

The higher topological complexity in digital images

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ABSTRACT

Y. Rudyak develops the concept of the topological complexity $TC(X)$ defined by M. Farber. We study this notion in digital images by using the fundamental properties of the digital homotopy. These properties can also be useful for the future works in some applications of algebraic topology besides topological robotics. Moreover, we show that the cohomological lower bounds for the digital topological complexity $TC(X, \kappa)$ do not hold.

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KEYWORDS: *topological complexity; digital topology; homotopy theory; digital topological complexity; image analysis.*

1. INTRODUCTION

One of the most popular topics in the field of applications of algebraic topology in recent years arises from the interpretation of algebraic topological elements in the area of robotics. After M. Farber put it in the literature for the first time [17], this subject has been studied by many researchers [13], [18], [19], [20], [23] and [28] since that time. This is very important because these studies can shed light on further engineering and navigation problems. For this purpose, it is necessary to understand how tools of algebraic topology take a place in the problem of robot motion planning. The answer is configuration spaces, that is explained in [18]. One of the main consequence of this innovative approach to the robot motion planning problem in any configuration space X

is the calculation of the topological complexity $\text{TC}(X)$, which depends only on the homotopy type of X [17].

This numerical invariant plays a crucial role in the selection of the best suited route for the robot by measuring the navigational complexity of a system's configuration space. The concept of Schwarz genus of a fibration [29] and some properties of fibrations are united and it follows that a new definition of the topological complexity $\text{TC}(X)$ coincides with the first one. Farber develops remarkable methods to compute lower or upper bounds for the topological complexity $\text{TC}(X)$ [17]. Using cohomological cup-product is one of the most precious one from the view of topology. Farber and Grant extend this method to any cohomology operations [19]. Farber states that the topological complexity $\text{TC}(X)$ is greater than the zero-divisor-cup-length of the cohomology $H^*(X; k)$, where k is a field [Theorem 7, [17]]. In addition, Farber uses Künneth formulae to prove this theorem. In digital setting, Ege and Karaca introduce the digital explanation of cup-product and also show that Künneth formulae does not hold for digital images [15]. We examine in this study that whether it is valid or not for the digital cup-product or any other digital cohomology operations. It is important to consider the generalization of the topological complexity. Rudyak introduced the higher topological complexity $\text{TC}_n(X)$ in [28], where X is a path-connected topological space and n is a positive integer. It is also noted that the higher topological complexity $\text{TC}_n(X)$ is more significant for $n > 1$.

The subject of robotics can gain even more values in the future because the main areas in which digital topology is influenced are the issues directly related to the technology of our time such as robot designs, image processing, computer graphics algorithms and computer images. So Karaca and İ̇s give the digital version of the topological complexity $\text{TC}(X)$ [23]. We will show that it is also possible to generalize this digital version $\text{TC}(X, \kappa)$, where (X, κ) is a digital image. To construct the digital higher topological complexity, we need a good knowledge of digital homotopy theory. Then we sometimes refer to many articles or books from algebraic topology and digital topology such as [10], [11], [12], [27], [16], [24], [30], and [31]. We introduce the different version of some definitions. For example, we do not prefer using subdivisions in the definition of a cofibration defined in [27].

In this paper, we try to reveal the simple background of digital topology in the first part. After that we present some new results or reinterpretations about function spaces, evaluation maps, exponential law, fibrations, and cofibrations in digital topology using several definitions, in which take part in the same section, for reaching our main goal that is to give a new definition of the digital topological complexity. In topological setting, these are facts in the study of homotopy theory but in digital setting, they must be proved. We sometimes reinterpret some existing definitions from our own perspective. We all do it because in Section 4, we want to introduce digital higher topological complexity via this new definition includes digital fibrations. Also, we show some properties about the digital higher topological complexity $\text{TC}_n(X, \kappa)$.

For example, we address when the topological complexity TC coincides with the higher topological complexity TC_n in digital images. Finally, in Section 5, we go back to the study of the digital topological complexity for a short time. We are interested in a very important tool often used in topological robotics. It is a cohomological lower bound for the topological complexity TC. We show that it is invalid for digital images. We give a counter example to show our assertion in the last.

2. PRELIMINARIES

In this section, we present fundamental notions and some basic facts for digital topology and topological robotics.

