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F-n-resolvable spaces and compactifications

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Abstract

A topological space is said to be resolvable if it is a union of two disjoint dense subsets. More generally it is called n-resolvable if it is a union of n pairwise disjoint dense subsets.
In this paper, we characterize topological spaces such that their reflections (resp., compactifications) are n-resolvable (resp., exactly-n-resolvable, strongly-exactly-n-resolvable), for some particular cases of reflections and compactifications.

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Introduction

Let $n > 1$ be an integer. Generalizing the concept of resolvable spaces introduced by Hewitt in [16], Ceder in [6] defined a topological space X to be n-resolvable space if it has a family of n pairwise disjoints dense subsets. The latter is called exactly n-resolvable if it is n-resolvable but not $(n+1)$-resolvable and it is called strongly exactly n-resolvable denoted by SE_nR if it is n-resolvable and no empty subset of X is $(n+1)$-resolvable. SE_1R space is commonly said strongly irresolvable space (abbreviated as SI-space) or hereditarily irresolvable (see [7] and [13]).

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The theory of categories and functors play an enigmatic role in topology, specially the notion of reflective subcategories. Recently, some authors have been interested by particular functors like $\mathbf{T_0}$, $\mathbf{S}$, ρ and $\mathbf{FH}$.

In [10], [11] and [8], the authors have characterized topological spaces whose F-reflections are door, submaximal, nodec and resolvable.

Some papers, as [5] and [3] were interested in spaces such that their compactifications are submaximal, door and nodec. Specially in [2], K. Belaid and M. Al-Hajri have characterized topological spaces such that their one point compactifications (resp., Wallman compactifications) are resolvable.

In the first section of this paper, we characterize topological spaces such that their T_0-reflections are n-resolvable (resp., exactly n-resolvable, strongly exactly n-resolvable).

In the second section, topological spaces, such that their Tychonoff reflections and functionally Hausdorff reflections are n-resolvable (resp., exactly n-resolvable), have been characterized.

The third section of this paper is devoted to a characterization of topological spaces such that their one point compactifications (resp., Wallman compactifications) are n-resolvable (resp., exactly n-resolvable, strongly exactly n-resolvable).

1. T_0-n-resolvable spaces, T_0-exactly-n-resolvable spaces and T_0-strongly-exactly-n-resolvable spaces.

Let X be a topological space. The $\mathbf{T}_0$-reflection of X denoted by $\mathbf{T}_0(X)$ is defined as follow.

Consider the equivalence relation $\sim$ on X by: $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$, for $x, y \in X$. Then the resulting quotient space $\mathbf{T}_0(X) := X/\sim$ is a Kolmogroff space called the $\mathbf{T}_0$-reflection of X.

Recall that a continuous map $q : X \longrightarrow Y$ is said to be a quasihomeomorphism if $U \longmapsto q^{-1}(U)$ (resp., $C \longmapsto q^{-1}(C)$) defines a bijection $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ (resp., $\mathcal{F}(Y) \longrightarrow \mathcal{F}(X)$), where $\mathcal{O}(X)$ (resp., $\mathcal{F}(X)$) is the collection of all open sets (resp., closed sets) of X)[15].
In particular the canonical surjection $\mu_X : X \longrightarrow \mathbf{T}_0(X)$ is an onto *quasihomeomorphism* and consequently a closed (resp., open) map, (see [4]).

In order to give the main result of this section we recall the following results introduced in [10].

Notation 1.1 ([10, Notations 2.2]). Let X be a topological space, $a \in X$ and $A \subseteq X$. We denote by:

(1) $d_0(a) := \{x \in X : \overline{\{x\}} = \overline{\{a\}}\}$.
(2) $d_0(A) = \cup[d_0(a); a \in A]$.

Remark 1.2 ([10, Remarks 2.3]). Let X be a topological space and A be a subset of X. The following properties hold.

(i) $d_0(A) = \mu_X^{-1}(\mu_X(A))$.

(ii) $d_0(d_0(A)) = d_0(A)$.
(iii) $A \subseteq d_0(A) \subseteq \overline{A}$ and consequently $\overline{d_0(A)} = \overline{A}$.
(iv) In particular if A is open (resp., closed), then $d_0(A) = A$.

