Applied General Topology

Appl. Gen. Topol. 19, no. 2 (2018), 269-280
doi:10.4995/agt.2018.9737

More on the cardinality of a topological space

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Communicated by D. Georgiou

Abstract

In this paper we continue to investigate the impact that various separation axioms and covering properties have onto the cardinality of topological spaces. Many authors have been working in that field. To mention a few, let us refer to results by Arhangel'skii, Alas, Hajnal-Juhász, Bell-Gisburg-Woods, Dissanayake-Willard, Schröder and to the excellent survey by Hodel "Arhangel'skiĭ's Solution to Alexandroff's problem: A survey".
Here we provide improvements and analogues of some of the results obtained by the above authors in the settings of more general separation axioms and cardinal invariants related to them. We also provide partial answer to Arhangel'skii's question concerning whether the continuum is an upper bound for the cardinality of a Hausdorff Lindelöf space having countable pseudo-character (i.e., points are G_δ). Shelah in 1978 was the first to give a consistent negative answer to Arhangel'skii's question; in 1993 Gorelic established an improved result; and further results were obtained by Tall in 1995. The question of whether or not there is a consistent bound on the cardinality of Hausdorff Lindelöf spaces with countable pseudo-character is still open. In this paper we introduce the Hausdorff point separating weight $Hpw(X)$, and prove that (1) $|X| \leq Hpsw(X)^{aL_c(X)\psi(X)}$, for Hausdorff spaces and (2) $|X| \leq Hpsw(X)^{wL_c(X)\psi(X)}$, where X is a Hausdorff space with a π-base consisting of compact sets with non-empty interior. In 1993 Schröder proved an analogue of Hajnal and Juhasz inequality $|X| \leq 2^{c(X)\chi(X)}$ for Hausdorff spaces, for Urysohn spaces by considering weaker invariant - Urysohn cellularity $Uc(X)$ instead of cellularity $c(X)$. We introduce the n-Urysohn cellularity n-$Uc(X)$ (where $n \geq 2$) and prove that the previous inequality is true in the class of n-Urysohn spaces replacing $Uc(X)$ by the weaker n-$Uc(X)$. We also show that $|X| \leq 2^{Uc(X)\pi\chi(X)}$ if X is a power homogeneous Urysohn space.

2010 MSC: *54D25; 54D10; 54D15; 54D30.*

Received 24 February 2018 – Accepted 19 June 2018

Keywords: *n-Hausdorff space; n-Urysohn space; homogeneous spaces; cardinal invariants.*

1. Introduction

We will follow the terminology and notation in [16].

Firstly, we will discuss the classical Hajnal and Juhasz's inequality $|X| \leq 2^{c(X)\chi(X)}$ proven for Hausdorff spaces [13]. An improvement of this inequality is obtained by Bonanzinga in [6] for the general class of n-Hausdorff spaces. A space X is defined to be n-Hausdorff (where $n \geq 2$) if $H(X) = n$ where $H(X)$ is the Hausdorff number of X, i.e. the smallest cardinal τ such that for every subset $A \subset X, |A| \geq \tau$, there exist neighborhoods $U_a, a \in A$, such that $\bigcap_{a\in A} U_a = \varnothing$. For every n-Hausdorff space X, the n-Hausdorff pseudo-character of X, denoted $n\text{-}H\psi(X)$, is defined as the smallest κ such that for each point x there is a collection $\{V(\alpha, x) : \alpha < \kappa\}$ of open neighborhoods of x such that if $x_1, x_2, .., x_n$ are distinct points from X, then there exist $\alpha_1, \alpha_2, ..., \alpha_n < \kappa$ such that $\bigcap_{i=1}^{n} V(\alpha_i, x_i) = \varnothing$. It was then proved that $|X| \leq 2^{c(X)\chi(X)}$ holds if replacing the character with the Hausdorff pseudo-character, and that for every 3- Hausdorff space the inequality $|X| \leq 2^{c(X)3\text{-}H\psi(X)}$ holds. In [12] Gotchev proved that the latter inequality is true for every space X having finite Hausdorff number. In [16] Schröder investigated the inequality of Hajnal and Juhasz for Urysohn spaces replacing cellularity $c(X)$ with the weaker invariant Urysohn cellularity $Uc(X)$ (as $Uc(X) \leq c(X)$). In Section 2 we prove that Schröder's inequality $|X| \leq 2^{Uc(X)\chi(X)}$, for a Urysohn space X, can be restated for n-Urysohn spaces provided the Urysohn cellularity is replaced by the n-Urysohn cellularity (Theorem 2.11 below).

An analogue of the Hajnal-Juhasz inequality in the setting of homogeneous spaces was established in [8] where Carlson and Ridderbos use the Erdös-Rado theorem to show that if X is a power homogeneous Hausdorff space then $|X| \leq 2^{c(X)\pi_\chi(X)}$. In Section 2 we prove that this result can be modified in the setting of Urysohn spaces to give the homogeneous analogue of Schröder's result. In particular, we prove that if X is Urysohn power homogeneous space then $|X| \leq 2^{Uc(X)\pi\chi(X)}$.

In Section 3 we give a partial solution to Arhangel'skii's problem [3] concerning whether the continuum is an upper bound for the cardinality of a Hausdorff Lindelöf space having countable pseudo-character.

