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τ-metrizable spaces

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Abstract

In [1], A. A. Borubaev introduced the concept of τ-metric space, where τ is an arbitrary cardinal number. The class of τ-metric spaces as τ runs through the cardinal numbers contains all ordinary metric spaces (for $\tau = 1$) and thus these spaces are a generalization of metric spaces. In this paper the notion of τ-metrizable space is considered.

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1. Preliminaries and notations

Our notation and terminology is standard and generally follows [2]. The cardinality of a set X is denoted by $|X|$. Throughout, we denote by τ an arbitrary nonzero cardinal number. The cardinalities of the natural numbers and of the real numbers are denoted by $\aleph_0$ and $\mathfrak{c}$, respectively. The character, the weight and the density of a topological space X are denoted by $\chi(X)$, $w(X)$ and $d(X)$, respectively. As usual I denotes the closed unit interval $[0, 1]$ with the Euclidean metric topology.

By $\mathbb{R}^\tau_+$ we denote the topological product of τ copies of the space $\mathbb{R}_+ = [0, +\infty)$ (with the natural topology). On the space $\mathbb{R}^\tau_+$, the operations of addition, multiplication, and multiplication by a scalar, as well as a partial ordering, are defined in a natural way (coordinatewise).

Now, we present the notion of τ-metric space [1]. Let X be a nonempty set. A mapping $\rho_\tau : X \times X \to \mathbb{R}^\tau_+$ is called a τ-metric on X if the following axioms hold:

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(1) $\rho_\tau(x,y) = \theta$ if and only if $x = y$, where θ is the point of the space $\mathbb{R}^\tau_+$ whose all coordinates are zeros.
(2) $\rho_\tau(x,y) = \rho_\tau(y,x)$ for all $x, y \in X$.
(3) $\rho_\tau(x,z) \leqslant \rho_\tau(x,y) + \rho_\tau(y,z)$ for all $x, y, z \in X$.

The pair (X, ρ_τ) is called a τ-metric space and the elements of X are called points.

Every τ-metric space (X, ρ_τ) generates a Tychonoff (that is, completely regular and Hausdorff) topological space (X, T_{ρ_τ}). The topology T_{ρ_τ} on X defined by the local basis consisting of the sets of the form

$$G(x) = \{y \in X : \rho_\tau(x,y) \in O(\theta)\},$$

where $O(\theta)$ runs through all open neighbourhoods of the point θ in the space $\mathbb{R}^\tau_+$, of each point $x \in X$ is called the topology induced by the τ-metric ρ_τ.

In this paper the notion of τ-metrizable space is introduced. The paper is organized as follows. Section 2 contains the basic concepts of τ-metrizable spaces. Generally, τ-metrizable spaces may be not metrizable. We prove that if $\tau \leqslant \aleph_0$, then every τ-metrizable space is metrizable. In section 3 we obtain a generalization of the classical metrization theorem of Urysohn. More precisely, we prove that every Tychonoff space of weight $\tau > \aleph_0$ is τ-metrizable. Finally, in section 4 we prove that every compact τ-metrizable space has density less than or equal to τ.

2. Basic concepts

The notion of a τ-metric space leads to the notion of a τ-metrizable space which is inserted in the following definition.

Definition 2.1. A topological space (X, T) is called τ-metrizable if there exists a τ-metric ρ_τ on the set X such that the topology T_{ρ_τ} induced by the τ-metric ρ_τ coincides with the original topology T of X. τ-metrics on the set X which induce the original topology of X will be called τ-metrics on the space X.

Note that τ-metrizable spaces are useful because only such spaces can be presented as limits of τ-long projective systems of metric spaces [1, Theorem 3].