Let \mathbb{Z}^n be the set of lattice points in the n -dimensional Euclidean space. Then (X, κ) is called a *digital image*, where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation for the elements of X , [4]. Two distinct points p and q in \mathbb{Z}^n are c_l -adjacent for a positive integer l with $1 \leq l \leq n$, if there are at most l indices i such that $|p_i - q_i| = 1$ and for all other indices i such that $|p_i - q_i| \neq 1$, $p_i = q_i$, [4]. For instance, consider the situations $n = 1$, $n = 2$ and $n = 3$, respectively. Then we have 2-adjacency in \mathbb{Z} since $c_1 = 2$ in the first case; 4 and 8 adjacencies in \mathbb{Z}^2 since $c_1 = 4$ and $c_2 = 8$ in the second case and last 6, 18, and 26 adjacencies in \mathbb{Z}^3 since $c_1 = 6$, $c_2 = 18$ and $c_3 = 26$ as well.

Given the adjacency relation κ on \mathbb{Z}^n , a κ -neighbor of $p \in \mathbb{Z}^n$ is a point of \mathbb{Z}^n that is κ -adjacent to p , [22]. A digital image $X \subset \mathbb{Z}^n$ is κ -connected if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \dots, x_r\}$ of points of the digital image X such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ -neighbors, where $i = 0, 1, \dots, r - 1$, [22].

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$. Assume that $f : X \rightarrow Y$ is a function and κ_i is an adjacency relation defined on \mathbb{Z}^{n_i} , for $i = \{0, 1\}$. f is called *digitally (κ_0, κ_1) -continuous* if, when any κ_0 -connected subset of X is taken, its image under f is also κ_1 -connected, [4].

Proposition 2.1 ([4]). *Assume that two digital images (X, κ) and (X', λ) are given. Then the map $h : X \rightarrow X'$ is (κ, λ) -continuous if and only if for any x and $x' \in X$ such that x and x' are κ -adjacent, either $f(x) = f(x')$ or $f(x)$ and $f(x')$ are λ -adjacent.*

Proposition 2.2 ([4]). *If $f : (X, \kappa_1) \rightarrow (Y, \kappa_2)$ and $g : (Y, \kappa_2) \rightarrow (Z, \kappa_3)$ are digitally continuous maps, then the composite map*

$$g \circ f : (X, \kappa_1) \rightarrow (Z, \kappa_3)$$

is digitally continuous as well.

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0 -adjacency and κ_1 -adjacency, respectively. $f : X \rightarrow Y$ is a *digital (κ_0, κ_1) -isomorphism* if f is bijective and digital (κ_0, κ_1) -continuous and $f^{-1} : Y \rightarrow X$ is digital (κ_1, κ_0) -continuous, [7].

Proposition 2.3 ([3]). *A digital isomorphism relation is equivalence on digital images.*

A *digital interval* is defined by $[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} : a \leq z \leq b\}$, [6]. Let (X, κ) be a digital image. We say that f is a *digital path* from x to y in (X, κ) if $f : [0, m]_{\mathbb{Z}} \rightarrow X$ is a $(2, \kappa)$ -continuous function such that $f(0) = x$, and $f(m) = y$, [6]. If $f(0) = f(m)$, then the κ -path is said to be closed, and the function f is called a κ -loop.

Let $(X, \kappa_0) \in \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \in \mathbb{Z}^{n_1}$ be two digital images. For two (κ_0, κ_1) -continuous functions $f, g : X \rightarrow Y$, if there is a positive integer m and a function $H : X \times [0, m]_{\mathbb{Z}} \rightarrow Y$ such that

- for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, m) = g(x)$;
- for all $x \in X$, $H_x : [0, m]_{\mathbb{Z}} \rightarrow Y$, defined by $H_x(t) = H(x, t)$ for all $t \in [0, m]_{\mathbb{Z}}$, is $(2, \kappa_1)$ -continuous;
- for all $t \in [0, m]_{\mathbb{Z}}$, $H_t : X \rightarrow Y$, defined by $H_t(x) = H(x, t)$ for all $x \in X$, is (κ_0, κ_1) -continuous,

they are said to be *digitally (κ_0, κ_1) -homotopic* in Y and this is denoted by $f \simeq_{(\kappa_0, \kappa_1)} g$, [4]. The function H is called a *digital (κ_0, κ_1) -homotopy* between f and g .

Proposition 2.4 ([4]). *A digital homotopy relation is equivalence on digitally continuous functions.*

A digitally continuous function $f : X \rightarrow Y$ is *digitally nullhomotopic* in Y if f is digitally homotopic in Y to a constant function, [4]. A (κ_0, κ_1) -continuous function f from a digital image X to another image Y is a (κ_0, κ_1) -*homotopy equivalence* if there exists a (κ_1, κ_0) -continuous function g from Y to X such that $g \circ f$ is (κ_0, κ_0) -homotopic to the identity function 1_X and $f \circ g$ is (κ_1, κ_1) -homotopic to the identity function 1_Y , [5].