In [9] Charlesworth proved that $|X| \leq psw(X)^{L(X)\psi(X)}$ for every T_1 space X, where $psw(X)$ is the mininum infinite cardinal κ such that X has an open cover $\mathcal{S}$ (called separating open cover) having the property that for each distinct x and y in X there is an $S \in \mathcal{S}$ such that $x \in S$ and $y \notin S$ and such that each point of X is in at most κ elements of $\mathcal{S}$. Charlesworth's result is one of the few that provided partial answer to both of the above Arhangel'skii's problem and another one formulated in the same paper: "Is continuum an upper bound for T_1 Lindelöf space having countable character?". Shelah, in an unpublished paper in 1978, was the first to provide a consistent negative answer to the question of Arhangel'skii's (whether or not $2^{\aleph_0}$ is an upper bound for the cardinality of a Hausdorff Lindelöf space of countable pseudo-characher) by constructing a model of $ZFC + CH$ in which there is a Lindelöf regular space of countable pseudo-character with cardinality $\mathfrak{c}^+ = \aleph_2$. Shelah's paper was eventually published in 1996 [17]. Then, Gorelic [11] proved that is consistent with CH that 2^{ω_1} is arbitrarily large and there is a Lindelöf, 0-dimensional Hausdorff space X of countable pseudo-character with $|X| = 2^{\omega_1}$, and thus improving Shelah's result. The question of whether or not there is a consistent bound on the cardinality of Hausdorff Lindelöf spaces with countable pseudo-character is still open.

We introduce an analogue of $psw(X)$ in the class of Hausorff spaces, denoted $Hpsw(X)$, and prove that $|X| \leq Hpsw(X)^{aL_c(X)\psi(X)}$ for a Hausdorff space X thus giving a partial answer to Arhangel'skii's problem in ZFC by even replacing $L(X)$ with the weaker invariant $aL_c(X)$. This is also a partial answer to a question in [10] if in the main result which states that for Hausdorff spaces X, $|X| \leq 2^{aL_c(X)\chi(X)}$, $\chi(X)$ can be replaced by $\psi(X)$.

We also prove that $|X| \leq Hpsw(X)^{wL_c(X)\psi(X)}$, for a Hausdorff space X with a π-base consisting of compact sets with non-empty interior. This result is closely related to results in [5], [4] and [1].

2. A generalization of Schröder's inequality

In [16], Schröder gives the following definition:

Definition 2.1 ([16])**.** Let X be a topological space. A collection $\mathcal{V}$ of open subsets of X is called Urysohn-cellular, if O_1, O_2 in $\mathcal{V}$ and $O_1 \neq O_2$ implies $\overline{O_1} \cap \overline{O_2} = \varnothing$. The Urysohn-cellularity of X, $Uc(X)$, is defined by

$$Uc(X) = \sup\{|\mathcal{V}| : \mathcal{V} \text{ is Urysohn-cellular }\} + \aleph_0.$$

Recall that a topological space X is said to be quasiregular provided for every open set V, there is a nonempty open set U such that the closure of U is contained in V. We observe the following properties.

Lemma 2.2. If X is a quasiregular space, then for every cellular family $\mathcal{U}$ such that $|\mathcal{U}| = \kappa$ there exists a Urysohn cellular family $\mathcal{U}'$ such that $|\mathcal{U}'| = \kappa$.

Proof. Let X be a quasiregular space and $\mathcal{U}$ be a cellular family with $|\mathcal{U}| = \kappa$. For every $U \in \mathcal{U}$ there exists an open set $V_U \subset U$ such that $\overline{V_U} \subset U$. Clearly, if U_1 and U_2 are distinct elements of $\mathcal{U}$ such that $U_1 \cap U_2 = \varnothing$, we have $\overline{V_{U_1}} \cap \overline{V_{U_2}} = \varnothing$. Hence $\mathcal{U}' = \{V_U : U \in \mathcal{U}\}$ is a Urysohn cellular family for X such that $|\mathcal{U}'| = \kappa$. □

Property 2.3. If X is a quasiregular space, $c(X) = Uc(X)$.

Proof. Clearly, $Uc(X) \leq c(X)$. Let $Uc(X) = \kappa$ and suppose that $c(X) > \kappa$. Then by Lemma 2.2 there exists a Urysohn cellular family $\mathcal{U}$ such that $|\mathcal{U}| > \kappa$; a contradiction. □

Recall the following:

Theorem 2.4 ([12, Corollary 3.2])**.** *Let X be a space with $H(X)$ finite. Then* $|X| \leq 2^{c(X)\chi(X)}$.

The previous result together with Property 2.3 gives the following:

Corollary 2.5. If X is a quasiregular n-Hausdorff (where $n \geq 2$) space, $|X| \leq 2^{Uc(X)\chi(X)}$.

Recall that in [7] the authors define a space X to be n-Urysohn (where $n \geq 2$) if $U(X) = n$ where $U(X)$ is the Urysohn number of X, i.e. the smallest cardinal τ such that for every subset $A \subset X, |A| \geq \tau$, there exist neighborhoods $U_a, a \in A$, such that $\bigcap_{a\in A} \overline{U_a} = \varnothing$.

We introduce the following:

Definition 2.6. Let X be a topological space. A collection $\mathcal{V}$ of open subsets of X is called *n-Urysohn-cellular,* where $n \geq 2$, if $O_1, O_2, ..., O_n$ in $\mathcal{V}$ and $O_1 \neq O_2 \neq ... \neq O_n$ implies $\overline{O_1} \cap \overline{O_2} \cap ... \cap \overline{O_n} = \varnothing$. The *$n$-Urysohn-cellularity* of X, n-$Uc(X)$, is defined by

$$n\text{-}Uc(X) = \sup\{|\mathcal{V}| : \mathcal{V} \text{ is } n\text{-Urysohn-cellular }\} + \aleph_0.$$

Clearly, if $\mathcal{V}$ is a Urysohn cellular collection of open subsets, then $\mathcal{V}$ is n-Urysohn cellular for every $n \geq 2$. Also if $Uc(X) \leq \kappa$, then n-$Uc(X) \leq \kappa$ for every $n \geq 2$.