Proposition 2.2. *A metric space is τ-metrizable.*

Proof. Let (X, ρ) be a metric space, t_ρ be the topology induced by the metric ρ, and let τ be a cardinal number. Consider a set Λ such that $|\Lambda| = \tau$ and set $\rho_\lambda = \rho$ for each $\lambda \in \Lambda$. The mapping $\rho_\tau : X \times X \to \mathbb{R}^\tau_+$ defined by $\rho_\tau(x,y) = \{\rho_\lambda(x,y)\}_{\lambda \in \Lambda}$ for every $x, y \in X$ is a τ-metric on X. It is easy to see that $t_\rho = T_{\rho_\tau}$. □

Proposition 2.3. *A τ-metrizable space is τ'-metrizable for every cardinal number $\tau' > \tau$.*

Proof. Let X be a τ-metrizable space, ρ_τ be a τ-metric on the space X and τ' be a cardinal number such that $\tau' > \tau$. Consider two sets K and Λ such that $K \subset \Lambda$, $|K| = \tau$ and $|\Lambda| = \tau'$, and set $\rho_\tau(x,y) = \{\rho_\tau^k(x,y)\}_{k\in K}$ for every $x, y \in X$. Let k_0 be one fixed element of K. The mapping $\rho_{\tau'} : X \times X \to \mathbb{R}_+^{\tau'}$ defined by $\rho_{\tau'}(x,y) = \{\rho_{\tau'}^\lambda(x,y)\}_{\lambda\in\Lambda}$ for every $x, y \in X$, where

$$\rho_{\tau'}^\lambda(x,y) = \begin{cases} \rho_\tau^\lambda(x,y), & \text{if } \lambda \in K \\ \rho_\tau^{k_0}(x,y), & \text{if } \lambda \in \Lambda \setminus K, \end{cases}$$

is a τ'-metric on X such that $T_{\rho_{\tau'}} = T_{\rho_\tau}$. □

The following examples show that τ-metrizable spaces may be not metrizable.

Example 2.4. The product $\mathbb{R}^{\mathfrak{c}} = \prod_{\lambda\in\Lambda} X_\lambda$, where $X_\lambda = \mathbb{R}$ for every $\lambda \in \Lambda$ and $|\Lambda| = \mathfrak{c}$, of uncountably many copies of the real line $\mathbb{R}$ is not metrizable, since it is not first-countable. However, the space $\mathbb{R}^{\mathfrak{c}}$ is $\mathfrak{c}$-metrizable. Assuming each copy X_λ of $\mathbb{R}$ has its usual metric d_λ, the mapping $\rho_{\mathfrak{c}} : \mathbb{R}^{\mathfrak{c}} \times \mathbb{R}^{\mathfrak{c}} \to \mathbb{R}_+^{\mathfrak{c}}$ defined by $\rho_{\mathfrak{c}}(x,y) = \{d_\lambda(x_\lambda, y_\lambda)\}_{\lambda\in\Lambda}$ for every $x = \{x_\lambda\}_{\lambda\in\Lambda} \in \mathbb{R}^{\mathfrak{c}}$ and $y = \{y_\lambda\}_{\lambda\in\Lambda} \in \mathbb{R}^{\mathfrak{c}}$ is a $\mathfrak{c}$-metric on $\mathbb{R}^{\mathfrak{c}}$ and the topology induced by $\rho_{\mathfrak{c}}$ coincides with the product topology.

Example 2.5. Let $\mathbb{R}$ be the set of real numbers with the discrete topology $\mathcal{D}$ and $(\mathbb{R}_\infty, \mathcal{D}_\infty)$ be the Alexandroff's one-point compactification of the space $(\mathbb{R}, \mathcal{D})$, that is $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ and $\mathcal{D}_\infty = \mathcal{D} \cup \{\mathbb{R}_\infty \setminus K : K \text{ is a finite subset of } \mathbb{R}\}$. The space $(\mathbb{R}_\infty, \mathcal{D}_\infty)$ is not metrizable (because it is not separable). We prove that the space $(\mathbb{R}_\infty, \mathcal{D}_\infty)$ is $\mathfrak{c}$-metrizable. Let $Fin(\mathbb{R})$ be the collection of all the nonempty finite subsets of $\mathbb{R}$ with $|Fin(\mathbb{R})| = \mathfrak{c}$. For every $F \in Fin(\mathbb{R})$ we define:

(1) $\rho_F(x,x) = 0$ for each $x \in \mathbb{R}_\infty$.

