Carpitella, S.; Certa, A.; Izquierdo Sebastián, J.; La Fata, CM. (2018). k-out-of-n systems: an exact formula for the stationary availability and multi-objective configuration design based on mathematical programming and TOPSIS. Journal of Computational and Applied Mathematics. 330:1007-1015. https://doi.org/10.1016/j.cam.2017.01.006

k-out-of-n systems: an exact formula for the stationary availability and multi-objective configuration design based on mathematical programming and TOPSIS

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Abstract

Reliability and availability analyses are recognized as essential for guiding decision makers in the implementation of actions addressed to improve the technical and economical performance of complex systems. For industrial systems with reparable components, the most interesting parameter used to drive maintenance is the stationary availability. In this regard, the present paper proposes an exact formula for computing the system stationary availability of a k-out-of-n system. Such a formula is proved to be in agreement with the fundamental theorem of Markov chains. Then, a multi-objective mathematical model is formulated for choosing the optimal system configuration design. The Pareto front is developed using the Lexicographic Goal Programming (LGP) method, and the TOPSIS method is successively implemented to choose the k-out-of-n configuration that represents the best compromise between the considered objective functions. A numerical example is provided.

Keywords: k-out-of-n system; stationary availability; Markov chains; multi-objective optimization; TOPSIS.

1. Introduction and literature review

Reliability and availability are two common measures of complex systems' performance that play an essential role in product and service quality [1]. Therefore, reliability and availability analyses are fundamental to support the analyst in the implementation of actions addressed to the improvement of the technical and economical performance of the system under investigation. Since reliability and availability analyses are based on the identification of the major system criticalities, reliability relations among components need to be firstly established, and then the set of components to be maintained detected.

Management of maintenance activities should be based on the combined optimization of reliability and availability parameters. In this regard, numerous contributions propose the use of mathematical programming. Vasili *et al.* [2] present a detailed literature review and focus on optimization models for preventive maintenance policies, risk-based optimization models, and models constrained to ensure safety conditions. According to Yssaad and Abene [3], maintenance optimization can be effectively pursued using a reliability-centered maintenance (RCM) approach [4]. The authors demonstrate the global improvement of reliability and availability parameters of power distribution systems arising from the implementation of an RCM approach to results of a Failure Mode, Effects and Criticality Analysis (FMECA) [5, 6]. Francese *et al*. [7] and Curcurù *et al.* [8-10] deal with reliability analyses of complex systems under the presence of epistemic uncertainty on input data. Martón *et al.* [11] propose a model for the simultaneous optimization of testing and maintenance activities on ageing equipment with multiple items. The authors emphasize that the available literature proposes numerous models for assessing Reliability, Availability and Maintainability (RAM) of safety equipments. Most of such models assess the risk level of technological systems or defines appropriate design

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and/or surveillance and maintenance policies that ensure an optimum level of safety during the plant's operational life. Shariatkhah *et al.* [12] propose a model that takes into account the dynamic behavior of an energy conversion system to evaluate its availability. The authors mostly stress the need to consider the dependence of different forms of energy and propose a combined Monte Carlo and Markov chain-based approach [13]. Sabouhi *et al.* [14] refer to power plants to present a reliability model aimed at optimizing maintenance strategies and also highlight how system reliability must take into account data related to system critical components. Pang *et al.* [15] apply a failure mechanism analysis to the main critical components of an aircraft to understand how the reliability of the system could be affected. As proposed by Lu and Wu [16], a useful approach to perform a reliability analysis is to consider the decomposition of the general activity of the investigated system into its various working phases. Specifically, the authors do not decompose the system into its basic components or subsystems, but they perform a reliability analysis that considers the success of the overall mission and breaks down such a state into the various system behaviors during each working phase. In such a way, the failure and repair behaviors of each component are characterized. In [17], the authors use the same approach to analyze the reliability of an aircraft when separately considering climbing, cruising, and landing phases.

