APPLIED GENERAL TOPOLOGY

A note about various types of sensitivity in general semiflows

ALICA MILLER

Department of Mathematics, University of Louisville, USA (`alica.miller@louisville.edu`)

Communicated by F. Balibrea

ABSTRACT

We discuss the implications between various types of sensitivity in general semiflows (sensitivity, syndetic sensitivity, thick sensitivity, thick syndetic sensitivity, multisensitivity, periodic sensitivity, thick periodic sensitivity), including the weak mixing as a very strong type of sensitivity and the strong mixing as the strongest of all type of sensitivity.

1. INTRODUCTION

In this note we discuss the implications between various types of sensitivity in general semiflows (sensitivity, syndetic sensitivity, thick sensitivity, thick syndetic sensitivity, multisensitivity, periodic sensitivity, thick periodic sensitivity), including the weak mixing as a very strong type of sensitivity and the strong mixing as the strongest of all type of sensitivity. Under *general semiflow* we assume a semiflow (T, X) where T is a commutative noncompact Hausdorff acting topological monoid (with additive operation) and X is a metric space with at least two points. (We do not assume neither the compactness of the phase space X, nor that the transition maps $x \mapsto tx$ have dense images.) So from now on we assume that every monoid T (including groups) and every phase space X are as described above.

All notions that we mention, but not define, are standard and can be, for example found in [1, 4, 6]. Stronger forms of sensitivity were introduced in [8], and discussed, for example, in [3, 5, 8, 9].

A nonempty open subset is called a *nopen*, or a *nopen subset*. If (X, d) is a metric space, $x \in X$ and $r > 0$, the *open ball* with center x and radius r is denoted by $B(x, r)$. It consists of all points $y \in X$ such that $d(x, y) < r$. The *closed ball* with center x and radius r is denoted by $B^-(x, r)$. It consists of all points $y \in X$ such that $d(x, y) \leq r$. It is a closed subset of X.

We say that a subset A of T is *syndetic* if there is a compact $K \subseteq T$ such that for every $t \in T$, $(t + K) \cap A \neq \varnothing$. We say that a subset B of T is *thick* if for every compact $K \subseteq T$ there is a $t \in T$ such that $t + K \subseteq B$. We say that a subset C of T is *thickly syndetic* if for every compact $K \subseteq T$ there is a syndetic subset $S \subseteq T$ such that $S + K \subseteq C$.

We say that a subset D of T is *periodic* if it contains a translate $t + S$ of a closed syndetic submonoid of T. We say that a subset E of T is *thickly periodic* if for every compact $K \subseteq T$ there is a periodic subset $P \subseteq T$ such that $P + K \subseteq E$.

Definition 1.1. We say that a monoid T satisfies the *syndetic property*, shortly *sp property*, or that T is an *sp monoid*, if no syndetic subset of T is compact. We say that a monoid T satisfies the *dual syndetic property*, shortly *dsp property*, or that T is a *dsp monoid*, if for every compact subset K of T, the set $T \setminus K$ is a syndetic subset of T.

The condition (sp) can be equivalently formulated in the following way:

(sp') *For any two compact subsets K and K' of T there is an element $t \in T$ such that $(t + K) \cap K' = \varnothing$.*

Another equivalent way is the following one:

(sp") *For every compact subset K of T, the set $T \setminus K$ is a thick subset of T.*

Let us show that the conditions (sp) and (sp') are equivalent. Suppose (sp) holds. Let K, K' be two compact subsets of T. By (sp) none of them is sysndetic, hence there is an element $t \in T$ such that $(t + K) \cap K' = \varnothing$. Conversely, suppose that (sp') holds. Let S be a syndetic subset of T and let K be a corresponding compact for S. Suppose S is compact. Then there is a $t \in T$ such that $(t + K) \cap S = \varnothing$ (it exists by (sp')). This contradicts to the syndeticity of S.

It is easy to see that the conditions (sp') and (sp") are equivalent.

The condition (dsp) can be equivalently formulated in the following way:

(dsp') *For any compact subset K of T there exists a compact subset K' of T such that no translate $t + K'$, $t \in T$, is contained in K.*

Another equivalent formulation is the following one.