(2) $\rho_F(x,\infty) = \rho_F(\infty,x) = \begin{cases} 0, & \text{if } x \notin F \\ 1, & \text{otherwise} \end{cases}$ for each $x \in \mathbb{R}$.

(3) $\rho_F(x,y) = \begin{cases} 0, & \text{if } x \notin F \text{ and } y \notin F \\ 1, & \text{otherwise} \end{cases}$ for each $x, y \in \mathbb{R}$ with $x \neq y$.

The mapping $\rho_{\mathfrak{c}} : \mathbb{R}_\infty \times \mathbb{R}_\infty \to \mathbb{R}_+^{\mathfrak{c}}$ defined by $\rho_{\mathfrak{c}}(x,y) = \{\rho_F(x,y)\}_{F\in Fin(\mathbb{R})}$ for every $x, y \in \mathbb{R}_\infty$ is a $\mathfrak{c}$-metric on $\mathbb{R}_\infty$. We prove that the topology $T_{\rho_{\mathfrak{c}}}$ induced by the $\mathfrak{c}$-metric $\rho_{\mathfrak{c}}$ coincides with the topology $\mathcal{D}_\infty$.

Let $x \in \mathbb{R}$. If $G(x) = \{y \in \mathbb{R}_\infty : \rho_{\mathfrak{c}}(x,y) \in O(\theta)\}$, where $O(\theta)$ is an open neighbourhood of the point θ in the space $\mathbb{R}_+^{\mathfrak{c}}$, then $\{x\} \in \mathcal{D}_\infty$ and $\{x\} \subseteq G(x)$. Moreover, for the open neighbourhood $\prod_{F\in Fin(\mathbb{R})} W_F$ of the point θ, where

$$W_F = \begin{cases} [0, \frac{1}{2}), & \text{if } F = \{x\} \\ \mathbb{R}_+, & \text{otherwise} \end{cases}$$

we have $G(x) = \{y \in \mathbb{R}_\infty : \rho_{\mathfrak{c}}(x,y) \in \prod_{F\in Fin(\mathbb{R})} W_F\} \subseteq \{x\}$.

Now, we consider the point ∞ of $\mathbb{R}_\infty$. If $\{\infty\} \cup (\mathbb{R} \setminus K)$, where $K \in Fin(\mathbb{R})$ is an open neighbourhood of the point ∞ in the space $\mathbb{R}_\infty$, then for the open neighbourhood $\prod_{F\in Fin(\mathbb{R})} W_F$ of the point θ, where

$$W_F = \begin{cases} [0, \frac{1}{2}), & \text{if } F = K \\ \mathbb{R}_+, & \text{otherwise} \end{cases}$$

we have $G(\infty) = \{y \in \mathbb{R}_\infty : \rho_{\mathfrak{c}}(\infty, y) \in \prod_{F\in Fin(\mathbb{R})} W_F\} \subseteq \{\infty\} \cup (\mathbb{R} \setminus K)$. Finally, let $\prod_{F\in Fin(\mathbb{R})} U_F$ be an open neighbourhood of the point θ in the space $\mathbb{R}_+^{\mathfrak{c}}$ and suppose that $\{F \in Fin(\mathbb{R}) : U_F \neq \mathbb{R}_+\} = \{K_1, \ldots, K_m\}$. Then,

$$\{\infty\} \cup (\mathbb{R} \setminus (K_1 \cup \ldots \cup K_m)) \subseteq G(\infty) = \{y \in \mathbb{R}_\infty : \rho_{\mathfrak{c}}(\infty, y) \in \prod_{F\in Fin(\mathbb{R})} U_F\}.$$

However, a τ-metrizable space may be metrizable considering addition conditions as the following assertions show.

