf{R}} \sigma(t)$ whenever $s \mathrel{\mathsf{R}} t$. We say that the relation is *monotonic* if, for all $f \in \Sigma$, and $s, t, t_1, \ldots, t_k \in \mathcal{T}(\Sigma, \mathcal{X})$, $f(t_1, \ldots, t_{i-1}, s, t_{i+1}, \ldots, t_k) \mathrel{\mathsf{R}} f(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_k)$ whenever $s \mathrel{\mathsf{R}} t$.

A rewrite rule is an ordered pair $(l, r)$, written $l \to r$, with $l, r \in \mathcal{T}(\Sigma, \mathcal{X})$, $l \notin \mathcal{X}$ and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. The left-hand side (*lhs*) of the rule is $l$, and the right-hand side (*rhs*) is $r$. A rewrite rule $l \to r$ is said to be *collapsing* if $r \in \mathcal{X}$. A *Term Rewriting System* (TRS) is a pair $\mathcal{R} = (\Sigma, R)$, where $R$ is a set of rewrite rules. An instance $\sigma(l)$ of a *lhs* $l$ of a rule is called a *redex*. Given $\mathcal{R} = (\Sigma, R)$, we consider $\Sigma$ as the disjoint union $\Sigma = \mathcal{C} \uplus \mathcal{D}$ of symbols $c \in \mathcal{C}$ (called *constructors*) and symbols $f \in \mathcal{D}$ (called *defined functions*), where $\mathcal{D} = \{root(l) \mid l \to r \in R\}$ and $\mathcal{C} = \mathcal{F} - \mathcal{D}$.

A term $s \in \mathcal{T}(\Sigma, \mathcal{X})$ rewrites to $t$ (at position $p$), written $s \xrightarrow{p}_R t$ (or just $s \to_R t$, or $s \to t$), if $s|_p = \sigma(l)$ and $t = s[\sigma(r)]_p$, for some rule $l \to r \in R$, $p \in \mathcal{P}os(s)$ and substitution $\sigma$. We write $s \xrightarrow{>p}_{\mathcal{R}} t$ if $s \xrightarrow{q}_{\mathcal{R}} t$ for some $q > p$. A TRS $\mathcal{R}$ is terminating if its one step rewrite relation $\to_R$ is terminating.

## 3 Rewriting Modulo Equational Theories

Given a set of equations $E$, we write $s \overset{p}{\vdash\!\dashv}_E t$ (a single 'equational step') if there is a position $p \in \mathcal{P}os(s)$, an equation $u = v \in E$ and a substitution $\sigma$ such that $s|_p = \sigma(u)$ and $t|_p = \sigma(v)$, or $s|_p = \sigma(v)$ and $t|_p = \sigma(u)$ (we write $s \vdash\!\dashv_E t$ if position $p$ is not relevant). Note that $\vdash\!\dashv_E$ is a symmetric relation. Then, $\sim_E$ is the reflexive and transitive closure of $\vdash\!\dashv_E$; we have the following equivalence that will be useful in our development:

$$\sim_E = \vdash\!\dashv_E^* = (\overset{\Lambda}{\vdash\!\dashv}_E \cup \overset{>\Lambda}{\vdash\!\dashv}_E)^*.$$

We also write $s \overset{>\Lambda}{\sim}_E t$ iff $s = f(s_1, \ldots, s_k)$, $t = f(t_1, \ldots, t_k)$ and $s_i \sim_E t_i$ for all $i$, $1 \leq i \leq k$. We define $s \overset{\Lambda}{\sim}_E t$ as the reflexive and transitive closure of $\overset{\Lambda}{\vdash\!\dashv}_E$.

Given a rewrite theory $\mathcal{R} = (\Sigma, E, R)$, where $R$ is a set of rewrite rules and $E$ is a set of equational axioms, we write $s \to_{R/E} t$ if there exist $u, v$ such that $s \sim_E u$, $u \to_R v$, and $v \sim_E t$.We say that a rewrite theory $\mathcal{R} = (\Sigma, E, R)$ is *$E$-terminating*, iff $\to_{R/E}$ is terminating. In general, given terms $s$ and $t$, the problem of checking whether $s \to_{R/E} t$ holds is undecidable: in order to check whether $s \to_{R/E} t$ we have to search through the possibly infinite equivalence classes $[s]_E$ and $[t]_E$ to see whether a matching is found for a subterm of some $u \in [s]_E$ and the result of rewriting $u$ belongs to the equivalence class $[t]_E$. For this reason, a much simpler relation $\to_{R,E}$ is defined, which becomes decidable if an $E$-matching algorithm exists. For any terms $s, t$, $s \to_{R,E} t$ holds iff there is a position $p$ in $s$, a rule $l \to r$ in $R$, and a substitution $\sigma$ such that $s|_p \sim_E \sigma(l)$ and $t = s[\sigma(r)]_p$ (see [23]). This relation only allows applying rules from $R$ in redexes at positions equal or above of positions of terms where equations from $E$ have been applied. We say that a rewrite theory $\mathcal{R} = (\Sigma, E, R)$ is *$(R, E)$-terminating*, if $\to_{R,E}$ is terminating. In the following, we assume that $E$ and $R$ are finite sets of equations and rules, respectively.

Regarding $E$-termination analysis using *dependency pairs* (DPs), Kusakari and Toyama observed that there is no simple extension of DPs to directly deal with $\to_{R/E}$-computations [18, 16]. In contrast, several approaches have been developed for $\to_{R,E}$-computations [10, 18, 20]. Since $\to_{R,E} \subseteq \to_{R/E}$ (but the opposite inclusion does not hold, in general), $E$-termination cannot be concluded from $(R, E)$-termination. Actually, Marché and Urbain showed that there are $(R, E)$-terminating rewrite theories $\mathcal{R}$ which are *not* $E$-terminating.

**Example 2** *Consider the following rewrite theory $\mathcal{R} = (\Sigma, E, R)$, where '$+$' is an AC symbol [20]:*

$$a + b \to a + (b + c).$$

*Note that $t = a + (b + c)$ is an $\to_{R,E}$-normal form (hence $(R, E)$-terminating). However, $t \sim_{AC} (a + b) + c$ which is $E$-nonterminating.*

Giesl and Kapur [10] proved the equivalence of both notions of termination with respect to a notion of *extension completion* $\mathcal{E}xt_E(R)$ (see below) of a rewrite theory $\mathcal{R} = (\Sigma, E, R)$ for $E$ *regular* (i.e., $\mathcal{V}ar(u) = \mathcal{V}ar(v)$ for all $u = v$ in $E$), and *linear* (neither $u$ nor $v$ have repeated variables). For $E$ being a set containing associative or commutative axioms, this notion of extension goes back to Peterson and Stickel [23].

**Theorem 1** [10, Theorem 11] *Let $\mathcal{R} = (\Sigma, E, R)$ be a rewrite theory with $E$ a regular and linear equational theory and $t \in \mathcal{T}(\Sigma, \mathcal{X})$. Then, $t$ starts an infinite $\to_{R/E}$-reduction if and only if $t$ starts an infinite $\to_{\mathcal{E}xt_E(R),E}$-reduction. Therefore, $\mathcal{R}$ is $E$-terminating if and only if $\to_{\mathcal{E}xt_E(R),E}$ is terminating.*

## 3.1 Combination of Associative and Commutative Theories

Let $E$ be a set of equations that has the modular decomposition $E = \bigcup_{f \in \Sigma} E_f$, where if $k = ar(f) \neq 2$, then $E_f = \varnothing$, and if $k = 2$, then $E_f \subseteq \{A_f, C_f\}$, where:

- $A_f$ is the associativity axiom $f(f(x, y), z) = f(x, f(y, z))$,
- $C_f$ is the commutativity axiom $f(x, y) = f(y, x)$.

We also define $\Sigma = \Sigma_A \uplus \Sigma_C \uplus \Sigma_{AC} \uplus \Sigma_\varnothing$ where $f \in \Sigma_A \Leftrightarrow E_f = \{A_f\}$, $f \in \Sigma_C \Leftrightarrow E_f = \{C_f\}$, $f \in \Sigma_{AC} \Leftrightarrow E_f = \{A_f, C_f\}$, $f \in \Sigma_\varnothing \Leftrightarrow E_f = \varnothing$. In the following, we often say that a symbol $f \in \Sigma$ *is associative* iff $f \in \Sigma_A \cup \Sigma_{AC}$.

**Definition 1 ($A\vee C$-rewrite theory)** *An equational theory $E = \bigcup_{f \in \Sigma} E_f$, where if $k = ar(f) \neq 2$, then $E_f = \varnothing$, and if $k = 2$, then $E_f \subseteq \{A_f, C_f\}$ is called an $A\vee C$-*theory*. A rewrite theory $\mathcal{R} = (\Sigma, E, R)$ such that $E$ is an $A\vee C$-theory, is called an $A\vee C$-*rewrite theory*.

To deal with rewriting modulo $A\vee C$-theories by using $(R, E)$-rewriting we have to extend $R$ by following [23, Definition 10.4]:

$$\begin{array}{rcl} \mathcal{E}xt_{AC}(R) & = & R \cup \{f(l, w) \to f(r, w) \mid l \to r \in R, f = root(l) \in \Sigma_{AC}\} \\ \mathcal{E}xt_A(R) & = & R \cup \{f(l, w) \to f(r, w), f(w, l) \to f(w, r), f(z, f(l, w)) \to f(z, f(r, w)) \\ & & \mid l \to r \in R, f = root(l) \in \Sigma_A\} \\ \mathcal{E}xt_C(R) & = & R \end{array}$$

where $w$ and $z$ are fresh variables which do not occur in the original rule of $R$. Therefore, given an $A\vee C$ theory $E$, we let:

$$\mathcal{E}xt_E(R) = \mathcal{E}xt_{AC}(R) \cup \mathcal{E}xt_A(R) \cup \mathcal{E}xt_C(R).$$

Note that $R \subseteq \mathcal{E}xt_E(R)$.

**Example 3** *Consider the following TRS $\mathcal{R}$:*

$$f(x, x) \quad \to \quad f(0, 0)$$

*where $f \in \Sigma_{AC}$.*

*Hence, $\mathcal{E}xt_{AC}(R)$ only adds the following rule to $\mathcal{R}$:*

$$f(f(x, x), y) \quad \to \quad f(f(0, 0), y)$$

## 3.2 Minimal Terms and Infinite Rewrite Sequences

Given a TRS $\mathcal{R} = (\mathcal{C} \uplus \mathcal{D}, R)$, with $\mathcal{C}$ a subsignature of constructors and $\mathcal{D}$ a subsignature of defined symbols, so that each rule in $R$ is of the form $f(t_1, \ldots, t_n) \to r$ with $f \in \mathcal{D}$, the *minimal* nonterminating terms associated to $\mathcal{R}$ are those nonterminating terms $t$ whose proper subterms $u$ (i.e., $t \rhd u$) are terminating. Let $\mathcal{T}_\infty$ denote the set of minimal nonterminating terms associated to $\mathcal{R}$ [14]. Minimal nonterminating terms have two important properties:

1. Every nonterminating term $s$ contains a minimal nonterminating subterm $t \in \mathcal{T}_\infty$ (i.e., $s \trianglerighteq t$), and

2. minimal nonterminating terms $t$ are always rooted by a *defined* symbol $f \in \mathcal{D}$: $\forall t \in \mathcal{T}_\infty, root(t) \in \mathcal{D}$.

Considering the structure of the infinite rewrite sequences starting from a minimal nonterminating term $t = f(t_1, \ldots, t_k) \in \mathcal{T}_\infty$ is helpful to arrive at the notion of dependency pair. Such sequences proceed as follows (see, e.g., [14]):

1. a finite number of reductions can be performed *below* the root of $t$, thus rewriting $t$ into $t'$; then

2. a rule $f(l_1, \ldots, l_k) \to r$ applies *at the root* of $t'$ (i.e., $t' = \sigma(f(l_1, \ldots, l_k))$ for some substitution $\sigma$); and

3. there is a minimal nonterminating term $u \in \mathcal{T}_\infty$ (hence $root(u) \in \mathcal{D}$) at some position $p$ of $\sigma(r)$ which is a *nonvariable position of* $r$ which 'continues' the infinite sequence initiated by $t$ in a similar way.

This means that considering the occurrences of defined symbols in the right-hand sides of the rewrite rules suffices to 'catch' *every possible infinite rewrite sequence starting from* $\sigma(r)$. In particular, no infinite sequence can be issued from $t'$ *below* the variables of $r$ (more precisely: all bindings $\sigma(x)$ are *terminating* terms). The standard definition of dependency pair [1] and (minimal) chain of dependency pairs [13] rely on (1)–(3) above [14]. These facts are formalized as follows:

**Proposition 1** [14, Lemma 1] *Let $\mathcal{R} = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS. For all $t \in \mathcal{T}_\infty$, there exist $l \to r \in R$, a substitution $\sigma$ and a term $u \in \mathcal{T}_\infty$ such that $root(u) \in \mathcal{D}$, $t \xrightarrow{>\Lambda}{}^{*} \sigma(l) \xrightarrow{\Lambda} \sigma(r) \trianglerighteq u$ and there is a nonvariable subterm $v$ of $r$, $r \trianglerighteq v$, such that $u = \sigma(v)$.*

The following auxiliary results will be used later.

**Proposition 2** *Let $\mathcal{R} = (\Sigma, E, R)$ be a rewrite theory and $t, s \in \mathcal{T}(\Sigma, \mathcal{X})$. If $t$ is $(R, E)$-terminating, then*

1. *If $t \trianglerighteq s$, then $s$ is $(R, E)$-terminating.*

2. *If $t \to^*_{R,E} s$ then $s$ is $(R, E)$-terminating.*

PROOF. Trivial. □

**Proposition 3 ($E$-Termination Preserved under $E$-Equivalence)** *Let $\mathcal{R} = (\Sigma, E, R)$ be a rewrite theory and $t, s \in \mathcal{T}(\Sigma, \mathcal{X})$. If $t \sim_E s$, then $t$ is $E$-terminating if and only if $s$ is $E$-terminating.*

PROOF. Trivial. $\square$

Proposition 3 does *not* hold if we change $E$-termination by $(R, E)$-termination (see Example 2). However, as a consequence of Theorem 1 and Proposition 3, we have:

**Corollary 1** *Let $\mathcal{R} = (\Sigma, E, R)$ be a rewrite theory such that $E$ is a set of regular and linear equations and $t, s \in \mathcal{T}(\Sigma, \mathcal{X})$. If $t \sim_E s$, then $t$ is $(\mathcal{E}xt_E(R), E)$-terminating if and only if $s$ is $(\mathcal{E}xt_E(R), E)$-terminating.*

As a corollary of Theorem 1, we have the following.

**Corollary 2** *Let $\mathcal{R} = (\Sigma, E, R)$ be an $A{\vee}C$-rewrite theory and $t \in \mathcal{T}(\Sigma, \mathcal{X})$. Then, $t$ is $E$-terminating if and only if it is $(\mathcal{E}xt_E(R), E)$-terminating.*

In the following section we begin the analysis of infinite $E$-rewrite sequences according to the schema in [14]. We aim at providing an appropriate notion of minimal $E$-nonterminating term (for $A{\vee}C$-theories $E$) which allows us to reach a result similar to Proposition 1.

# 4 Stably Minimal $E$-nonterminating Terms

In the dependency pair approach [1, 14, 13], the analysis of infinite rewrite sequences is restricted to those starting from *minimal nonterminating terms* $t \in \mathcal{T}_\infty$. The following notion of minimal $E$-nonterminating term is implicit in [10, proof of Theorem 16]. Similar definitions can be found in [17, 18, 16, 21].

**Definition 2 (Minimal $E$-nonterminating Term** [10]**)** *Let $\mathcal{R} = (\Sigma, E, R)$ be a rewrite theory. An $E$-nonterminating term $t \in \mathcal{T}(\Sigma, \mathcal{X})$ is said to be* minimal *(written $t \in \mathcal{T}_{\infty,R,E}$) if every strict subterm $s$ of $t$ (i.e., $t \rhd s$) is $(\mathcal{E}xt_E(R), E)$-terminating.*

**Remark 1** *In Definition 2, if we assume that $E$ is linear and regular (like $A{\vee}C$-theories), then, by Theorem 1, we could equivalently start by saying that $t$ is $(\mathcal{E}xt_E(R), E)$-nonterminating. This leads to a more symmetric definition, which we often use in the following without further comment.*

Every $E$-nonterminating term $s$ contains a minimal $E$-nonterminating subterm $t \in \mathcal{T}_{\infty,R,E}$ (this is stated without proof in [10, proof of Theorem 16]).

**Proposition 4** *Let $\mathcal{R} = (\Sigma, E, R)$ be a rewrite theory and $s \in \mathcal{T}(\Sigma, \mathcal{X})$. If $s$ is $E$-nonterminating, then there is a subterm $t$ of $s$ ($s \unrhd t$) such that $t \in \mathcal{T}_{\infty,R,E}$.*

PROOF. By structural induction. If $s$ is a constant symbol, it is obvious: take $t = s$. If $s = f(s_1, \ldots, s_k)$, then we proceed by contradiction. If there is no (not necessarily strict) subterm $t$ of $s$ such that $t$ is minimal, then in particular $s$ is not minimal. Therefore, there is a strict subterm $t$ of $s$ ($s \rhd t$) which is

$E$-nonterminating. By the Induction Hypothesis, there is $t'$ which is minimal and such that $t \trianglerighteq t'$. Then, we have $s \triangleright t'$, thus leading to a contradiction. $\Box$

Note that Giesl and Kapur's minimality of terms is preserved under $\to_{\mathcal{E}xt_E(R),E}$-reductions below the root.

**Proposition 5** *Let $\mathcal{R} = (\Sigma, E, R)$ be a rewrite theory and $s \in \mathcal{T}_{\infty,R,E}$. If $s \xrightarrow{>\Lambda}{}^{*}_{\mathcal{E}xt_E(R),E} t$ and $t$ is $E$-nonterminating, then $t \in \mathcal{T}_{\infty,R,E}$.*

Proof. Since $s$ is rewritten below the root, we can write $s = f(s_1, \ldots, s_k)$, where, by minimality of $s$, we know that $s_1, \ldots, s_k$ are all $(\mathcal{E}xt_E(R), E)$-terminating. Furthermore, since $\xrightarrow{>\Lambda}_{\mathcal{E}xt_E(R),E}$ performs no rewriting or $\sim_E$-steps at the root, we have that $t = f(t_1, \ldots, t_k)$ with $s_i \to^*_{\mathcal{E}xt_E(R),E} t_i$ for all $i$, $1 \le i \le k$. By Proposition 2, $t_i$ is $(\mathcal{E}xt_E(R), E)$-terminating for all $i$, $1 \le i \le k$. And since $t$ is assumed to be $E$-nonterminating, $t \in \mathcal{T}_{\infty,R,E}$. $\Box$

**Remark 2 (Root Symbols of Minimal Terms)** *Note that if $E$ is an $A\vee C$-equational theory, then $root(t) \in \mathcal{D}$ whenever $t \in \mathcal{T}_{\infty,R,E}$. As remarked by Giesl and Kapur (see also Example 8 below) this is not true for arbitrary equational theories.*

The problem with Giesl and Kapur's Definition 2 is that minimality is *not* preserved under $E$-equivalence.

**Example 4** *Consider again the TRS $\mathcal{R}$ in Example 3.*
*Following [10], the term $f(f(1,0),0) \in \mathcal{T}_{\infty,R,E}$ since is $AC$-nonterminating*

$$f(f(1,0),0) \sim_{AC} f(1, f(0,0)) \sim_{AC} f(f(0,0),1) \xrightarrow{\Lambda}_R f(f(0,0),1) \cdots$$

*but its strict subterms $f(1,0)$, $1$ and $0$ are $AC$-terminating. However, the root step with $\sigma(l) = \sigma(f(f(x,x),y) = f(f(0,0),1)$ shows that $\sigma(l) \notin \mathcal{T}_{\infty,R,E}$ since $f(0,0)$ is $AC$-nonterminating.*

**Example 5** *Consider the following TRS $\mathcal{R}$:*

$$f(x,x) \quad \to \quad f(0, f(1,2)) \tag{1}$$

*where $f \in \Sigma_{AC}$. Hence, $\mathcal{E}xt_{AC}(R)$ only adds the following rule to $\mathcal{R}$:*

$$f(f(x,x),y) \quad \to \quad f(f(0, f(1,2)), y) \tag{2}$$

*Note that $t = f(f(0,1), f(0, f(1,2)))$ is $(\mathcal{E}xt_{AC}(R), AC)$-nonterminating:*

$$\begin{array}{lcl}
\underline{f(f(0,1), f(0, f(1,2)))} & \sim_A & f(0, \underline{f(1, f(0, f(1,2)))}) \\
& \sim_A & f(0, \underline{f(\overline{f(1,0)}, f(1,2))}) \\
& \sim_C & f(0, \underline{f(\overline{f(0,1)}, f(1,2))}) \\
& \sim_A & \underline{f(0, f(0, f(1, f(1,2))))} \\
& \sim_A & \underline{f(f(0,0), f(1, f(1,2)))} \\
& \xrightarrow{\Lambda}_{\mathcal{E}xt_{AC}(R)} & \underline{f(f(0, f(1,2)), f(1, f(1,2)))} \\
& \to_{\mathcal{E}xt_{AC}(R),AC} & \cdots
\end{array}$$

*Since $f(0,1)$ and $f(0,f(1,2))$ are in $(\mathcal{E}xt_{AC}(R),AC)$-normal form, we have that $t \in \mathcal{T}_{\infty,R,AC}$. However, $t' = f(f(0,0),f(1,f(1,2)))$, which is $AC$-equivalent to $t$ (i.e., $t \sim_{AC} t'$), is $AC$-nonterminating, but it is* not *minimal because its strict subterm $f(1,f(1,2)))$ is $(\mathcal{E}xt_{AC}(R),AC)$-nonterminating:*

$$
\begin{array}{ccl}
\underline{f(1,f(1,2))} & \sim_A & \underline{f(f(1,1),2)} \\
 & \stackrel{\Lambda}{\to}_{\mathcal{E}xt_{AC}(R)} & \underline{f(f(0,f(1,2)),2)} \\
 & \sim_A & \underline{f(0,\overline{f(f(1,2),2)})} \\
 & \sim_A & \underline{f(0,\overline{f(1,f(2,2))})} \\
 & \sim_A & \underline{f(f(0,1),f(2,2))} \\
 & \sim_C & \underline{f(f(2,2),f(0,1))} \\
 & \stackrel{\Lambda}{\to}_{\mathcal{E}xt_{AC}(R)} & f(f(0,f(1,2)),f(0,1)) \\
 & \sim_A & \underline{f(\overline{f(f(0,1),2)}),f(0,1))} \\
 & \sim_C & \underline{f(f(0,1),\overline{f(f(0,1),2)}))} \\
 & \sim_A & \underline{f(f(0,1),\overline{f(0,f(1,2))}))} \\
 & \to_{\mathcal{E}xt_{AC}(R),AC} & \cdots
\end{array}
$$

Example 5 shows that an essential property of minimal terms when considered as part of infinite $(\mathcal{E}xt_E(R),E)$-rewriting sequences for $A{\vee}C$-theories $E$ gets lost: the application of $(\mathcal{E}xt_E(R),E)$-rewrite steps *at the root* of a minimal term $s$ by means of a rule $l \to r$ (i.e., $s \sim_{AC} \sigma(l) \stackrel{\Lambda}{\to}_{\mathcal{E}xt_E(R)} \sigma(r)$) does *not* guarantee that there is a *nonvariable subterm* $v$ of the right-hand side $r$ which is a prefix of the 'next' minimal term in the infinite sequence. In the following proposition, we prove that the problem illustrated in Example 5 is due to the application of associative steps at the root of a minimal term.