(dsp") *For any compact subset K of T there exists a compact subset K' of T such that no translate $k + K'$, $k \in K$, is contained in K.*

Note that the condition (dsp') is just a reformulation of the condition (dsp). Let us show that (dsp') is equivalent with (dsp"). Clearly, (dsp') implies (dsp"). We will show the contrapositive of the converse, i.e., that the negation ($\sim$ dsp') of (dsp) implies the negation ($\sim$ dsp") of (dsp"). Assume that there is a compact $K \subseteq T$ such that for every compact $K' \subseteq T$ there is a $t \in T$ with $t + K' \subseteq K$. For this K and any compact $K' \subseteq T$ there is a $t^* \in T$ such that $(t^* + (K \cup \{0\})) \subseteq K$. Hence $t^* \in K$. Thus $t^* + K' \subseteq K$ with $t^* \in K$, i.e., ($\sim$ dsp") holds. The statement is proved.

The condition (sp) was introduced in our paper [7], where we discussed chaos-related properties on the product of semiflows. The property (dsp) is for the first time considered in this paper.

Example 1.2.

(1) Every topological group is sp.
   Suppose to the contrary, i.e., that there are two compact subsets K, K' of the topological group T such that for every $t \in T$ we have $(t + K) \cap K' \neq \varnothing$. Then every $t \in T$ is of the form $t = k' - k$ for some $k \in K$ and $k' \in K'$. Since $K' - K$ is compact and T is noncompact, if we select $t \notin K' - K$, we get a contradiction.
   The groups $\mathbb{R}$ and $\mathbb{Z}$ are dsp.
(2) Every directional monoid is both sp and dsp. (A topological monoid T is said to be *directional* if for every compact subset K of T there is a $t \in T$ such that $(t + T) \cap K = \varnothing$. This notion was introduced in our paper [7].) In particular, $\mathbb{N}_0^n$ and $\mathbb{R}_+^n$ are both sp and dsp. (Here $\mathbb{N}_0$ denotes the additive monoid of nonnegative integers, while $\mathbb{R}_+$ denotes the additive monoid of nonnegative real numbers, both sets with the topology induced from $\mathbb{R}$.)
(3) The monoid $T = [0,1)$ with the topology induced from $\mathbb{R}$ and the operation $x + y = \max\{x, y\}$ is both sp and dsp.
(4) The monoid $T = \{0\} \cup (1/2, 1]$ with the topology induced from $\mathbb{R}$ and the operation $x + y = \max\{x, y\}$ is neither sp nor dsp.
   Indeed, to show that T is not sp we can consider the compacts $K = [2/3, 1]$ and $K' = [5/8, 5/6]$. To show that T is not dsp we consider the compact subset $K = \{1\}$.

Proposition 1.3. *The condition (sp) for topological monoids is stronger than the condition (dsp).*

Proof. Suppose to the contrary, i.e., that there is a topological monoid T in which (sp) and ($\sim$ dsp") hold. Let $K \subseteq T$ be a compact such that for every compact $K' \subseteq T$ there is a $k \in K$ with $k + K' \subseteq K$. By (sp'), there is a $t^* \in T$ such that $(t^* + K) \cap K = \varnothing$. However, for $K' = \{t^*\}$ there is a $k \in K$ such that $k + t^* \in K$. This contradicts to $(t^* + K) \cap K = \varnothing$. $\qquad\square$

Definition 1.4. A semiflow (T, X) is:

(a) *strongly mixing (StrM)* if for any two nopens U, V in X the set $D(U, V) = \{t \in T \mid tU \cap V \neq \varnothing\}$ contains $T \setminus K$ for some compact $K \subseteq T$;

(b) *weakly mixing (WM)* if for any nopens U_1, V_1, U_2, V_2 in X, $D(U_1, V_1) \cap D(U_2, V_2) \neq \varnothing$;

(c) *sensitive (S)* if there is a sensitivity constant $c > 0$ such that for any nopen $U \subseteq X$, $D(U, c) = \{t \in T \mid (\exists\, x, y \in U)\ d(tx, ty) > c\} \neq \varnothing$;

(d) *strongly sensitive (StrS)* if there is a sensitivity constant $c > 0$ such that for every nopen U in X the set $D(U, c)$ contains $T \setminus K$ for some compact $K \subseteq T$;

(e) *multisensitive (MulS)* if there is a sensitivity constant $c > 0$ such that for any integer $n \geq 1$ and any nopens $U_1, U_2, \ldots, U_n$ in X, $D(U_1, c) \cap \cdots \cap D(U_n, c) \neq \varnothing$;