Proposition 2.6. *A n-metric space is metrizable for every finite cardinal number n.*

Proof. Let (X, ρ_n) be a n-metric space and T_{ρ_n} be the topology induced by ρ_n. Consider a vector expression of the form $\rho_n(x, y) = (\rho_n^1(x, y), \ldots, \rho_n^n(x, y))$ for every $x, y \in X$. The mapping $\rho : X \times X \to \mathbb{R}_+$ defined by

$$\rho(x, y) = \max\{\rho_n^1(x, y), \ldots, \rho_n^n(x, y)\}$$

for every $x, y \in X$ is a metric on X. It is easy to see that the metric topology is the same as T_{ρ_n}. □

Definition 2.7. Two τ-metrics ρ_{1_τ} and ρ_{2_τ} on a set X are called equivalent if they induce the same topology on X, that is $T_{\rho_{1_\tau}} = T_{\rho_{2_\tau}}$.

Example 2.8. Let ρ_τ be a τ-metric on X. Consider a set Λ such that $|\Lambda| = \tau$ and let us set $\rho_\tau(x, y) = \{\rho_\tau^\lambda(x, y)\}_{\lambda\in\Lambda}$ for every $x, y \in X$. The mapping $\rho_\tau^* : X \times X \to \mathbb{R}_+^\tau$ defined by $\rho_\tau^*(x, y) = \big\{\min\{1, \rho_\tau^\lambda(x, y)\}\big\}_{\lambda\in\Lambda}$ for every $x, y \in X$ is a τ-metric on X equivalent to ρ_τ.

Proposition 2.9. *An $\aleph_0$-metric space is metrizable.*

Proof. Let $(X, \rho_{\aleph_0})$ be an $\aleph_0$-metric space. Consider the equivalent $\aleph_0$-metric $\rho_{\aleph_0}^*$ to $\rho_{\aleph_0}$ of Example 2.8. Let $\rho_{\aleph_0}^*(x, y) = (\rho_{\aleph_0}^{*1}(x, y), \rho_{\aleph_0}^{*2}(x, y), \ldots)$ for every $x, y \in X$. The mapping $\rho : X \times X \to \mathbb{R}_+$ defined by

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \rho_{\aleph_0}^{*i}(x, y)$$

for every $x, y \in X$ is a metric on X. The process of proving that the topology induced by the metric ρ coincides with the topology $T_{\rho_{\aleph_0}}$ is similar to the proof of the Theorem 4.2.2 of [2]. □

Corollary 2.10. *If $\tau \leqslant \aleph_0$, then every τ-metrizable space is metrizable.*

Proof. Follows directly from Propositions 2.6 and 2.9. □

Proposition 2.11. *For each $\tau > \aleph_0$ there is a τ-metrizable space X_τ with $w(X_\tau) = \tau$, which is not metrizable.*

Proof. Let X_τ be the Alexandroff's one-point compactification of a discrete space X of cardinality τ, where $\tau > \aleph_0$. The space X_τ is not metrizable (because it is not separable). It is known that $|Fin(X)| = |X| = \tau$. Therefore, in the same manner as in Example 2.5, we can prove that the space X_τ is τ-metrizable. Let us note that $w(X_\tau) = \tau$. □

Proposition 2.12. *For every $\tau \geqslant \aleph_0$ and every τ-metrizable space X, we have $\chi(X) \leqslant \tau$.*