**Proposition 6** *Let $\mathcal{R} = (\Sigma, E, R)$ be a rewrite theory and $t \in \mathcal{T}_{\infty,R,E}$.*

1. *If $E$ is regular and linear and $t' \stackrel{>\Lambda}{\sim}_E t$, then $t' \in \mathcal{T}_{\infty,R,E}$.*

2. *If $C_{root(t)} \in E_{root(t)}$ and $t' \stackrel{\Lambda}{\sim}_C t$, then $t' \in \mathcal{T}_{\infty,R,E}$.*

Proof. Let $t = f(t_1,\ldots,t_k)$. By minimality of $t$, $t_i$ is $(\mathcal{E}xt_E(R),E)$-terminating for all $1 \leq i \leq k$.

1. If $t' \stackrel{>\Lambda}{\sim}_E t$, then $t' = f(t'_1,\ldots,t'_k)$ and $t'_i \sim_E t_i$ for all $1 \leq i \leq k$. By Proposition 3, $t'$ is $(\mathcal{E}xt_E(R),E)$-nonterminating and by Corollary 1, $t'_i$ is $(\mathcal{E}xt_E(R),E)$-terminating for all $1 \leq i \leq k$. Hence $t' \in \mathcal{T}_{\infty,R,E}$.

2. If $C_{root(t)} \in E_{root(t)}$, then $k = 2$ and $t = f(t_1,t_2)$, where both $t_1$ and $t_2$ are $(\mathcal{E}xt_E(R),E)$-terminating. Since $t' = f(t_2,t_1) \stackrel{\Lambda}{\sim}_C t$, by Proposition 3 $t'$ is $(\mathcal{E}xt_E(R),E)$-nonterminating and we have $t' \in \mathcal{T}_{\infty,R,E}$.

□

**Example 6** *Term $t$ in Example 5 can be rewritten at the root* only *by rule* (2) *of $\mathcal{E}xt_{AC}(R)$. We can apply this rule to $t'$ in Example 5 (for instance) to obtain $s' = \sigma(r) = f(f(0, f(1,2)), f(1, f(1,2)))$ (where $r = f(f(0, f(1,2)), y)$), which is $(\mathcal{E}xt_{AC}(R), AC)$-nonterminating. Note that $s'$ contains a minimal term $u \in \mathcal{T}_{\infty,R,E}$. Since $s'|_2 = f(1, f(1,2))$ is $(\mathcal{E}xt_{AC}(R), AC)$-nonterminating, it follows that $s'$ is* not *minimal. Since $s'|_1 = f(0, f(1,2))$ is $(\mathcal{E}xt_{AC}(R), AC)$-terminating, the only possibility is that $u$ occurs in $s'|_2$. Actually, $s'|_2$ is minimal already; hence, $u = s'|_2$. But note the absence of any nonvariable position $p \in \mathcal{P}os(r)$ in the right-hand side of the considered rule such that $\sigma(r|_p) = u = f(1, f(1,2))$.*

*This is in sharp contrast with the situation of the DP-approach for ordinary rewriting. Furthermore, it is not difficult to see that for all $t'' \sim_{AC} t$ such $t'' = \sigma'(l)$ for some substitution $\sigma'$, we have a similar situation. Thus, the problem illustrated here cannot be solved by using a different $\sim_{AC}$ sequence before performing the $\mathcal{E}xt_{AC}(R)$-root-step.*

In the following we introduce a new notion of minimality which solves these problems.

## 4.1 A New Notion of Minimal $E$-nonterminating Terms

The following definition solves the problems discussed above by explicitly requiring that the condition defining minimality is preserved under $E$-equivalence.

**Definition 3 (Stably Minimal $E$-nonterminating Term)** *Let $\mathcal{R} = (\Sigma, E, R)$ be a rewrite theory. Let $\mathcal{M}_{\infty,R,E}$ be a set of stably minimal $E$-nonterminating terms in the following sense: $t \in \mathcal{T}(\Sigma, \mathcal{X})$ belongs to $\mathcal{M}_{\infty,R,E}$ iff $t$ is $E$-nonterminating, and for all $t' \sim_E t$ and every proper subterm $s'$ of $t'$ (i.e., $t' \rhd s'$), $s'$ is $(\mathcal{E}xt_E(R), E)$-terminating.*

We have the following useful characterization of minimality.

**Proposition 7 (Characterization of Stably Minimal Terms)** *Let $\mathcal{R} = (\Sigma, R, E)$ be a rewrite theory and $t \in \mathcal{T}(\Sigma, \mathcal{X})$. Then, $t \in \mathcal{M}_{\infty,R,E}$ if and only if $[t]_E \subseteq \mathcal{T}_{\infty,R,E}$. Therefore,*

$$\mathcal{M}_{\infty,R,E} = \{t \in \mathcal{T}(\Sigma, \mathcal{X}) \mid [t]_E \subseteq \mathcal{T}_{\infty,R,E}\}$$

The problem in Example 5 disappears now: $t$ is *not* (stably) minimal according to Definition 3. The same situation happens with the problem in Example 4: $f(f(1,0),0) \in \mathcal{T}_{\infty,R,E}$ but $f(f(1,0),0) \notin \mathcal{M}_{\infty,R,E}$ since $f(f(1,0),0) \sim_E f(f(0,0),1)$ and $f(0,0)$ is $E$-nonterminating. In fact, $f(0,0) \in \mathcal{M}_{\infty,R,E}$.

The following result shows how to *find* stably minimal $E$-nonterminating terms associated to a given $E$-nonterminating term. This is essential in our development. A set of equations $E$ is *size-preserving* if and only if for each equation $u = v$ the length of $u$ and $v$ are the same, i.e. $|u| = |v|$ and the multiset of the variables in $u$ coincides with the multiset of the variables in $v$ [22].

**Proposition 8** *Let $\mathcal{R} = (\Sigma, E, R)$ be a rewrite theory such that $E$ is regular and size-preserving. Let $s \in \mathcal{T}(\Sigma, \mathcal{X})$. If $s$ is $E$-nonterminating, then there is a subterm $t$ of some $s' \sim_E s$ (i.e., $s' \trianglerighteq t$) such that $t \in \mathcal{M}_{\infty,R,E}$.*

PROOF. By structural induction. If $s$ is a constant symbol or a variable, then since $s$ has no strict subterms, then $s \in \mathcal{M}_{\infty,R,E}$, so in this case, we can choose $t = s$. If $s = f(s_1, \ldots, s_k)$, then we proceed by contradiction. If there is no subterm $t$ of some $s' \sim_E s$ ($s' \trianglerighteq t$) such that $t \in \mathcal{M}_{\infty,R,E}$, then in particular $s \notin \mathcal{M}_{\infty,R,E}$, (and thus $s' \notin \mathcal{M}_{\infty,R,E}$ for all $s' \sim_E s$) i.e., (since $s$ is $E$-nonterminating) there is an E-equivalent term $s' \sim s$ containing a strict $(\mathcal{E}xt_E(R), E)$-nonterminating subterm $t'$ ($s' \triangleright t'$). Therefore, $t'$ is $E$-nonterminating as well. By the Induction Hypothesis, there is $t \in \mathcal{M}_{\infty,R,E}$ such that $t' \trianglerighteq t$. Then, $s' \triangleright t$, thus leading to a contradiction. □

Clearly, Proposition 8 holds whenever $\mathcal{R}$ is an $A\vee C$-rewrite theory.

**Example 7** *Consider the term $t$ in Example 5. Although $t \in \mathcal{T}_{\infty,R,E}$, $t \notin \mathcal{M}_{\infty,R,E}$: the term $t' = f(f(0,0), f(1, f(1,2)))$, which is AC-equivalent to $t$, contains a subterm $u = f(1, f(1,2))$ which is $E$-nonterminating. It is not difficult to see that actually $u \in \mathcal{M}_{\infty,R,E}$.*

In general, Proposition 8 does *not* hold for arbitrary sets of equations $E$.

**Example 8** *Consider the following example [10, Example 13]:*

$$R : f(x) \;\to\; x \qquad\qquad E : f(a) \;=\; a$$

*Note that $a \in \mathcal{T}_{\infty,R,E}$. However, $a$ is* not *stably minimal because $a \sim_E f(a)$ but $f(a) \notin \mathcal{T}_{\infty,R,E}$. Thus, Proposition 8 does not hold.*

Since $\mathcal{M}_{\infty,R,E} \subseteq \mathcal{T}_{\infty,R,E}$, for $A\vee C$-rewrite theories $E$ we have the following corollary of Proposition 5.

**Corollary 3** *Let $\mathcal{R} = (\Sigma, E, R)$ be an $A\vee C$-rewrite theory and $s \in \mathcal{M}_{\infty,R,E}$. If $s \xrightarrow{>\Lambda}{}^{*}_{\mathcal{E}xt_E(R),E} t$ and $t$ is $E$-nonterminating, then $t \in \mathcal{T}_{\infty,R,E}$.*

In general, Corollary 3 does *not* hold if we require that $t \in \mathcal{M}_{\infty,R,E}$.

**Example 9** *Term $u = f(f(1,1),2)$ in Example 6 is stably minimal: $u \in \mathcal{M}_{\infty,R,E}$. We have that $f(\underline{f(1,1)},2) \xrightarrow{>\Lambda}_R f(f(0,f(1,2)),2)$. Note that $f(f(0,f(1,2)),2) \notin \mathcal{M}_{\infty,R,E}$. We have*

$$\underline{f(f(0,f(1,2)),2)} \sim_A f(0, \underline{f(f(1,2),2)}) \sim_A f(0, f(1, f(2,2)))$$

*where $f(0, f(1, f(2,2)))$ contains a subterm $f(1, f(2,2))$ which is $(\mathcal{E}xt_E(R), E)$-nonterminating.*

The following results show that the problem arises when $s \in \mathcal{M}_{\infty,R,E}$ is such that $root(s)$ includes *associativity* among its axioms, that is, $A_f \in E_f$. In the following, we focus on $A\vee C$-rewrite theories. Hence, we will often implicitly use Corollary 2 to speak about $E$-termination rather than $(\mathcal{E}xt_E(R), E)$-termination (see Remark 1).

**Proposition 9** *Let $\mathcal{R} = (\Sigma, E, R)$ be an A$\vee$C-rewrite theory. If $t \in \mathcal{T}_{\infty,R,E}$ is such that* (1) $A_{root(t)} \notin E_{root(t)}$ *or* (2) $t = f(t_1, t_2)$, $A_f \in E_f$, $root(t_1) \neq f$, *and* $root(t_2) \neq f$, *then* $t \in \mathcal{M}_{\infty,R,E}$.

PROOF. Let $t = f(t_1, \ldots, t_k)$, where, since $t \in \mathcal{T}_{\infty,R,E}$, $t_i$ is $E$-terminating for all $i$, $1 \leq i \leq k$. We consider two main cases according to $f$:

1. If $f$ does not have the associativity axiom, i.e., $A_f \notin E_f$, then we consider two cases:
   (a) $f$ is commutative, i.e., $E_f = \{C_f\}$. Then, $k = 2$ and we can write $t = f(t_1, t_2)$, where $t_1$ and $t_2$ are $E$-terminating. For all $u \sim_E t$ given by $u = f(u_1, u_2)$ we have two possibilities: either $u_1 \sim_E t_1$ and $u_2 \sim_E t_2$, or $u_1 \sim_E t_2$ and $u_2 \sim_E t_1$. In both cases, since $E$-equivalence preserves $E$-termination (Proposition 3), we conclude that $u_1$ and $u_2$ are $E$-terminating and hence $t \in \mathcal{M}_{\infty,R,E}$.
   (b) $f$ is not commutative, i.e., $E_f = \varnothing$. Then, for all $u \sim_E t$ we have $u \overset{>\Lambda}{\sim}_E t$ and the result follows from Proposition 6-(1).
2. If $t = f(t_1, t_2)$, $A_f \in E_f$, $root(t_1) \neq f$, and $root(t_2) \neq f$, then no associativity axiom can be applied at the root of $t$. Then, we can treat $t$ as in one of the cases 1a or 1b above.

$\square$

**Proposition 10** *Let $\mathcal{R} = (\Sigma, E, R) = (\mathcal{C} \uplus \mathcal{D}, E, R)$ be an A$\vee$C-rewrite theory and $s \in \mathcal{M}_{\infty,R,E}$ be such that* (1) $A_{root(s)} \notin E_{root(s)}$ *or* (2) $s = f(s_1, s_2)$, $A_f \in E_f$, *and* $root(s_1), root(s_2) \in \mathcal{C}$. *If $s \xrightarrow{>\Lambda}{}^{*}_{\mathcal{E}xt_E(R),E} t$ and $t$ is $E$-nonterminating, then $t \in \mathcal{M}_{\infty,R,E}$.*

PROOF. Since $s \in \mathcal{M}_{\infty,R,E}$, we have that, for all $s' \sim_E s$, all proper subterms $u$ of $s'$ are $E$-terminating. We can write $s = f(s_1, \ldots, s_k)$, where, by stable minimality of $s$, all the $s_1, \ldots, s_k$ are $E$-terminating. Furthermore, since $\xrightarrow{>\Lambda}_{\mathcal{E}xt_E(R),E}$ performs no rewriting or $E$-steps at the root, we have that $t = f(t_1, \ldots, t_k)$ with $s_i \to^{*}_{\mathcal{E}xt_E(R),E} t_i$ for all $i$, $1 \leq i \leq k$. By Proposition 2, $t_i$ is $E$-terminating for all $i$, $1 \leq i \leq k$. Therefore, $t \in \mathcal{T}_{\infty,R,E}$. If $root(s) = root(t)$ is such that $A_f \notin E_f$, by Proposition 9, $t \in \mathcal{M}_{\infty,R,E}$. On the other hand, if $s = f(s_1, s_2)$, $A_f \in E_f$, and $root(s_1), root(s_2) \in \mathcal{C}$, then reductions on $s_1$ and $s_2$ do not change $root(s_1)$ nor $root(s_2)$. Thus, $t = f(t_1, t_2)$ and $root(t_1) = root(s_1)$ and $root(t_2) = root(s_2)$. Since $f \in \mathcal{D}$ (due to minimality of $s$), by Proposition 9, $t \in \mathcal{M}_{\infty,R,E}$. $\square$

Now we provide a more precise result about where we can find stably minimal subterms within an $E$-nonterminating term for A$\vee$C-rewrite theories $\mathcal{R} = (\Sigma, E, R)$. In the following theorem, given a term $s$ and a symbol $f$, by an *$f$-subterm* $t$ of $s$ (written $s \trianglerighteq_f t$) we mean a subterm $t$ of $s$ such that $t = s|_p$ and for all $q < p$, $root(s|_q) = f$. We also write $s \vartriangleright_f t$ if $s \trianglerighteq_f t$ and $s \neq t$. This notion is similar to the one used in [18] called *head subterm* but taking into account $s$ instead of $[s]_E$ to get the subterm.

**Theorem 2** *Let $\mathcal{R} = (\Sigma, E, R)$ be an $A\vee C$-rewrite theory. If $s$ is $E$-nonterminating, then there is a subterm $t \in \mathcal{T}_{\infty,R,E}$ of $s$ ($s \trianglerighteq t$) and*

1. *If (1) $A_{root(t)} \notin E_{root(t)}$ or (2) $t = f(t_1, t_2)$, $A_f \in E_f$, $root(t_1) \neq f$, and $root(t_2) \neq f$, then $t \in \mathcal{M}_{\infty,R,E}$.*

2. *If $t = f(t_1, t_2)$, $A_f \in E_f$, and $root(t_1) = f$ or $root(t_2) = f$, and $t \notin \mathcal{M}_{\infty,R,E}$, then there is $s' \sim_E t$ and a strict $f$-subterm $u$ of $s'$ (i.e., $s' \triangleright_f u$) such that $root(u) = f$ and $u \in \mathcal{M}_{\infty,R,E}$.*

Proof. By Proposition 4, $s$ contains a subterm $t \in \mathcal{T}_{\infty,R,E}$. If (1) $A_{root(t)} \notin E_{root(t)}$ or (2) $t = f(t_1, t_2)$, $A_f \in E_f$, $root(t_1) \neq f$, and $root(t_2) \neq f$, then, by Proposition 9, $t \in \mathcal{M}_{\infty,R,E}$. Otherwise, we know that $t = f(t_1, t_2)$, $A_f \in E_f$, and $root(t_1) = f$ or $root(t_2) = f$. If $t \in \mathcal{M}_{\infty,R,E}$, then we are done. If $t \notin \mathcal{M}_{\infty,R,E}$, then there must be a term $t' \sim_E t$, $t' \neq t$, which contains a strict subterm $t''$ (i.e., $t' \triangleright t''$) which is $E$-nonterminating. By Proposition 8, there are terms $t''' \sim_E t''$ and $u \in \mathcal{M}_{\infty,R,E}$ such that $t''' \trianglerighteq u$. If $root(u) \neq f$, then, since $t \sim_E t' \triangleright t'' \sim_E t''' \trianglerighteq u$, there must be a strict subterm $v$ of $t$ (i.e., $t \triangleright v$) satisfying $u \sim_E v$. By Proposition 3, $v$ is $E$-nonterminating. This contradicts that $t \in \mathcal{T}_{\infty,R,E}$. Thus, $root(u) = f$ as desired. Furthermore, we note that $t' = C[t'']$ for some *nonempty* context $C$ and hence $t \sim_E t' \sim_E C[t''']$. Thus, if we let $s' = C[t''']$, then $s' \triangleright u$. We can further conclude that $s' \triangleright_f u$: first note that $root(s') = f$ because $s' \sim_E t$, $root(t) = f$, and $\sim_E$-steps do not change the root of $t$ (because $E$ is an $A\vee C$-theory). Assume that $s''$ is such that $s' \triangleright s'' \triangleright u$ and $root(s'') \neq f$. Then by reasoning as above, we would conclude that $t$ contains a subterm $v'' \sim_E s''$, which contradicts $t \in \mathcal{T}_{\infty,R,E}$. Thus, $s' \triangleright_f u$. This completes the proof. $\Box$

The following result is just a convenient reformulation of the previous one.

**Corollary 4** *Let $\mathcal{R} = (\Sigma, E, R)$ be an $A\vee C$-rewrite theory. If $s$ is $E$-nonterminating, then either there is a subterm $t \in \mathcal{M}_{\infty,R,E}$ of $s$ ($s \trianglerighteq t$), or there is a subterm $t \in \mathcal{T}_{\infty,R,E}$ of $s$ satisfying that $t = f(t_1, t_2)$, $A_f \in E_f$, and $root(t_1) = f$ or $root(t_2) = f$, and such that there is $s' \sim_E t$ and a strict $f$-subterm $u$ of $s'$ ($s' \triangleright_f u$) such that $root(u) = f$ and $u \in \mathcal{M}_{\infty,R,E}$.*

# 5 Structure of (Stably) Minimal Infinite $A\vee C$-Rewrite Sequences

Now we analyze $A\vee C$-rewrite sequences starting from stably minimal $A\vee C$-nonterminating terms. First we consider a restricted case.