A digital image (X, κ) is said to be κ -*contractible* if its identity map is (κ, κ) -homotopic to a constant function c for some $c_0 \in X$, where the constant function $c : X \rightarrow X$ is defined by $c(x) = c_0$ for all $x \in X$, [4]. Let (X, κ) and (A, λ) be two digital images with the inclusion map $i : (A, \lambda) \rightarrow (X, \kappa)$. A is called a *digitally κ -retract* of the image X if and only if there is a digitally continuous map $r : (X, \kappa) \rightarrow (A, \lambda)$ such that $r(a) = a$ for all $a \in A$, [4]. Then the function r is called a *digital retraction* of X onto A . The product of two digital paths defined in [25]: If $f : [0, m_1]_{\mathbb{Z}} \rightarrow X$ and $g : [0, m_2]_{\mathbb{Z}} \rightarrow X$ are digital κ -paths with the condition $f(m_1) = g(0)$, then define *the product* $(f * g) : [0, m_1 + m_2]_{\mathbb{Z}} \rightarrow X$ by

$$(f * g)(t) = \begin{cases} f(t), & t \in [0, m_1]_{\mathbb{Z}} \\ g(t - m_1), & t \in [m_1, m_1 + m_2]. \end{cases}$$

Adjacency for the cartesian product of any two digital images is defined in [27]. Let (X, κ) and (Y, λ) be two digital images and consider the cartesian product $X \times Y$ of these sets. Then $X \times Y$ has the following adjacency relation: for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, we say that (x_1, y_1) and (x_2, y_2) are adjacent

on $X \times Y$ if x_1 and x_2 are κ -adjacent and y_1 and y_2 are λ -adjacent. This adjacency relation on $X \times Y$ is generally denoted by κ_* .

Definition 2.5 ([24]). Let (E, κ_1) , (B, κ_2) and (F, κ_3) be any digital images, where B is a κ_2 -connected space. Let $p : (E, \kappa_1) \rightarrow (B, \kappa_2)$ be a (κ_1, κ_2) -continuous surjection map. Then

- the triple (E, p, B) is called a *digital bundle*. The digital set B is called a *digital base set*, the digital set E is called a *digital total set* and the digital map p is called a *digital projection* of the bundle.
- the quadruple $\xi = (E, p, B, F)$ is called a *digital fiber bundle* with a digital base set B , a digital total set E , a digital fiber set F if p satisfies the following two conditions:
 - 1) For all $b \in B$, the map $p^{-1}(b) \rightarrow F$ is a (κ_1, κ_3) -isomorphism.
 - 2) For every point $b \in B$, there exists a κ_2 -connected subset U of B such that $\varphi : p^{-1}(U) \rightarrow U \times F$ is (κ_1, κ_*) -isomorphism making following diagram commute:

$$\begin{array}{ccc}
 p^{-1}(U) & \xrightarrow{\varphi} & U \times F \\
 & \searrow p & \swarrow \\
 & & U
 \end{array}$$

Definition 2.6 ([16]). Let (E, κ_1) and (B, κ_2) be two digital images. A digital map $p : (E, \kappa_1) \rightarrow (B, \kappa_2)$ is said to have the *homotopy lifting property with respect to a digital image* (X, κ_3) if for any digital map $\tilde{f} : X \rightarrow E$ and any digital homotopy $G : X \times [0, m]_{\mathbb{Z}} \rightarrow B$ such that $p \circ \tilde{f} = G \circ i$, where i is an inclusion map and m is a positive integer, there exists a (κ_*, κ_0) -continuous map $\tilde{G} : X \times [0, m]_{\mathbb{Z}} \rightarrow E$ making both triangles below commute:

$$\begin{array}{ccc}
 X & \xrightarrow{\tilde{f}} & E \\
 \downarrow i & \searrow \tilde{G} & \downarrow p \\
 X \times [0, m]_{\mathbb{Z}} & \xrightarrow{G} & B
 \end{array}$$

Definition 2.7 ([16]). A digital map $p : (E, \kappa_1) \rightarrow (B, \kappa_2)$ is a *digital fibration* if it has the homotopy lifting property with respect to every digital image. If $b_0 \in B$, then $p^{-1}(b_0) = F$ is called the *digital fiber*.

Ege and Karaca prove that the composition of two digital fibrations is again a digital fibration, [16]. They also show that if the digital maps $p_1 : (E_1, \kappa_1) \rightarrow (B_1, \kappa_2)$ and $p_2 : (E_2, \kappa'_1) \rightarrow (B_2, \kappa'_2)$ are digital fibrations, then

$$p_1 \times p_2 : (E_1 \times E_2, \kappa_*) \rightarrow (B_1 \times B_2, \kappa'_*)$$

is a digital fibration, where κ_* and κ'_* are adjacency relations on cartesian products $(E_1 \times E_2)$ and $(B_1 \times B_2)$, respectively.