As highlighted by Billington and Allan [18], relations among system components could often be represented by block diagrams. In particular, the simplest configurations are related to series and parallel systems. Obviously, configurations with redundant components are commonly designed to increase the overall level of the system reliability/availability. Chambari *et al.* [19] underline the important role of redundancy in both reliability and cost optimization. They deal with a Redundant Allocation Problem (RAP) to find out the best redundancy strategy that improves the system operating conditions. A further RAP is solved by Yeh and Hsieh [20] who propose a penalty guided artificial bee colony algorithm to investigate the optimal number of redundant components in design problems. Garg and Sharma [21] present a fuzzy multi-objective method to undertake the RAP development and make the model more flexible and suitable for decision making. SureshBabu *et al.* [22] agree with the need to use redundancy of component to optimize systems reliability. However, Sharma *et al.* [23] emphasize the need to find a trade-off between the maximization of the system reliability and the minimization of resource utilization. Referring to the last point, Swetha *et al.* [24] notice the general underutilization of resources in redundancy techniques, and so they apply the algorithm Resource Reclaimed Scheme (RRS) to allocate and schedule the critical and non-critical tasks of an avionic mission system. Alebrant Mendes *et al.* [25] focus on the preventive maintenance of redundant systems and propose a Markov model for determining the time interval between two consecutive maintenance inspections to optimize system availability and maintenance costs. Markov models are also used by Hellmich and Berg [26] to organize the repair activities of standby safety systems. Huang *et al.* [27] state that the standby redundancy is a helpful practice. Montoro-Cazorla and Pérez-Ocón [28] deal with the possibility of including standby units to increase the system operational time. In particular, they illustrate the calculation of availability, reliability, and rate of occurrence of failures when considering a system with one online component and $n - 1$ cold standby components.

Moreover, partial redundancy is a significant configuration to improve systems' reliability/availability. Partial redundancy is implemented in a k-out-of-n configuration [29], for which a system is comprised of n components out of which at least k (with $k \leq n$) have to run simultaneously to assure the functioning state of the system, namely if $n - k + 1$ components fail then the whole system fails. For partially redundant systems, the available literature presents a wide variety of mathematical programming models where costs, reliability, and availability are commonly considered as objectives and/or constraints. Arulmozhi [30] focuses on k-out-of-n systems and proposes an equation to calculate the value of the reliability function by means of a recursive algorithm. Lu and Lewis [31] observe that the k-out-of-n configuration enables safety objectives based on increasing the system reliability level to be met. Kang and Kim [32] develop a method to quantify the unavailability of a *k*-out-of-*n* reactor protection system in a nuclear power plant. The method also enables an investigation into the most dangerous situations related to the entire system. Referring to a *k*-out-of-*n* surveillance system, Zhang and Pham [33] formulate an optimization model where the cost minimization is the objective function, and apply an algorithm to finally select the best maintenance policy. As for the optimal design of *k*-out-of-*n* systems, Moghaddass and Zuo [34] research the need for finding an effective trade-off between the system configuration to be designed and the maintenance strategy to be implemented.

For industrial systems with reparable components, such as production systems, the most interesting parameter used to drive the maintenance management is the stationary availability A_S, whose meaning is the expected time percentage in which the system is in a functioning state [35]. The improvement of the stationary availability is widely recognized in the literature as a strategic objective to be pursued [36]. With this recognition, the main aim of the present paper is to investigate on the stationary availability as a parameter for optimally designing a *k*-out-of-*n* system. To this purpose, a multi-objective mathematical formulation is proposed wherein the stationary availability is one of the objective functions to be optimized. Firstly, referring to a *k*-out of-*n* configuration, the related stationary availability $A_{S\binom{n}{k}}$ is computed by means of a novel exact formula that actually represents the main contribution of the work. Such a formula is proved through the fundamental theorem of Markov chains. Then, the TOPSIS [37] multi-criteria decision method is proposed to select the most suitable solution among those belonging to the Pareto front obtained from a multi-objective setting of the problem, the latter solved using the Lexicographic Goal Programming (LGP) technique. Specifically, the decision maker (DM) must choose the *k*-out-of-*n* configuration that represents the best compromise between the objectives to be simultaneously optimized. Depending on the decisional context, the objective functions can be differently weighted. Therefore, TOPSIS is used here as a decision support tool to select the best *k*-out-of-*n* configuration because of its ability to take into account DMs' perceptions and its easy implementation, as well.