Proof. Let X be a τ-metrizable space and ρ_τ be a τ-metric on the space X with $\tau \geqslant \aleph_0$. Consider a set Λ such that $|\Lambda| = \tau$. The family $\mathcal{B}_\theta$ of all products $\prod_{\lambda\in\Lambda} W_\lambda$, where finitely many W_λ are intervals of the form $[0, b)$ with rational b and the remaining $W_\lambda = \mathbb{R}_+$, form a local basis of the point θ in the space $\mathbb{R}^\tau_+$. Hence, for every $x \in X$, the family

$$\mathcal{B}(x) = \{G(x) = \{y \in X : \rho_\tau(x, y) \in B\} : B \in \mathcal{B}_\theta\}$$

is a local basis of the point x in the space X. Since $|\mathcal{B}_\theta| = \tau$, we have $|\mathcal{B}(x)| \leqslant \tau$. □

3. A τ-metrization theorem

Metrization theorems are theorems that give sufficient conditions for a topological space to be metrizable (see [2,5]). In this section we obtain a generalization of the classical metrization theorem of Urysohn.

Lemma 3.1. *If (X, ρ_τ) is a τ-metric space and A is a subspace of X, then the topology induced by the restriction of the τ-metric ρ_τ to $A \times A$ is the same as the subspace topology of A in X.*

Theorem 3.2. *Every Tychonoff space of weight $\tau > \aleph_0$ is τ-metrizable.*

Proof. Let X be a Tychonoff space such that $w(X) = \tau > \aleph_0$. The space $I^\tau = \prod_{\lambda\in\Lambda} X_\lambda$, where $X_\lambda = I$ for every $\lambda \in \Lambda$ and $|\Lambda| = \tau$ is τ-metrizable (see Example 2.4). Assuming each copy X_λ of I has its usual metric d_λ, the mapping $d_\tau : I^\tau \times I^\tau \to \mathbb{R}^\tau_+$ defined by $d_\tau(x, y) = \{d_\lambda(x_\lambda, y_\lambda)\}_{\lambda\in\Lambda}$ for every $x = \{x_\lambda\}_{\lambda\in\Lambda} \in I^\tau$ and $y = \{y_\lambda\}_{\lambda\in\Lambda} \in I^\tau$ is a τ-metric on I^τ. We shall prove that X is τ-metrizable by imbedding X into the τ-metrizable space I^τ, i.e. by showing that X is homeomorphic with a subspace of I^τ. But this follows immediately from the fact that the Tychonoff cube I^τ is universal for all Tychonoff spaces of weight τ (see [2, Theorem 2.3.23]). By Lemma 3.1, the space X is τ-metrizable. □

As every τ-metrizable space is Tychonoff (see [1]), we get the following result.

Corollary 3.3. *A space of weight $\tau > \aleph_0$ is τ-metrizable if and only if it is Tychonoff.*

Remark 3.4. We can use Theorem 3.2 to find τ-metrizable spaces, where $\tau > \aleph_0$, that are not metrizable. Below we consider some examples. Example 3.5 is a $\mathfrak{c}$-metrizable space which is not second-countable, Example 3.6 is a $\mathfrak{c}$-metrizable space which is not normal and Example 3.7 is a 2^τ-metrizable space, where $\tau \geqslant \mathfrak{c}$, which is not metrizable.

Example 3.5. Let S be the Sorgenfrey line, that is the real line with the topology in which local basis of x are the sets $[x, y)$ for $y > x$. Since X is separable but not second-countable, it cannot be metrizable. Furthermore, S is Tychonoff and $w(S) = \mathfrak{c}$. From Theorem 3.2 it follows that the Sorgenfrey line is a $\mathfrak{c}$-metrizable space.