The following definitions are natural.

Definition 1.3. A topological space X is called *T_0-n-resolvable* (resp., *T_0-exactly-n-resolvable*, *T_0-strongly-exactly-n-resolvable*) if its T_0-reflection is n-resolvable (resp., exactly-n-resolvable, strongly-exactly-n-resolvable).

Before giving the characterization of T_0-n-resolvable spaces, let us introduce the following definition.

Definition 1.4. A family $\{A_i : i \in I\}$ of subsets of a topological space X is called pairwise d_0-disjoint if and only if $d_0(A_i) \cap d_0(A_j) = \varnothing$, for any $i \neq j \in I$.

By Remarks 1.2 (iii), a pairwise d_0-disjoint family is a pairwise disjoint family.

The following result characterise T_0-n-resolvable spaces.

Theorem 1.5. *Let X be a topological space. Then the following statements are equivalent:*
(1) *X is a T_0-n-resolvable space;*
(2) *X have a dense pairwise d_0-disjoint family with cardinality n.*

Proof. (1)$\Longrightarrow$(2)

Suppose that X is a T_0-n-resolvable space. Then $T_0(X)$ has a dense pairwise disjoint family $\{\mu_X(A_i);\ 1 \leq i \leq n\}$, where A_1,..., A_n are subsets in X. So applying μ_X^{-1}, one can see easily that $\{d_0(A_i) :\ 1 \leq i \leq n\}$ is a family of pairwise disjoint subsets of X.

Now since μ_X is an onto quasihomeomorphism then, by [10, Lemma 2.16], we have:

$$\forall 1 \leq i \leq n \ \ X = \mu_X^{-1}(\mathbf{T}_0(X)) = \mu_X^{-1}\left(\overline{\mu_X(A_i)}\right) = \overline{\mu_X^{-1}(\mu_X(A_i))} = \overline{d_0(A_i)}.$$

Therefore $\{A_i;\ 1 \leq i \leq n\}$ is a dense pairwise d_0-disjoint family of X.

(2)$\Longrightarrow$(1)

Suppose that X has a dense pairwise d_0-disjoint family $\{A_i;\ 1 \leq i \leq n\}$ with cardinality n. Then, for any $1 \leq i \neq j \leq n$, the condition $d_0(A_i) \cap d_0(A_j) = \varnothing$ implies immediately that $\mu_X(A_i) \cap \mu_X(A_j)) = \varnothing$.

Now, let $1 \leq i \leq n$. The density of $d_0(A_i)$ in X shows that:

$$T_0(X) = \mu_X(X) = \mu_X(\overline{d_0(A_i)}) = \mu_X(\overline{\mu_X^{-1}(\mu_X(A_i))}) = \mu_X(\mu_X^{-1}\left(\overline{\mu_X(A_i)}\right)) = \overline{\mu_X(A_i)}.$$

Therefore, $\{\mu_X(A_i) :\ 1 \leq i \leq n\}$ is a dense pairwise disjoint family of subsets of $T_0(X)$. $\square$

Remark 1.6. Clearly every T_0-n-resolvable space is a n-resolvable space. The converse does not hold, indeed:

Let X be a subset of cardinality n ($n > 1$) equipped with the indiscreet topology. Clearly the family $\{\{x\};\ \ x \in X\}$ is composed by disjoint dense subsets of X and thus X is n-resolvable. But $T_0(X)$ is a one point which is not 2-resolvable. Remark that in this case $d_0(\{x\}) = X$, for any $x \in X$ and consequently, $d_0(A) = X$ for any subset A of X, therefore there is no d_0-disjoint family of X with cardinality greater or equal to 2.

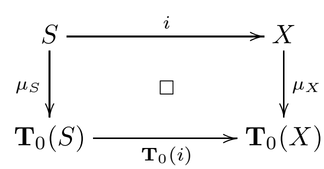
The following result is an immediate consequence of the previous theorem.

Corollary 1.7. *Let X be a topological space. X is a T_0-exactly-n-resolvable space if and only if* $\max\{|\ \mathcal{F}\ |\ \ \mathcal{F}$ *is a dense* d_0- *disjoint family of* $X\} = n$.