**Proposition 11** *Let $\mathcal{R} = (\Sigma, E, R) = (\mathcal{C} \uplus \mathcal{D}, E, R)$ be an $A\vee C$-rewrite theory. Let $s \in \mathcal{M}_{\infty,R,E}$ be such that $f = root(s)$ and either (1) $A_f \notin E_f$, or (2) $s =$*

*$f(s_1, s_2)$, $A_f \in E_f$, and $root(s_1)$, $root(s_2) \in \mathcal{C}$. Assume that for all $l \to r \in R$ such that $root(l) = f$ and all subterms $v$ of $r$ ($r \trianglerighteq v$) such that $v = g(v_1, v_2)$ for some associative symbol $g$, we have that $root(v_1), root(v_2) \notin \mathcal{X} \cup \{g\}$. Then, there exist $l \to r \in R$, a substitution $\sigma$ and terms $t \in \mathcal{T}(\Sigma, \mathcal{X})$ and $u \in \mathcal{M}_{\infty,R,E}$ such that*

$$s \xrightarrow{>\Lambda}{}^{*}_{\mathcal{E}xt_E(R),E}\; t \sim_E \sigma(l) \xrightarrow{\Lambda}_R \sigma(r) \trianglerighteq u$$

*and there is a nonvariable subterm $v$ of $r$, $r \trianglerighteq v$, such that $u = \sigma(v)$.*

PROOF. Let $S$ be an infinite $(\mathcal{E}xt_E(R), E)$-rewrite sequence starting from $s$. Since $s \in \mathcal{M}_{\infty,R,E}$, $s$ is $(\mathcal{E}xt_E(R), E)$-nonterminating and all its proper subterms are $(\mathcal{E}xt_E(R), E)$-terminating, $S$ must contain a possibly empty sequence of *inner* $(\mathcal{E}xt_E(R), E)$-rewrite steps followed by a root step. Therefore, there exists a rule $l \to r \in \mathcal{E}xt_E(R)$ such that $s \xrightarrow{>\Lambda}{}^{*}_{\mathcal{E}xt_E(R),E} t \sim_E \sigma(l) \xrightarrow{\Lambda}_{\mathcal{E}xt_E(R)} \sigma(r)$. By Proposition 10, we know that $t \in \mathcal{M}_{\infty,R,E}$. Furthermore, due to our assumptions (1) or (2) on $s$, and taking into account the shape of rules in $\mathcal{E}xt_E(R) - R$ for $A{\vee}C$-theories $E$, we can conclude that the rule $l \to r$ actually belongs to $R$. Since stable minimality is preserved under $\sim_E$, we also have $\sigma(l) \in \mathcal{M}_{\infty,R,E}$. Since $\sim_E$-steps do not change the root symbol of terms for $A{\vee}C$-theories $E$, $root(s) = root(t) = root(l) \in \mathcal{D}$. Let $l = f(l_1, \ldots, l_k)$ for some $k$-ary defined symbol $f \in \mathcal{D}$. Since $\sigma(l) \in \mathcal{M}_{\infty,R,E}$, $\sigma(l_i)$ is $(\mathcal{E}xt_E(R), E)$-terminating for all $i$, $1 \leq i \leq k$. In particular, $\sigma(x)$ is $(\mathcal{E}xt_E(R), E)$-terminating for all $x \in \mathcal{V}ar(l)$. Since $\sigma(r)$ is $(\mathcal{E}xt_E(R), E)$-nonterminating, by Theorem 2 there is a subterm $u \in \mathcal{T}_{\infty,R,E}$ of $\sigma(r)$. Therefore, there must be a nonvariable subterm $v$ of $r$ (i.e., $r \trianglerighteq v$ and $root(v) \in \mathcal{D}$), such that $u = \sigma(v)$. Let $g = root(v)$. We consider two cases according to Theorem 2:

1. If $A_g \notin E_g$, then $u \in \mathcal{M}_{\infty,R,E}$.
2. If $A_g \in E_g$, then there must be $v = g(v_1, v_2)$ for terms $v_1$ and $v_2$ such that $root(v_1), root(v_2) \notin \mathcal{X} \cup \{g\}$. Therefore, $u = g(u_1, u_2)$ with $u_i = \sigma(v_i)$ satisfying $root(u_i) \neq g$ for $i = 1, 2$. Then, $u \in \mathcal{M}_{\infty,R,E}$.

□

Unfortunately, stable minimality of (arbitrary) $E$-nonterminating terms $s$ for $A{\vee}C$-theories $E$ is not preserved under inner $(\mathcal{E}xt_E(R), E)$-rewritings (see Example 9). As Proposition 10 shows, the problem arises when $s$ is rewritten into a term like, e.g., $t = f(f(t_1, t_2), t_3)$ on which associative steps can be issued to rearrange $t$ and possibly introducing an $E$-nonterminating term below the root, thus *losing* stable minimality.

However, as a consequence of previous results, the following theorem establishes the desired property for stable minimal $A{\vee}C$-nonterminating terms.

**Theorem 3** *Let $\mathcal{R} = (\Sigma, E, R)$ be an $A{\vee}C$-rewrite theory. For all $s \in \mathcal{M}_{\infty,R,E}$, there exist $l \to r \in \mathcal{E}xt_E(R)$ and a substitution $\sigma$ such that*

$$s \xrightarrow{>\Lambda}{}^{*}_{\mathcal{E}xt_E(R),E}\; t \;\sim_E\; t' \trianglerighteq_f\; t'' \sim_E \sigma(l) \xrightarrow{\Lambda}_{\mathcal{E}xt_E(R)} \sigma(r)$$

*$t'' \in \mathcal{M}_{\infty,R,E}$ and there is a nonvariable subterm $v$ of $r$ ($r \trianglerighteq v$), such that either*

*1.* $v = f(v_1, v_2)$ *for some associative symbol* $f$*,* $root(v_1) \in \mathcal{X} \cup \{f\}$ *or* $root(v_2) \in \mathcal{X} \cup \{f\}$*,* $root(\sigma(v_1)) = f$ *or* $root(\sigma(v_2)) = f$*,* $\sigma(v) \in \mathcal{T}_{\infty,R,E}$ *and there is a term* $t' \sim_E \sigma(v)$ *containing a strict* $f$*-subterm* $u = f(u_1, u_2)$ $(t' \rhd_f u)$ *such that* $u \in \mathcal{M}_{\infty,R,E}$*, or*

*2.* $\sigma(v) \in \mathcal{M}_{\infty,R,E}$ *otherwise.*

PROOF. Let $S$ be an infinite $(\mathcal{E}xt_E(R), E)$-rewrite sequence starting from $s$. Since $s \in \mathcal{M}_{\infty,R,E}$, $s$ is $(\mathcal{E}xt_E(R), E)$-nonterminating and all its proper subterms are $(\mathcal{E}xt_E(R), E)$-terminating, $S$ must contain a root step after possibly many $(\mathcal{E}xt_E(R), E)$-rewrite steps below the root.

Therefore, $s \xrightarrow{>\Lambda}{}^{*}_{\mathcal{E}xt_E(R),E} t$ and by Corollary 3 , $t \in \mathcal{T}_{\infty,R,E}$. By Theorem 2, we have that,

1. If $A_{root(t)} \notin E_{root(t)}$ or $t = f(t_1, t_2)$, $A_f \in E_f$, $root(t_1) \neq f$, and $root(t_2) \neq f$, then $t \in \mathcal{M}_{\infty,R,E}$.
2. If $t = f(t_1, t_2)$, $A_f \in E_f$, and $root(t_1) = f$ or $root(t_2) = f$, and $t \notin \mathcal{M}_{\infty,R,E}$, then there is $t' \sim_E t$ and a strict $f$-subterm $t''$ of $t'$ (i.e., $t' \rhd_f t''$) such that $root(t'') = f$ and $t'' \in \mathcal{M}_{\infty,R,E}$.

Therefore, the sequence can proceed as follows:

$$t \sim_E t' \unrhd_f t'' \sim_E \sigma(l) \xrightarrow{\Lambda}_{\mathcal{E}xt_E(R)} \sigma(r)$$

Where, if (1) holds, $t \in \mathcal{M}_{\infty,R,E}$ and therefore $t = t' = t''$. Otherwise, in case (2), if $t \in \mathcal{M}_{\infty,R,E}$ we are done as before. If not, there is $t' \sim_E t$ and a strict $f$-subterm $t''$ of $t'$ (i.e., $t' \rhd_f t''$) such that $root(t'') = f$ and $t'' \in \mathcal{M}_{\infty,R,E}$.

Since stably minimality is preserved by $\sim_E$, therefore, $\sigma(l) \in \mathcal{M}_{\infty,R,E}$. Since $A \vee C$ axioms cannot change $root(s)$ or $root(t)$, we have $f = root(s) = root(t) = root(l) \in \mathcal{D}$. Write $l = f(l_1, \ldots, l_k)$. Since $\sigma(l) \in \mathcal{M}_{\infty,R,E}$, $\sigma(l_i)$ is $(\mathcal{E}xt_E(R), E)$-terminating. In particular, $\sigma(x)$ is $(\mathcal{E}xt_E(R), E)$-terminating for all $x \in \mathcal{V}ar(l)$. Since $\sigma(r)$ is $(\mathcal{E}xt_E(R), E)$-nonterminating, by Proposition 4 there is a subterm $u \in \mathcal{T}_{\infty,R,E}$ of $\sigma(r)$ $(\sigma(r) \unrhd u)$. Since $\sigma(x)$ is $(\mathcal{E}xt_E(R), E)$-terminating for all $x \in \mathcal{V}ar(r)$, there must be a nonvariable subterm $v$ of $r$ $(r \unrhd v)$, such that $u = \sigma(v)$. By Theorem 2, we only need to carefully consider the case when $u = f(u_1, u_2) \notin \mathcal{M}_{\infty,R,E}$ for some associative symbol $f$ such that $A_f \in E_f$, $root(u_1) = f$ or $root(u_2) = f$. Therefore, we must have $v = f(v_1, v_2)$ for some terms $v_1$ and $v_2$. Since $u_1 = \sigma(v_1)$ and $u_2 = \sigma(v_2)$, we must have $v_1 \in \mathcal{X} \cup \{f\}$ or $v_2 \in \mathcal{X} \cup \{f\}$. Theorem 2 also ensures that there is $s' \sim_E \sigma(v)$ such that $s' \rhd_f u'$ and $u' \in \mathcal{M}_{\infty,R,E}$. □

Example 5 shows that Theorem 3 does not hold for Giesl and Kapur's minimal terms $s \in \mathcal{T}_{\infty,R,E}$.

## 6 $A{\vee}C$-Dependency Pairs and Chains

Propositions 8 and 11 together with Theorem 3 are the basis for our definition of *$A{\vee}C$-Dependency Pairs* and the corresponding *chains*. Together, they show that given an $A{\vee}C$-rewrite theory $\mathcal{R} = (\Sigma, E, R)$, every $E$-nonterminating term $s$ has an associated infinite $(\mathcal{E}xt_E(R), E)$-rewrite sequence starting from a stably minimal subterm $t \in \mathcal{M}_{\infty,R,E}$. Such a sequence proceeds as described in Proposition 11 and Theorem 3, depending on the shape of $t$.

This process is abstracted in the following definition of $A{\vee}C$-dependency pairs (Definition 4) and in the definition of chain below (Definition 5).

Given a signature $\Sigma$ and $f \in \Sigma$, we let $f^\sharp$ denote a fresh new symbol (often called *tuple* symbol or DP-symbol) associated to a symbol $f$ [1]. Let $\Sigma^\sharp$ be the set of tuple symbols associated to symbols in $\Sigma$. As usual, for $t = f(t_1, \ldots, t_k) \in \mathcal{T}(\Sigma, \mathcal{X})$, we write $t^\sharp$ to denote the *marked* term $f^\sharp(t_1, \ldots, t_k)$ (written sometimes $F(t_1, \ldots, t_k)$). Given a set of rules $R$ and a symbol $f \in \Sigma$, we let $R_f = \{l \to r \in R \mid root(l) = f\}$.

**Definition 4 ($A{\vee}C$-Dependency Pairs)** *Let $\mathcal{R} = (\Sigma, E, R) = (\mathcal{C} \uplus \mathcal{D}, E, R)$ be an $A{\vee}C$-rewrite theory. Then, $\mathsf{DP}_E(R) = \{l^\sharp \to s^\sharp \mid l \to r \in \mathcal{E}xt_E(R), r \trianglerighteq s, root(s) \in \mathcal{D}, l \not\triangleright v \sim_E s\}$ is the set of $A{\vee}C$-dependency pairs ($A{\vee}C$-DPs) of $\mathcal{R}$.*

Requiring $l \not\triangleright v \sim_E s$ for $\mathsf{DP}_{AC}(\mathcal{R})$ in Definition 4 follows Dershowitz's criteria [6] extended to $A{\vee}C$ rewrite theories . In general, the set of $A{\vee}C$-DPs which is obtained from Definition 4 is a subset of those which are obtained by particularizing Giesl and Kapur's definitions to the $A{\vee}C$ case [10].

**Example 10** *Consider the AC-rewrite theory $\mathcal{R} = (\Sigma, E, R)$ in Example 5. The set $DP_E(R)$ consists of the following pairs:*

$$
\begin{array}{rcll}
F(x,x) & \to & F(0, f(1,2)) & (3)\\
F(x,x) & \to & F(1,2) & (4)\\
F(f(x,x),y) & \to & F(f(0,f(1,2)),y) & (5)\\
F(f(x,x),y) & \to & F(0,f(1,2)) & (6)\\
F(f(x,x),y) & \to & F(1,2) & (7)
\end{array}
$$

### 6.1 Chains of $A{\vee}C$-DPs

An essential property of the dependency pair method is that it provides a *characterization* of termination of TRSs $\mathcal{R}$ as the absence of infinite (minimal) *chains of dependency pairs* [1, 13]. If we want to prove the same for $A{\vee}C$-rewrite theories, we have to introduce a suitable notion of chain which can be used with $A{\vee}C$-DPs. As in the DP-framework, where the origin of *pairs* does not matter, we should rather think of another rewrite theory $\mathcal{P} = (\Gamma, F, P)$ which is used together with $\mathcal{R}$ to build the chains. According to the usual terminology [13], we often call *pairs* to the rules $u \to v \in P$.

In the following definition, given sets of equations $E$ and $F$, we let $\simeq_{F,E} = (\overset{\Lambda}{\vdash\!\dashv}_F \cup \overset{>\Lambda}{\vdash\!\dashv}_E)^*$. Moreover, we define $\xrightarrow{\Lambda}^*_{\mathcal{S}_{f_i}}$ as the application of rules $l \to r \in \mathcal{S}$ such that $root(l) = f$.

**Definition 5 (Chain of Pairs - Minimal Chain)** *Let $\mathcal{P} = (\Gamma, F, P)$ be a rewrite theory, $\mathcal{R} = (\Sigma, E, R)$ be an $A\vee C$- rewrite theory, and $\mathcal{S} = (\mathcal{F}, S)$ be a TRS. An $(F, P, E, R, S)$-chain is a finite or infinite sequence of pairs $u_i \to v_i \in P$, together with substitutions $\sigma$ and $\theta_i$ satisfying that, for all $i \geq 1$:*

1. *If $\sigma(v_i) = f_i(v_{i1}, v_{i2})$ satisfies $\sigma(v_i) = \theta_i(u'_i)$ for some $u'_i = v'_i \in F$ or $v'_i = u'_i \in F$ such that $u'_i = f_i(u'_{i1}, u'_{i2})$ satisfies $u'_{i1} \notin \mathcal{X}$ or $u'_{i2} \notin \mathcal{X}$, then*

$$\sigma(v_i) \simeq_{F,E} \circ \xrightarrow{\Lambda}^*_{\mathcal{S}_{f_i}} t_i \ \to^*_{\mathcal{E}xt_E(R),E} \circ \simeq_{F,E} \circ \xrightarrow{\Lambda}^*_{\mathcal{S}_{f_i}} \circ \simeq_{F,E} \sigma(u_{i+1})$$

2. *and $\sigma(v_i) = t_i \to^*_{\mathcal{E}xt_E(R),E} \circ \simeq_{F,E} \sigma(u_{i+1})$, otherwise.*

*An $(F, P, E, R, S)$-chain is called* minimal *if for all $i \geq 1$, and $t'_i \simeq_{F,E} t_i$, $t'_i$ is $(\mathcal{E}xt_E(R), E)$-terminating.*

As usual, in Definition 5 we assume that different occurrences of dependency pairs do not share any variable (renaming substitutions are used if necessary).

Note that the definition derives directly from Theorem 3: First we have to look for the minimal term of $\sigma(v_i)$, i.e. $t_i$, (see Theorem 2) which can be rewritten by using $\xrightarrow{>\Lambda}^*_{\mathcal{E}xt_E(R),E}$ and again, since minimality can be lost we have to apply again Theorem 2 to connect with the next pair in the chain. This more abstract notion of chain can be particularized to be used with $A\vee C$-DPs, by just taking

1. $P = \mathsf{DP}_E(R)$,
2. $F = E^\sharp$, where $E^\sharp = \{s^\sharp = t^\sharp \mid s = t \in E\}$, and
3. $\mathcal{S} = \{f^\sharp(f(x, y), z) \to f^\sharp(x, y), f^\sharp(x, f(y, z)) \to f^\sharp(y, z) \mid f \in \Sigma_A \cup \Sigma_{AC}\}$.

We have the following:

**Proposition 12** *Let $\Sigma$ be a signature and $E$ be a set of noncollapsing equations over $\Sigma$. Let $s, t \in \mathcal{T}(\Sigma, \mathcal{X})$. Then, $s \sim_E t$ if and only if $s^\sharp \simeq_{E^\sharp,E} t^\sharp$.*

Proof. We have $s \sim_E t$ if and only if $s\ (\overset{\Lambda}{\vdash\!\dashv}_E \cup \overset{>\Lambda}{\vdash\!\dashv}_E)^*\ t$. We proceed by induction on the length of the $(\overset{\Lambda}{\vdash\!\dashv}_E \cup \overset{>\Lambda}{\vdash\!\dashv}_E)$-sequence from $s$ to $t$. If $n = 0$, then $s = t$ and $s^\sharp = t^\sharp$. By reflexivity of $\simeq_{E^\sharp,E}$, we have $s^\sharp \simeq_{E^\sharp,E} t^\sharp$. If $n > 0$, then $s\ (\overset{\Lambda}{\vdash\!\dashv}_E \cup \overset{>\Lambda}{\vdash\!\dashv}_E)\ s_0\ (\overset{\Lambda}{\vdash\!\dashv}_E \cup \overset{>\Lambda}{\vdash\!\dashv}_E)^*\ t$ and, by the induction hypothesis, we know that $s_0^\sharp \simeq_{E^\sharp,E} t^\sharp$. Now, we consider two cases for the step $s\ (\overset{\Lambda}{\vdash\!\dashv}_E \cup \overset{>\Lambda}{\vdash\!\dashv}_E)\ s_0$:

1. If $s \overset{\Lambda}{\vdash\!\dashv}_E s_0$, then there is a substitution $\sigma$ and an equation $u = v \in E$ such that $s = \sigma(u)$ and $s_0 = \sigma(v)$ or $s = \sigma(v)$ and $s_0 = \sigma(u)$. Therefore, since $E$ is not collapsing, we have that $u, v \notin \mathcal{X}$. Then $s^\sharp = \sigma(u^\sharp)$ and $s_0^\sharp = \sigma(v^\sharp)$ (resp. $s^\sharp = \sigma(v^\sharp)$ and $s_0^\sharp = \sigma(u^\sharp)$). Therefore, since $u^\sharp = v^\sharp \in E^\sharp$, we have $s^\sharp \overset{\Lambda}{\vdash\!\dashv}_{E^\sharp} s_0^\sharp$. Hence, $s^\sharp \simeq_{E^\sharp,E} s_0$.

2. If $s \overset{>\Lambda}{\vdash\!\dashv}_E s_0$, then, since marking only affects the root symbol of $s$ and $s_0$, we also have $s^\sharp \overset{>\Lambda}{\vdash\!\dashv}_E s_0^\sharp$. Hence, $s^\sharp \simeq_{E^\sharp,E} s_0$.

Thus, by transitivity of $\simeq_{E^\sharp,E}$, we conclude that $s^\sharp \simeq_{E^\sharp,E} t^\sharp$ as desired. We similarly prove that $s^\sharp \simeq_{E^\sharp,E} t^\sharp$ implies $s \sim_E t$. □

**Proposition 13** *Let $\Sigma$ be a signature $f \in \Sigma$ and $s, t \in \mathcal{T}(\Sigma, \mathcal{X})$. Then, $s \trianglerighteq_f t$ if and only if $s^\sharp \xrightarrow{\Lambda}{}^{*}_{\mathcal{S}_f} t^\sharp$.*

**Theorem 4 (Soundness)** *Let $\mathcal{R} = (\Sigma, E, R)$ be an A∨C-rewrite theory with $\Sigma = \mathcal{C} \uplus \mathcal{D}$. Let $\mathcal{S} = (\Sigma \cup \mathcal{D}^\sharp, S)$ be a TRS such that*

$$S = \{f^\sharp(f(x,y),z) \to f^\sharp(x,y), f^\sharp(x, f(y,z)) \to f^\sharp(y,z) \mid f \in \Sigma_A \cup \Sigma_{AC}\}.$$

*If there is no infinite minimal $(E^\sharp, \mathsf{DP}_E(R), E, R, \mathcal{S})$-chain, then $\mathcal{R}$ is $(\mathcal{E}xt_E(R), E)$-terminating*

Proof. In the remainder of the proof, we let $F = E^\sharp$. We proceed by contradiction. If $\mathcal{R}$ is not $(\mathcal{E}xt_E(R), E)$-terminating, then, by Proposition 8, for each $(\mathcal{E}xt_E(R), E)$-nonterminating term there is an associated stably minimal term $s \in \mathcal{M}_{\infty,R,E}$. Let $f = root(s)$. We consider two cases for $s$.