Question 2.7. Is there a space X such that $(n+1)$-$Uc(X) = \kappa$ and n-$Uc(X) \neq \kappa$?

Recall that the θ-closure of a set A in the space X is the set $cl_\theta(A) = \{x \in X :$ for every neighborhood $U \ni x, \overline{U} \cap A \neq \varnothing\}$ [19].

Proposition 2.8. Let $\{A_\alpha\}_{\alpha\in A}$ be a collection of subsets of X, then $\bigcup_{\alpha\in A} cl_\theta(A_\alpha) \subseteq cl_\theta(\bigcup_{\alpha\in A} A_\alpha)$.

Proof. If $x \in \bigcup_{\alpha\in A} cl_\theta(A_\alpha)$, then there exists $\alpha \in A$ such that $x \in cl_\theta(A_\alpha)$. Therefore for every neighborhood U_x we have $\overline{U}_x \cap A_\alpha \neq \varnothing$. Then $\overline{U}_x \cap (\bigcup_{\alpha\in A} A_\alpha) \neq \varnothing$. This implies $x \in cl_\theta(\bigcup_{\alpha\in A} A_\alpha)$. □

The next lemma represents a modification of Lemma 7 in [16]:

Lemma 2.9. Let X be a topological space and $\mu = n$-$Uc(X)$. Let $(U_\alpha)_{\alpha\in A}$ be a collection of open sets. Then there are $B_1, B_2, ..., B_{n-1} \subseteq A$ with $|B_i| \leq \mu \,\forall i = 1, 2, ..., n-1$ and $\bigcup_{\alpha\in A} U_\alpha \subseteq cl_\theta(\bigcup_{\alpha\in B_1\cup B_2\cup ...\cup B_{n-1}} \overline{U_\alpha})$.

Proof. Let $\mathcal{V} = \{V \subset X : V$ is open and $\exists \alpha \in A$ such that $V \subseteq U_\alpha\}$. By Zorn's Lemma, take a maximal n-Urysohn-cellular family $\mathcal{W} \subseteq \mathcal{V}$ and $|\mathcal{W}| \leq \mu$.

For every $W \in \mathcal{W}$ take $\beta = \beta(W)$ such that $U_{\beta(W)} \in \{U_\alpha : \alpha \in A\}$ and $W \subseteq U_{\beta(W)}$. We may assume $\beta \in B = B_1 \sqcup B_2 \sqcup ... \sqcup B_{n-1}, B_i \subseteq A$ and $|B_i| \leq \mu, \forall i = 1, 2, ..., n-1$.

We want to prove that $\bigcup_{\alpha\in A} U_\alpha \subseteq cl_\theta((\bigcup_{\alpha\in B_1} \overline{U_\alpha}) \cup ... \cup (\bigcup_{\alpha\in B_{n-1}} \overline{U_\alpha}))$.

Assume the contrary, then there exists $x \in \bigcup_{\alpha\in A} U_\alpha$ and $x \notin cl_\theta((\bigcup_{\alpha\in B_1}\overline{U_\alpha})\cup ... \cup (\bigcup_{\alpha\in B_{n-1}}\overline{U_\alpha}))$.

Then we can find $\alpha_0 \in A$ and a neighborhood U_x of x such that $x \in U_{\alpha_0}$ and $\overline{U_x} \cap ((\bigcup_{\alpha\in B_1}\overline{U_\alpha}) \cup ... \cup (\bigcup_{\alpha\in B_{n-1}}\overline{U_\alpha})) = \varnothing$.

Then $\overline{(U_{\alpha_0} \cap U_x)} \subseteq \overline{U_x}$ and $(U_{\alpha_0} \cap U_x) \cup \mathcal{W}$ is a n-Urysohn cellular family containing $\mathcal{W}$; this contradicts the maximality of $\mathcal{W}$. □

Corollary 2.10 ([16]). *Let X be a topological space and $\mu = Uc(X)$. Let $(U_\alpha)_{\alpha\in A}$ be a collection of open sets. Then there is $B \subseteq A$ with $|B| \leq \mu$ and $\bigcup_{\alpha\in A} U_\alpha \subseteq cl_\theta \bigcup_{\alpha\in B}\overline{U_\alpha}$.*

Theorem 2.11. Let X be a n-Urysohn space. Then $|X| \leq 2^{n\text{-}Uc(X)\chi(X)}$.