(f) *strongly multisensitive (StrMulS)* if there is a sensitivity constant $c > 0$ such that for any integer $n \geq 1$ and any nopens $U_1, U_2, \ldots, U_n$ in X, $D(U_1, c) \cap \cdots \cap D(U_n, c)$ contains $T \setminus K$ for some compact $K \subseteq T$;

(g) *thickly sensitive (TS)* if there is a sensitivity constant $c > 0$ such that for every nopen U in X the set $D(U, c)$ is a thick subset of T;

(h) *syndetically sensitive (SyndS)* if there is a sensitivity constant $c > 0$ such that for every nopen U in X the set $D(U, c)$ is a syndetic subset of T;

(i) *thickly syndetically sensitive (TSyndS)* if there is a sensitivity constant $c > 0$ such that for every nopen U in X the set $D(U, c)$ is a thickly syndetic subset of T;

(j) *periodically sensitive (PerS)* if there is a sensitivity constant $c > 0$ such that for every nopen U in X the set $D(U, c)$ is a periodic subset of T;

(k) *thickly periodically sensitive (TPerS)* if there is a sensitivity constant $c > 0$ such that for every nopen U in X the set $D(U, c)$ is a thickly periodic subset of T.

The next lemma is well-known, see, for example, [1].

Lemma 1.5. *Let (T, X) be a weakly mixing semiflow and $U_1, \ldots, U_n, V_1, \ldots, V_n$ nonempty open subsets of X ($n \geq 1$). Then there is a $t \in T$ such that $tU_i \cap V_i \neq \varnothing$ for all $i = 1, \ldots, n$.*

2. Relations between various types of sensitivity in general semiflows

In this section we will justify the implication diagram below. In the diagram a crossed implication arrow between two conditions means that that implication does not hold, i.e., that there is a counterexample for that implication. If the condition sp or dsp is given by the arrow, that means that the implication holds when that condition is assumed.

Proposition 2.1. *Every strongly mixing semiflow is strongly sensitive.*

Proof. Let a, b be two points of X with $d(a, b) = \Delta > 0$ and let $B_a = B(a, \Delta/4)$, $B_b = B(b, \Delta/4)$. Let $c = \Delta/4$. Then for any $a' \in B_a$ and any $b' \in B_b$, $d(a', b') > c$. Let U be a nopen in X. Since (T, X) is strongly mixing, there is a compact $K \subseteq T$ such that for every $t \in T \setminus K$ there are $x, y \in U$ with

$tx \in B_a$, $ty \in B_b$. Hence $d(tx, ty) > c$. Thus $D(U, c) \supseteq T \setminus K$, so that (T, X) is strongly sensitive. $\qquad \square$
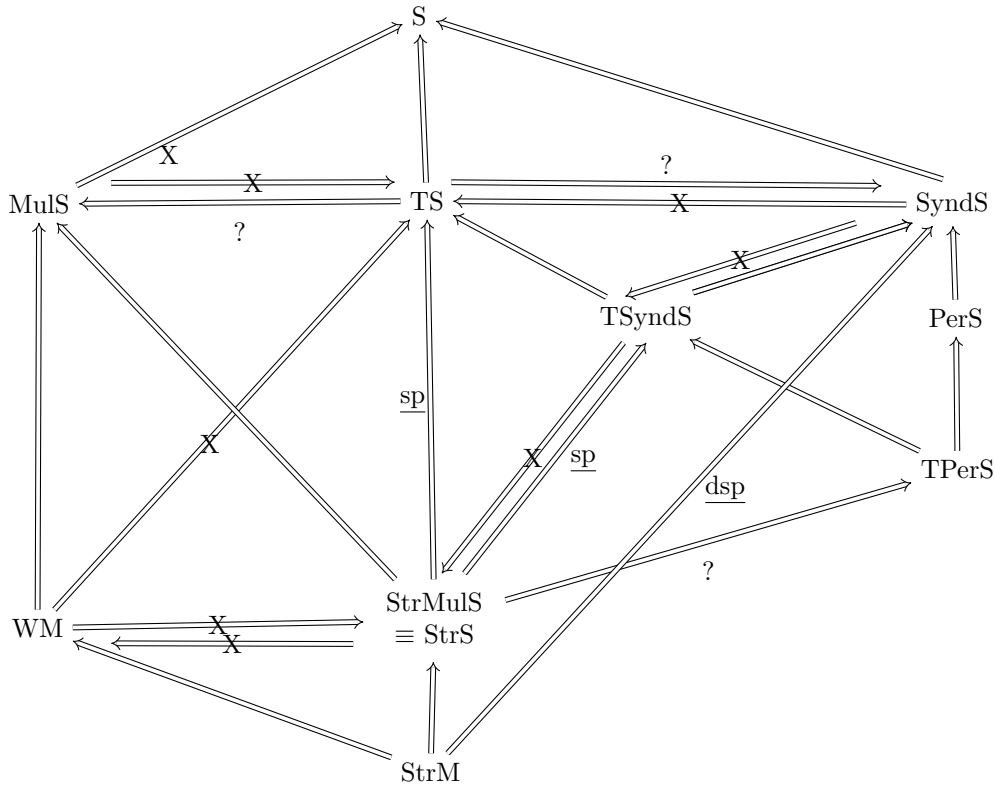
Proposition 2.2. *If T is a dsp monoid, then every strongly mixing semiflow is syndetically sensitive.*