Before giving a characterization of a T_0-strongly-exactly-n-resolvable space we need the following lemma.

Lemma 1.8. *Let X be a topological space and S a subset of X. Then $\mu_X(S) \simeq \mu_S(S)$.*

Proof. S is a subset of X then, the following diagram is commutative.



- $T_0(\mathbf{i}) : T_0(S) \longrightarrow T_0(\mathbf{i})(T_0(S))$ is bijective. In fact it is enough to show that $T_0(\mathbf{i})$ is one-to-one.

Let x, y two elements of S such that $T_0(\mathbf{i})(\mu_S(x)) = T_0(\mathbf{i})(\mu_S(y))$. Then, $\mu_X(\mathbf{i}(x)) = \mu_X(\mathbf{i}(y))$ and thus $\mu_X(x) = \mu_X(y)$. Hence, we get $\overline{\{x\}}^S = \overline{\{x\}} \cap S = \overline{\{y\}} \cap S = \overline{\{y\}}^S$, as desired.

- $T_0(\mathbf{i})$ is an open map. Indeed, let $\widetilde{U}$ be an open set of $T_0(S)$. Then, there exists V an open set of X such that $\mu_S^{-1}(\widetilde{U}) = V \cap S$. Thus

$$\begin{array}{rcl} T_0(\mathbf{i})(\widetilde{U}) & = & T_0(\mathbf{i})(\mu_S(V \cap S)) \\ & = & \mu_X(\mathbf{i}(V \cap S)) \\ & = & \mu_X(V \cap S) \end{array}$$

So, let us show that $\mu_X(V \cap S) = \mu_X(V) \cap T_0(\mathbf{i})(T_0(S))$. Indeed:

$$\begin{array}{rcl} \mu_X(V \cap S) & \subseteq & \mu_X(V) \cap \mu_X(S) \\ & = & \mu_X(V) \cap \mu_X(\mathbf{i}(S)) \\ & = & \mu_X(V) \cap T_0(\mathbf{i})(\mu_S(S)) \\ & = & \mu_X(V) \cap T_0(\mathbf{i})(T_0(S)) \end{array}$$

Which gives the first inclusion.

Conversely, let $x \in \mu_X(V) \cap T_0(\mathbf{i})(T_0(S))$. Then there exist $y \in V$ and $t \in S$ such that $\mu_X(y) = x = T_0(\mathbf{i})(\mu_S(t)) = \mu_X(\mathbf{i}(t)) = \mu_X(t)$. Thus, $\overline{\{y\}} = \overline{\{t\}}$.

Since $y \in V$, then $V \cap \{t\} \neq \varnothing$. So, $x = \mu_X(t) \in \mu_X(V \cap S)$ which proves that $\mu_X(V) \cap T_0(\mathbf{i})(T_0(S)) \subseteq \mu_X(V \cap S)$ which gives the second inclusion as desired.

- $\mu_X(S) \simeq \mu_S(S)$.

According to the above, we conclude that $T_0(\mathbf{i})$ is an homeomorphism from $T_0(S)$ to $T_0(\mathbf{i})(T_0(S))$. Then, $\mu_X(S) = \mu_X(\mathbf{i}(S)) = T_0(i)(\mu_S(S)) = T_0(i)(T_0(S))$ $\simeq T_0(S) = \mu_S(S)$. □

Theorem 1.9. *Let X be a topological space. Then the following statements are equivalent:*

(1) *X is a T_0-strongly-exactly-n-resolvable space.*
(2) *X is T_0-n-resolvable and for any subset S of X, S is not T_0-$(n+1)$-resolvable.*

Proof. (1)$\Longrightarrow$(2)

Let S be a subset of X. Since X is T_0-strongly-exactly-n-resolvable, $\mu_X(S)$ is not $(n+1)$-resolvable. Then, by Lemma 1.8, $\mu_S(S) = T_0(S)$ is not $(n+1)$-resolvable. Therefore, X is a T_0-n-resolvable space in which every subset S of X, is not T_0-$(n+1)$-resolvable.