The remainder of the paper is organized as follows. The stationary availability formula proposed for *k*-out-of-*n* systems is presented in Section 2, where a numerical example is also provided to validate the proposed equation. In Section 3, the multi-objective mathematical model is formulated and the resolution approach is described. A case study is presented in Section 4, and the conclusions close the work.

2. Proposed stationary availability formula

Let us consider a system *S* of *n* identical redundant components each one characterized by constant failure and repair rates λ and μ, respectively. The difficulty to know the trend of the failure rate over the time strongly justifies the assumption of its constancy. As the repair rate concerns, the main part of reliability studies is grounded upon the assumption of its constancy over time to simplify the computation of the reliability and availability values of systems constituted by reparable components. Without such a hypothesis, several systems, such as the *k*-out-of-*n* ones analyzed in the present paper, could be investigated only by simulation. Furthermore, electronic components are always characterized by a constant failure rate, whereas mechanical components have a slightly increasing failure rate.

Therefore, the individual stationary availability of a repairable component is computed by the well known equation (1):

$$A_S = \frac{\mu}{\lambda + \mu}. \quad (1)$$

Let us also consider the following hypotheses regarding the entire system:

– all components are repairable as well as the whole system;
– all components are stochastically independent and identical from a reliability point of view;

– there are no constraints about the maximum availability of maintenance crews.

These hypotheses guarantee the possibility of executing a generic maintenance operation and the possibility of easily aggregating different system states. The following proposition gives an exact formula to calculate the stationary availability of the *k*-out-of-*n* system, i.e. $A_{S_{\binom{n}{k}}}$.

Proposition. Under the stated hypotheses, the exact stationary availability $A_{S_{\binom{n}{k}}}$ of a *k*-out-of-*n* system is given by the following formula (2):

$$A_{S_{\binom{n}{k}}} = \frac{\sum_{i=k}^{n} \binom{n}{i} \cdot \mu^{i} \cdot \lambda^{(n-i)}}{\sum_{i=k}^{n} \binom{n}{i} \cdot \mu^{i} \cdot \lambda^{(n-i)} + \binom{n}{k-1} \cdot \mu^{(k-1)} \cdot \lambda^{(n-k+1)}}, \forall k \leq n. \quad (2)$$

Proof

In a specific time instant, the possible states of a component are the functioning state (noted by C) and the failure state (noted by its complementary state, $\overline{\mathrm{C}}$). The probability of being in a functioning state coincides with the component availability, whereas the probability of being in a failure state coincides with the component unavailability. Referring to the entire system, the probability of being in a specific state is the probability of the intersection of the states of its components. Under the aforementioned hypotheses, component states are stochastically independent so that the probability of their intersection can be calculated by means of their product.

The stationary availability of the systems is calculated in equation (2) by computing the ratio between the probability of the functioning states and the probability of all the possible states (both functioning and failure states in which the system *S* may be in a generic time instant). The ratio is obtained by means of the natural partition of the event space. The numerator represents the probability of the union of the system functioning states. Being the latter mutually exclusive, the probability of the union is precisely the sum of the probabilities of each functioning state. In the denominator of (2), all the possible states are considered. In fact, in addition to all the possible functioning states, one has to consider the failure states of the system that may occur when $n - k + 1$ components fail. The number of configurations that imply the system failure is $\binom{n}{k-1}$. Therefore, the denominator of formula (2) is the probability of the union of events considered in the numerator and the events representing the failure of the system. This finishes the proof.

Formula (2) may also be proved to be in agreement with the fundamental theorem of Markov chains. In fact, the finite number of states (nodes) of the system may be represented by a strongly directed graph. Under the hypotheses previously described, all the weights associated to transitions (links) between any two Markov states can be straightforwardly derived in terms of λ and μ. As a result, the process is governed by a stochastic regular matrix, which has a unique associated stationary probability, by virtue of the fundamental theorem of Markov chains.

Moreover,

- The ratio between each possible functioning state and the denominator of (2) gives back the probability of the analogue state represented in the Markov chain method. As a result, the sum of all those probabilities is the functioning probability or stationary availability of the system.