Proof. Set $\mu = n\text{-}Uc(X)\chi(X)$. For every $x \in X$ let $\mathcal{B}(x)$ denote an open neighbourhood base of x with $|\mathcal{B}(x)| \leq \mu$. Construct an increasing sequence $\{C_\alpha : \alpha < \mu^+\}$ of subsets of X and a sequence $\{\mathcal{V}_\alpha : \alpha < \mu^+\}$ of open collections of open subsets of X such that:

(1) $|C_\alpha| \leq 2^\mu$ for all $\alpha < \mu^+$.
(2) $\mathcal{V}_\alpha = \bigcup\{\mathcal{B}(c) : c \in \bigcup_{\tau<\alpha} C_\tau\}, \alpha < \mu^+$.
3 If $\{G_{\gamma_1,\gamma_2,...,\gamma_{n-1}} : (\gamma_1,\gamma_2,...,\gamma_{n-1}) \subseteq \mu\}$ is a collection of subsets of X and each $G_{\gamma_1,\gamma_2,...,\gamma_{n-1}}$ is the union of closures of $\leq \mu$ many elements of $\mathcal{V}_\alpha$ and

$$\bigcup_{\{\gamma_1,\gamma_2,...,\gamma_{n-1}\}\subseteq\mu} cl_\theta G_{\gamma_1,\gamma_2,...,\gamma_{n-1}} \neq X$$

then

$$C_\alpha \setminus \bigcup_{\{\gamma_1,\gamma_2,...,\gamma_{n-1}\}\subseteq\mu} cl_\theta G_{\gamma_1,\gamma_2,...,\gamma_{n-1}} \neq \varnothing.$$

The construction is by transfinite induction. Let x_0 be a point of X and put $C_0 = \{x_0\}$. Let $0 < \alpha < \mu^+$ and assume that C_β has been constructed for each $\beta < \alpha$. Note that $\mathcal{V}_\alpha$ is defined by (2) and $\mathcal{V}_\alpha \leq 2^\mu$. For each collection $\{G_{\gamma_1,\gamma_2,...,\gamma_{n-1}} : (\gamma_1,\gamma_2,...,\gamma_{n-1}) \subseteq \mu\}$ of subsets of X where each $G_{\gamma_1,\gamma_2,...,\gamma_{n-1}}$ is the union of closures of $\leq \mu$ many elements of $\mathcal{V}_\alpha$ and

$$\bigcup_{\{\gamma_1,\gamma_2,...,\gamma_{n-1}\}\subseteq\mu} cl_\theta G_{\gamma_1,\gamma_2,...,\gamma_{n-1}} \neq X,$$

choose a point of $X \setminus \bigcup_{\{\gamma_1,\gamma_2,...,\gamma_{n-1}\}\subseteq\mu} cl_\theta G_{\gamma_1,\gamma_2,...,\gamma_{n-1}}$. Let H_α be the set of points chosen in this way, (clearly, $|H_\alpha| \leq 2^\mu$) and let $C_\alpha = H_\alpha \cup (\bigcup_{\beta<\alpha} C_\beta)$. It is clear that the family $\{C_\alpha : 0 < \alpha < \mu^+\}$ constructed in this way satisfies condition $(1), (2)$ and (3).

Let $C = \bigcup_{\alpha<\mu^+} C_\alpha$. We shall show that $C = X$. Assume there is $y \in X \setminus C$. For every $B_{\gamma_1}, B_{\gamma_2}, ..., B_{\gamma_{n-1}} \in \mathcal{B}(y)$, with $|B_{\gamma_i}| > 1 \forall i = 1,2,...,n-1$ and $\gamma_1,\gamma_2,...,\gamma_{n-1} \subseteq \mu$ define

$$\mathcal{F}_{\gamma_1,\gamma_2,...,\gamma_{n-1}} = \{V_c : c \in C, V_c \in \mathcal{B}(c), \overline{V_c} \cap \overline{B_{\gamma_1}} \cap \overline{B_{\gamma_2}} \cap ... \cap \overline{B_{\gamma_{n-1}}} = \varnothing\}.$$

Since X is n-Urysohn, we have

$$C \subseteq \bigcup_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}\subseteq\mu} \bigcup \mathcal{F}_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}}.$$

By Lemma 2.9 we find for every $\{\gamma_1, \gamma_2, \ldots\gamma_{n-1}\} \subseteq \mu$ subcollections

$$\mathcal{G}_{\gamma_1}, \mathcal{G}_{\gamma_2}, \ldots, \mathcal{G}_{\gamma_{n-1}} \subseteq \mathcal{F}_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}}, |\mathcal{G}_{\gamma_i}| \leq \mu \,\forall i = 1, 2, \ldots, n-1$$

such that

$$\bigcup \mathcal{F}_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}} \subseteq cl_\theta \bigcup_{i=1}^{n-1} (\bigcup \overline{\mathcal{G}_{\gamma_i}}).$$

Note $y \notin cl_\theta \bigcup_{i=1}^{n-1}(\bigcup \overline{\mathcal{G}_{\gamma_i}})$. Indeed, since

$$(\bigcup_{i=1}^{n-1}(\bigcup \overline{\mathcal{G}_{\gamma_i}})) \cap \overline{B_{\gamma_1}} \cap \overline{B_{\gamma_2}} \cap \ldots \cap \overline{B_{\gamma_{n-1}}} = \varnothing$$

and then

$$(\bigcup_{i=1}^{n-1}(\bigcup \overline{\mathcal{G}_{\gamma_i}})) \cap \overline{(B_{\gamma_1} \cap B_{\gamma_2} \cap \ldots \cap B_{\gamma_{n-1}})} = \varnothing.$$

Find $\alpha < \mu^+$ such that $\bigcup_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}\subseteq\mu}(\mathcal{G}_{\gamma_1} \cup \mathcal{G}_{\gamma_2} \cup \ldots \cup \mathcal{G}_{\gamma_{n-1}}) \subseteq \mathcal{V}_\alpha$. Then

$$y \notin \bigcup_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}\subseteq\mu} cl_\theta \bigcup_{i=1}^{n-1}(\bigcup \overline{\mathcal{G}_{\gamma_i}})$$

but

$$C_\alpha \subseteq C \subseteq \bigcup_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}\subseteq\mu} \bigcup \mathcal{F}_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}} \subseteq \bigcup_{\{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}\}\subseteq\mu} cl_\theta \bigcup_{i=1}^{n-1}(\bigcup \overline{\mathcal{G}_{\gamma_i}}).$$