Proof. Let T be a dsp monoid and (T, X) a strongly mixing semiflow. Let p, q be two distinct points of X, and let $d = d(p, q)$ and $c = d/3$. We claim that the constant c can serve as a sensitivity constant such that for every nopen U of X the set $D(U, c)$ is a syndetic subset of T. Indeed, let $O_1 = B(p, c)$ and $O_2 = B(q, c)$. Then for any two points $x_1 \in O_1$ and $x_2 \in O_2$ we have $d(x_1, x_2) > c$. Fix a nopen $U \subseteq X$. Since (T, X) is strongly mixing, there is a compact $K_1 \subseteq T$ such that for every $t \in T \setminus K_1$ there is a point $x \in U$ with $tx \in O_1$. Also there is a compact $K_2 \subseteq T$ such that for every $t \in T \setminus K_2$ there is a point $y \in U$ with $ty \in O_2$. Hence for every $t \in T \setminus (K_1 \cup K_2)$ there is a pair of points (x, y) from U such that $tx \in O_1$ and $ty \in O_2$, so that $d(tx, ty) > c$. Since T is dsp, $T \setminus (K_1 \cup K_2)$ is syndetic and so the proposition is proved. $\quad \square$

Proposition 2.3. *A semiflow is strongly sensitive if and only if it is strongly multisensitive.*

Proof. Suppose a semiflow (T, X) is strongly sensitive. Let $n \geq 1$ and let the U_i, $i = 1, \ldots, n$ be nopens in X. For each $i \in \{1, \ldots, n\}$ there is a compact K_i such that for every $t \in T \setminus K_i$ there are $x_i, y_i \in U_i$ with $d(tx_i, ty_i) > c_i$. Let $K = K_1 \cup \cdots \cup K_n$ and let $c = \min\{c_1, \ldots, c_n\}$. Then for every $t \in T \setminus K$ and every $i \in \{1, \ldots, n\}$ there are $x_i, y_i \in U_i$ with $d(tx_i, ty_i) > c$. Hence (T, X) is strongly munltisensitive. The other direction is clear. $\square$

Proposition 2.4. *There is a strongly multisensitive semiflow which is not weakly mixing.*

Proof. Let $X = [0, \infty)$ with the metric $d(x, y) = |e^x - e^y|$. Let $T = [0, \infty)$ act on X by $t.x = t + x$. Then (T, X) is strongly multisensitive. Indeed, for any n pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ of elements of X (for any $n \in \mathbb{N}$) the distances $d(x_i + t, y_i + t)$, $i = 1, 2, \ldots, n$, will be bigger than any $c > 0$ for all $t \in \mathbb{R}_+$ from some point on as the function $f(x) = e^x$ tends to infinity as x tends to infinity. However (T, X) it is not weakly mixing since for $U = (a, b)$ and $V = (c, d)$ with $d < a$ there is no $t \in T$ with $tU \cap V \neq \varnothing$. $\square$

Proposition 2.5. *Every strongly sensitive semiflow with an sp acting monoid is thickly sensitive.*