(2)$\Longrightarrow$(1)

Let $\mu_X(S)$ be a subset of $T_0(X)$, where S be a subset of X. By hypothesis, S is not $T_0 - (n+1)$-resolvable that is $T_0(S) = \mu_S(S)$ is not $(n+1)$-resolvable. Using Lemma 1.8, $\mu_X(S)$ is not $(n+1)$-resolvable. So that every subset $\mu_X(S)$ of $T_0(X)$ is not $(n+1)$-resolvable and thus $T_0(X)$ is strongly-exactly-n-resolvable. □

2. $\boldsymbol{\rho}$-n-resolvable spaces and $\boldsymbol{FH}$-n-resolvable spaces

Recall that a T_1 topological space X is called Tychonoff if for any closed subset F of X and for any $x \in X$ not in F there exists a real continuous map f from X to $\mathbb{R}$ (we write $f \in \mathbf{C}(X)$) such that $f(x) = 0$ and $f(F) = \{1\}$. We say that F and x are completely separated. In particular two distinct points in a given Tychonoff space X are said to be completely separated if x and $\{y\}$ are completely separated.

A T_1 topological space in which every two distinct points are completely separated, is called functionally Hausdorff space.

Give a topological space X. We define the equivalence relation $\sim$ on X by $x \sim y$ if and only if $f(x) = f(y)$ for all $f \in \mathbf{C}(X)$.

On the one hand, the set of equivalence classes $X/\sim$ equipped with the quotient topology, is a functionally Hausdorff space called the **FH**-reflection of X.

On the other hand, consider ρ_X the canonical surjection map from X to $X/\sim$. Then for any continuous map f_α from X to $\mathbb{R}$, there exists a unique map $\rho(f_\alpha)$ from $X/\sim$ to $\mathbb{R}$ satisfying $\rho(f_\alpha)(\rho_X(x)) = f(x)$, for any $x \in X$. So, $X/\sim$ equipped with the the topology whose closed sets are of the form $\cap[\boldsymbol{\rho}(f_\alpha)^{-1}(F_\alpha) : \alpha \in I]$, where $f_\alpha : X \longrightarrow \mathbb{R}$ (resp., F_α) is a continuous map (resp., a closed subset of $\mathbb{R}$), is a a Tychonoff space (see for instance [22]) called the ρ-reflection of X.

We need to introduce and recall some definitions, notations and results.

Notation 2.1 ([10, Notation 3.1]). Let X be a topological space, $a \in X$ and A a subset of X. We denote by:

(1) $d_{\boldsymbol{\rho}}(a) := \cap[f^{-1}(f(\{a\})) : \ f \in \mathbf{C}(X)]$.
(2) $d_{\boldsymbol{\rho}}(A) := \cup[d_{\boldsymbol{\rho}}(a) : a \in A]$.

Definition 2.2. Let X be a topological space. X is called:

(1) *ρ-n-resolvable* (resp., **FH**-*n-resolvable*) space if its ρ-reflection (resp., **FH**-reflection) is a n-resolvable space.
(2) *ρ-exactly-n-resolvable* (resp., **FH**-*exactly-n-resolvable*) space if its ρ-reflection (resp., **FH**-reflection) is an exactly-n-resolvable space.
(3) *ρ-strongly-exactly-n-resolvable* (resp., **FH**-*strongly-exactly-n-resolvable*) space if its ρ-reflection (resp., **FH**-reflection) is a strongly-exactly-n-resolvable space.

Recall that for a given topological space X and $A \subseteq X$, A is called a zero-set if there exists $f \in C(X)$ such that $A = f^{-1}(\{0\})$. The complement of a zero-set is called a cozero-set.

A space is Tychonoff if and only if the family of zero-sets of the space is a base for the closed sets (equivalently, the family of cozero-sets of the space is a base for the open sets)(see [22, Proposition 1.7]). In [10] it is showen that a closed (resp., open) subset of $\boldsymbol{\rho}(X)$ is of the form $\cap[\boldsymbol{\rho}(f)^{-1}(\{0\}) : f \in H]$ (resp., $\cup[\boldsymbol{\rho}(f)^{-1}(\mathbb{R}^\star) : f \in H]$) , where H is a collection of continuous maps from X to $\mathbb{R}$.