Put $G_{\gamma_1,\gamma_2,\ldots,\gamma_{n-1}} = \bigcup_{i=1}^{n-1}(\bigcup \overline{\mathcal{G}_{\gamma_i}})$. This contradicts 3. □

Corollary 2.12 ([16]). *Let X be a Urysohn space. Then $|X| \leq 2^{Uc(X)\chi(X)}$.*

We end this section with a new cardinality bound for power homogeneous Urysohn spaces involving the Urysohn cellularity $Uc(X)$. Recall that a space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$. X is *power homogeneous* if there exists a cardinal κ for which X^κ is homogeneous. It is well established that cardinality bounds on a topological space can be improved if the space possesses homogeneous-like properties. For example, while $|X| \leq 2^{c(X)\chi(X)}$ holds for any Hausdorff space X, Carlson and Ridderbos [8] have shown that if X is additionally power homogeneous then $|X| \leq 2^{c(X)\pi\chi(X)}$, where $\pi\chi(X)$ denote the π-character of the space X. By modifying this result, we show below that an analogous result holds for Urysohn power homogeneous spaces when $Uc(X)$ is used in place of $c(X)$.

It is first helpful to establish this result in the homogeneous setting. To prove the following result we will use the well-known Erdös-Rado Theorem which states that if $f : [X]^2 \to \kappa$ is a function and $|X| > 2^\kappa$, then there is some subset Y of X with $|Y| \geq \kappa^+$ such that $f(y) = f(z)$ for all $y, z \in [Y]^2$, where $[Y]^2$ denotes the family of all subsets of Y of cardinality $= 2$.

Theorem 2.13. If X is a homogeneous Hausdorff space that is Urysohn or quasiregular then $|X| \leq 2^{Uc(X)\pi\chi(X)}$.

Proof. If X is quasiregular, then, by Property 2.3 (following Lemma 2 of this section), $Uc(X) = c(X)$ and the result follows from Proposition 2.1 in [8]. So we assume X is Urysohn.

Let $\kappa = Uc(X)\pi\chi(X)$, fix $p \in X$, and let $\mathcal{B}$ be a local π-base at p such that $|\mathcal{B}| \leq \kappa$. As X is homogeneous, for all $x \in X$ there exists a homeomorphism $h_x : X \to X$ such that $h_x(p) = x$.

As X is Urysohn, for all $x \neq y \in X$ there exist open sets U and V such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \varnothing$. Then $p \in h_x^{-1}[U] \cap h_y^{-1}[V]$, an open set. As $\mathcal{B}$ is a local π-base at p, there exists $B(x,y) \in \mathcal{B}$ such that $B(x,y) \subseteq h_x^{-1}[U] \cap h_y^{-1}[V]$. Thus, $h_x[B(x,y)] \subseteq U$, $h_y[B(x,y)] \subseteq V$, and

$$\overline{h_x[B(x,y)]} \cap \overline{h_y[B(x,y)]} = \varnothing.$$

The existence of $B(x,y)$ for each $x \neq y \in X$ defines a function $B : [X]^2 \to \mathcal{B}$.

Suppose by way of contradiction that $|X| > 2^\kappa$. As $|\mathcal{B}| \leq \kappa$, we can apply the Erdös-Rado Theorem to the function B. Thus, there exists a subset Y of X with $|Y| = \kappa^+$ and $A \in \mathcal{B}$ such that $B(x,y) = A$ for all distinct $x, y \in Y$.

Observe that for every $x \neq y \in Y$, we have

$$\overline{h_x[A]} \cap \overline{h_y[A]} = \overline{h_x[B(x,y)]} \cap \overline{h_y[B(x,y)]} = \varnothing.$$

This shows $\{h_x[A] : x \in Y\}$ is a Urysohn cellular family. However,

$$|\{h_x[A] : x \in Y\}| = |Y| = \kappa^+ > Uc(X),$$

which is a contradiction. Thus, $|X| \leq 2^\kappa = 2^{Uc(X)\pi\chi(X)}$. □

To establish the more general theorem in the power homogeneous case, we adapt the proof of Theorem 2.3 in [8]. Importantly, we adopt the following notation: If X is a power homogeneous space, let μ be a cardinal such that X^μ is homogeneous. Fix a projection $\pi : X^\mu \to X$ and a point p in the diagonal $\Delta(X, \mu)$. Let κ be a cardinal such that $\pi_\chi(X) \leq \kappa$ and fix a local π-base $\mathcal{U}$ at $\pi(p)$ in X such that $|\mathcal{U}| \leq \kappa$. We may assume without loss of generality that $\kappa \leq \mu$. For any $B \subseteq A \subseteq \mu$, let $\pi_{A \to B}$ be the projection of X^A to X^B, and for $A \subseteq \mu$, define $\mathcal{U}(A)$ by

$$\mathcal{U}(A) = \left\{ \pi_{A \to B}^{-1} \left[\prod_{b \in B} U_b \right] : B \in [A]^{<\omega}, \text{ and } U_b \in \mathcal{U} \text{ for all } b \in B \right\}.$$

Observe that the family $\mathcal{A}$ is a local π-base at p_A in X^A.