1. If (1) $A_f \notin E_f$ or (2) $s = f(s_1, s_2)$, $A_f \in E_f$, and $root(s_1), root(s_2) \in \mathcal{C}$, and for all $l \to r \in R_f$ and all subterms $v$ of $r$ ($r \trianglerighteq v$) such that $v = g(v_1, v_2)$ for some associative symbol $g$, we have that $root(v_1), root(v_2) \notin \mathcal{X} \cup \{g\}$. Then by Proposition 11, there exist $l \to r \in R$, a substitution $\sigma$ and terms $t \in \mathcal{T}(\Sigma, \mathcal{X})$ and $u \in \mathcal{M}_{\infty,R,E}$ such that

$$s \xrightarrow{>\Lambda}{}^{*}_{\mathcal{E}xt_E(R),E} t \sim_E \sigma(l) \xrightarrow{\Lambda}_R \sigma(r) \trianglerighteq u$$

and there is a nonvariable subterm $v$ of $r$, $r \trianglerighteq v$, such that $u = \sigma(v)$. Hence, $u^\sharp = \sigma(v)^\sharp = \sigma(v^\sharp)$. By using Proposition 12 and, since $l^\sharp \to v^\sharp \in \mathsf{DP}(R)$ and $\mathsf{DP}(R) \subseteq \mathsf{DP}_E(R)$, we have

$$s^\sharp \xrightarrow{>\Lambda}{}^{*}_{\mathcal{E}xt_E(R),E} t^\sharp \simeq_{F,E} \sigma(l)^\sharp = \sigma(l^\sharp) \to_{\mathsf{DP}_E(R)} \sigma(v^\sharp) = u^\sharp$$

2. Otherwise, by Theorem 3, there is a rule $l \to r \in \mathcal{E}xt_E(R)$, a matching substitution $\sigma$ and terms $t''$ and $u \in \mathcal{M}_{\infty,R,E}$ such that:

$$s \xrightarrow{>\Lambda}{}^{*}_{\mathcal{E}xt_E(R),E} t \ \sim_E \ t' \trianglerighteq_f \ t'' \sim_E \sigma(l) \xrightarrow{\Lambda}_{\mathcal{E}xt_E(R)} \sigma(r)$$

Furthermore, by Theorem 3, there is a nonvariable subterm $v$ of $r$ for which we have two possibilities:

(a) We have $v = g(v_1, v_2)$ for some associative symbol $g$, where $root(v_1) \in \mathcal{X} \cup \{g\}$ or $root(v_2) \in \mathcal{X} \cup \{g\}$, $root(\sigma(v_1)) = g$ or $root(\sigma(v_2)) = g$, $\sigma(v) \in \mathcal{T}_{\infty,R,E}$ and there is a term $w \sim_E \sigma(v)$ containing a strict $g$-subterm $u = g(u_1, u_2)$ ($w \rhd_g u$) such that $u \in \mathcal{M}_{\infty,R,E}$. If we assume that there is an $v' \sim_E v$ which is a replacing subterm of $l$, i.e., $l \rhd v' \sim_E v$, then $\sigma(l) \rhd \sigma(v') \sim_E \sigma(v)$. Since $\sigma(v) \sim_E w \rhd_g u$ such that $u \in \mathcal{M}_{\infty,R,E}$, this contradicts that $\sigma(l) \in \mathcal{M}_{\infty,R,E}$. Thus, $l \not\rhd v' \sim_E v$. Since $l^\sharp \to v^\sharp \in \mathsf{DP}(\mathcal{E}xt_E(R))$ we have $\mathsf{DP}(\mathcal{E}xt_E(R)) \subseteq \mathsf{DP}_E(R)$. By using Propositions 12 and 13, we can write:

$$s^\sharp \xrightarrow{>\Lambda}{}^{*}_{\mathcal{E}xt_E(R),E} \circ \simeq_{F,E} \circ \xrightarrow{\Lambda}{}^{*}_{\mathcal{S}_f} t''^\sharp \simeq_{F,E} \sigma(l)^\sharp = \sigma(l^\sharp) \xrightarrow{\Lambda}_{\mathsf{DP}_E(R)} \sigma(v^\sharp) = \sigma(v)^\sharp \simeq_{F,E} w^\sharp \xrightarrow{\Lambda}{}^{+}_{\mathcal{S}_f} u^\sharp$$

(b) Otherwise, $\sigma(v) \in \mathcal{M}_{\infty,R,E}$. If we assume that there is an $v' \sim_E v$ which is a replacing subterm of $l$, i.e., $l \rhd v' \sim_E v$, then $\sigma(l) \rhd \sigma(v') \sim_E \sigma(v)$. Since $\sigma(v) = u$ such that $u \in \mathcal{M}_{\infty,R,E}$, this contradicts that $\sigma(l) \in \mathcal{M}_{\infty,R,E}$. Thus, $l \not\rhd v' \sim_E v$. Hence, $l^\sharp \to v^\sharp \in \mathsf{DP}(\mathcal{E}xt_E(R)) \subseteq \mathsf{DP}_E(R)$, and, by using Propositions 12 and 13, we can write:

$$s^\sharp \xrightarrow{>\Lambda}{}^{*}_{\mathcal{E}xt_E(R),E} \circ \simeq_{F,E} \circ \xrightarrow{\Lambda}{}^{*}_{\mathcal{S}_f} t''^\sharp \simeq_{F,E} \sigma(l)^\sharp = \sigma(l^\sharp) \xrightarrow{\Lambda}_{\mathsf{DP}_E(R)} \sigma(v^\sharp) = u^\sharp$$

Note that, since $u \in \mathcal{M}_{\infty,R,E}$, we have that $u^\sharp$ is $(\mathcal{E}xt_E(R), E)$-terminating. Thus, $s^\sharp$ starts a minimal $(E^\sharp, \mathsf{DP}_E(R), E, R, \mathcal{S})$-chain which could be infinitely extended from $u^\sharp$ in a similar way (as usual, in order to fit the requirement of variable-disjointness among two arbitrary pairs in a chain of pairs, we assume that appropriately renamed $A\vee C$-DPs are available when necessary). This contradicts our initial assumption. □

Now we prove that the previous $A\vee C$-dependency pairs approach is not only correct but also complete for proving $A\vee C$-termination.

**Theorem 5 (Completeness)** *Let $\mathcal{R} = (\Sigma, E, R)$ be an $A\vee C$-rewrite theory. Let $\mathcal{S} = (\Sigma \cup \mathcal{D}^\sharp, S)$ be a TRS such that*

$$S = \{f^\sharp(f(x,y),z) \to f^\sharp(x,y), f^\sharp(x, f(y,z)) \to f^\sharp(y,z) \mid f \in \Sigma_A \cup \Sigma_{AC}\}.$$

*If $\mathcal{R}$ is $(\mathcal{E}xt_E(R), E)$-terminating, then there is no infinite minimal $(E^\sharp, \mathsf{DP}_E(R), E, R, \mathcal{S})$-chain.*

Proof. By contradiction. If there is an infinite minimal $(E^\sharp, \mathsf{DP}_E(R), E, R, \mathcal{S})$-chain, then there are substitutions $\sigma_i$ and $A\vee C$-dependency pairs $u_i \to v_i \in \mathsf{DP}_E(R)$ such that :

1. If $\sigma(v_i) = f_i(v_{i1}, v_{i2})$ satisfies $\sigma(v_i) = \theta_i(u'_i)$ for some $u'_i = v'_i \in F$ or $v'_i = u'_i \in F$ such that $u'_i = f_i(u'_{i1}, u'_{i2})$ satisfies $u'_{i1} \notin \mathcal{X}$ or $u'_{i2} \notin \mathcal{X}$, then

$$\sigma(v_i) \simeq_{F,E} \circ \xrightarrow{\Lambda}{}^{*}_{\mathcal{S}_{f_i}} t_i \to^{*}_{\mathcal{E}xt_E(R),E} \circ \simeq_{F,E} \circ \xrightarrow{\Lambda}{}^{*}_{\mathcal{S}_{f_i}} \circ \simeq_{F,E} \sigma(u_{i+1})$$

2. and $\sigma(v_i) = t_i \to^{*}_{\mathcal{E}xt_E(R),E} \circ \simeq_{F,E} \sigma(u_{i+1})$, otherwise.

Now, consider the first dependency pair $u_1 \to v_1$ in the sequence:

- If (1) holds, $u_1^\natural$ is the left-hand side of a rule $l_1 \to r_1 \in \mathcal{E}xt_E(R)$ and $v_1^\natural$ is a subterm of $r_1$. Therefore, $r_1 = C_1[v_1^\natural]_{p_1}$ for some $p_1 \in \mathcal{P}os_\Sigma(r_1)$ and we can perform the $A\vee C$-rewriting step $s_1 = \sigma_1(u_1^\natural) = \sigma_1(l_1) \to_{\mathcal{E}xt_E(R)} \sigma_1(r_1) = (C_1)[\sigma_1(v_1^\natural)]_{p_1}$, where, $\sigma_1(v_1^\natural)^\sharp = \sigma_1(v_1) \simeq_{F,E} \circ \xrightarrow{\Lambda}{}^*_{\mathcal{S}_{f_1}} t_1 \to^*_{\mathcal{E}xt_E(R),E} \circ \simeq_{F,E} \circ \xrightarrow{\Lambda}{}^*_{\mathcal{S}_{f_i}} \circ \simeq_{F,E} \sigma_2(u_2)$ and $\sigma_2(u_2)$ initiates an infinite minimal $(E^\sharp, \mathsf{DP}_E(R), E, R, \mathcal{S})$-chain.

  By Theorem 3 we have that $t_1 \xrightarrow{>\Lambda}{}^*_{\mathcal{E}xt_E(R),E} \circ \sim_E \circ \trianglerighteq_f \circ \sim_E s_2[\sigma_2(u_2^\natural)]_{p_1} = s_2$. Therefore, we can build in that way an infinite $A\vee C$-rewrite sequence

  $$s_1 \to_{\mathcal{E}xt_E(R)} \circ \sim_E \circ \trianglerighteq_f t_1 \xrightarrow{>\Lambda}{}^*_{\mathcal{E}xt_E(R),E} \circ \sim_E \circ \trianglerighteq_f \circ \sim_E s_2 \to_{\mathcal{E}xt_E(R)} \cdots$$

  which contradicts the $(\mathcal{E}xt_E(R), E)$-termination of $\mathcal{R}$.

- If (2) holds, $u_1^\natural$ is the left-hand side of a rule $l_1 \to r_1 \in R$ and $v_1^\natural$ is a subterm of $r_1$. Therefore, $r_1 = C_1[v_1^\natural]_{p_1}$ for some $p_1 \in \mathcal{P}os_\Sigma(r_1)$ and we can perform the $A\vee C$-rewriting step $s_1 = \sigma_1(u_1^\natural) = \sigma_1(l_1) \to_R \sigma_1(r_1) = (C_1)[\sigma_1(v_1^\natural)]_{p_1}$, where, $t_1^\sharp = \sigma_1(v_1^\natural)^\sharp = \sigma_1(v_1) \to^*_{\mathcal{E}xt_E(R),E} \circ \simeq_{F,E} \sigma_2(u_2)$ and $\sigma_2(u_2)$ initiates an infinite minimal $(E^\sharp, \mathsf{DP}_E(R), E, R, \mathcal{S})$-chain.

  By Proposition 11 we have that $t_1 \xrightarrow{>\Lambda}{}^*_{\mathcal{E}xt_E(R),E} \circ \sim_E s_2[\sigma_2(u_2^\natural)]_{p_1} = s_2$. Therefore, we can build in that way an infinite $A\vee C$-rewrite sequence

  $$s_1 \to_R t_1 \xrightarrow{>\Lambda}{}^*_{\mathcal{E}xt_E(R),E} \circ \sim_E s_2 \to_{\mathcal{E}xt_E(R)} \cdots$$

  which contradicts the $(\mathcal{E}xt_E(R), E)$-termination of $\mathcal{R}$.

□

As a corollary of Theorems 4 and 5, we have:

**Corollary 5 (Characterization of $A\vee C$-Termination)** *Let $\mathcal{R} = (\Sigma, E, R)$ be an $A\vee C$-rewrite theory. Let $\mathcal{S} = (\Sigma \cup \mathcal{D}^\sharp, S)$ be a TRS such that $S = \{f^\sharp(f(x,y),z) \to f^\sharp(x,y), f^\sharp(x, f(y,z)) \to f^\sharp(y,z) \mid f \in \Sigma_A \cup \Sigma_{AC}\}$. Then, $\mathcal{R}$ is $(\mathcal{E}xt_E(R), E)$-terminating if and only if there is no infinite minimal $(E^\sharp, \mathsf{DP}_E(R), E, R, \mathcal{S})$-chain.*

# 7 An $A\vee C$-Dependency Pair Framework

In the following, we extend Giesl et al.'s DP-framework to provide a suitable framework for mechanizing proofs of $A\vee C$-termination using $A\vee C$-DPs.

**Definition 6 ($A\vee C$ Problem)** *An $A\vee C$ problem $\tau$ is a tuple $\tau = (F, P, E, R, S)$, where $\mathcal{R} = (\Sigma, E, R)$ is an $A\vee C$-rewrite theory, $\mathcal{P} = (\Gamma, F, P)$ is a rewrite theory, and $\mathcal{S} = (\mathcal{F}, S)$ is a TRS. An $A\vee C$ problem is finite if there is no infinite minimal $(F, P, E, R, S)$-chain. An $A\vee C$ problem $\tau$ is infinite if $\mathcal{R}$ is $E$-nonterminating or there is an infinite minimal $(F, P, E, R, S)$-chain.*

The following definition extends the notion of *DP-processor* [13] to prove termination of $A{\vee}C$-rewrite theories.

**Definition 7 ($A{\vee}C$ Processor)** *An $A{\vee}C$ processor* $\mathsf{Proc}$ *is a mapping from $A{\vee}C$ problems into sets of $A{\vee}C$ problems. Alternatively, it can also return* "$\mathsf{no}$"*. An $A{\vee}C$ processor* $\mathsf{Proc}$ *is*

- sound *if for all $A{\vee}C$ problems $\tau$, $\tau$ is finite whenever* $\mathsf{Proc}(\tau) \neq \mathsf{no}$ *and* $\forall \tau' \in \mathsf{Proc}(\tau)$*, $\tau'$ is finite.*
- complete *if for all $A{\vee}C$ problems $\tau$, $\tau$ is infinite whenever* $\mathsf{Proc}(\tau) = \mathsf{no}$ *or* $\exists \tau' \in \mathsf{Proc}(\tau)$ *such that $\tau'$ is infinite.*

Similar to [13] for the DP-framework, we construct a tree whose nodes are labeled with $A{\vee}C$ problems or "$\mathsf{yes}$" or "$\mathsf{no}$", and whose root is labeled with $(E^\sharp, \mathsf{DP}_E(R), E, R, S)$. Now we have the following result which extends [13, Corollary 5] to $A{\vee}C$-rewrite theories.

**Theorem 6 ($A{\vee}C$-DP Framework)** *Let $\mathcal{R} = (\Sigma, E, R)$ be an $A{\vee}C$-theory. We construct a tree whose nodes are labeled with $A{\vee}C$ problems or* "$\mathsf{yes}$" *or* "$\mathsf{no}$"*, and whose root is labeled with $(E^\sharp, \mathsf{DP}_E(R), E, R, S)$, where*

$$S = \{f^\sharp(f(x,y),z) \to f^\sharp(x,y), f^\sharp(x,f(y,z)) \to f^\sharp(y,z) \mid f \in \Sigma_A \cup \Sigma_{AC}\}.$$

*For every inner node labeled with $\tau$, there is a sound processor* $\mathsf{Proc}$ *satisfying one of the following conditions:*

1. $\mathsf{Proc}(\tau) = \mathsf{no}$ *and the node has just one child, labeled with* "$\mathsf{no}$".
2. $\mathsf{Proc}(\tau) = \varnothing$ *and the node has just one child, labeled with* "$\mathsf{yes}$".
3. $\mathsf{Proc}(\tau) \neq \mathsf{no}$*,* $\mathsf{Proc}(\tau) \neq \varnothing$*, and the children of the node are labeled with the $A{\vee}C$ problems in* $\mathsf{Proc}(\tau)$.

*If all leaves of the tree are labeled with* "$\mathsf{yes}$"*, then $\mathcal{R}$ is $E$-terminating. Otherwise, if there is a leaf labeled with* "$\mathsf{no}$" *and if all processors used on the path from the root to this leaf are complete, then $\mathcal{R}$ is not $E$-terminating.*

## 7.1 Preprocessing

A simply technique that can be useful when dealing with proofs of termination in the DP-framework is to try to remove rules from the original system before building the DP problem. In this way, we will start the proof with less rules and therefore less pairs, which can simplify the proof of termination. We extend here its use for proving $E$-termination. A reduction pair $(\gtrsim, \sqsupset)$ consists of a stable and monotonic quasi-ordering $\gtrsim$, and a stable and well-founded ordering $\sqsupset$ satisfying either $\gtrsim \circ \sqsupset \subseteq \sqsupset$ or $\sqsupset \circ \gtrsim \subseteq \sqsupset$. We say that $(\gtrsim, \sqsupset)$ is monotonic if $\sqsupset$ is monotonic. $\sim$ is the stable, reflexive, transitive, and symmetric equivalence induced by $\gtrsim$, i.e., $\sim = \gtrsim \cap \lesssim$.

**Proposition 14 (Removing strict rewrite rules)** *Let $\mathcal{R} = (\Sigma, E, R)$ be a rewrite theory. Let $(\gtrsim, \sqsupset)$ be a monotonic reduction pair such that $l\,(\gtrsim \cup \sqsupset)\,r$ for all $l \to r \in R$ and $u \sim v$ for all $u = v \in E$. Let $R_{\sqsupset} = \{l \to r \in R \mid l \sqsupset r\}$ and $R' = R - R_{\sqsupset}$. Then, $\mathcal{R}$ is $E$-terminating if and only if $\mathcal{R}' = (\Sigma, E, R')$ is $E$-terminating.*

Proof. Since $R' \subseteq R$, the *only if* part is obvious. For the *if* part, we proceed by contradiction. If $\mathcal{R}$ is not $E$-terminating, then there is an infinite E-rewrite sequence $A$:

$$t_1 \to_{R/E} t_2 \to_{R/E} \cdots t_n \to_{R/E} \cdots$$

that can be written in the following way:

$$t_1 \sim_E \circ \to_R \circ \sim_E t_2 \sim_E \circ \to_R \circ \sim_E \cdots t_n \sim_E \circ \to_R \circ \sim_E \cdots$$

where an infinite number of rules in $R_{\sqsupset}$ have been used; otherwise, there would be an infinite tail $t_m \to_{R'/E} t_{m+1} \to_{R'/E} \cdots$ for some $m \geq 1$ where only rules in $R'$ are applied, contradicting the $E$-termination of $\mathcal{R}'$. Let $J = \{j_1, j_2, \ldots\}$ be the infinite set of indices indicating $E$-rewrite steps $t_j \to_{R/E} t_{j+1}$ in $A$, for all $j \in J$, where rules in $R_{\sqsupset}$ have been used to perform the $E$-rewriting step. Since $l \sqsupset r$ for all $l \to r \in R_{\sqsupset}$ and $u \sim v$ for all $u = v \in E$, by stability and monotonicity of $\sqsupset$ and $\sim$ (since $\sim = \gtrsim \cap \lesssim$), we have that $t_{j_i} \sqsupset t_{j_i+1}$. Since $l \gtrsim r$ for all $l \to r \in R'$, by stability and monotonicity of $\gtrsim$, we have that $t_{j_i+1} \gtrsim t_{j_{i+1}}$. By compatibility between $\gtrsim$ and $\sqsupset$ (and since $\sim = \gtrsim \cap \lesssim$), we have $t_{j_i} \sqsupset t_{j_{i+1}}$ for all $i \geq 1$. We obtain an infinite sequence $t_{j_1} \sqsupset t_{j_2} \sqsupset \cdots$ which contradicts well-foundedness of $\sqsupset$. $\square$

## 7.2 $A \vee C$-Dependency Graph

$A \vee C$ problems focus our attention on the analysis of *infinite minimal chains*. Our aim here is obtaining a notion of graph which is able to represent all infinite *minimal* chains of pairs as given in Definition 5.

**Definition 8 ($A \vee C$-Graph of Pairs)** *Let $\mathcal{P} = (\Gamma, F, P)$ be a rewrite theory, $\mathcal{R} = (\Sigma, E, R)$ be an $A \vee C$- rewrite theory and $\mathcal{S} = (\mathcal{F}, S)$ be a TRS. The $A \vee C$-*graph *associated to them (denoted $\mathsf{G}(F, P, E, R, S)$) has $P$ as the set of nodes. There is an arc from $u \to v \in P$ to $u' \to v' \in P$ if $u \to v, u' \to v'$ is an $(F, P, E, R, S)$-chain.*

In termination proofs, we are concerned with the so-called *strongly connected components* (SCCs) of the dependency graph, rather than with the cycles themselves (which are exponentially many) [15]. A strongly connected component in a graph is a *maximal cycle*, i.e., a cycle which is not contained in any other cycle. In the following result, given two sets of rules $S$ and $Q$, we let $S_Q$ be the least subset of $S$ satisfying that whenever there is a rule $u \to v \in Q$, such that $v$ unifies with $s$ for some $s = t \in F$ or $t = s \in F$ such that $s = f(s_1, s_2)$ and $s_1 \notin \mathcal{X}$ or $s_2 \notin \mathcal{X}$, then $S_f \subseteq S_Q$.

**Theorem 7 (SCC Processor)** *Let $\mathcal{P} = (\Gamma, F, P)$ be a rewrite theory, $\mathcal{R} = (\Sigma, E, R)$ be an $A{\vee}C$- rewrite theory and $\mathcal{S} = (\mathcal{F}, S)$ be a TRS. Then, the processor* $\mathsf{Proc}_{SCC}$ *given by*

$$\mathsf{Proc}_{SCC}(F, P, E, R, S) = \{(F, Q, E, R, S_Q) \mid Q \textit{ are the pairs of an SCC in } \mathsf{G}(F, P, E, R, S)\}$$

*is sound and complete.*

As a consequence, we can *separately* work with the SCCs of $\mathsf{G}(F, P, E, R, S)$, disregarding other parts of the graph. Now we can use these notions to introduce the $A{\vee}C$-dependency graph, i.e., the $A{\vee}C$-graph whose nodes are the $A{\vee}C$-DPs instead of an arbitrary set of pairs.

**Definition 9 ($A{\vee}C$-Dependency Graph)** *Let $\mathcal{R} = (\Sigma, E, R)$ be an $A{\vee}C$-rewrite theory with $\Sigma = \mathcal{C} \uplus \mathcal{D}$. Let $\mathcal{S} = (\Sigma \cup \mathcal{D}^\sharp, S)$ be a TRS such that $S = \{f^\sharp(f(x, y), z) \to f^\sharp(x, y), f^\sharp(x, f(y, z)) \to f^\sharp(y, z) \mid f \in \Sigma_A \cup \Sigma_{AC}\}$. The* $A{\vee}C$-Dependency Graph *associated to $\mathcal{R}$ is:*

$$\mathsf{DG}(\mathcal{R}) = \mathsf{G}(E^\sharp, \mathsf{DP}_E(R), E, R, S)$$

## 7.3 Estimating the $A{\vee}C$-Dependency Graph

As in standard rewriting, the $A{\vee}C$-dependency graph of an $A{\vee}C$-rewrite theory is in general *not* computable. So, we need to use some approximation of it. For any term $t \in \mathcal{T}(\Sigma, \mathcal{X})$ let $\textsc{Cap}(t)$ result from replacing all proper subterms rooted by a defined symbol by fresh variables and let $\textsc{Ren}(t)$ which *independently* renames all *occurrences* of variables in $t$ by using new fresh variables [1].