Example 3.6. Let $P = \{(\alpha, \beta) \in \mathbb{R}^2 : \beta > 0\}$ be the open upper half-plane with the Euclidean topology and $L = \{(\alpha, \beta) \in \mathbb{R}^2 : \beta = 0\}$. We set $X = P \cup L$. For every $x \in P$ let $B(x)$ be the family of all open discs in P centered at x. For every $x \in L$ let $B(x)$ be the family of all sets of the form $\{x\} \cup D$, where D is an open disc in P which is tangent to L at the point x. The family T of all subsets of X that are unions of subfamilies of $\cup\{B(x) : x \in X\}$ is a topology on X and the family $\{B(x) : x \in X\}$ is a neighbourhood system for the topological space (X, T). The space X is called the Niemytzki plane (see, for example, [2, 4]). X is a Tychonoff space with $w(X) = \mathfrak{c}$, which is not normal. Therefore, by Theorem 3.2, X is a $\mathfrak{c}$-metrizable space, but not metrizable.

Example 3.7. Let $\beta D(\tau)$ be the Čech-Stone compactification of the discrete space $D(\tau)$ of cardinality $\tau \geqslant \mathfrak{c}$. Then, $w(\beta D(\tau)) = 2^\tau$ (see [2, Theorem 3.6.11]). Since $\beta D(\tau)$ is zero-dimensional (see [2, Theorem 3.6.13]), it is Tychonoff. The space $D(\tau)$ is not compact. Therefore, $\beta D(\tau)$ is not metrizable (see [3, Exercise 9, §38, Ch.5]). From Theorem 3.2 it follows that $\beta D(\tau)$ is 2^τ-metrizable. Particularly, if one assumes the continuum hypothesis, the Čech-Stone compactification $\beta\omega$ of the discrete space of the non-negative integers $\omega = \{0, 1, 2, \ldots\}$ is $\mathfrak{c}$-metrizable.

Remark 3.8. A space X may be τ-metrizable for some infinite cardinal number $\tau < w(X)$, as shown in the following example.

Example 3.9. Let Λ be a set of cardinality $\tau > \aleph_0$, $D(\kappa)$ the discrete space of cardinality $\kappa > \tau$, and $F = \prod_{\lambda \in \Lambda} X_\lambda$, where $X_\lambda = D(\kappa)$ for every $\lambda \in \Lambda$, with the Tychonoff product topology. We note that the points of F are functions from Λ to $D(\kappa)$. The space F is not metrizable for $\chi(F) = \tau$ (see [2, Exercise 2.3.F(b)]). Moreover, $w(F) = \kappa$ (see [2, Exercise 2.3.F(a)]). We prove that the space F is τ-metrizable. For every $\lambda \in \Lambda$ we define:

(1) $\rho_\lambda(f, f) = 0$ for each $f \in F$.

(2) $\rho_\lambda(f, g) = \begin{cases} 0, & \text{if } f(\lambda) = g(\lambda) \\ 1, & \text{otherwise} \end{cases}$ for each $f, g \in F$ with $f \neq g$.

The mapping $\rho_\tau : F \times F \to \mathbb{R}^\tau_+$ defined by $\rho_\tau(f, g) = \{\rho_\lambda(f, g)\}_{\lambda \in \Lambda}$ for every $f, g \in F$ is a τ-metric on F. We prove that the topology T_{ρ_τ} induced by the τ-metric ρ_τ coincides with the Tychonoff product topology.

Let $f \in F$, $\prod_{\lambda\in\Lambda} U_\lambda$ be an open neighbourhood of the point θ in the space $\mathbb{R}_+^\tau$, and suppose that $\{\lambda \in \Lambda : U_\lambda \neq \mathbb{R}_+\} = \{\lambda_1, \ldots, \lambda_m\}$. For the open neighbourhood $\prod_{\lambda\in\Lambda} W_\lambda$ of the point f, where

$$W_\lambda = \begin{cases} \{f(\lambda)\}, & \text{if } \lambda \in \{\lambda_1, \ldots, \lambda_m\} \\ D(\kappa), & \text{otherwise} \end{cases}$$

we have $\prod_{\lambda\in\Lambda} W_\lambda \subseteq G(f) = \{g \in F : \rho_\tau(f,g) \in \prod_{\lambda\in\Lambda} U_\lambda\}$.