Proof. Let (T, X) be strongly sensitive and let $c > 0$ be its sensitivity constant. Let $U \subseteq X$ be nopen. Then there is a compact $K \subseteq T$ such that for every $t \in T \setminus K$, $t \in D(U, c)$. Let K' be a compact in T. We need to show that there is a $t \in T$ such that $t + K' \subseteq D(U, c)$. It is enough to show that there is a $t \in T$ such that $t + K' \subseteq T \setminus K$. Otherwise, for every $t \in T$, $(t + K') \cap K \neq \varnothing$, contradicting the assumption that T is sp. $\square$

Proposition 2.6. *There is a weakly mixing semiflow which is not thickly sensitive.*

Proof. Let $\mathbb{T}$ be a one-dimensional torus $\mathbb{R}/\mathbb{Z}$, i.e., $\mathbb{T} = [0, 1)$ with the metric $d(x, y) = \min\{|x - y|, 1 - |x - y|\}$. Define a continuous function $f : \mathbb{T} \to \mathbb{T}$ by $f(x) = 2x \pmod 1$ for every $x \in \mathbb{T}$. A point $x \in \mathbb{T}$ in the cascade $(\mathbb{T}, f)$ is said to be *eventually fixed* if there is an $n \geq 0$ such that $f^n(x) = 0$. The set of all eventually fixed points is $X = \{k/2^n \mid k, n \geq 0 \text{ integers}, k < 2^n\}$, which is a dense subset of $\mathbb{T}$. Note that $f(X) \subseteq X$, so that we can consider the restricted semiflow (X, f). Each point in this semiflow has a finite orbit whose last term is 0. The point 0 is the only fixed point. As shown in [6], (X, f) is weakly mixing.

Let now $T_1 = \{0, 1\}$ be a discrete monoid with the operation $0+0 = 0$, $1+0 = 0 + 1 = 1 + 1 = 1$. Let $T = \mathbb{N}_0 \times T_1 = \{(n, t) \mid n \in \mathbb{N}_0, t \in T_1\}$ be the product monoid of the discrete monoids $\mathbb{N}_0$ and T_1 (with componentwise addition). Define a monoid action of T on X by

$$(n, t) . x = \begin{cases} f^n(x) & \text{if } t = 0, \\ 0 & \text{if } t = 1. \end{cases}$$

It is easy to verify that this is indeed a monoid action. All transition maps are clearly continuous, so we have a topological semi-flow (T, X). This semiflow is weakly mixing since (X, f) is weakly mixing.

It was shown in [6] that (T, X) is not thickly sensitive. □

Example 2.7. An example of a weakly mixing semiflow which is not strongly sensitive is the semiflow from the proof of Proposition 2.6. Indeed, by Propositions 2.6 and 2.5, it is enough to see that the monoid T in the proof of Proposition 2.6 is sp, i.e., that for any two compact subsets K and K' of T there is an element $(n, t) \in T$ such that $((n, t) + K) \cap K' = \varnothing$. Since T is discrete, compacts are finite, so that for any sufficiently big $n \in \mathbb{N}_0$ the element $(n, 0)$ will work.

Proposition 2.8. *Every weakly mixing semiflow is multi-sensitive.*

Proof. We will follow [6]. If the diameter of X is infinite let D be any positive real number, otherwise let $\operatorname{diam}(X) = 12D > 0$. Then for any ball $B(x, 4D)$, $x \in X$, we have $X \setminus B(x, 4D) \neq \varnothing$. We will show that (T, X) is multi-sensitive with sensitivity constant $c = D$. Let $m \geq 1$ be an integer and let $U_1, U_2, \ldots, U_m$ be nonempty open subsets of X. Let $x_i \in U_i$ $(i = 1, 2, \ldots, m)$. For each $i = 1, 2, \ldots, m$ let $B_i = B(x_i, r_i)$, where $r_i < D$ is such that $B_i \subseteq U_i$. Let also $C_i^- = B^-(x_i, 2D)$. Then each $V_i = X \setminus C_i^-$ is a nonempty open subset of X. Note that for any $a \in B_i$ and $b \in V_i$, $d(a, b) > D$. By Lemma 1.5 there is a $t \in T$ such that at the same time $tB_i \cap B_i \neq \varnothing$ and $tB_i \cap V_i \neq \varnothing$ for $i = 1, 2, \ldots, m$. Let $y_i, z_i \in B_i \subseteq U_i$ $(i = 1, 2, \ldots, m)$ be such that $ty_i \in B_i$ and $tz_i \in V_i$ for $i = 1, 2, \ldots, m$. Then $d(ty_i, tz_i) > D$ $(i = 1, 2, \ldots, m)$. □