As usual, we should not talk about a *mgu* when dealing with rewriting modulo equations. Instead, the appropriate notion is that of complete set of $E$-unifiers. However, although in theory, all these $E$-unifiers have to be considered, for our results of reachability it is enough to check the existence of one.

**Proposition 15** *Let $\mathcal{R} = (\Sigma, E, R)$ be an $A{\vee}C$-rewrite theory with $\Sigma = \mathcal{C} \uplus \mathcal{D}$. Let $u, t \in \mathcal{T}(\Sigma, \mathcal{X})$ be such that $\mathcal{V}ar(u) \cap \mathcal{V}ar(t) = \varnothing$ and $\theta, \theta'$ be substitutions. If $\theta(t) \to^*_{\mathcal{E}xt_E(R),E} \circ \sim_E \theta'(u)$, then $\textsc{Ren}(\textsc{Cap}(t))$ and $u$ $E$-unify.*

Proof. In the following, we let $s = \textsc{Ren}(\textsc{Cap}(t))$. Clearly, $t = \sigma(s)$ for some substitution $\sigma$. We proceed by induction on the length $n$ of the sequence from $\theta(t) \to^*_{\mathcal{E}xt_E(R),E} t'$ in $\theta(t) \to^*_{\mathcal{E}xt_E(R),E} t' \sim_E \theta'(u)$.

1. If $n = 0$, then $\theta(t) = t' \sim_E \theta'(u)$. Since $t = \sigma(s)$, we have $\theta(\sigma(s)) \sim_E \theta'(u)$. Since $\mathcal{V}ar(s) \cap \mathcal{V}ar(u) = \varnothing$, we conclude that $s$ and $u$ $E$-unify.

2. If $n > 0$, then we have $t \to_{\mathcal{E}xt_E(R),E} t'' \to^*_{\mathcal{E}xt_E(R),E} t' \sim_E \theta'(u)$.

   Let $p \in \mathcal{P}os(t)$ be the position where the $E$-rewrite step $t \to_{\mathcal{E}xt_E(R),E} t''$ is performed. By definition of $\textsc{Cap}$ and $\textsc{Ren}$ we have that $s = s[z]_q$ for some (fresh) variable $z$ and position $q$ such that $q \leq p$. Let $s' = \textsc{Ren}(\textsc{Cap}(t''))$. Since $t'' = \sigma'(s')$ for some substitution $\sigma'$, by the induction hypothesis,

$s'' = \textsc{Ren}(\textsc{Cap}(s'))$ (which is just a renaming of the fresh variables in $s'$, i.e., $s'' = \rho(s')$ for some renaming substitution $\rho$ for such fresh variables) and $u$ E-unify, i.e., there is a substitution $\nu$ such that $\nu(s'') \sim_E \nu(u)$. Note that we can write $s' = \tau(s)$ for some substitution $\tau$ such that $\tau(x) = x$ for all $x \neq z$ and $\tau(z) = s'|_q$. Therefore, $\nu(\rho(\tau(s))) \sim_E \nu(u)$, i.e., $s$ and $u$ $E$-unify.

□

Now, we are ready to provide a correct estimation of our graph of pairs. Correctness of our definition relies on Proposition 15.

**Definition 10 (Estimated $A{\vee}C$-Graph of Pairs)** *Let $\mathcal{P} = (\Gamma, F, P)$ be a rewrite theory, $\mathcal{R} = (\Sigma, E, R)$ be an $A{\vee}C$- rewrite theory and $\mathcal{S} = (\mathcal{F}, S)$ be a TRS. The* estimated $A{\vee}C$-graph *associated to them (denoted $\mathsf{EG}(F, P, E, R, S)$) has $P$ as the set of nodes and arcs which connect them as follows:*

1. *If $v$ unifies with $s$ for some $s = t \in F$ or $t = s \in F$ such that $s = f(s_1, s_2)$ and $s_1 \notin \mathcal{X}$ or $s_2 \notin \mathcal{X}$, then, there is an arc from $u \to v \in P$ to $u' \to v' \in P$ if $root(u') = f$.*
2. *Otherwise, there is an arc from $u \to v \in P$ to $u' \to v' \in P$ if $\textsc{Ren}(\textsc{Cap}(v))$ and $u'$ $(F \cup E)$-unify (where equations in $F$ can only be applied at root position).*

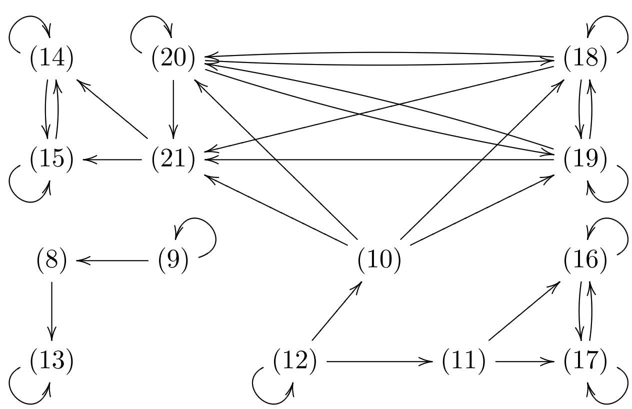
According to Definition 8, we would have the corresponding one for the *estimated $A{\vee}C$-DG*: $\mathsf{EDG}(\mathcal{R}) = \mathsf{EG}(E^\sharp, \mathsf{DP}_E(R), E, R, S)$, where

$$S = \{f^\sharp(f(x, y), z) \to f^\sharp(x, y), f^\sharp(x, f(y, z)) \to f^\sharp(y, z) \mid f \in \Sigma_A \cup \Sigma_{AC}\}.$$

**Example 11** *For the A∨C-rewrite theory in Figure 1, the set* $\mathsf{DP}_E(R)$ *is*[1]*:*

$$
\begin{array}{rcll}
\texttt{LIST2SET}(\texttt{cons}(N,L)) & \to & \texttt{UNION}(N,\texttt{list2set}(L)) & (8)\\
\texttt{LIST2SET}(\texttt{cons}(N,L)) & \to & \texttt{LIST2SET}(L) & (9)\\
\texttt{IN}(N,\texttt{union}(M,S)) & \to & \texttt{EQ}(N,M) & (10)\\
\texttt{IN}(N,\texttt{union}(M,S)) & \to & \texttt{OR}(\texttt{eq}(N,M),\texttt{in}(N,S)) & (11)\\
\texttt{IN}(N,\texttt{union}(M,S)) & \to & \texttt{IN}(N,S) & (12)\\
\texttt{UNION}(\texttt{union}(N,N),Z) & \to & \texttt{UNION}(N,Z) & (13)\\
\texttt{AND}(\texttt{and}(\texttt{true},B),Z) & \to & \texttt{AND}(B,Z) & (14)\\
\texttt{AND}(\texttt{and}(\texttt{false},B),Z) & \to & \texttt{AND}(\texttt{false},Z) & (15)\\
\texttt{OR}(\texttt{or}(\texttt{true},B),Z) & \to & \texttt{OR}(\texttt{true},Z) & (16)\\
\texttt{OR}(\texttt{or}(\texttt{false},B),Z) & \to & \texttt{OR}(B,Z) & (17)\\
\texttt{EQ}(\texttt{s}(N),\texttt{s}(M)) & \to & \texttt{EQ}(N,M) & (18)\\
\texttt{EQ}(\texttt{cons}(N,L),\texttt{cons}(M,L')) & \to & \texttt{EQ}(N,M) & (19)\\
\texttt{EQ}(\texttt{cons}(N,L),\texttt{cons}(M,L')) & \to & \texttt{EQ}(L,L') & (20)\\
\texttt{EQ}(\texttt{cons}(N,L),\texttt{cons}(M,L')) & \to & \texttt{AND}(\texttt{eq}(N,M),\texttt{eq}(L,L')) & (21)
\end{array}
$$

*The (estimated) A∨C-DG is:*




*By using Theorem 7 we transform the A∨C problem* $(E^\sharp,\mathsf{DP}(R),E,R,S)$ *into a set of A∨C problems* $\mathsf{Proc}_{SCC}(E^\sharp,\mathsf{DP}(R),E,R,S)$ *given by*

$$\{(E^\sharp,\{(9)\},E,R,\varnothing),(E^\sharp,\{(12)\},E,R,\varnothing),(E^\sharp,\{(13)\},E,R,S_{union}),$$
$$(E^\sharp,\{(14),(15)\},E,R,S_{and}),(E^\sharp,\{(16),(17)\},E,R,S_{or}),(E^\sharp,\{(18),(19),(20)\},E,R,\varnothing)\}$$

*which contains six new (but simpler) A∨C problems.*

[1] We have introduced new 'prefix' symbols *eq*, *cons* and *union* instead of the original 'infix' ones `_==_`, `_;_`, `__`.

## 7.4 Use of Reduction Pairs

In the dependency pair framework reduction pairs are used to obtain *smaller* sets of pairs $\mathcal{P}' \subseteq \mathcal{P}$ by removing the *strict* pairs, i.e., those pairs $u \to v \in \mathcal{P}$ such that $u \sqsupset v$. Stability is required both for $\gtrsim$ and $\sqsupset$ because, although we only check the left- and right-hand sides of the rewrite rules $l \to r$ (with $\gtrsim$) and pairs $u \to v$ (with $\gtrsim$ or $\sqsupset$), the chains of pairs involve *instances* $\sigma(l)$, $\sigma(r)$, $\sigma(u)$, and $\sigma(v)$ of rules and pairs and we aim at concluding $\sigma(l) \gtrsim \sigma(r)$, and $\sigma(u) \gtrsim \sigma(v)$ or $\sigma(u) \sqsupset \sigma(v)$, respectively. Monotonicity is required for $\gtrsim$ to deal with the application of rules $l \to r$ to an arbitrary depth in terms. Since the pairs are 'applied' only at the root level, no monotonicity is required for $\sqsupset$ (but, for this reason, we cannot compare the rules in $\mathcal{R}$ using $\sqsupset$). Dealing with associative and/or commutative axioms, we will compare them with the equivalence relation defined by the stable, reflexive, transitive, and symmetric equivalence $\sim$ induced by $\gtrsim$, i.e., $\sim = \gtrsim \cap \lesssim$, since we need to impose compatibility with the equational theories $E$ and $F$. The following theorem formalizes a generic processor to remove pairs from $\mathcal{P}$ by using reduction pairs.

**Theorem 8 (Reduction Pair Processor)** *Let $\mathcal{P} = (\Gamma, F, P)$ be a rewrite theory, $\mathcal{R} = (\Sigma, E, R)$ be an A$\vee$C-rewrite theory, and $\mathcal{S} = (\mathcal{F}, S)$ be a TRS. Let $(\gtrsim, \sqsupset)$ be a reduction pair such that*

1. *$R \subseteq \gtrsim$,*
2. *$P \cup S \subseteq \gtrsim \cup \sqsupset$, and*
3. *$E \cup F \subseteq \sim$.*

*Let $P_\sqsupset = \{u \to v \in P \mid u \sqsupset v\}$ and $S_\sqsupset = \{s \to t \in S \mid s \sqsupset t\}$. Then, the processor $\mathsf{Proc}_{RP}$ given by*

$$\mathsf{Proc}_{RP}(F, P, E, R, S) = \begin{cases} \{(F, P - P_\sqsupset, E, R, S - S_\sqsupset)\} & \textit{if (1), (2), and (3) hold} \\ \{(F, P, E, R, S)\} & \textit{otherwise} \end{cases}$$

*is sound and complete.*

Proof. Since $P - P_\sqsupset \subseteq P$ and $S - S_\sqsupset \subseteq S$, completeness is assured. Regarding soundness, we proceed by contradiction. Assume that there is an infinite minimal $(F, P, E, R, S)$-chain $A$, but that there is no infinite minimal $(F, P - P_\sqsupset, E, R, S - S_\sqsupset)$-chain. Due to the finiteness of $P$ and $S$, we can assume that there is $Q \subseteq P$ and $T \subseteq S$ such that $A$ has a tail $B$ where all pairs in $Q$ and rules in $T$ are infinitely often used. We distinguish two kinds of elementary steps in $B$, according to Definition 5.

1. If $\sigma(v_i) = f_i(v_{i1}, v_{i2})$ satisfies $\sigma(v_i) = \theta_i(u'_i)$ for some $u'_i = v'_i \in F$ or $v'_i = u'_i \in F$ such that $u'_i = f_i(u'_{i1}, u'_{i2})$ satisfies $u'_{i1} \notin \mathcal{X}$ or $u'_{i2} \notin \mathcal{X}$, then

$$\sigma(v_i) \simeq_{F,E} \circ \xrightarrow{\Lambda}{}^{*}_{\mathcal{S}_{f_i}} t_i \ \to^{*}_{\mathcal{E}xt_E(R),E} \circ \simeq_{F,E} \circ \xrightarrow{\Lambda}{}^{*}_{\mathcal{S}_{f_i}} \circ \simeq_{F,E} \sigma(u_{i+1})$$

Note that, due to the requirements imposed for the rules in $R$ and $S$ and equations in $E$ and $F$, and by stability and transitivity of $\gtrsim$ (hence of $\sim$), monotonicity and transitivity of $\gtrsim$, we have

$$\sigma(v_i) \sim \circ (\gtrsim \cup \sqsupset)\, t_i \gtrsim \circ \sim \circ (\gtrsim \cup \sqsupset) \circ \sim \sigma(u_{i+1})$$

Here, it is important to specifically consider the case when the rules $l \to r$ involved in $\to^*_{\mathcal{E}xt_E(R),E}$-steps are taken from $\mathcal{E}xt_E(R) - R$, i.e, $l \to r \notin R$. In this case, we do not have an explicit compatibility requirement of $l \to r$ with $\gtrsim$, i.e., $l \gtrsim r$ is not explicitly required. However, since $\mathcal{R}$ is an $A \vee C$ theory, such rules are connected with rules rule $l' \to r' \in R$ in a simple way. For instance if $l = f(l', w) \to f(r', w) = r$ for some $l' \to r' \in R$ such that $root(l') = f$, then, since $l' \gtrsim r'$ holds, by monotonicity of $\gtrsim$, we also have $l = f(l', w) \gtrsim f(r', w) = r$. With other rules included in $\mathcal{E}xt_E(R) - R$ (see Section 3.1) we would proceed in a similar way. Now, taking into account that $\sim \circ (\gtrsim \cup \sqsupset) = \gtrsim \cup \sqsupset$ and $\sim \circ \gtrsim = \gtrsim$, we have

$$\sigma(v_i)\, (\gtrsim \cup \sqsupset)\, t_i\, (\gtrsim \cup \sqsupset) \sigma(u_{i+1})$$

Note that, by the compatibility condition required for $\gtrsim$ and $\sqsupset$, this means that $\sigma(v_i) \gtrsim \sigma(u_{i+1})$ or $\sigma(v_i) \sqsupset \sigma(u_{i+1})$.

2. If $\sigma(v_i) = t_i \to^*_{\mathcal{E}xt_E(R),E} \circ \simeq_{F,E} \sigma(u_{i+1})$, then we analogously have $\sigma(v_i) \gtrsim \sigma(u_{i+1})$.

Since $u_i\, (\gtrsim \cup \sqsupset)\, v_i$ for all $u_i \to v_i \in Q \subseteq P$, by stability of $\gtrsim$ and $\sqsupset$, we have $\sigma(u_i)\, (\gtrsim \cup \sqsupset)\, \sigma(v_i)$ for all $i \geq 1$. No pair $u \to v \in Q$ satisfies that $u \sqsupset v$ and no rule $s \to t \in T$ satisfies $s \sqsupset t$. Since $u \to v$ and $s \to t$ occurs infinitely often in $B$, and taking into account that $\sigma(v_i) \gtrsim \sigma(u_{i+1})$ or $\sigma(v_i) \sqsupset \sigma(u_{i+1})$ for all $i \geq 1$, there would be an infinite set $\mathcal{I} \subseteq \mathbb{N}$ such that $\sigma(u_i) \sqsupset \sigma(u_{i+1})$ for all $i \in \mathcal{I}$ or there would be an infinite set $\mathcal{J} \subseteq \mathbb{N}$ such that $\sigma(s_j) \sqsupset \sigma(t_{j+1})$ for all $j \in \mathcal{J}$. And we have $\sigma(u_i)\, (\gtrsim \cup \sqsupset)\, \sigma(u_{i+1})$ for all other $u_i \to v_i \in Q$ or $\sigma(s_j)(\gtrsim \cup \sqsupset)\sigma(t_{j+1})$ for all other $s_j \to t_j \in T$. Thus, by using the compatibility conditions of the reduction pair, we obtain an infinite decreasing $\sqsupset$-sequence which contradicts well-foundedness of $\sqsupset$.

Therefore, $Q \subseteq (P - P_\sqsupset)$ and $T \subseteq (S - S_\sqsupset)$, which means that $B$ is an infinite minimal $(F, P - P_\sqsupset, E, R, S - S_\sqsupset)$-chain, thus leading to a contradiction. □

## 7.5 Other Processors

Many times, the set of $F$ axioms can be reduced to those equations that are really involved in minimal $A \vee C$-chains. The following processor shows a trivial method to eliminate them.

**Theorem 9 (F Usable Equations Processor)** *Let $\mathcal{P} = (\Gamma, F, P)$ be a rewrite theory, $\mathcal{R} = (\Sigma, E, R)$ be an $A \vee C$-rewrite theory, and $\mathcal{S} = (\mathcal{F}, S)$ be a TRS such that*

*1. $root(u), root(v) \in \Gamma - \Sigma$ for all $u \to v \in P$,*

*2. $root(s) = root(t) \in \Gamma - \Sigma$ for all $s = t \in F$, and*

*3. $root(l) = root(r) \in \Gamma - \Sigma$ for all $l \to r \in S$, and*

$$Let\ \hat{F} \quad = \quad \{s = t \in F \mid root(s) = root(u) \text{ or } root(s) = root(v) \text{ for some } u \to v \in P\}$$

*Then, the processor* $\mathsf{Proc}_{FUEq}$ *given by*

$$\mathsf{Proc}_{FUEq}(F, P, E, R, S) = \{(\hat{F}, P, E, R, S)\}$$

*is sound and complete.*

PROOF. Regarding soundness, we proceed by contradiction. Assume that there is an infinite minimal $(F, P, E, R, S)$-chain $A$, but that there is no infinite minimal $(\hat{F}, P, E, R, S)$-chain. Due to the finiteness of $P$, we can assume that there is $Q \subseteq P$ and $F' \subseteq F$ such that $A$ has a tail $B$ where all pairs in $Q$ and equations in $F'$ are infinitely often used. We distinguish two kinds of elementary steps in $B$, according to Definition 5.

1. If $\sigma(v_i) = f_i(v_{i1}, v_{i2})$ satisfies $\sigma(v_i) = \theta_i(u'_i)$ for some $u'_i = v'_i \in F'$ or $v'_i = u'_i \in F'$ such that $u'_i = f_i(u'_{i1}, u'_{i2})$ satisfies $u'_{i1} \notin \mathcal{X}$ or $u'_{i2} \notin \mathcal{X}$, then

$$\sigma(v_i) \simeq_{F',E} t_i \xrightarrow{\Lambda}{}^{*}_{\mathcal{S}_{f_i}} t'_i \ \to^{*}_{\mathcal{E}xt_E(R),E} \circ \simeq_{F',E} \circ \xrightarrow{\Lambda}{}^{*}_{\mathcal{S}_{f_i}} \circ \simeq_{F',E} \sigma(u_{i+1})$$

   For this sequence we have:

   - $root(v_i) = f_i \in \Gamma - \Sigma$ (by (1)),
   - that means that in the step $\sigma(v_i) \simeq_{F',E} t'_i$ we can apply equations below the root by using $E$ and if we apply an equation $s = t \in F'$, then $root(s) = root(t) = root(v_i) = f_i$ (by (2)) since we only use them at root position. Then, $s = t \in \hat{F}$.
   - In the step $t_i \xrightarrow{\Lambda}{}^{*}_{\mathcal{S}_{f_i}} t'_i$, again we proceed in a similar way. Since for all $l \to r \in S$ we know that $root(l) = root(r)$ (by (3)), then we have that $root(t_i) = root(t'_i) = root(v_i)$.
   - The application of $\to^{*}_{\mathcal{E}xt_E(R),E}$-steps are below the root (since $root(t'_i) \in \Gamma - \Sigma$) and therefore the root symbol remains untouched.
   - In the next steps, since the root symbol remains unchanged proceeding like in previous steps, again, if a equation $s = t \in F'$ is applied at the root position has to be such that $root(s) = root(t) = root(v_i) = root(t_i) = root(t'_i) = \ldots = root(u_{i+1})$, therefore $s = t \in \hat{F}$.

2. If $\sigma(v_i) = t_i \to^{*}_{\mathcal{E}xt_E(R),E} \circ \simeq_{F',E} \sigma(u_{i+1})$, then we analogously have that the equations that can be used to connect with the next pair $u_{i+1}$ are the equations in $E$ and those from $F'$ such that $root(s) = root(t) = root(v_i) = root(u_{i+1})$. Then, $s = t \in \hat{F}$.

Therefore, $root(v_i) = root(t_i) = root(t'_i) = \ldots = root(u_{i+1}) = f_i$ and $F' \subseteq \hat{F}$, which means that $B$ is an infinite $(\hat{F}, P, E, R, S)$-chain. Since $\{s_i \mid s_i \doteq_{\hat{F},E} t_i\} \subseteq \{s_i \mid s_i \doteq_{F,E} t_i\}$ and by minimality, for all $w \doteq_{F,E} t_i$, $w$ is $(\mathcal{E}xt_E(R), E)$-terminating therefore for all $w \doteq_{\hat{F},E} t_i$, $w$ is $(\mathcal{E}xt_E(R), E)$-terminating. Therefore $B$ is an infinite minimal $(\hat{F}, P, E, R, S)$-chain thus leading to a contradiction.