- Likewise, dividing the term $\begin{pmatrix} n \\ k-1 \end{pmatrix} \cdot \mu^{(k-1)} \cdot \lambda^{(n-k+1)}$ by the entire denominator, the system unavailability $U_{S_{\binom{n}{k}}}$ can be obtained, that is $U_{S_{\binom{n}{k}}} = 1 - A_{S_{\binom{n}{k}}}$, which coincides with the probability of the system being in a failure state.

We illustrate this in the following subsection by means of a simple numerical example.

2.1 Formula validation using Markov chains: a numerical example

Let us consider a 2-out-of-3 system. By applying the proposed equation (2), the following stationary availability is obtained:

$$A_{S_{\binom{n}{k}}} = \frac{\mu^2 + 3 \cdot \lambda \cdot \mu}{\mu^2 + 3 \cdot \lambda \cdot \mu + 3 \cdot \lambda^2}. \tag{3}$$

To validate the proposed equation (2), let us consider the Markov chain associated with the analyzed system. Figure 1 represents the associated directed Markov graph where 0 and 1 are the system functioning states (and 2, obviously, the failure state). In particular, 0 is the state where all the three components are available (3C). Since $\overline{\mathrm{C}}$ stands for the failure state of a component, the meanings of states 1 and 2 are now evident.

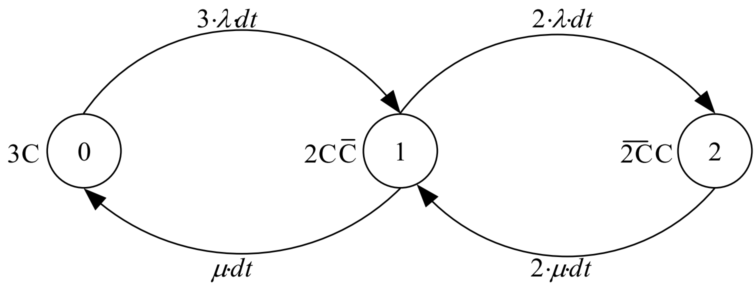

Figure 1. Markov graph for a 2-out-of-3 system

As aforementioned, λ and μ are the constant failure and repair rates of each component, respectively. The transaction probabilities from one state to another, as reported in Figure 1, only depend on the current state and not on the preceding sequence of events. Since λ and μ are assumed to be constant over time, the transaction probabilities are constant as well, and the relative stochastic process is a homogeneous Markov process.

Considering the Markov graph of Figure 1, let $P_0(t)$, $P_1(t)$ and $P_2(t)$ be the probability of being at states 0, 1 and 2 at time t. Therefore, the following differential equations (4)-(6) hold:

$$\frac{\mathrm{d}P_0(\mathrm{t})}{\mathrm{dt}} = -3 \cdot \lambda \cdot P_0(\mathrm{t}) + \mu \cdot P_1(\mathrm{t}), \tag{4}$$

$$\frac{\mathrm{d}P_1(\mathrm{t})}{\mathrm{dt}} = +3 \cdot \lambda \cdot P_0(\mathrm{t}) - (2 \cdot \lambda + \mu) \cdot P_1(\mathrm{t}) + 2 \cdot \mu \cdot P_2(\mathrm{t}) \tag{5}$$

$$\frac{\mathrm{d}P_2(\mathrm{t})}{\mathrm{dt}} = +2 \cdot \lambda \cdot P_1(\mathrm{t}) - 2 \cdot \mu \cdot P_2(\mathrm{t}) \tag{6}$$

The time-dependent availability value is obtained by the resolution of the previous system of differential equations.

To obtain the value of the parameter of interest, *i.e.* the stationary availability, the following system of linearly dependent equations (7)-(9) needs to be solved. It arises from the previous system (4)-(6) by considering that for t tending to infinity the probability of being at each possible state i is constant, and thus $\frac{dP_i(t)}{dt} = 0$.