Corollary 2.9. *There is a multisensitive semiflow which is not thickly sensitive.*

Proof. Otherwise using Proposition 2.8 we would be able to conclude that every weakly mixing semiflow is thickly sensitive, which would contradict to Propositon 2.6. □

Proposition 2.10. *Every strongly sensitive semiflow whose acting monoid is sp is thickly syndetically sensitive.*

Proof. Let (T, X) be strongly sensitive with sensitivity constant c and let U be a nopen subset of X. Since (T, X) is strongly sensitive, there is a compact $K \subseteq T$ such that $D(U, c) \supseteq T \backslash K$. Hence, since T is sp and since, by Proposition 1.3, sp implies dsp, $D(U, c)$ is thick and syndetic, or, equivalently, thickly syndetic. □

Proposition 2.11. *There is a syndetically sensitive semiflow which is not thickly sensitive, nor thickly syndetically sensitive.*

Proof. An example is given in [5, Example 10]. □

Proposition 2.12. *There is thickly syndetically sensitive semiflow which is not strongly sensitive.*

Proof. An example is given in [5, Example 11]. It is a uniformly rigid weakly mixing minimal semiflow. As stated in [5], the existence of such semiflows follows from the paper [2], with a general acting monoid T in place of $\mathbb{N}_0$. □

Proposition 2.13. *Every strongly mixing semiflow is weakly mixing. Every strongly multisensitive semiflow is multi sensitive. Every thickly syndetically sensitive semiflow is thickly sensitive. Every thickly syndetically sensitive semiflow is syndetically sensitive. Every thickly periodically sensitive semiflow is periodically sensitive. Every thickly periodically sensitive semiflow is thickly syndetically sensitive. Every periodically sensitive semiflow is syndetically sensitive. Every multisensitive (resp. thickly sensitive; syndetically sensitive) semiflow is sensitive.*

Proof. The first statement is well-known and easy to see. The remaining ones follow from the definitions. □

3. Concluding remarks

We analyzed a variety of "sensitivity-properties", starting with the strong mixing as the strongest one and ending with the sensitivity as the weakest one. We organized them into an implication diagram and proved that some of those implications are true, some are not true, and some are left as open questions. In the process we introduced the properties (sp) and (dsp) of topological monoids. Here are the remaining questions.

Question 1. Is (sp) a strictly stronger property than (dsp), i.e., is there a topological monoid which is dsp but not sp?

Question 2. Find examples showing that in the implications StrS ⇒ TS and StrS ⇒ TSyndS the condition (sp) is indeed needed, and that in the implication SM ⇒ SyndS the condition (dsp) is indeed needed.

Question 3. Investigate if, in general, (StrS) implies (TPerS), (TS) implies (MulS), and (SyndS) implies (TS).

References

[1] E. Glasner, Ergodic Theory via Joinings, Mathematical Surveys and Monographs, American Mathematical Society, 2003.

[2] E. Glasner and D. Maon, Rigidity in topological dynamics, Ergod. Th. & Dynam. Sys. 9 (1989),309–320.

[3] L. He, X. Yan and L. Wang, Weak-mixing implies sensitive dependence, J. Math. Anal. Appl. 299 (2004), 300–304.

[4] E. Kontorovich, M. Megrelishvili, A note on sensitivity of semigroup actions, Semigroup Forum 76 (2008), 133–141.

[5] H. Liu, L. Liao and L. Wang, Thickly syndetical sensitivity of topological dynamical system, Discrete Dyn. Nature Soc. 2014, Article ID 583431.

[6] A. Miller, Weak mixing in general semiflows implies multi-sensitivity, but not thick sensitivity, J. Nonlinear Sci. Appl., to appear.

[7] A. Miller and C. Money, Chaos-related properties on the product of semiflows, Turkish J. Math. 41 (2017), 1323–1336.

[8] T. S. Moothathu, Stronger forms of sensitivity for dynamical systems, Nonlinaerity 20 (2007), 2115–2126.

[9] T. Wang, J. Yin and Q. Yan, The sufficient conditions for dynamical systems of semigroup actions to have some stronger forms of sensitivities, J. Nonlinear Sci. Appl. 9 (2016), 989–997.