Regarding completeness, we proceed by contradiction. Assume that there is an infinite minimal $(\hat{F}, P, E, R, S)$-chain $A$, but that there is no infinite minimal $(F, P, E, R, S)$-chain. Due to the finiteness of $P$, we can assume that there is $Q \subseteq P$ and $F' \subseteq \hat{F}$ such that $A$ has a tail $B$ where all pairs in $Q$ and equations in $F'$ are infinitely often used. Since $\hat{F} \subseteq F$, every infinite $(\hat{F}, P, E, R, S)$-chain is also an infinite $(F, P, E, R, S)$-chain. Reasoning as in the correctness part over the infinite sequence, we know that $root(v_i) = root(t_i) = root(t'_i) = \ldots = root(u_{i+1}) = f_i$ and therefore, we conclude that the only equations from $F$ that can be used in the infinite $(F, P, E, R, S)$-chain belong to $\hat{F}$. By minimality, for all $w(\overset{\Lambda}{\vdash\!\dashv}_{\hat{F}} \cup \overset{>\Lambda}{\vdash\!\dashv}_{E})^* t_i$, $w$ is $(\mathcal{E}xt_E(R), E)$-terminating, since the only equations that can be applied to $t_i$ are those $\{s = t \in F \mid root(s) = root(t_i) \text{ or } root(t) = root(t_i)\}$ which correspond with $\hat{F}$, we can conclude that for all $w(\overset{\Lambda}{\vdash\!\dashv}_{F} \cup \overset{>\Lambda}{\vdash\!\dashv}_{E})^* t_i$, $w$ is $(\mathcal{E}xt_E(R), E)$-terminating. Therefore $B$ is an infinite minimal $(F, P, E, R, S)$-chain thus leading to a contradiction.

$\square$

**Example 12** *By Example 11 we have* $\tau_0 = (E^\sharp, \mathsf{DP}(R), E, R, S)$ *by applying the SCC processor into* $\mathsf{Proc}_{SCC}(\tau_0) = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6\}$ *where*

- $\tau_1 = (E^\sharp, \{(9)\}, E, R, \varnothing)$,
- $\tau_2 = (E^\sharp, \{(12)\}, E, R, \varnothing)$,
- $\tau_3 = (E^\sharp, \{(13)\}, E, R, S_{union})$,
- $\tau_4 = (E^\sharp, \{(14), (15)\}, E, R, S_{and})$,
- $\tau_5 = (E^\sharp, \{(16), (17)\}, E, R, S_{or})$ *and*
- $\tau_6 = (E^\sharp, \{(18), (19), (20)\}, E, R, \varnothing)$.

*For the each of these A∨C problem,we can apply* $\mathsf{Proc}_{FUEq}$.

- *For* $\tau_1$*, we have* $\mathsf{Proc}_{FUEq}(\tau_1) = (\varnothing, \{(9)\}, E, R, \varnothing)$,
- *For* $\tau_2$*, we have* $\mathsf{Proc}_{FUEq}(\tau_2) = (\varnothing, \{(12)\}, E, R, \varnothing)$,
- *For* $\tau_3$*, we have* $\mathsf{Proc}_{FUEq}(\tau_3) = (E^\sharp_{union}, \{(13)\}, E, R, S_{union})$,
- *For* $\tau_4$*, we have* $\mathsf{Proc}_{FUEq}(\tau_4) = (E^\sharp_{and}, \{(14), (15)\}, E, R, S_{and})$,
- *For* $\tau_5$*, we have* $\mathsf{Proc}_{FUEq}(\tau_5) = (E^\sharp_{or}, \{(16), (17)\}, E, R, S_{or})$ *and*
- *For* $\tau_6$*, we have* $\mathsf{Proc}_{FUEq}(\tau_6) = (E^\sharp_{eq}, \{(18), (19), (20)\}, E, R, \varnothing)$.

# 8 Usable Rules and Equations for $A\vee C$ Problems

Usable rules are widely used in the DP framework to improve the power of DP processors. In this section we show how to obtain the set of usable rules and usable equations for a given $A\vee C$ problem and how to use them to define a new reduction pair processor. We follow the approach and techniques developed in [14, 26]. We assume that all all our rewrite theories are finite (they have no infinite rules or equations). Our first intuition was to define a proper notion of $A\vee C$-dependency that not only take into account the symbols occurring in the rules, but also the symbols occurring in the equations. But, since $A\vee C$ equations do not introduce new symbols in their left- and right-hand sides, we can use the standard notion of dependency that only considers symbols occurring in the rules. We use some auxiliar definitions. Let $Rls_R(f) = \{l \to r \in R \mid root(l) = f\}$, $Eqs_E(f) = \{u = v \in E \mid root(u) = f \vee root(v) = f\}$. Let $Fun(t) = \{f \mid \exists p \in \mathcal{P}os_{\mathcal{F}}(t), f = root(t|_p)\}$.

**Definition 11 (Dependency [28])** *Let $\mathcal{R} = (\Sigma, R)$ be a TRS. We say that $f \in \Sigma$ has a* dependency *on $h \in \Sigma$ (written $f \rhd_R h$) if $f = h$ or there is a function symbol $g$ with $g \rhd_R h$ and a rule $l \to r \in Rls_R(f)$ with $g \in Fun(r)$.*

To obtain the correct notions of usable rule and equation, we have to look at the structure of the chains. We have two possible ways to proceed in an $(F, P, E, R, S)$-chain. Given $u_i \to v_i \in P$ either

$$\sigma(v_i) \simeq_{F,E} \circ \xrightarrow{\Lambda}{}^{*}_{\mathcal{S}_{f_i}} t_i \to^{*}_{\mathcal{E}xt_E(R),E} \circ \simeq_{F,E} \circ \xrightarrow{\Lambda}{}^{*}_{\mathcal{S}_{f_i}} \circ \simeq_{F,E} \sigma(u_{i+1})$$

or

$$\sigma(v_i) = t_i \to^{*}_{\mathcal{E}xt_E(R),E} \circ \simeq_{F,E} \sigma(u_{i+1})$$

Then, to obtain the set of usable rules and also usable equations we have to look for usable symbols not only in $P$, but also in $F$ and $S$.

**Definition 12 ($A\vee C$-Usable Rules and Equations)** *Let $\tau$ be an $A\vee C$ problem such that $\tau = (F, P, E, R, S)$ where $\mathcal{R} = (\Sigma, E, R)$ is an $A\vee C$-rewrite theory, $\mathcal{P} = (\Gamma, F, P)$ is a rewrite theory, and $\mathcal{S} = (\mathcal{F}, S)$ is a TRS. The set $\mathcal{U}_R(\tau)$ of* $A\vee C$-usable rules *of $\tau$ is*

$$\begin{array}{rcll}
\mathcal{U}_R(\tau) & = & \displaystyle\bigcup_{s\to t \,\in\, P, f \,\in\, Fun(t), f \rhd_R g} & Rls_R(g) \quad \cup \\
& & \displaystyle\bigcup_{u=v \,\in\, F, f \,\in\, Fun(u) \,\cup\, Fun(v), f \rhd_R g} & Rls_R(g) \quad \cup \\
& & \displaystyle\bigcup_{l\to r \,\in\, S, f \,\in\, Fun(r), f \rhd_R g} & Rls_R(g)
\end{array}$$

*The set* $\mathcal{U}_E(\tau)$ *of* $A\vee C$-usable equations *of* $\tau$ *is*

$$\begin{array}{rcll} \mathcal{U}_E(\tau) & = & \bigcup\limits_{s\to t \,\in\, P, f \,\in\, Fun(t), f \rhd_R g} & Eqs_E(g) \quad \cup \\ & & \bigcup\limits_{u=v \,\in\, F, f \,\in\, Fun(u) \,\cup\, Fun(v), f \rhd_R g} & Eqs_E(g) \quad \cup \\ & & \bigcup\limits_{l\to r \,\in\, S, f \,\in\, \cup Fun(r), f \rhd_R g} & Eqs_E(g) \end{array}$$

Note that, if the rules from $S$ are of the form $f^\sharp(f(x,y),z) \to f^\sharp(x,y)$ or $f^\sharp(x,f(y,z)) \to f^\sharp(y,z)$ do not introduce new rules as usable.

Now, we define an interpretation that, given an $A\vee C$ problem $\tau = (F,P,E,R,S)$, allows us to transform any infinite minimal $(F,P,E,R,S)$-chain into an infinite sequence of pairs from $P$ where, for all $i \geq 1$,

$$\sigma'(v_i) \simeq_{F,E'} \circ \to^*_{\mathcal{S}_{f_i} \cup \mathcal{C}_\varepsilon} t_i \to^*_{\mathcal{E}xt_{E'}(R'),E'} \circ \simeq_{F,E'} \circ \to^*_{\mathcal{S}_{f_i} \cup \mathcal{C}_\varepsilon} \circ \simeq_{F,E'} \circ \to^*_{\mathcal{C}_\varepsilon} \sigma'(u_{i+1})$$

or

$$\sigma'(v_i) = t_i \to^*_{\mathcal{E}xt_{E'}(R'),E'} \circ \simeq_{F,E'} \circ \to^*_{\mathcal{C}_\varepsilon} \sigma'(u_{i+1})$$

with $\mathcal{C}_\varepsilon = \{\mathtt{c}(x,y) \to x, \mathtt{c}(x,y) \to y\}$ being $\mathtt{c}$ a new fresh binary symbol, $E' = \mathcal{U}_E(\tau)$ and $R' = \mathcal{U}_R(\tau) \cup \mathcal{C}_\varepsilon$. We modify the original interpretation used in [14, 26] in such a way that, if a term is rooted by a non-usable $A\vee C$ symbol, then all its equivalent terms have exactly the same interpretation.

**Definition 13** (**$A\vee C$-Interpretation**) *Let* $\mathcal{R} = (\Sigma, E, R)$ *be an* $A\vee C$*-rewrite theory and* $\Delta \subseteq \Sigma$. *Let* $>$ *be an arbitrary total ordering over* $\mathcal{T}(\Sigma \cup \{\bot, \mathtt{c}\}, \mathcal{X})$ *where* $\bot$ *is a fresh constant symbol and* $\mathtt{c}$ *is a fresh binary symbol. The* $A\vee C$*-Interpretation* $\mathcal{I}_{\Delta,E}$ *is a mapping from* $E$*-terminating terms in* $\mathcal{T}(\Sigma, \mathcal{X})$ *to terms in* $\mathcal{T}(\Sigma \cup \{\bot, \mathtt{c}\}, \mathcal{X})$ *defined as follows:*

$$\mathcal{I}_{\Delta,E}(t) = \begin{cases} t & \text{if} \quad t \in \mathcal{X} \\ \mathtt{f}(\mathcal{I}_{\Delta,E}(t_1), \ldots, \mathcal{I}_{\Delta,E}(t_k)) & \text{if} \quad t = \mathtt{f}(t_1, \ldots, t_k) \\ & \qquad \text{and } \mathtt{f} \in \Delta \\ \mathit{order}(\{\mathtt{c}(\mathtt{f}(\mathcal{I}_{\Delta,E}(s_1), \ldots, \mathcal{I}_{\Delta,E}(s_k)), s') & \\ \qquad\qquad \mid s = \mathtt{f}(s_1, \ldots, s_n) \in [t]_E\}) & \text{if} \quad t = \mathtt{f}(t_1, \ldots, t_k) \\ & \qquad \text{and } \mathtt{f} \notin \Delta \end{cases}$$

$$\begin{array}{ll} \textit{where} & s' = \mathit{order}\left(\{\mathcal{I}_{\Delta,E}(u) \mid s \to_{\mathcal{E}xt_E(R),E} u\}\right) \\ & \mathit{order}(T) = \begin{cases} \bot, & \text{if } T = \varnothing \\ \mathtt{c}(t, \mathit{order}(T \setminus \{t\})) & \text{if } t \text{ is minimal in } T \text{ w.r.t. } > \end{cases} \end{array}$$

We have to ensure now that the adapted interpretation does not generate infinite terms. This is achieved thanks to the fact that for any $A\vee C$-equational theory $E$, $E$-equivalence classes are always finite, and reductions with $\to_{\mathcal{E}xt_E(R),E}$ are finitely branching due to finiteness of $R$ (assumed in Section 3).

**Lemma 1** *Let* $\mathcal{R} = (\Sigma, E, R)$ *be an* $A\vee C$*-rewrite theory and* $\Delta \subseteq \Sigma$ *and* $t \in \mathcal{T}(\Sigma, \mathcal{X})$. *If* $t$ *is* $E$*-terminating then* $\mathcal{I}_{\Delta,E}$ *is well-defined.*

Proof. According to Definition 13, to obtain an infinite term as result of $\mathcal{I}_{\Delta,E}(t)$, either: (1) we get a term $t'$ such that $[t']_E$ is infinite, or (2) we would have to perform an infinite number of applications of the function *order* $(\{\mathcal{I}_{\Delta,E}(u) \mid s \to_{\mathcal{R}} u\})$:

(1) We know that $\mathcal{R}$ is an $A{\vee}C$-rewrite theory. Therefore all the equations are of the form $f(f(x,y),z) = f(x,f(y,z))$ or $f(x,y) = f(y,x)$. Since equations are linear and no new symbols are added, $[t']_E$ is finite.

(2) We have an infinite sequence of the following way:

$$t = t_0 \sim_E t_1 \to_{\mathcal{E}xt_E(R),E} t_2 \sim_E t_3 \to_{\mathcal{E}xt_E(R),E} \cdots$$

that contradicts the $E$-termination of $t$.

$\square$

Now, to prove the main theorem, we need some auxiliar results that allow us to construct the new infinite sequence. The idea is that, for each relation in the chain $R$, if $s\ R\ t$ then $\mathcal{I}_{\Delta,E}(s)\ R\ \mathcal{I}_{\Delta,E}(t)$.

**Definition 14** *Let $\mathcal{R} = (\Sigma, E, R)$ be a rewrite theory, $\Delta \subseteq \Sigma$ and $\sigma$ be a substitution. We define $\sigma_{\mathcal{I}_{\Delta,E}}$ as $\sigma_{\mathcal{I}_{\Delta,E}}(x) = \mathcal{I}_{\Delta,E}(\sigma(x))$.*

**Lemma 2** *Let $\mathcal{R} = (\Sigma, E, R)$ be an $A{\vee}C$-rewrite theory and $\Delta \subseteq \Sigma$. Let $t$ be a term and $\sigma$ be a substitution. If $\sigma(t)$ is $E$-terminating, then $\mathcal{I}_{\Delta,E}(\sigma(t)) \to^*_{\mathcal{C}_\varepsilon} \sigma_{\mathcal{I}_{\Delta,E}}(t)$. If $t$ only contains $\Delta$ symbols, then $\mathcal{I}_{\Delta,E}(\sigma(t)) = \sigma_{\mathcal{I}_{\Delta,E}}(t)$.*

Proof. By structural induction on $t$:

- If $t = x$ is a variable then $\mathcal{I}_{\Delta,E}(\sigma(x)) = \sigma_{\mathcal{I}_{\Delta,E}}(x)$.
- If $t = f(t_1, \ldots, t_k)$ then
  - If $f \in \Delta$ then $\mathcal{I}_{\Delta,E}(\sigma(t)) = f(\mathcal{I}_{\Delta,E}(\sigma(t_1)), \ldots, \mathcal{I}_{\Delta,E}(\sigma(t_k)))$. By hypothesis, terms $\sigma(t_i)$ are $E$-terminating for $1 \leq i \leq ar(f)$. By induction hypothesis, for all terms $t_i$ we have $\mathcal{I}_{\Delta,E}(\sigma(t_i)) \to^*_{\mathcal{C}_\varepsilon} \sigma_{\mathcal{I}_{\Delta,E}}(t_i)$. This implies $f(\mathcal{I}_{\Delta,E}(\sigma(t_1)), \ldots, \mathcal{I}_{\Delta,E}(\sigma(t_k))) \to^*_{\mathcal{C}_\varepsilon} \sigma_{\mathcal{I}_{\Delta,E}}(t)$.
  - If $f \notin \Delta$, we have that for all $s = f(s_1, \ldots, s_n) \in [t]_E$ we obtain $\mathcal{I}_{\Delta,E}(\sigma(t)) \to^+_{\mathcal{C}_\varepsilon} c(f(\mathcal{I}_{\Delta,E}(\sigma(s_1)), \ldots, \mathcal{I}_{\Delta,E}(\sigma(s_k))), s'))$ using proper $\mathcal{C}_\varepsilon$-steps. Since $t \in [t]_E$, we can obtain $\mathcal{I}_{\Delta,E}(\sigma(t)) \to^+_{\mathcal{C}_\varepsilon} c(f(\mathcal{I}_{\Delta,E}(\sigma(t_1)), \ldots, \mathcal{I}_{\Delta,E}(\sigma(t_k))), t'))$. Therefore, we get $f(\mathcal{I}_{\Delta,E}(\sigma(t_1)), \ldots, \mathcal{I}_{\Delta,E}(\sigma(t_k)))$ applying again a $\mathcal{C}_\varepsilon$ rule. Then, we conclude that $f(\mathcal{I}_{\Delta,E}(\sigma(t_1)), \ldots, \mathcal{I}_{\Delta,E}(\sigma(t_k))) \to^*_{\mathcal{C}_\varepsilon} \sigma_{\mathcal{I}_{\Delta,E}}(t)$ reasoning as in the previous item.

Therefore, we have that $\mathcal{I}_{\Delta,E}(\sigma(t)) \to^*_{\mathcal{C}_\varepsilon} \sigma_{\mathcal{I}_{\Delta,E}}(t)$.

The second part of the lemma is proved similarly. By structural induction on $t$:

- If $t = x$ is a variable then $\mathcal{I}_{\Delta,E}(\sigma(x)) = \sigma_{\mathcal{I}_{\Delta,E}}(x)$.
- If $t = f(t_1, \ldots, t_k)$ and $f \in \Delta$ (because $t$ only contains $\Delta$ symbols), then $\mathcal{I}_{\Delta,E}(\sigma(t)) = f(\mathcal{I}_{\Delta,E}(\sigma(t_1)), \ldots, \mathcal{I}_{\Delta,E}(\sigma(t_k)))$. By hypothesis, terms $\sigma(t_i)$ are $E$-terminating for $1 \leq i \leq ar(f)$ and terms $t_i$ only contain $\Delta$ symbols. Therefore, by induction hypothesis, $\mathcal{I}_{\Delta,E}(\sigma(t_i)) = \sigma_{\mathcal{I}_{\Delta,E}}(t_i)$. This implies $f(\mathcal{I}_{\Delta,E}(\sigma(t_1)), \ldots, \mathcal{I}_{\Delta,E}(\sigma(t_k))) = \sigma_{\mathcal{I}_{\Delta,E}}(t)$.

□

**Lemma 3** *Let $\tau = (F, P, E, R, S)$ be an A∨C problem where $\mathcal{R} = (\Sigma, E, R)$ is an A∨C-rewrite theory, $\mathcal{P} = (\Gamma, F, P)$ is a rewrite theory, and $\mathcal{S} = (\mathcal{F}, S)$ is a TRS. Let $\Delta = (\Gamma \cup \Sigma \cup \mathcal{F}) - (\{root(l) \mid l \to r \in (R - \mathcal{U}_R(\tau))\} \cup \{root(u) \mid u = v \in (E - \mathcal{U}_E(\tau))$ or $v = u \in (E - \mathcal{U}_E(\tau))\})$. If $s$ and $t$ are $E$-terminating and $s \overset{\Lambda}{\vdash\!\dashv}_F t$ then $\mathcal{I}_{\Delta,E}(s) \overset{\Lambda}{\vdash\!\dashv}_F \mathcal{I}_{\Delta,E}(t)$.*

Proof. Let $s \overset{\Lambda}{\vdash\!\dashv}_{\{u=v\}} t$ and $s = \sigma(u) \overset{\Lambda}{\vdash\!\dashv}_{\{u=v\}} \sigma(v) = t$ or $s = \sigma(v) \overset{\Lambda}{\vdash\!\dashv}_{\{v=u\}} \sigma(u) = t$ for some substitution $\sigma$. Since $u, v \in \mathcal{T}(\Delta, \mathcal{X})$ by the construction of $\Delta$, by Lemma 2 we get $\mathcal{I}_{\Delta,E}(\sigma(u)) = \sigma_{\mathcal{I}_{\Delta,E}}(u) \overset{\Lambda}{\vdash\!\dashv}_{\{u=v\}} \sigma_{\mathcal{I}_{\Delta,E}}(v) = \mathcal{I}_{\Delta,E}(\sigma(v))$ or $\mathcal{I}_{\Delta,E}(\sigma(v)) = \sigma_{\mathcal{I}_{\Delta,E}}(v) \overset{\Lambda}{\vdash\!\dashv}_{\{v=u\}} \sigma_{\mathcal{I}_{\Delta,E}}(u) = \mathcal{I}_{\Delta,E}(\sigma(u))$. □

**Lemma 4** *Let $\tau = (F, P, E, R, S)$ be an A∨C problem where $\mathcal{R} = (\Sigma, E, R)$ is an A∨C-rewrite theory, $\mathcal{P} = (\Gamma, F, P)$ is a rewrite theory, and $\mathcal{S} = (\mathcal{F}, S)$ is a TRS. Let $\Delta = (\Gamma \cup \Sigma \cup \mathcal{F}) - (\{root(l) \mid l \to r \in (R - \mathcal{U}_R(\tau))\} \cup \{root(u) \mid u = v \in (E - \mathcal{U}_E(\tau))$ or $v = u \in (E - \mathcal{U}_E(\tau))\})$. If $s$ and $t$ are $E$-terminating and $s \overset{p}{\vdash\!\dashv}_E t$ then $\mathcal{I}_{\Delta,E}(s) \overset{p\ *}{\vdash\!\dashv}_{\mathcal{U}_E(\tau)} \mathcal{I}_{\Delta,E}(t)$.*

Proof. We proceed by induction on the position $p \in \mathcal{P}os(s)$ of the redex in the reduction $s \overset{p}{\vdash\!\dashv}_{\{u=v\}} t$.