$$-3 \cdot \lambda \cdot P_0 + \mu \cdot P_1 = 0, \tag{7}$$

$$3 \cdot \lambda \cdot P_0 - (2 \cdot \lambda + \mu) \cdot P_1 + 2 \cdot \mu \cdot P_2 = 0, \tag{8}$$

$$2 \cdot \lambda \cdot P_1 - 2 \cdot \mu \cdot P_2 = 0. \tag{9}$$

To solve the linearly dependent system of equations (7)-(9), equation (10) is introduced to replace one equation from (7)-(9), accounting for the global probability:

$$P_0 + P_1 + P_2 = 1. \tag{10}$$

As a result, the following probability values (11)-(13) are obtained:

$$P_0 = \frac{\mu^2}{\mu^2 + 3 \cdot \lambda \cdot \mu + 3 \cdot \lambda^2}, \tag{11}$$

$$P_1 = \frac{3 \cdot \lambda \cdot \mu}{\mu^2 + 3 \cdot \lambda \cdot \mu + 3 \cdot \lambda^2}, \tag{12}$$

$$P_2 = \frac{3 \cdot \lambda^2}{\mu^2 + 3 \cdot \lambda \cdot \mu + 3 \cdot \lambda^2}, \tag{13}$$

In contrast with the calculation of the Perron eigenvector of the system matrix given by (7)-(9), or with other methods found in the literature based on the resolution of sets of differential equations in the variables P_i, the results presented here stress the higher computational simplicity of the exact formula (2). The system steady state availability is thus given by the sum of the probabilities of occurrence of the system functioning states, namely (Eq. 14):

$$A_{S\binom{n}{k}} = P_0 + P_1 = \frac{\mu^2 + 3 \cdot \lambda \cdot \mu}{\mu^2 + 3 \cdot \lambda \cdot \mu + 3 \cdot \lambda^2}. \tag{14}$$

Also, one can observe that equation (3) coincides with (14), and it is the sum of two terms, each corresponding to P_0 and P_1 respectively. Furthermore, the system unavailability $U_{S\binom{n}{k}}$ (Eq. 15) is given by the value of P_2 and it is obviously the complement to one of the identical equations (3) or (14):

$$U_{S\binom{n}{k}} = P_2 = \frac{3 \cdot \lambda^2}{\mu^2 + 3 \cdot \lambda \cdot \mu + 3 \cdot \lambda^2}. \tag{15}$$

3. Design of a *k*-out-of-*n* system

3.1 Problem formulation

Let us consider a *k*-out-of-*n* system comprising *n* components having the same reliability features where at least *k* of them (with $k \leq n$) must function to guarantee the overall system functioning. The problem here is to determine the optimal number of components *n* that simultaneously optimize the two following conflicting objectives:

$$A_{S\binom{n}{k}},\ C. \tag{16}$$

$A_{S\binom{n}{k}}$ is the system stationary availability to be maximized computed by the proposed equation (2), whereas *C* represents the incremental cost to be minimized associated with the use of redundant components. *C* is expressed by equation (17) and depends on the number of redundant components:

$$C = (n - k) \cdot c_u. \tag{17}$$

Here c_u is the unitary cost of any additional operating component, and includes component purchase and maintenance costs.

3.2 Methodological resolution approach

In this paper, the search for the optimal solution is carried out in two stages. Firstly, the set of non-dominated solutions (*i.e.* the Pareto front) is obtained, and then the solutions belonging to the Pareto front are evaluated and compared to select the best one by means of the TOPSIS multi-criteria decisional approach. TOPSIS enables taking into account the relative priorities of the two objective functions of the Pareto front on the basis of the decision maker perceptions.

The available literature proposes different mathematical approaches for the identification of the Pareto front in multi-objective optimization problems. In [35] and [38] a description of some of these methods is supplied with a particular focus on the maintenance field, whereas in [39] a wide list of papers suggesting the use of the TOPSIS method in the maintenance field is provided.

3.2.1 Stage 1: determination of the Pareto front

The LGP method [40] is initially used to discover the extreme Pareto optimal solutions. This method separately considers the two objective functions, thereby reducing the multi-objective problem to a single-objective problem. The following sequential steps are implemented to determine the extreme solution of maximum availability:

a) maximizing $A_{S\binom{n}{k}}$ as a single-objective problem to obtain $A_{S\binom{n}{k}max}$;

b) minimizing C by forcing the value of $A_{S\binom{n}{k}}$ to be not smaller than $A_{S\binom{n}{k}max}$ to obtain thus C_{max}.

The procedure is analogously applied by swapping the objective hierarchy to find the other two coordinates C_{min} and $A_{S\binom{n}{k}min}$ of the point representing the extreme solution of minimum cost.