Definition 2.3 ([10, Definition 3.14])**.** Let X be a topological space, a subset V of X is called:

(i) a *functionally open* subset of X (for short F-open) if and only if $d_\rho(V)$ is open in X.
(ii) a *functionally dense* subset of X (for short F-dense) if and only if for any F-open subset W of X, $d_{\boldsymbol{\rho}}(V)$ meets $d_{\boldsymbol{\rho}}(W)$.
(iii) $\boldsymbol{\rho}$-dense, if $g(V) \neq \{0\}$ for every nonzero continuous map g from X to $\mathbb{R}$.

Definition 2.4. Let X be a topological space and $\{A_i : \ i \in I\}$ be a family of subsets of X. We say that this family is pairwise $d_{\boldsymbol{\rho}}$-disjoint if and only if $d_{\boldsymbol{\rho}}(A_i) \cap d_{\boldsymbol{\rho}}(A_j) = \varnothing$, for any $i \neq j \in I$.

Theorem 2.5. *Let X be a topological space. Then the following statements are equivalent:*

(i) X is **FH**-*n-resolvable.*
(ii) X have a F-dense pairwise $d_{\boldsymbol{\rho}}$-disjoint family with cardinality n.

Proof. $(i) \Longrightarrow (ii)$

Suppose that X is an **FH**-n-resolvable space. Then, there exists a family $\{\rho_X(A_1), ..., \rho_X(A_n)\}$ of dense pairwise disjoint subsets of $\mathbf{FH}(X)$.

Now, applying ρ_X^{-1}, we see easily that the family $\{A_1, ..., A_n\}$ is pairwise $d_{\boldsymbol{\rho}}$-disjoint. Finally, the equality $\overline{\boldsymbol{\rho}_X(A_i)} = \mathbf{FH}(X)$ means that A_i is a F-dense subset of X. Therefore, $\{A_1, ..., A_n\}$ is pairwise $d_{\boldsymbol{\rho}}$-disjoint family of X with cardinality n.

$(ii) \Longrightarrow (i)$

Conversely, let $\{A_i :\ 1 \leq i \leq n\}$ be a family of F-dense pairwise $d_{\boldsymbol{\rho}}$-disjoint subsets of X. Then on the one hand, for every $1 \leq i \leq n$, $\boldsymbol{\rho}_X(A_i)$ is a dense subset of $\mathbf{FH}(X)$ and on the other hand, $\forall\ 1 \leq i \neq j \leq n$, we have

$$\begin{array}{rcl} d_{\boldsymbol{\rho}}(A_i) \cap d_{\boldsymbol{\rho}}(A_j)) & = & \boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A_1)) \cap \boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A_j))) \\ & = & \boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A_i) \cap \boldsymbol{\rho}_X(A_j))) \\ & = & \varnothing \end{array}$$

Therefore, $\{\rho_X(A_1), ..., \rho_X(A_n)\}$ is a family of dense pairwise disjoint subsets of $\mathbf{FH}(X)$. □

By the same way as in Theorem 2.5, the following result is immediate.

Theorem 2.6. *Let X be a topological space. Then the following statements are equivalent:*

(i) X is $\boldsymbol{\rho}$-n-resolvable.

(ii) X have a $\boldsymbol{\rho}$-dense and pairwise $d_{\boldsymbol{\rho}}$-disjoint family of cardinality n.

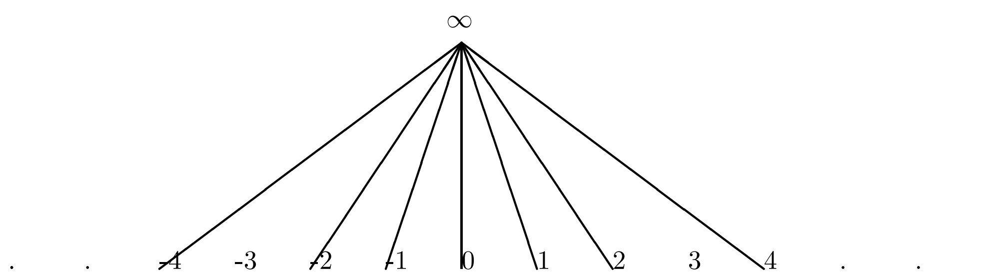
Remark 2.7. Since every F-dense subset is a $\boldsymbol{\rho}$-dense subset (see [10, Remarks 3.15]), then by Theorem 2.6, every $\mathbf{FH}$-n-resolvable space is $\boldsymbol{\rho}$-n-resolvable.