The following is Lemma 2.2 in [8]. This lemma establishes that if X^μ is homogeneous that not only are there homeomorphisms $h_x : X^\mu \to X^\mu$ such that $h_x(p) = x$ for all $x \in X$, but that we can guarantee these homeomorphisms have special properties.

Lemma 2.14. *For every $x \in \Delta(X,\mu)$ there is a homeomorphism $h_x : X^\mu \to X^\mu$ such that $h_x(p) = x$ and the following conditions are satisfied;*

(1) *For all $z \in X^\mu$, if $z_\kappa = p_\kappa$ then $\pi(h_x(z)) = \pi(x)$,*
(2) *For all $U \in \mathcal{U}(\kappa)$, there is a point $q(U) \in \pi_\kappa^{-1}[U]$ and a basic open neighbourhood U_x of $h_x(q(U))_\kappa$ in X^κ such that;*
   (a) *$q(U)_\alpha = p_\alpha$ for all $\alpha \in \mu \setminus \kappa$,*
   (b) *$\pi_\kappa^{-1}[U_x] \subseteq h_x[\pi_\kappa^{-1}[U]]$.*

We now prove a generalization of Theorem 2.13.

Theorem 2.15. If X is a power homogeneous Hausdorff space that is Urysohn or quasiregular then $|X| \leq 2^{Uc(X)\pi\chi(X)}$.

Proof. If X is quasiregular then again $Uc(X) = c(X)$ and the proof follows directly from Theorem 2.3 in [8]. So as in the last proof we assume X is Urysohn.

Let $\kappa = Uc(X)\pi_\chi(X)$. For every $x \in \Delta(X,\mu)$, fix a homeomorphism $h_x : X^\mu \to X^\mu$ as in Lemma 2.14 above. Now, for $x \in \Delta(X,\mu)$ and $U \in \mathcal{U}(\kappa)$, the set U_x obtained from Lemma 2.14 is a basic open set in X^κ. We may thus fix a collection $\{U_{x,\alpha} : \alpha < \kappa\}$ of open sets in X such that

$$U_x = \bigcap_{\alpha<\kappa} \pi_\alpha^{-1}[U_{x,\alpha}].$$

For every $\alpha \in \kappa$ we can fix a local π-base $\{V(x,U,\alpha,\beta) : \beta < \kappa\}$ of the point $h_x(q(U))_\alpha$ in X.

In Claim 1 in the proof of Theorem 2.3 in [8], it is shown that whenever $x \neq y \in \Delta(X,\mu)$ there exists $U \in \mathcal{U}(\kappa)$ and $\alpha,\beta < \kappa$ such that $V(x,U,\alpha,\beta) \subseteq U_{x,\alpha}\backslash\overline{U}_{y,\alpha}$. We make a related Claim in our proof. We omit the proof as it is very similar.

Claim. Whenever $x \neq y \in \Delta(X,\mu)$, there is $U \in \mathcal{U}(\kappa)$ and $\alpha,\beta < \kappa$ such that

$$\overline{V(x,U,\alpha,\beta)} \subseteq \overline{U_{x,\alpha}} \text{ and } \overline{V(x,U,\alpha,\beta)} \cap \overline{U_{y,\alpha}} = \varnothing.$$

Continuing with our main proof, by way of contradiction, assume that $|X| > 2^\kappa$. Define a a map $G : [X]^2 \to \mathcal{U}(\kappa) \times \kappa \times \kappa$ as follows. For $\{x,y\} \in [X]^2$ and $x \neq y$, we apply the Claim and let $G(\{x,y\}) = \langle U,\alpha,\beta\rangle$ be such that the conclusion of the Claim is satisfied. Here we have identified $\Delta(X,\mu)$ with X. As $|\mathcal{U}(\kappa)\times\kappa\times\kappa| = \kappa$ and $|X| > 2^\kappa$, we can apply the Erdös-Rado Theorem to find $Y \subset X$ and $\langle U,\alpha,\beta\rangle \in \mathcal{U}(\kappa)\times\kappa\times\kappa$ such that $|Y| = \kappa^+$ and for all $\{x,y\} \in [Y]^2$, $G(\{x,y\}) = \langle U,\alpha,\beta\rangle$. Thus for all $y \in Y$ we have $\overline{V(y,U,\alpha,\beta)} \subseteq \overline{U}_{y,\alpha}$.

Let $\mathcal{C} = \{V(x,U,\alpha,\beta) : x \in Y\}$ and note that $\mathcal{C}$ is a collection of open subsets of X. If $x \neq y \in Y$ then

$$\overline{V(x,U,\alpha,\beta)} \cap \overline{U_{y,\alpha}} = \varnothing \text{ and } \overline{V(y,U,\alpha,\beta)} \subseteq \overline{U_{y,\alpha}},$$

and therefore $\overline{V(x,U,\alpha,\beta)}$ and $\overline{V(y,U,\alpha,\beta)}$ are disjoint. This means $\mathcal{C}$ is a Urysohn cellular family. However, $|\mathcal{C}| = |Y| = \kappa^+ > Uc(X)$, which is a contradiction. Thus $|X| \leq 2^\kappa$.

□

This above result shows that Schröder's cardinality bound $2^{Uc(X)\chi(X)}$ for Urysohn spaces can be improved in the power homogeneous setting.