Once the two extreme points of the Pareto front are determined, the ε constraint method [41] is used to describe the entire Pareto front, also valid in the case of non-convex regions. By using such a method, the multi-objective optimization model described in Section 3.1 is modified into a single objective optimization model where one of the two objectives must be optimized and the other is constrained to ε values depending on the preference with respect the two objective functions. The ε constraint method is implemented as follows:

- Maximizing $A_{S\binom{n}{k}}$ and forcing C to be not smaller than C_{min}. In such a way, a non-dominated solution of the Pareto front is found. Let C^* be the value of the cost function related to the solution found.
- Maximizing $A_{S\binom{n}{k}}$ by forcing C to be not smaller than C^*.
- Iterating the procedure until the other extreme Pareto solution $\left(A_{S\binom{n}{k}max}, C_{max}\right)$ is obtained.

3.2.2 Stage 2: selection of the best configuration by the TOPSIS method

The TOPSIS method was originally developed by Hwang and Yoon [37] and later further developed by Hwang, Lai and Liu [41]. On the basis of different evaluation criteria, TOPSIS is a multi-criteria decision making method that provides an ordered ranking of alternatives. TOPSIS is grounded upon the concept of distance between each alternative and both the positive ideal solution and the negative ideal solution. Therefore, the best alternative is that characterized by the shortest distance from the positive ideal solution, and the farthest distance from the negative ideal solution.

The TOPSIS method is suitable to be applied in the problem selection of the best *k*-out-of-*n* configuration. In fact, TOPSIS firstly offers the typical advantages of multi-criteria decision methods such as the possibility of accounting for the analyst perception about the design problem by differently weighting the considered objective functions. Then, it allows to manage the Pareto solutions obtained in the previous step. In detail, differently from other multi-criteria methods, TOPSIS analyzes the quantitative scores of each alternative against the evaluation criteria, and turns back a ranking on the basis of a comparison between the alternative scores and the positive and the negative ideal solutions. In our context, each Pareto solution is characterized by a quantitative value under each criterion (representing an objective function), and the extreme Pareto optimal solutions translate the concept expressed by the ideal solutions. In the present work, the choice to be made concerns the optimal number of redundant components *n* in the design of a *k*-out-of-*n* system. The alternatives to be compared are the Pareto solutions, each one characterized by a specific number of $n-k$ redundant components which correspond to specific values of $A_{S\binom{n}{k}}$ and *C*.

The TOPSIS method requires as input data the decision matrix, where the assessment of each alternative with respect to the evaluation criteria is given, as well as a weight vector associated with the evaluation criteria that reflects the DM perceptions on the basis of the context under investigation. To select the best alternative (*i.e.* solution) among those integrating the Pareto front, the implementation of TOPSIS requires the following steps.

– Let g_{ij} be the evaluation of each Pareto solution *i* with relation to each objective (*i.e.* criteria) *j* (*i.e.* stationary availability and cost).
– Weighting the evaluation criteria.
– Computing the weighted and normalized decision matrix, where the generic element u_{ij} is:

$$u_{ij} = w_j \cdot z_{ij}, \forall i, \forall j, \tag{18}$$

where w_j is the weight associated with the criterion *j*, and z_{ij} is the score of the generic solution *i* under the criterion *j*, normalized by the following equation (19):

$$z_{ij} = \frac{g_{ij}}{\sqrt{\sum_{i=1}^{n} g_{ij}^2}}, \forall i, \forall j. \tag{19}$$

– Identification of the positive ideal point A^* and of the negative ideal point A^- by means of the following equations (20) and (21):

$$A^* = \left(u_1^*,...,u_k^*\right) = \left\{\left(max_i\, u_{ij} \mid j \in I'\right), \left(min_i\, u_{ij} \mid j \in I''\right)\right\}, \tag{20}$$

$$A^- = \left(u_1^-,...,u_k^-\right) = \left\{\left(min_i\, u_{ij} \mid j \in I'\right), \left(max_i\, u_{ij} \mid j \in I''\right)\right\}, \tag{21}$$

where I' and I'' are the sets of criteria to be maximized and minimized respectively.