The following results are immediate.

Corollary 2.8. *Let X be a topological space. X is a* $\mathbf{FH}$*-exactly-n-resolvable space if and only if* $\max\{|\,\mathcal{F}\,| \quad \mathcal{F}$ *is F-dense and $d_{\boldsymbol{\rho}}-$disjoint family of* $X\} = n$.

Corollary 2.9. *Let X be a topological space. X is a $\boldsymbol{\rho}$-exactly-n-resolvable space if and only if* $\max\{|\,\mathcal{F}\,| \quad \mathcal{F}$ *is $\boldsymbol{\rho}$-dense and $d_{\boldsymbol{\rho}}-$disjoint family of* $X\} = n$.

Remark 2.10. Regarding Lemma 1.8, this result does not subsist in the case of the functors FH and ρ as showing by the following example.
Consider the Alexandroff space $X = \mathbb{Z} \cup \{\infty\}$ such that $\overline{\{n\}} = \{n\}$, for every $n \in \mathbb{Z}$ and $\overline{\{\infty\}} = X$. It is clear that every real continuous map from X is constant and thus $FH(X) = \rho(X)$ is a one point space. Now, consider $S = \mathbb{Z}$, then $FH(S) = \rho(S) = S$, but $\rho_X(S)$ is a one point. One can illustrates this situation by the following picture.



Question 2.11. *The Theorem 1.9 is an immediate consequence of Lemma 1.8 which is not valuable in the case of the functors FH and ρ as showing by Remark 2.10. Hence the following question is immediate. Are* **FH***-strongly-exactly-n-resolvable (resp., ρ-strongly-exactly-n-resolvable) spaces equivalent to FH-n-resolvable (resp., ρ-n-resolvable) in which every subset S of X, is not FH-$(n+1)$-resolvable (resp., ρ-$(n+1)$-resolvable)?*

3. n-resolvable spaces and compactifications

Definition 3.1. A *compactification* of a topological space X is a pair $(K(X), e)$ where $K(X)$ is a compact space and e an embedding of X as a dense subset of $K(X)$.

Remark 3.2. In many cases, e will be an inclusion map, so that $X \subseteq K(X)$. In other cases , we can agree to write X when mean $e(X)$, so that we can again write $X \subseteq K(X)$. Whenever one of this situations occurs we say simply that $K(X)$ is a compactification of X, and think of $K(X)$ as containing X as a dense subspace.

Lemma 3.3 ([2, Lemma 2.1])**.** *Let X be a topological space and $K(X)$ be a compactification of X and A be a subset of $K(X)$. If X is an open set of $K(X)$ Then the following statements are equivalent:*

1. *A is a dense subset of $K(X)$.*
2. *$A \cap X$ is a dense subset of X.*

Using Lemma 3.3, the following proposition is immediate.

Proposition 3.4. *Let X be a topological space and $K(X)$ be a compactification of X. If X is an open set of $K(X)$ Then the following statements are equivalent:*

(1) *X is n-resolvable.*
(2) *$K(X)$ is n-resolvable.*

Recall that for a topological space X, the set $\widetilde{X} = X \cup \{\infty\}$ with the topology whose members are the open sets of X and all subsets U of $\widetilde{X}$ such that $\widetilde{X} \setminus U$ is a closed compact subset of X, is called the Alexandroff extension of X (or the one-point compactification of X).

Now, regarding Proposition 3.4, we get immediately the following result.

Corollary 3.5. *Let X be a non compact topological space Then the following statements are equivalent:*

(1) *The one point compactification $\widetilde{X}$ of X is n-resolvable.*
(2) *X is n-resolvable.*

We turn our attention to spaces such that their Wallman compactifications are n-resolvable spaces.

First, let us recall the construction of Wallman compactification of T_1-space (a concept introduced, in 1938, by Wallman [23]).

Let $\mathcal{P}$ be a class of subsets of a topological space X wich is closed under fnite intersections and finite unions.