Question 2.16. If X is power homogeneous and n-Urysohn, is

$$|X| \leq 2^{n-Uc(X)\pi\chi(X)}?$$

3. The Hausdorff point separating weight

Recall the following properties which represent weaker forms of the Lindelöf degree $L(X)$:

- the *almost Lindelöf degree of X with respect to closed sets*, denoted $aL_c(X)$, is the smallest infinite cardinal κ such that for every closed subset H of X and every collection $\mathcal{V}$ of open sets in X that covers H, there is a subcollection $\mathcal{V}'$ of $\mathcal{V}$ such that $|\mathcal{V}'| \leq \kappa$ and $\{\overline{V} : V \in \mathcal{V}'\}$ covers H;
- the *weak Lindelöf degree of X with respect to closed sets*, denoted $wL_c(X)$, is the smallest infinite cardinal κ such that for every closed subset H of X and every collection $\mathcal{V}$ of open sets in X that covers H, there is a subcollection $\mathcal{V}'$ of $\mathcal{V}$ such that $|\mathcal{V}'| \leq \kappa$ and $H \subseteq \overline{\bigcup \mathcal{V}'}$.

The following holds:

$$wL_c(X) \leq aL_c(X) \leq L(X)$$

Recall the following definition:

Definition 3.1 ([9]). A point separating open cover $\mathcal{S}$ for a space X is an open cover of X having the property that for each distinct points x and y in X there is S in $\mathcal{S}$ such that x is in S but y is not in S. The point separating weight of a space X is the cardinal

$$psw(X) = min\{\tau : X \text{ has a point separating cover } \mathcal{S}$$

$$\text{such that each point of } X \text{ is contained in at most } \tau \text{ elements of } \mathcal{S}\} + \aleph_0$$

Definition 3.2. A *Hausdorff point separating open cover* $\mathcal{S}$ for a space X is an open cover of X having the property that for each distinct points x and y in X there is S in $\mathcal{S}$ such that x is in S but y is not in $\overline{S}$. The *Hausdorff point separating weight* of a Hausdorff space X is the cardinal

$$Hpsw(X) = min\{\tau : X \text{ has a Hausdorff point separating cover } \mathcal{S}$$

$$\text{such that each point of } X \text{ is contained in at most } \tau \text{ elements of } \mathcal{S}\} + \aleph_0$$

Recall the following:

Theorem 3.3 ([9, Theorem 2.1]). *If X is T_1, then $nw(X) \leq psw(X)^{L(X)}$.*

Following the proof of Theorem 2.1 in [9], we prove the following:

Theorem 3.4. If X is a Hausdorff space, then $nw(X) \leq Hpsw(X)^{aL_c(X)}$.

Proof. Let $aL_c(X) = \kappa$ and let $\mathcal{S}$ be a Hausdorff point separating open cover for X such that for each $x \in X$ we have $|\mathcal{S}_x| \leq \lambda$, where $\mathcal{S}_x$ denotes the collection of members of $\mathcal{S}$ containing x. We first show that $d(X) \leq \lambda^\kappa$. For each $\alpha < \kappa^+$ construct a subset D_α of X such that:

(1) $|D_\alpha| \leq \lambda^\kappa$.
(2) If $\mathcal{U}$ is a subcollection of $\bigcup\{\mathcal{S}_x : x \in \bigcup_{\beta<\alpha} D_\beta\}$ such that $|\mathcal{U}| \leq \kappa$ and $X \setminus \bigcup\overline{\mathcal{U}} \neq \varnothing$, then $D_\alpha \setminus \bigcup\overline{\mathcal{U}} \neq \varnothing$.

Such a D_α can be constructed since the number of possible $\mathcal{U}$'s at the αth stage of construction is $\leq (\lambda^\kappa \cdot \kappa \cdot \lambda)^\kappa = \lambda^\kappa$. Let $D = \bigcup_{\alpha<\kappa^+} D_\alpha$. Clearly $|D| \leq \lambda^\kappa$. Furthermore D is a dense subset of X. Indeed, if there is a point $p \in X \setminus \overline{D}$, since $Hpsw(X) \leq \lambda$, for every $x \in \overline{D}$ there exists an open set $V_x \in \mathcal{S}_x$ such that $x \in V_x$ and $p \notin \overline{V_x}$. Since $x \in \overline{D}$, we have $V_x \cap D \neq \varnothing$. Fix $y \in V_x \cap D$. Then $V_x \in \bigcup\{\mathcal{S}_y : y \in D\}$. Put $\mathcal{W} = \{V_x : x \in \overline{D}\} \subseteq \bigcup\{\mathcal{S}_y : y \in D\}$. Clearly, $\mathcal{W}$ is an open cover of $\overline{D}$. Since $aL_c(X) \leq \kappa$, we can select a subcollection $\mathcal{W}' \subseteq \mathcal{W}$ with $|\mathcal{W}'| \leq \kappa$ such that $\overline{D} \subseteq \bigcup\{\overline{V} : V \in \mathcal{W}'\}$ and $p \notin \bigcup\{\overline{V} : V \in \mathcal{W}'\}$; this contradicts 2. Since $d(X) \leq \lambda^\kappa$ we have that $|\mathcal{S}| \leq \lambda^\kappa$. Let $\mathcal{N} = \{X \setminus S : S$ is the union of at most κ members of $\mathcal{S}\}$. Then $|\mathcal{N}| \leq \lambda^\kappa$ and $\mathcal{N}$ is a network for X.

□

Theorem 3.5. If X is Hausdorff space, then $|X| \leq Hpsw(X)^{aL_c(X)\psi(X)}$.