– For each alternative *i*, computation of the distance from the positive ideal point A^* and from the negative ideal A^- by equations (22) and (23) respectively:

$$S_i^* = \sqrt{\sum_{j=1}^{k}\left(u_{ij} - u_j^*\right)^2}, i = 1,...n \tag{22}$$

$$S_i^- = \sqrt{\sum_{j=1}^{k}\left(u_{ij} - u_j^-\right)^2} \quad ,i = 1,\ldots n \tag{23}$$

– Characterization of each solution i by the related closeness coefficient C_i^*, which represents how well i performs with respect to the positive ideal solution (Eq. 24):

$$C_i^* = \frac{s_i^-}{s_i^- + s_i^*} \qquad 0 \le C_i^* \le 1 \qquad \forall i\,. \tag{24}$$

– Ranking the Pareto solutions on the basis of the obtained closeness coefficients. Specifically, referring to two generic solutions i and z, if $C_i^* \ge C_z^*$ then solution i must be preferred to solution z.

4. Case study

Let us consider a k-out-of-n system and the design problem introduced in Section 3.1. The input data needed to carry out the first stage of the methodological resolution approach described in Section 3.2.1 is given in Table 1.

Table 1. Input data

c_u	λ	μ	k
unitary	0.0002	0.002	2

Table 2 shows the Pareto solutions for the two objective functions defined in (16), namely the stationary availability $A_{S\binom{n}{k}}$ computed by the proposed equation (2), and the incremental cost C associated with the use of redundant components. The range of possible values to be investigated to obtain the optimal number of redundant components n is assumed to be [k, 5].

Table 2. Pareto solutions

Pareto solution	n	$A_{S\binom{n}{k}}$	C
1st	2	0.833333	0
2nd	3	0.977444	1
3rd	4	0.995907	2
4th	5	0.99969	3

Assuming that the DM equally weights the two objectives (criteria), the closeness coefficients, computed by equation (24) provided by the TOPSIS method, are given in Table 3.

Table 3. Closeness coefficients of Pareto solutions

Pareto solution	n	C_i^*
1st	2	0.824256
2nd	3	0.674032
3rd	4	0.370925
4th	5	0.175744

Furthermore, Table 4 shows the ranking of the obtained Pareto solutions and the related optimal number of components *n* with relation to different values of criteria weights. The proposed mathematical model has been solved by using the Lingo software. For each scenario, the related Pareto solution has been found out in a fraction of second. Thus, even problems of bigger dimension (*i.e.* for *n* and *k* values greater than those used in the case study) may be easily solved.

Table 4. Pareto solution ranking

w_C	$w_{AS_{\binom{n}{k}}}$	**Pareto solution ranking**	**Optimal number *n* of components**
0.3	0.7	2nd 1st 3rd 4th	3
0.7	0.3	1st 2nd 3rd 4th	2

5. Conclusions

Referring to complex industrial systems, the *k*-out-of-*n* configuration enables improved reliability and availability performance by means of component redundancy. Since the stationary availability is one of the most relevant parameters which the management and planning of maintenance activities is based on, the main novelty of the paper was the proposal of an exact formula to compute the stationary availability of *k*-out-of-*n* systems. The proposed formula represents an effective alternative to the classical method based on Markov chains, which requires greater computational effort than the proposed formula. Then, the design problem of the optimal *k*-out-of-*n* configuration was faced. Specifically, the problem dealt with the determination of the optimal number of components *n* that simultaneously optimize a multi-objective function where the stationary availability and the incremental cost due to the use of redundant components were taken into account. To solve such a problem, a two-stage methodology was proposed: the set of Pareto optimal solutions was first obtained through the Lexicographic Goal Programming (LGP) method, and then evaluated and compared using the TOPSIS multi-criteria decision approach to select the best design configuration – namely, the best compromise solution among those belonging to the Pareto front. In the given case study, the TOPSIS method was specifically applied to rank the Pareto solutions by differently weighting the two objective functions on the basis of the DM's perceptions.

Future research developments may concern the design of standby configuration systems for which the proposal of an exact formula able to calculate the related value of stationary availability will be investigated. In order to take into account the uncertainty conditions that characterize the design scenario, the fuzzy TOPSIS multi-criteria approach could also be applied [42].

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