- First, we consider that $root(s) \in \Delta$.
  - If $p = \Lambda$ (therefore $u = v \in \mathcal{U}_E(\tau)$). So we have $s = \sigma(u) \overset{\Lambda}{\vdash\!\dashv}_{\{u=v\}} \sigma(v) = t$ or $s = \sigma(v) \overset{\Lambda}{\vdash\!\dashv}_{\{u=v\}} \sigma(u) = t$ for some substitution $\sigma$. Moreover, $u, v \in \mathcal{T}(\Delta, \mathcal{X})$ by the construction of $\Delta$. By Lemma 2, we get $\mathcal{I}_{\Delta,E}(\sigma(u)) = \sigma_{\mathcal{I}_{\Delta,E}}(u) \overset{\Lambda}{\vdash\!\dashv}_{\{u=v\}} \sigma_{\mathcal{I}_{\Delta,E}}(v) = \mathcal{I}_{\Delta,E}(\sigma(v))$ or $\mathcal{I}_{\Delta,E}(\sigma(v)) = \sigma_{\mathcal{I}_{\Delta,E}}(v) \overset{\Lambda}{\vdash\!\dashv}_{\{u=v\}} \sigma_{\mathcal{I}_{\Delta,E}}(u) = \mathcal{I}_{\Delta,E}(\sigma(u))$.
  - If $p \neq \Lambda$ then $s = f(s_1, \ldots, s_i, \ldots, s_n)$, $t = f(s_1, \ldots, t_i, \ldots, s_n)$, $s_i \overset{q}{\vdash\!\dashv}_{\{u=v\}} t_i$ and $p = i.q$. By the induction hypothesis, $\mathcal{I}_{\Delta,E}(s_i) \overset{q\ *}{\vdash\!\dashv}_{\mathcal{U}_E(\tau)} \mathcal{I}_{\Delta,E}(t_i)$, $\mathcal{I}_{\Delta,E}(s) \overset{i.q\ *}{\vdash\!\dashv}_{\mathcal{U}_E(\tau)} \mathcal{I}_{\Delta,E}(t)$ and, hence $\mathcal{I}_{\Delta,E}(s) \overset{p\ *}{\vdash\!\dashv}_{\mathcal{U}_E(\tau)} \mathcal{I}_{\Delta,E}(t)$.

- Finally, we consider the case $root(s) \notin \Delta$. Since we can infer that $t \in [s]_E$ and $[s]_E = [t]_E$ because $s \overset{p}{\longmapsto}_{\{u=v\}} t$ then, $\mathcal{I}_{\Delta,E}(s) = \mathcal{I}_{\Delta,E}(t)$ and, hence, $\mathcal{I}_{\Delta,E}(s) \overset{p}{\longmapsto}^{*}_{\mathcal{U}_E(\tau)} \mathcal{I}_{\Delta,E}(t)$.

$\square$

**Lemma 5** *Let $\tau = (F, P, E, R, S)$ be an A∨C problem where $\mathcal{R} = (\Sigma, E, R)$ is an A∨C-rewrite theory, $\mathcal{P} = (\Gamma, F, P)$ is a rewrite theory, and $\mathcal{S} = (\mathcal{F}, S)$ is a TRS. Let $\Delta = (\Gamma \cup \Sigma \cup \mathcal{F}) - (\{root(l) \mid l \to r \in (R - \mathcal{U}_R(\tau))\} \cup \{root(u) \mid u = v \in (E - \mathcal{U}_E(\tau))$ or $v = u \in (E - \mathcal{U}_E(\tau))\})$. If $s$ and $t$ are E-terminating and $s \simeq_{F,E} t$ then $\mathcal{I}_{\Delta,E}(s) \simeq_{F,\mathcal{U}_E(\tau)} \mathcal{I}_{\Delta,E}(t)$.*

Proof. We can write $s \simeq_{F,E} t$ as $s(\overset{\Lambda}{\longmapsto}_F \cup \overset{>\Lambda}{\longmapsto}_E)^* t$ and we know:

1. If $s' = t'$ trivially $\mathcal{I}_{\Delta,E}(s') = \mathcal{I}_{\Delta,E}(t')$.
2. If $s' \overset{\Lambda}{\longmapsto}_F t'$ for any two terms $s', t'$ then $\mathcal{I}_{\Delta,E}(s') \overset{\Lambda}{\longmapsto}_F \mathcal{I}_{\Delta,E}(t')$ by Lemma 3.
3. If $s' \overset{>\Lambda}{\longmapsto}_E t'$ for any two terms $s', t'$ then $\mathcal{I}_{\Delta,E}(s') \overset{>\Lambda}{\longmapsto}^{*}_{\mathcal{U}_E(\tau)} \mathcal{I}_{\Delta,E}(t')$ by Lemma 4.
4. If $s' \overset{\Lambda}{\longmapsto}_F \cup \overset{>\Lambda}{\longmapsto}_E t'$ for any two terms $s', t'$ then $\mathcal{I}_{\Delta,E}(s') \overset{\Lambda}{\longmapsto}_F \cup \overset{>\Lambda}{\longmapsto}^{*}_{\mathcal{U}_E(\tau)} \mathcal{I}_{\Delta,E}(t')$ by (2) and (3), which is equivalent to $\mathcal{I}_{\Delta,E}(s')(\overset{\Lambda}{\longmapsto}_F \cup \overset{>\Lambda}{\longmapsto}_{\mathcal{U}_E(\tau)})^* \mathcal{I}_{\Delta,E}(t')$.

Now, we proceed on the length $n$ of the sequence $s(\overset{\Lambda}{\longmapsto}_F \cup \overset{>\Lambda}{\longmapsto}_E)^* t$.

- If $n = 0$ then $s = t$ and $\mathcal{I}_{\Delta,E}(s) = \mathcal{I}_{\Delta,E}(t)$.
- If $n > 0$ then $s(\overset{\Lambda}{\longmapsto}_F \cup \overset{>\Lambda}{\longmapsto}_E)u(\overset{\Lambda}{\longmapsto}_F \cup \overset{>\Lambda}{\longmapsto}_E)^* t$. By the induction hypothesis, $\mathcal{I}_{\Delta,E}(u)(\overset{\Lambda}{\longmapsto}_F \cup \overset{>\Lambda}{\longmapsto}_{\mathcal{U}_E(\tau)})^* \mathcal{I}_{\Delta,E}(t)$, and by (4) we have $\mathcal{I}_{\Delta,E}(s)(\overset{\Lambda}{\longmapsto}_F \cup \overset{>\Lambda}{\longmapsto}_{\mathcal{U}_E(\tau)})^* \mathcal{I}_{\Delta,E}(u)$.

Hence, if $s \simeq_{F,E} t$ then $\mathcal{I}_{\Delta,E}(s) \simeq_{F,\mathcal{U}_E(\tau)} \mathcal{I}_{\Delta,E}(t)$. $\square$

**Lemma 6** *Let $\tau = (F, P, E, R, S)$ be an A∨C problem where $\mathcal{R} = (\Sigma, E, R)$ is an A∨C-rewrite theory, $\mathcal{P} = (\Gamma, F, P)$ is a rewrite theory, and $\mathcal{S} = (\mathcal{F}, S)$ is a TRS. Let $\Delta = (\Gamma \cup \Sigma \cup \mathcal{F}) - (\{root(l) \mid l \to r \in (R - \mathcal{U}_R(\tau))\} \cup \{root(u) \mid u = v \in (E - \mathcal{U}_E(\tau))$ or $v = u \in (E - \mathcal{U}_E(\tau))\})$. If $s$ and $t$ are E-terminating and $s \to_{\mathcal{E}xt_E(R),E} t$ then $\mathcal{I}_{\Delta,E}(s) \to^{+}_{\mathcal{E}xt_{\mathcal{U}_E(\tau)}(\mathcal{U}_R(\tau) \cup \mathcal{C}_\varepsilon),\mathcal{U}_E(\tau)} \mathcal{I}_{\Delta,E}(t)$.*

Proof. We proceed by induction on the position $p \in \mathcal{P}os(s)$ of the redex in the reduction $s \overset{p}{\sim}_E s' \xrightarrow{p}_{l \to r} t$ where $l \to r \in \mathcal{E}xt_E(R)$. By recursively applying Lemma 4 to $s = s_1 \longmapsto_E s_2 \longmapsto_E \cdots \longmapsto_E s_n = s'$, we have that $\mathcal{I}_{\Delta,E}(s) = \mathcal{I}_{\Delta,E}(s_1) \longmapsto_{\mathcal{U}_E(\tau)} \mathcal{I}_{\Delta,E}(s_2) \longmapsto_{\mathcal{U}_E(\tau)} \cdots \longmapsto_{\mathcal{U}_E(\tau)} \mathcal{I}_{\Delta,E}(s_n) = \mathcal{I}_{\Delta,E}(s')$.

- First, let $root(s) = root(s') \in \Delta$.
  - If $p = \Lambda$ ($l \to r \in \mathcal{E}xt_{\mathcal{U}_E(\tau)}(\mathcal{U}_R(\tau))$), we have $s' = \sigma(l) \xrightarrow{\Lambda}_{\{l\to r\}} \sigma(r) = t$ for some substitution $\sigma$. Moreover, $r \in \mathcal{T}(\Delta, \mathcal{X})$ by the construction of $\Delta$. By Lemma 2, we get

$$\mathcal{I}_{\Delta,E}(\sigma(l)) \to^*_{\mathcal{C}_\varepsilon} \sigma_{\mathcal{I}_{\Delta,E}}(l) \to_{\{l\to r\}} \sigma_{\mathcal{I}_{\Delta,E}}(r) = \mathcal{I}_{\Delta,E}(\sigma(r))$$

  - If $p \neq \Lambda$ then $s = f(s_1, \ldots, s_i, \ldots, s_n)$, $t = f(s_1, \ldots, t_i, \ldots, s_n)$ and $s_i \to_{\{l\to r\},E} t_i$. By the induction hypothesis,

$$\mathcal{I}_{\Delta,E}(s_i) \to^+_{\mathcal{E}xt_{\mathcal{U}_E(\tau)}(\mathcal{U}_R(\tau)\cup\mathcal{C}_\varepsilon),\mathcal{U}_E(\tau)} \mathcal{I}_{\Delta,E}(t_i)$$

    and, hence also $\mathcal{I}_{\Delta,E}(s) \to^+_{\mathcal{E}xt_{\mathcal{U}_E(\tau)}(\mathcal{U}_R(\tau)\cup\mathcal{C}_\varepsilon),\mathcal{U}_E(\tau)} \mathcal{I}_{\Delta,E}(t)$.

- Finally, let $root(s) = root(s') \notin \Delta$. Then,

$$\mathcal{I}_{\Delta,E}(t) \in order\left(\{\mathcal{I}_{\Delta,E}(u) \mid s \to_{\mathcal{E}xt_E(R),E} u\}\right)$$

  because $s \to_{\mathcal{E}xt_E(R),E} t$. By applying $\mathcal{C}_\varepsilon$ rules, we get $\mathcal{I}_{\Delta,E}(s) \to^+_{\mathcal{C}_\varepsilon} \mathcal{I}_{\Delta,E}(t)$.

Therefore, $\mathcal{I}_{\Delta,E}(s) \to^*_{\mathcal{E}xt_{\mathcal{U}_E(\tau)}(\mathcal{U}_R(\tau)\cup\mathcal{C}_\varepsilon),\mathcal{U}_E(\tau)} \mathcal{I}_{\Delta,E}(t)$. □

**Lemma 7** *Let $\tau = (F, P, E, R, S)$ be an $A{\vee}C$ problem where $\mathcal{R} = (\Sigma, E, R)$ is an $A{\vee}C$-rewrite theory, $\mathcal{P} = (\Gamma, F, P)$ is a rewrite theory, and $\mathcal{S} = (\mathcal{F}, S)$ is a TRS. Let $\Delta = (\Gamma \cup \Sigma \cup \mathcal{F}) - (\{root(l) \mid l \to r \in (R - \mathcal{U}_R(\tau))\} \cup \{root(u) \mid u = v \in (E - \mathcal{U}_E(\tau))$ or $v = u \in (E - \mathcal{U}_E(\tau))\})$. If $s$ and $t$ are $E$-terminating and $s \xrightarrow{\Lambda}_{S_{f_i}} t$ then $\mathcal{I}_{\Delta,E}(s) \to^+_{S_{f_i}\cup\mathcal{C}_\varepsilon} \mathcal{I}_{\Delta,E}(t)$.*

Proof. We know that $p = \Lambda$ and $l \to r \in S_{f_i}$. So we have $s = \sigma(l) \xrightarrow{\Lambda}_{\{l\to r\}} \sigma(r) = t$ for some substitution $\sigma$. Moreover, $r \in \mathcal{T}(\Delta, \mathcal{X})$ by the construction of $\Delta$. By Lemma 2 we get $\mathcal{I}_{\Delta,E}(\sigma(l)) \to^*_{\mathcal{C}_\varepsilon} \sigma_{\mathcal{I}_{\Delta,E}}(l) \to_{\{l\to r\}} \sigma_{\mathcal{I}_{\Delta,E}}(r) = \mathcal{I}_{\Delta,E}(\sigma(r))$. Therefore, $\mathcal{I}_{\Delta,E}(s) \to^+_{S_{f_i}\cup\mathcal{C}_\varepsilon} \mathcal{I}_{\Delta,E}(t)$. □

A relation $\gtrsim$ is $\mathcal{C}_\varepsilon$-compatible iff $\mathsf{c}(x, y) \gtrsim x$ and $\mathsf{c}(x, y) \gtrsim y$ for a new binary fresh symbol $\mathsf{c}$.

**Theorem 10 (RP Processor with $A{\vee}C$-Usable Rules and Equations)** *Let $\tau = (F, P, E, R, S)$ be an $A{\vee}C$ problem where $\mathcal{R} = (\Sigma, E, R)$ is an $A{\vee}C$-rewrite theory, $\mathcal{P} = (\Gamma, F, P)$ is a rewrite theory, and $\mathcal{S} = (\mathcal{F}, S)$ is a TRS. Let $(\gtrsim, \sqsupset)$ be a reduction pair such that $\gtrsim$ is $\mathcal{C}_\varepsilon$-compatible and*

1. *$\mathcal{U}_R(\tau) \subseteq \gtrsim$,*
2. *$(P \cup S) \subseteq \gtrsim \cup \sqsupset$, and*
3. *$F \cup \mathcal{U}_E(E) \subseteq \sim$.*

*Let $P_{\sqsupset} = \{u \to v \in P \mid u \sqsupset \; v\}$ and $S_{\sqsupset} = \{s \to t \in S \mid s \sqsupset \; t\}$. Then, the processor $\mathsf{Proc}_{RP}$ given by*

$$\mathsf{Proc}_{RP}(F, P, E, R, S) = \begin{cases} \{(F, P - P_{\sqsupset}, E, R, S - S_{\sqsupset})\} & \text{if (1), (2), and (3) hold} \\ \{(F, P, E, R, S)\} & \text{otherwise} \end{cases}$$

*is sound and complete.*

Proof. Since $P - P_{\sqsupset} \subseteq P$ and $S - S_{\sqsupset} \subseteq S$, completeness is assured. Regarding soundness, we proceed by contradiction. Assume that there is an infinite minimal $(F, P, E, R, S)$-chain $A$, but that there is no infinite minimal $(F, P - P_{\sqsupset}, E, R, S - S_{\sqsupset})$-chain. Due to the finiteness of $P$ and $S$, we can assume that there is $Q \subseteq P$ and $T \subseteq S$ such that $A$ has a tail $B$ where all pairs in $Q$ and rules in $T$ are infinitely often used. We distinguish two kinds of elementary steps in $B$, according to Definition 5.

1. If $\sigma(v_i) = f_i(v_{i1}, v_{i2})$ satisfies $\sigma(v_i) = \theta_i(u'_i)$ for some $u'_i = v'_i \in F$ or $v'_i = u'_i \in F$ such that $u'_i = f_i(u'_{i1}, u'_{i2})$ satisfies $u'_{i1} \notin \mathcal{X}$ or $u'_{i2} \notin \mathcal{X}$, then

$$\sigma(v_i) \simeq_{F,E} s_i \xrightarrow{\Lambda}{}^{*}_{\mathcal{S}_{f_i}} t_i \to^{*}_{\mathcal{E}xt_E(R),E} t'_i \simeq_{F,E} w_i \xrightarrow{\Lambda}{}^{*}_{\mathcal{S}_{f_i}} w'_i \simeq_{F,E} \sigma(u_{i+1})$$

We apply $\mathcal{I}_{\Delta,E}$ in Definition 13 to the initial term in the sequence. We let $\Delta = (\Gamma \cup \Sigma \cup \mathcal{F}) - (\{root(l) \mid l \to r \in (R - \mathcal{U}_R(\tau))\} \cup \{root(u) \mid u = v \in (E - \mathcal{U}_E(\tau))$ or $v = u \in (E - \mathcal{U}_E(\tau))\})$, $E' = \mathcal{U}_E(\tau)$ and $R' = \mathcal{U}_R(\tau) \cup \mathcal{C}_\varepsilon$. Sequentially, we obtain the following results:

- Since $v_i$ only contains $\Delta$ symbols, by Lemma 2 we have that $\sigma_{\mathcal{I}_{\Delta,E}}(v_i) = \mathcal{I}_{\Delta,E}(\sigma(v_i))$
- By Lemma 5, $\mathcal{I}_{\Delta,E}(\sigma(v_i)) \simeq_{F,E'} \mathcal{I}_{\Delta,E}(s_i)$.
- By induction on the length of the sequence $s_i \xrightarrow{\Lambda}{}^{*}_{\mathcal{S}_{f_i}} t_i$ and using Lemma 7, $\mathcal{I}_{\Delta,E}(s_i) \to^{*}_{\mathcal{S}_{f_i} \cup \mathcal{C}_\varepsilon} \mathcal{I}_{\Delta,E}(t_i)$.
- By induction on the length of the sequence $t_i \to^{*}_{\mathcal{E}xt_E(R),E} t'_i$ and using Lemma 6, $\mathcal{I}_{\Delta,E}(t_i) \to^{*}_{\mathcal{E}xt_{E'}(R'),E'} \mathcal{I}_{\Delta,E}(t'_i)$.
- By Lemma 5, $\mathcal{I}_{\Delta,E}(t'_i) \simeq_{F,E'} \mathcal{I}_{\Delta,E}(w_i)$.
- By induction on the length of the sequence $w_i \xrightarrow{\Lambda}{}^{*}_{\mathcal{S}_{f_i}} w'_i$ and using Lemma 7, $\mathcal{I}_{\Delta,E}(w_i) \to^{*}_{\mathcal{S}_{f_i} \cup \mathcal{C}_\varepsilon} \mathcal{I}_{\Delta,E}(w'_i)$.
- By Lemma 5, $\mathcal{I}_{\Delta,E}(w'_i) \simeq_{F,E'} \mathcal{I}_{\Delta,E}(\sigma(u_{i+1}))$.
- By Lemma 2, $\mathcal{I}_{\Delta,E}(\sigma(u_{i+1})) \to^{*}_{\mathcal{C}_\varepsilon} \sigma_{\mathcal{I}_{\Delta,E}}(u_{i+1})$.

Therefore, we obtain the following chain:

$$\begin{array}{c} \sigma_{\mathcal{I}_{\Delta,E}}(v_i) \simeq_{F,E'} \circ \to^{*}_{\mathcal{S}_{f_i} \cup \mathcal{C}_\varepsilon} \circ \to^{*}_{\mathcal{E}xt_{E'}(R'),E'} t''_i \\ t''_i \simeq_{F,E'} \circ \to^{*}_{\mathcal{S}_{f_i} \cup \mathcal{C}_\varepsilon} \circ \simeq_{F,E'} \circ \to^{*}_{\mathcal{C}_\varepsilon} \sigma_{\mathcal{I}_{\Delta,E}}(u_{i+1}) \end{array}$$

Note that, due to the requirements imposed for the rules in $\mathcal{U}_R(\tau)$ and $S$ and equations in $\mathcal{U}_E(\tau)$ and $F$, and by stability, transitivity and $\mathcal{C}_\varepsilon$-compatibility of $\gtrsim$ (hence of $\sim$), monotonicity and transitivity of $\gtrsim$, we have

$$\sigma_{\mathcal{I}_{\Delta,E}}(v_i) \sim \circ(\gtrsim \cup \sqsupset)\circ \gtrsim \circ \sim \circ(\gtrsim \cup \sqsupset)\circ \sim \circ \gtrsim \sigma_{\mathcal{I}_{\Delta,E}}(u_{i+1})$$

Here, it is important to specifically consider the case when the rules $l \to r$ involved in $\to_{\mathcal{E}xt_{E'}(R'),E'}$-steps are taken from $\mathcal{E}xt_{E'}(R') - (R')$, i.e, $l \to r \notin R'$. In this case, we do not have an explicit compatibility requirement of $l \to r$ with $\gtrsim$, i.e., $l \gtrsim r$ is not explicitly required. However, since $\mathcal{R}' = (\Sigma, E', R')$ is an $A\vee C$ rewrite theory, such rules are connected with rules rule $l' \to r' \in R'$ in a simple way. For instance if $l = f(l', w) \to f(r', w) = r$ for some $l' \to r' \in R'$ such that $root(l') = f$, then, since $l' \gtrsim r'$ holds, by monotonicity of $\gtrsim$, we also have $l = f(l', w) \gtrsim f(r', w) = r$. With other rules included in $\mathcal{E}xt_{E'}(R') - (R')$ (see Section 3.1) we would proceed in a similar way. Now, taking into account that $\sim \circ (\gtrsim \cup \sqsupset) = \gtrsim \cup \sqsupset$ and $\sim \circ \gtrsim = \gtrsim$, we have

$$\sigma_{\mathcal{I}_{\Delta,E}}(v_i)\ (\gtrsim \cup \sqsupset)\ \sigma_{\mathcal{I}_{\Delta,E}}(u_{i+1})$$

Note that, by the compatibility condition required for $\gtrsim$ and $\sqsupset$, this means that $\sigma_{\mathcal{I}_{\Delta,E}}(v_i) \gtrsim \sigma_{\mathcal{I}_{\Delta,E}}(u_{i+1})$ or $\sigma_{\mathcal{I}_{\Delta,E}}(v_i) \sqsupset \sigma_{\mathcal{I}_{\Delta,E}}(u_{i+1})$.