Proof. It is known that if X is a T_1 space, $|X| \leq nw(X)^{\psi(X)}$. Then by Theorem 3.4, we have $|X| \leq Hpsw(X)^{aL_c(X)\psi(X)}$. □

Corollary 3.6. If X is a Hausdorff space with $L(X) = \omega, \psi(X) = \omega$ and $Hpsw(X) \leq \mathfrak{c}$, then $|X| \leq \mathfrak{c}$.

The previous corollary gives a partial solution to Arhangel'skii's problem [2, Problem 5.2] concerning whether the continuum is an upper bound for the cardinality of a Hausdorff Lindelöf space having countable pseudo-character.

Remark 3.7. Using Remark 2.5 in [9] we note that countable pseudo-character is essential in Corollary 3.6: if X is the product of 2^ω copies of the two point discrete space, then X is Hausdorff, Lindelöf and $\psi(X) > \omega$ but $|X| > 2^\omega$.

The following theorem, under additional hypothesis, gives a result similar to Theorem 3.4 in which the weakly Lindelöf degree with respect to closed sets takes the place of the almost Lindelöf degree with respect to closed sets.

Theorem 3.8. If X is Hausorff space with a π-base consisting of compact sets with non-empty interior, then $nw(X) \leq Hpsw(X)^{wL_c(X)}$.

Proof. Let $wL_c(X) = \kappa$ and let $\mathcal{S}$ be a Hausdorff point separating open cover for X such that for each $x \in X$ we have $|\mathcal{S}_x| \leq \lambda$, where $\mathcal{S}_x$ denotes the collection of members of $\mathcal{S}$ containing x. Without loss of generality, we can suppose that the family $\mathcal{S}_x$ is closed under finite intersection. We first show that $d(X) \leq \lambda^\kappa$. For each $\alpha < \kappa^+$ construct a subset D_α of X such that:

(1) $D_\alpha \leq \lambda^\kappa$.
(2) If $\mathcal{U}$ is a subcollection of $\bigcup\{\mathcal{S}_x : x \in \bigcup_{\beta<\alpha} D_\beta\}$ such that $|\mathcal{U}| \leq \kappa$ and $X \setminus \overline{\bigcup \mathcal{U}} \neq \varnothing$, then $D_\alpha \setminus \overline{\bigcup \mathcal{U}} \neq \varnothing$.

Such a D_α can be constructed since the number of possible $\mathcal{U}$'s at the αth stage of construction is $(\leq \lambda^\kappa \cdot \kappa \cdot \lambda)^\kappa = \lambda^\kappa$. Let $D = \bigcup_{\alpha<\kappa^+} D_\alpha$. Clearly $|D| \leq \lambda^\kappa$. Furthermore D is a dense subset of X. Indeed if $\overline{D} \neq X$, $X \setminus \overline{D}$ is a non-empty open set. Since X has a π-base consisting of compact sets with non-empty interior, we can find a non empty open subset $W \subseteq X$ such that $\overline{W}$ is compact and $\overline{W} \subset X \setminus \overline{D}$, hence $\overline{W} \cap \overline{D} = \varnothing$. Fix $x \in \overline{D}$. For every $p \in \overline{W}$ there exists an open subset $V_p \in \mathcal{S}_x$ such that $p \notin \overline{V_p}$. Then, we can find a family $\{V_p : p \in \overline{W}\}$ of open subsets of X such that $\bigcap\{\overline{V_p} : p \in \overline{W}\} \cap \overline{W} = \varnothing$. So, for the compactness of $\overline{W}$ the family $\{\overline{V_p} \cap \overline{W} : p \in \overline{W}\}$ can not have the finite intersection property. So put $F_x = \overline{V_{p_1}} \cap ... \cap \overline{V_{p_k}}$, where $p_1, ..., p_\kappa \in \overline{W}$ are such that $F_x \cap \overline{W} = \varnothing$. Put $G_x = V_{p_1} \cap ... \cap V_{p_k}$. Since $\mathcal{S}_x$ is closed under finite intersection, $G_x \in \mathcal{S}_x$ and $G_x \cap \overline{W} = \varnothing$. Since $G_x \in \mathcal{S}_x$ then $G_x \in \mathcal{S}_y$ for some $y \in D$. Clearly, $\mathcal{V} = \{G_x : x \in \overline{D}\}$ is an open cover of $\overline{D}$. Using $wL_c(X) \leq \kappa$ we can select a subcollection $\mathcal{V}' \subseteq \mathcal{V}, |\mathcal{V}'| \leq \kappa$ such that $\overline{D} \subseteq \overline{\bigcup\{V : V \in \mathcal{V}'\}}$. For every $U \in \bigcup \mathcal{V}'$, $U \cap \overline{W} = \varnothing$, hence $\bigcup \mathcal{V}' \cap \overline{W} = \varnothing$. Since W is a nonempty open set, $\overline{\bigcup \mathcal{V}'} \cap W = \varnothing$ and then $X \setminus \overline{\bigcup \mathcal{V}'} \neq \varnothing$. This contradicts 2. Since $d(X) \leq \lambda^\kappa$ we have that $|\mathcal{S}| \leq \lambda^\kappa$. Let $\mathcal{N} = \{X \setminus S | S$ is the union of at most κ members of $\mathcal{S}\}$. Then $|\mathcal{N}| \leq \lambda^\kappa$ and $\mathcal{N}$ is a network for X. □

Then we have the following result:

Theorem 3.9. If X is a Hausdorff space with a π-base consisting of compact sets with non-empty interior, then $|X| \leq Hpsw(X)^{wL_c(X)\psi(X)}$.

Acknowledgements. *The authors are strongly indebted to the referee for the very careful reading of the paper.*

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