Now, let $f \in F$ and $\prod_{\lambda\in\Lambda} W_\lambda$ be an open neighbourhood of the point f in the space F, and suppose that $\{\lambda \in \Lambda : W_\lambda \neq D(\kappa)\} = \{\lambda_1, \ldots, \lambda_m\}$. For the open neighbourhood $\prod_{\lambda\in\Lambda} U_\lambda$ of the point θ, where

$$U_\lambda = \begin{cases} [0, \frac{1}{2}), & \text{if } \lambda \in \{\lambda_1, \ldots, \lambda_m\} \\ \mathbb{R}_+, & \text{otherwise} \end{cases}$$

we have $G(f) = \{g \in F : \rho_\tau(f,g) \in \prod_{\lambda\in\Lambda} U_\lambda\} \subseteq \prod_{\lambda\in\Lambda} W_\lambda$.

4. Compact τ-metrizable spaces

It is well known that every compact metrizable space is separable. An analogous result for τ-metrizable spaces is stated in this section.

Let us consider a set Λ such that $|\Lambda| = \tau \geqslant \aleph_0$ and let $\mathcal{B}_\varepsilon$ be the family of all open subsets $\prod_{\lambda\in\Lambda} W_\lambda$ of the product $\mathbb{R}_+^\tau$, where finitely many W_λ are intervals of the form $[0, \varepsilon)$ and the remaining $W_\lambda = \mathbb{R}_+$.

Definition 4.1. Let (X, ρ_τ) be a τ-metric space. A subset A of X is called O_ε-dense in (X, ρ_τ), where $O_\varepsilon \in \mathcal{B}_\varepsilon$, if for every $x \in X$ there exists $a \in A$ such that $\rho_\tau(x, a) \in O_\varepsilon$.

Definition 4.2. A τ-metric space (X, ρ_τ) is called ε-totally bounded if for every $O_\varepsilon \in \mathcal{B}_\varepsilon$ there exists a finite subset A of X which is O_ε-dense in (X, ρ_τ). The τ-metric space (X, ρ_τ) is called totally bounded if it is ε-totally bounded for every $\varepsilon > 0$.

Recall that the density $d(X)$ of a topological space X, is defined to be $d(X) = \min\{|D| : D \text{ is a dense subset of } X\}$.

Proposition 4.3. *For every totally bounded τ-metric space X, the inequality $d(X) \leqslant \tau$ holds.*

Proof. Let $n \in \{1, 2, \ldots\}$. For each $O_{1/n} \in \mathcal{B}_{1/n}$, let $A(O_{1/n})$ be a finite $O_{1/n}$-dense subset of X and consider the subset $A_n = \cup\{A(O_{1/n}) : O_{1/n} \in \mathcal{B}_{1/n}\}$ of X with $|A_n| \leqslant \tau$. The subset $A = \cup_{n=1}^\infty A_n$ of X is dense and $|A| \leqslant \tau$. □

Proposition 4.4. *Every compact τ-metric space X is totally bounded.*

Proof. Let $\varepsilon > 0$. For every $O_\varepsilon \in \mathcal{B}_\varepsilon$ the family

$$\{G(x) = \{y \in X : \rho_\tau(x, y) \in O_\varepsilon\} : x \in X\}$$

forms an open cover of X. By compactness of X, there exists a finite subset A of X such that $\bigcup_{a\in A} G(a) = X$. For every $x \in X$ there exists $a \in A$ with

$x \in G(a)$. Therefore, $\rho_\tau(x, a) \in O_\varepsilon$ and the subset A of X is O_ε-dense in (X, ρ_τ). □

Theorem 4.5. *For every compact τ-metrizable space X we have $d(X) \leqslant \tau$.*

Proof. Let X be a compact τ-metrizable space. According to Proposition 4.4, the space X is totally bounded. Therefore, by virtue of Proposition 4.3, $d(X) \leqslant \tau$. □

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