A $\mathcal{P}$-filter on X is a collection $\mathcal{F}$ of nonempty elements of $\mathcal{P}$ with the properties:

(a) $P_1, P_2 \in \mathcal{F}$ implies $P_1 \cap P_2 \in \mathcal{F}$.
(b) $P_1 \in \mathcal{F}$ $P_1 \subseteq P_2$ implies $P_2 \in \mathcal{F}$.

A $\mathcal{P}$-ultrafilter is a maximal $\mathcal{P}$-filter. When $\mathcal{P}$ is the class of closed sets of X, then the $\mathcal{P}$-filters are called closed filters.

The points of the Wallman compactification wX of a space X are the closed ultrafilters on X. For each closed set $D \subseteq X$, define D^* to be the set $D^* = \{\mathcal{A} \in wX : D \in \mathcal{A}\}$, if $D \neq \varnothing$ and $\varnothing^* = \varnothing$. Thus $\mathcal{C} = \{D^* : D$ is a closed set of $X\}$ is a base for the closed sets of a topology on wX.

Let U be an open subset of X. We define $U^* = \{\mathcal{A} \in wX : F \subseteq U$ for some F in $\mathcal{A}\}$. It is easily seen that the class $\{U^* : U$ is an open set of $X\}$ is a base for the open sets of the topology of wX. The following properties of wX are frequently useful:

Proposition 3.6. *For $x \in X$, let $w_X(x) = \{A \mid A$ is a closed set of X and $x \in A\}$. Then w_X is an embedding of X into wX. Thus, if $x \in X$, then $w_X(x)$ will be identified to x.*

Proposition 3.7. *If $U \subset X$ is open, then $wX \setminus U^* = (X \setminus U)^*$.*

Proposition 3.8. *If $D \subset X$ is closed, then $wX \setminus D^* = (X \setminus D)^*$.*

Proposition 3.9. *If U_1 and U_2 are open in X, then $(U_1 \cap U_2)^* = U_1^* \cap U_2^*$ and $(U_1 \cup U_2)^* = U_1^* \cup U_2^*$.*

In [19], Kovar has characterized space with finite Wallman compactification remainder as following:

Proposition 3.10. *Let X be a T_1-space, wX the Wallman compactification of X and k a finite number. Then the following statements are equivalent:*

1. *$Card(wX - X) = k$.*
2. *There exists a collection of k pairwise disjoint non compact closed sets of X and every family of non compact pairwise disjoint closed sets of X contain at most k elements.*

The following proposition follows immediately from Proposition 3.10.

Proposition 3.11. *Let X be a T_1-space and $k \in \mathbb{N}$ such that every family of non compact pairwise disjoint closed sets of X contain at most k elements. Then X is n-resolvable if and only if wX is n-resolvable.*

Corollary 3.12 ([2, corollary 3.5])**.** *Let X be a T_1-space, wX be the Wallman compactification of X and U be an open set of X. Then the following statements are equivalent:*

1. *$U \subsetneq U^*$.*
2. *There exists a non compact closed set F of X such that $F \subseteq U$.*

Definition 3.13. Let X be a T_1-topological space. Then X is said to be w-n-resolvable, if its Wallman compactification is n-resolvable.

Before characterizing w-n-resolvable spaces, let us introduce the useful definition.

Definition 3.14. We said that a finite family of subsets $\{D_i,\ i \in I\}$ of a topological space $(X, \mathcal{O}(X))$ satisfies the property $(\mathcal{P})$ if:

for every $(i, O) \in J = I \times \{O \in \mathcal{O}(X) :\ O \cap D_i = \varnothing\}$, there exists a non compact closed subset $F_{O,i} \subset O$ with $\{F_{O,i} :\ (i, O) \in J\}$ is a family of pairwise disjoint subsets of X.

Now, let us give one of the main result of this section.

Theorem 3.15. *Let X be a T_1- topological space, Then the following statements are equivalent:*

1. *X is w-n-resolvable.*
2. *X is a partition of a family of n subsets satisfying $(\mathcal{P})$.*

Proof. Let X be a w-n-resolvable space. Then there exist n pairwise disjoint dense subsets A_1,A_2,..., A_n of wX such that $wX = A_1 \cup A_2 \cup ... \cup A_n$. We denote $D_i = A_i \cap X$. It is clear that the family $\{D_i; 1 \leq i \leq n\}$ is a partition of X.