2. If $\sigma(v_i) = t_i \to^*_{\mathcal{E}xt_E(R),E} \circ \simeq_{F,E} \sigma(u_{i+1})$, then we analogously, applying Lemma 6 and Lemma 5, have

$$\sigma_{\mathcal{I}_{\Delta,E}}(v_i) \to^*_{\mathcal{E}xt_{E'}(R'),E'} \circ \simeq_{F,E'} \sigma_{\mathcal{I}_{\Delta,E}}(u_{i+1})$$

and, hence, $\sigma_{\mathcal{I}_{\Delta,E}}(v_i) \gtrsim \sigma_{\mathcal{I}_{\Delta,E}}(u_{i+1})$.

Since $u_i\ (\gtrsim \cup \sqsupset)\ v_i$ for all $u_i \to v_i \in Q \subseteq P$, by stability of $\gtrsim$ and $\sqsupset$, we have $\sigma_{\mathcal{I}_{\Delta,E}}(u_i)\ (\gtrsim \cup \sqsupset)\ \sigma_{\mathcal{I}_{\Delta,E}}(v_i)$ for all $i \geq 1$. No pair $u \to v \in Q$ satisfies that $u \sqsupset v$, and no rule $s \to t \in T$ satisfies $s \sqsupset t$. Since $u \to v$ and $s \to t$ occurs infinitely often in $B$, and taking into account that $\sigma_{\mathcal{I}_{\Delta,E}}(v_i) \gtrsim \sigma_{\mathcal{I}_{\Delta,E}}(u_{i+1})$ or $\sigma_{\mathcal{I}_{\Delta,E}}(v_i) \sqsupset \sigma_{\mathcal{I}_{\Delta,E}}(u_{i+1})$ for all $i \geq 1$, there would be an infinite set $\mathcal{J} \subseteq \mathbb{N}$ such that $\sigma_{\mathcal{I}_{\Delta,E}}(u_i) \sqsupset \sigma_{\mathcal{I}_{\Delta,E}}(u_{i+1})$ for all $i \in \mathcal{J}$ or there would be an infinite set $\mathcal{K} \subseteq \mathbb{N}$ such that $\sigma_{\mathcal{I}_{\Delta,E}}(s_j) \sqsupset \sigma_{\mathcal{I}_{\Delta,E}}(t_{j+1})$ for all $j \in \mathcal{K}$. And we have $\sigma_{\mathcal{I}_{\Delta,E}}(u_i)\ (\gtrsim \cup \sqsupset)\ \sigma_{\mathcal{I}_{\Delta,E}}(u_{i+1})$ for all other $u_i \to v_i \in Q$, and $\sigma_{\mathcal{I}_{\Delta,E}}(s_j)\ (\gtrsim \cup \sqsupset)\ \sigma_{\mathcal{I}_{\Delta,E}}(t_{j+1})$ for all other $u_j \to v_j \in T$. Thus, by using the compatibility conditions of the reduction pair, we obtain an infinite decreasing $\sqsupset$-sequence which contradicts well-foundedness of $\sqsupset$.

Therefore, $Q \subseteq (P - P_\sqsupset)$ and $T \subseteq (S - S_\sqsupset)$, which means that $B$ is an infinite chain, thus leading to a contradiction. □

**Example 13** *For the $A\vee C$-rewrite theory in Figure 1, we have the following rules in $R$ (with prefix symbols again):*

$$\begin{array}{rcll}
\texttt{list2set}(N) & \to & N & (22)\\
\texttt{list2set}(\texttt{cons}(N,L)) & \to & \texttt{union}(N,\texttt{list2set}(L)) & (23)\\
\texttt{in}(N,\texttt{null}) & \to & \texttt{false} & (24)\\
\texttt{in}(N,\texttt{union}(M,S)) & \to & \texttt{or}(\texttt{eq}(N,M),\texttt{in}(N,S)) & (25)\\
\texttt{union}(N,N) & \to & N & (26)\\
\texttt{and}(\texttt{true},B) & \to & B & (27)\\
\texttt{and}(\texttt{false},B) & \to & \texttt{false} & (28)\\
\texttt{or}(\texttt{true},B) & \to & \texttt{true} & (29)\\
\texttt{or}(\texttt{false},B) & \to & B & (30)\\
\texttt{eq}(\texttt{0},\texttt{s}(N)) & \to & \texttt{false} & (31)\\
\texttt{eq}(\texttt{s}(N),\texttt{s}(M)) & \to & \texttt{eq}(N,M) & (32)\\
\texttt{eq}(\texttt{cons}(N,L),M) & \to & \texttt{false} & (33)\\
\texttt{eq}(\texttt{cons}(N,L),\texttt{cons}(M,L')) & \to & \texttt{and}(\texttt{eq}(N,M),\texttt{eq}(L,L')) & (34)\\
\texttt{eq}(L,L) & \to & \texttt{true} & (35)
\end{array}$$

*By example 12, we have the following $A\vee C$ problems:*

- $\tau'_1 = (\varnothing, \{(9)\}, E, R, \varnothing)$,
- $\tau'_2 = (\varnothing, \{(12)\}, E, R, \varnothing)$,
- $\tau'_3 = (E^{\sharp}_{union}, \{(13)\}, E, R, S_{union})$,
- $\tau'_4 = (E^{\sharp}_{and}, \{(14),(15)\}, E, R, S_{and})$,
- $\tau'_5 = (E^{\sharp}_{or}, \{(16),(17)\}, E, R, S_{or})$ *and*
- $\tau'_6 = (E^{\sharp}_{eq}, \{(18),(19),(20)\}, E, R, \varnothing)$.

*For the each of these $A\vee C$ problem,we can apply* $\mathsf{Proc}_{UR}$.

- *In the case of $\tau'_1$ we have:*
  $\mathcal{U}_R(\tau'_1) = \varnothing$, $\mathcal{U}_E(\tau'_1) = \varnothing$ *and the following polynomial interpretation*[2]*:*

  $$[\texttt{LIST2SET}](x) \;=\; x * x + x \qquad [\texttt{cons}](x,y) \;=\; y + 1$$

  *to conclude finiteness of $\tau'_1$.*

[2]The quasi-orderings $\gtrsim$ induced by a polynomial interpretation can always be made compatible with the rules of the TRS $\mathcal{C}_\varepsilon$, i.e., $\mathcal{C}_\varepsilon \subseteq \gtrsim$.

- *For* $\tau'_2$ *we have:*
  $\mathcal{U}_R(\tau'_2) = \varnothing$, $\mathcal{U}_E(\tau'_2) = \varnothing$ *and the following polynomial interpretation:*

$$[\mathtt{IN}](x, y) = x * y + y \qquad [\mathtt{union}](x, y) = y + 1$$

  *to conclude finiteness of* $\tau'_2$.

- *For* $\tau'_3$ *we have:*
  $\mathcal{U}_R(\tau'_3) = \{(26)\}$, $\mathcal{U}_E(\tau'_3) = \{(E_{union})\}$ *and the following polynomial interpretation:*

$$[\mathtt{UNION}](x, y) = x + y + 1 \qquad [\mathtt{union}](x, y) = x + y$$

  *to conclude finiteness of* $\tau'_3$.

- *For* $\tau'_4$ *we have:*
  $\mathcal{U}_R(\tau'_4) = \{(27), (28)\}$, $\mathcal{U}_E(\tau'_4) = \{(E_{and})\}$ *and the following polynomial interpretation:*

$$\begin{array}{rclcrcl} [\mathtt{AND}](x, y) &=& x + y & \qquad & [\mathtt{and}](x, y) &=& x + y + 1 \\ [\mathtt{false}] &=& 1 & & [\mathtt{true}] &=& 1 \end{array}$$

  *This processor eliminate one strict pair and generate a new A∨C problem* $\tau_{4.1} = (E^\sharp_{and}, \{(14)\}, E, R, S_{and})$ *where again we have:*
  $\mathcal{U}_R(\tau_{4.1}) = \{(27), (28)\}$, $\mathcal{U}_E(\tau_{4.1}) = \{(E_{and})\}$ *and the following polynomial interpretation:*

$$\begin{array}{rclcrcl} [\mathtt{AND}](x, y) &=& x + y & \qquad & [\mathtt{and}](x, y) &=& x + y \\ [\mathtt{false}] &=& 1 & & [\mathtt{true}] &=& 1 \end{array}$$

  *to conclude finiteness of* $\tau_{4.1}$ *and therefore of* $\tau'_4$.

- *For* $\tau'_5$ *we have:*
  $\mathcal{U}_R(\tau'_5) = \{(29), (30)\}$, $\mathcal{U}_E(\tau'_5) = \{(E_{or})\}$ *and the following polynomial interpretation:*

$$\begin{array}{rclcrcl} [\mathtt{OR}](x, y) &=& x * y + x + y & \qquad & [\mathtt{or}](x, y) &=& x * y + x + y \\ [\mathtt{false}] &=& 1 & & [\mathtt{true}] &=& 1 \end{array}$$

  *This processor eliminate one strict pair and generate a new A∨C problem* $\tau_{5.1} = (E^\sharp_{or}, \{(16)\}, E, R, S_{or})$ *where again we have:*

$\mathcal{U}_R(\tau_{5.1}) = \{(29), (30)\}$, $\mathcal{U}_E(\tau_{5.1}) = \{(E_{and})\}$ *and the following polynomial interpretation:*

$$\begin{array}{rclcrcl} [\mathtt{OR}](x,y) & = & x+y & & [\mathtt{or}](x,y) & = & x+y+1 \\ [\mathtt{false}] & = & 1 & & [\mathtt{true}] & = & 1 \end{array}$$

*to conclude finiteness of* $\tau_{5.1}$ *and therefore of* $\tau'_5$.

- *Finally, for* $\tau'_6$ *we have:*
  $\mathcal{U}_R(\tau'_6) = \varnothing$, $\mathcal{U}_E(\tau'_6) = \varnothing$ *and the following polynomial interpretation:*

$$\begin{array}{rclcrcl} [\mathtt{EQ}](x,y) & = & x*y+x+y & & [\mathtt{cons}](x,y) & = & x+y+1 \\ [\mathtt{s}](x) & = & x+1 & & & & \end{array}$$

*This processor eliminate one strict pair and generate a new A∨C problem* $\tau_{6.1} = (E^{\sharp}_{eq}, \{(18), (19)\}, E, R, \varnothing)$ *where we have:*

$\mathcal{U}_R(\tau_{6.1}) = \varnothing$, $\mathcal{U}_E(\tau_{6.1}) = \varnothing$ *and the following polynomial interpretation:*

$$\begin{array}{rclcrcl} [\mathtt{EQ}](x,y) & = & x*y+x+y & & [\mathtt{cons}](x,y) & = & x+1 \\ [\mathtt{s}](x) & = & x+1 & & & & \end{array}$$

*This application eliminate another strict pair and generate a new A∨C problem* $\tau_{6.2} = (E^{\sharp}_{eq}, \{(18)\}, E, R, \varnothing)$. *We have:*

$\mathcal{U}_R(\tau_{6.2}) = \varnothing$, $\mathcal{U}_E(\tau_{6.2}) = \varnothing$ *and the following polynomial interpretation:*

$$\begin{array}{rclcrcl} [\mathtt{EQ}](x,y) & = & x*y+x+y & & [\mathtt{s}](x) & = & x+1 \end{array}$$

*to conclude finiteness of* $\tau_{6.2}$ *and therefore of* $\tau'_6$.

*Therefore, after showing the finiteness of all the A∨C problems generated from Example 1, we can conclude its E-termination.*

# 9 Benchmarks

We have implemented all techniques described in this paper in the termination tool MU-TERM. MU-TERM is a tool which can be used to verify a number of termination properties of (variants of) Term Rewriting Systems (TRSs): termination of rewriting, termination of innermost rewriting, termination of order-sorted rewriting, termination of context-sensitive rewriting, termination of innermost context-sensitive rewriting and, thanks to this new approach, termination of rewriting modulo specific axioms. With these new features implemented, MU-TERM has been able to participate in the International Competition of Termination Tools[3] in the category of *TRS Equational*. This is not the first implementation for proving termination of rewriting modulo axioms: C*i*ME [5] is able to prove AC-termination of TRSs, and APROVE [11] is able to deal with termination of rewriting modulo equations satisfying some restrictions. However, in the last editions of the competition C*i*ME has not participated and APROVE is the only termination tool that participates in this category from its first edition in 2004. There exists a *Termination Problem Data Base*[4] (TPDB) which contains 71 examples in the equational category[5]. In the 2010 edition there were only two participants: APROVE and MU-TERM. The organization selected randomly a subset of 34 examples from the entire set. MU-TERM was able to solve 16 out of them whereas APROVE solved 24. We considered this result as a good one since only a few techniques had been implemented to deal with termination modulo axioms and APROVE implements specific techniques since 2004. These include an *AC*-recursive path order (RPO) with status (3 examples out of them are solved with it) and processors based on usable rules (the remaining 5 examples are solved using them). There is no formal publication of any of these techniques. In the case of the AC-RPO, we suppose that they implement the master thesis of Stephan Falke [9] although [24] was published before. Recently, we have found out that this work was adapted to the dependency pair framework in the master thesis of Christian Stein ([25], in german and not available publicly). However, both papers are based on the notion of minimality presented in [10] which we have shown that is not appropriate. In the case of processors for managing usable rules is essential to deal with a correct notion of minimality [14, 26].

Now, with only the techniques described in this paper, MU-TERM is able to solve 59 examples out of 71. Two examples more than APROVE[6]. For full details see:

`http://zenon.dsic.upv.es/muterm/benchmarks/benchmarks-avc/benchmarks.html`

In comparison with the implementation of the techniques developed in [3], where MU-TERM were able to solve 39 examples[7], now, thanks to the new techniques, MU-TERM has become a powerful and competitive tool for proving termination of $A\vee C$-rewrite theories. The practical results are summarized in Table 1.

# 10 Related Work and Conclusions

This paper is an extended and revised version of [3]. We provide complete proofs for all results, and also present more examples about the use of the theory. The main conceptual differences between [3] and this paper can be summarized as follows:

[3] See `http://www.lri.fr/~marche/termination-competition/`

[4] See `http://termination-portal.org/wiki/TPDB`

[5] We have used version 7.0.2 of the TPDB.

[6] See the 2008 edition of the termination competition where the entire set of examples from the category where considered.

[7] see `http://www.dsic.upv.es/~balarcon/WRLA10/benchmarks.html`

| | MU-TERM $A\vee C$-DPs | APROVE | MU-TERM [3] |
|---|---|---|---|
| YES score | 59 | 57 | 39 |
| YES average time | 6.83 sec. | 5.12 sec. | 40.13 sec. |

Table 1: Comparative in proofs of termination of $A\vee C$-rewrite theories

- We have refined the notion of $A\vee C$- dependency pairs integrating an equational extension of Dershowitz's refinement of standard dependency pairs (see [6]).
- We have refined the notion of $A\vee C$-chain by allowing the application of $F$ axioms only at the root position.
- We have developed a preprocessing technique which is often able to remove rules from the original system before starting the proof in the $A\vee C$ DP-framework, thus simplifying the whole proof of $A\vee C$-termination.
- We have refined the $A\vee C$ processor of reduction pairs which is now able to eliminate rules, not only from $R$, but also from the set $S$.
- We have developed a new $A\vee C$ processor that restricts the set of $F$ axioms to those that are really used in the $A\vee C$ problem.
- We have extended the well-known technique of usable rules to $A\vee C$-termination and we have developed the corresponding $A\vee C$ processor to eliminate pairs and rules by means of reduction orders.
- We have implemented the techniques presented in [3] and the ones developed here. We have made some benchmarks showing the performance of them.

As remarked in the introduction, this is not the first work which tries to use dependency pairs for proving termination of rewriting modulo an equational theory, see [9, 10, 16, 17, 18, 20, 21, 25]. Our work, however, is, as far as we know, the first one which provides a satisfactory notion of minimal non-terminating term for an $A\vee C$-rewrite theory $\mathcal{R} = (\Sigma, E, R)$ which can be used to provide a suitable definition of minimal chain of dependency pairs, which can in turn be used to characterize $A\vee C$-termination (Corollary 5). In order to substantiate this claim, consider the AC-rewrite theory $\mathcal{R} = (\Sigma, E, R)$ in Example 5 again. The $A\vee C$-DPs for $\mathcal{R}$ are enumerated in Example 10. Such dependency pairs coincide with the ones which would be computed by, e.g., [9, 10, 17, 18, 25]. Remember that $t$ in Example 5 is *minimal* in Giesl and Kapur's sense (Definition 2); and also according to [9, 25] which inherit this notion. We should, then, be able to find an infinite *minimal* chain of DPs starting from $t^\sharp$. According to [9, 10, 17, 18, 25], 'minimal' means that $\sigma(v_i)$ is $(\mathcal{E}xt_E(R), E)$-terminating for all pairs $u_i \to v_i \in \mathsf{DP}_E(R)$ in the chain of dependency pairs induced by the substitution $\sigma$. However, this is *not* possible:

the marked version $t^\sharp$ of $t$ is $F(f(0,1), f(0, f(1,2)))$, which is an $(\mathcal{E}xt_E(R), E)$-terminating term. After some $E^\sharp \cup E$-equivalence steps (where $E^\sharp$ is applied only at root position) we would be able to apply one of the rules in $\mathsf{DP}_E(R)$. Note, however, that *no rule* $u \to v \in \mathsf{DP}_E(R)$ except (5) has a right-hand side $v$ which can be rewritten (after instantiation into $\sigma(v)$) into an instance $\sigma(u')$ of the left-hand side $u'$ of any other pair in $\mathsf{DP}_E(R)$ by means of $(\mathcal{E}xt_E(R), E^\sharp \cup E)$-rewriting steps. This means that only the dependency pair (5) could be used in any infinite minimal chain of dependency pairs starting from $t^\sharp$. But such a chain would start as follows:

$$F(f(0,1), f(0, f(1,2))) \simeq_{E^\sharp, E} F(f(0,0), f(1, f(1,2))) \to_{(5)} F(f(0, f(1,2)), f(1, f(1,2)))$$

where $F(f(0, f(1,2)), f(1, f(1,2)))$ contains a subterm $f(1, f(1,2))$ which, as showed in Example 5, is $(\mathcal{E}xt_E(R), E)$-nonterminating. Therefore, this chain of dependency pairs is *not* minimal. We conclude that, according to the notion of minimal chain in the aforementioned papers, *there is no minimal chain of pairs starting from* $t^\sharp$. This means that no *sound* approach to proving AC-termination on the basis of such notion of minimal chain is possible. In this paper we have introduced the notion of *stably minimal term* (Definition 3) which overcomes these problems (Proposition 11 and Theorem 3) and leads to an appropriate characterization of $A{\vee}C$-termination as the absence of infinite minimal chains of $A{\vee}C$-DPs (Definitions 4 and 5, and Corollary 5).

Furthermore, we note that [17, 18] deal with *AC-rewrite theories* only, and that [10], which considers more general rewrite theories $E$ including $A{\vee}C$-theories do not cover our work in a second respect: when purely associative theories are considered (i.e., rewrite theories $\mathcal{R} = (\Sigma, E, R)$ such that $E_f \subseteq \{A_f\}$ for all $f \in \Sigma$), then Giesl and Kapur's technique requires the computation of *instances* of the rules in $\mathcal{E}xt_E(R)$ for which the computation of *all* the $E$-unifiers $uni_E(v, l)$ of $v$ and $l$ for the rules $l \to r$ in $\mathcal{E}xt_E(R)$ and equations $u = v \in E$ or $v = u \in E$ is required. It is well-known, however, that the $E$-unification problem for associative theories $E$ is *infinitary*, which means that $uni_E(v, l)$ is not guaranteed to be finite, in general. In sharp contrast, we do not have to do that for dealing with purely associative rewrite theories $\mathcal{R}$.

Our second main (and novel) contribution is the formalization of an $A{\vee}C$-dependency pair framework (Definitions 6 and 7) which, on the basis of the previously developed theory, can be used to develop automatic tools for proving termination of $A{\vee}C$-rewrite theories (Theorem 6). Several important processors have been developed as well: the SCC processor (Theorem 7), the reduction pair processor (Theorem 8), the processor that restricts the set of $F$ axioms (Theorem 9), and the reduction pair processor with usable rules and equations (Theorem 10). We have implemented the techniques described in this paper in the termination tool MU-TERM and we have developed some benchmarks, showing that our $A{\vee}C$-DP Framework is currently the most powerful approach for proving termination of $A{\vee}C$-rewrite theories. As we have commented, the implementation of the techniques in [3] allowed us to participate in the termination competition in the equational category in the TPDB and therefore providing MU-TERM the ability of proving termination modulo axioms. Thanks to these new improvements, MU-TERM is a powerful tool for proving termination of $A \vee C$- rewrite theories and as far as we know, no tool is able to solve more examples from the equational category. Much work remains ahead in terms of further developing the new $A \vee C$-dependency pair framework. Appropriate reduction orderings which are well-suited for being used in the reduction pair processor should be investigated. It would also be very useful to explore how the requirements on $E$ can be relaxed to handle even more general sets of axioms. Regarding tool support for the method we have presented, we have integrated it within the tool MU-TERM [2]. In this way, our termination technique modulo arbitrary combinations of associative and/or commutative axioms is applicable to an even wider range of rewrite theories, which can be transformed into $A \vee C$-theories by non-termination-preserving transformations [7, 8, 19].

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