Let O be a nonempty open subset of X such that $O \cap D_i = \varnothing$. The density of A_i in wX gives an element $\mathcal{F}_i \in O^* \cap A_i$. By Corollary 3.12, there exists a non compact closed subset $G_{(i,O)} \subset O$ such that $G_{(i,O)} \in \mathcal{F}_i$.

Now, if i' is distinct from i and O' is a given nonempty open subset of X such that $O' \cap D_{i'} = \varnothing$, by the same way, there exists an element $\mathcal{F}_{i'} \in O'^* \cap A_{i'}$ and consequently there exists a non compact closed subset $G_{(i',O')} \subset O'$ such that $G_{(i',O')} \in \mathcal{F}_{i'}$. Since $A_i \cap A_{i'} = \varnothing$, then $\mathcal{F}_i \neq \mathcal{F}_{i'}$. Thus, there exist a closed subsets $F_i \in \mathcal{F}_i$ and $F_{i'} \in \mathcal{F}_{i'}$ such that $F_i \cap F_{i'} = \varnothing$. Let $F_{(i,O)} = G_{(i,O)} \cap F_i$ and $F_{(i',O')} = G_{(i',O')} \cap F'_i$. It is clear that $F_{(i,O)} \in \mathcal{F}_i \in wX \setminus X$ and $F_{(i',O')} \in \mathcal{F}_{i'} \in wX \setminus X$. Hence, $F_{(i,O)}$ and $F_{(i',O')}$ are non compact closed subsets (see [2, Lemma 3.4]), which are disjoint.

Conversely, let $\{D_i; 1 \leq i \leq n\}$ be a partition of X by n subsets satisfying $(\mathcal{P})$. For every $1 \leq i \leq n$, set $A_i = D_i \cup \{\mathcal{F} \in wX - X : F_{(i,O)} \in \mathcal{F}\}$, where O is an open subset of X such that $O \cap D_i = \varnothing$ (it is clearly seen that if $A_i = D_i$, then D_i is dense in wX). Clearly, by construction, A_i is a dense subset of wX for every $1 \leq i \leq n$.

To finish, let us show that the family $\{A_i : 1 \leq i \leq n\}$ are pairwise disjoint. So, suppose the existence of $1 \leq i \neq j \leq n$ such that $A_i \cap A_j \neq \varnothing$. Since $D_i \cap D_j = \varnothing$, then $A_i \cap A_j \cap (wX - X) \neq \varnothing$. By construction of A_i and A_j, there exist an ultrafilter $\mathcal{F}_i \in A_i$ and $\mathcal{F}_j \in A_j$ such that $\mathcal{F}_i = \mathcal{F}_j$. Furthermore, there exist open subsets O, O' and non compact closed subsets $F_{(i,O)} \in \mathcal{F}_i$ and $F_{(j,O')} \in \mathcal{F}_j$ such that $O \cap D_i = \varnothing$, $O' \cap D_j = \varnothing$, $F_{(i,O)} \subset O$ and $F_{(j,O')} \subset O'$. Hence, by the property $(\mathcal{P})$, $F_{(i,O)} \cap F_{(j,O')} = \varnothing$ and consequently $\mathcal{F}_i \neq \mathcal{F}_j$, which leads to a contradiction. □

As an immediate consequence of Theorem 3.15, for the particular case when $n = 2$, we have the following corollary.

Corollary 3.16 ([2, Theorem 3.6])**.** *Let X be a T_1- topological space, Then the following statements are equivalent:*

(1) *X is w-resolvable.*
(2) *X is a partition of two subsets $\{D_1, D_2\}$ and for each nonempty open subset $O \subseteq D_i$ ($i \in \{1, 2\}$), there exists a non compact closed subset F such that $F \subseteq O$.*

To close this section the following result is immediate.

Corollary 3.17. *Let X be a T_1-topological space. X is w-exactly-n-resolvable if and only if*
$\max\{| \mathcal{F} | \text{ ; } \mathcal{F} \text{ is a partition of } X, \text{ of } n \text{ dense subsets satisfying } (\mathcal{P})\} = n$.

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