Let (X, κ) and (Y, λ) be any digital images. The *digital function space* Y^X is defined as the set of all digitally continuous functions $X \rightarrow Y$ and has an adjacency relation as follows, [27]: for all $f, g \in Y^X$, the statement x and x' are κ -adjacent on X implies that $f(x)$ and $g(x')$ are λ -adjacent. Given two digital paths $f, g \in (X, \kappa_1)^{([0, m]_{\mathbb{Z}}, 2)}$, we say that the *paths f and g are γ -connected* if for all t times, they are γ -connected, where γ is an adjacency relation on $(X, \kappa_1)^{([0, m]_{\mathbb{Z}}, 2)}$, [23]. Suppose now (Y, κ_2) and (Z, κ_3) are digital images. *Digital function space map* $(Z, \kappa_3)^{(Y, \kappa_2)}$ is the set of all maps from (Y, κ_2) to (Z, κ_3) with an adjacency as follows, [27]: for $f, g \in (Z, \kappa_3)^{(Y, \kappa_2)}$, f and g are adjacent if $f(y)$ and $g(y')$ are κ_3 -adjacent whenever y and y' are κ_2 -adjacent, for any $y, y' \in (Y, \kappa_2)$.

We recall something about a digital image PX of all digital paths from [23]. To construct a digitally continuous digital motion planning algorithm $s : X \times X \rightarrow PX$, it is needed to be define the digital connectedness on PX by the definition of the digital continuity. Let χ be an adjacency relation on the digital image PX . Let also α and β be two digital paths in the digital image PX . The paths α and β are χ -connected if for all t times, they are χ -connected, where $t \in [0, m]_{\mathbb{Z}}$. It is also noted that if we have different steps (t times) for the digital paths, then we equalize the number of steps by increasing the less one of the paths with the endpoint of it. For example, two paths $\alpha = xyzw$ and $\beta = xy$ are given in PX . Then we consider β as $xyyy$ and now we can have an idea that whether α and β are adjacent or not.

Definition 2.8 ([23]). The *digital topological complexity* $TC(X, \kappa)$ is the minimal number k such that

$$X \times X = U_1 \cup U_2 \cup \dots \cup U_k$$

with the property that π admits a digitally continuous map $s_j : U_j \rightarrow PX$ such that $\pi \circ s_j = 1$ over each $U_j \subset X \times X$. If no such k exists, then we will set $TC(X, \kappa) = \infty$.

Karaca and İ̇s state that when the digital image (X, κ) is κ -contractible, $TC(X, \kappa_1)$ is equal to 1 and the converse of this statement is also true, [23]. In addition, they show that the digital image X with two different adjacencies κ_1 and κ_2 yields the result that if $\kappa_1 < \kappa_2$, then $TC(X, \kappa_1) \geq TC(X, \kappa_2)$. Another outstanding statement in [23] is that the digital topological complexity $TC(X, \kappa)$ of a digital image (X, κ) is an invariant up to the digital homotopy.

3. SOME DEFINITIONS AND PROPERTIES IN THE DIGITAL HOMOTOPY THEORY

In this section, we give some useful definitions and properties that we need in the next section. These are commonly related to the homotopy theory in digital topology.

Definition 3.1. Let $f : (X, \kappa_1) \rightarrow (Y, \kappa_2)$ be a map in digital images with digitally connected spaces (X, κ_1) and (Y, κ_2) . A digital fibrational substitute

of f is defined as a digital fibration $\widehat{f} : (Z, \kappa_3) \rightarrow (Y, \kappa_2)$ such that there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ f \downarrow & & \downarrow \widehat{f} \\ Y & \xlongequal[1_Y]{} & Y, \end{array}$$

where h is a digital homotopy equivalence.

Example 3.2. Let $X \subset \mathbb{Z}$ be a one-point digital image and Y be a digital image $[0, 1]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}} \subset \mathbb{Z}^2$ with 4-adjacency such that $f : X \rightarrow Y$ is a digital map. Consider another digital map $g : MSS'_6 \rightarrow Y$, where $MSS'_6 = [0, 1]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}} \subset \mathbb{Z}^3$ with 6-adjacency. We know that g is a digital fibration, [16]. Moreover, there is a digital homotopy equivalence $h : X \rightarrow MSS'_6$ because MSS'_6 is 6-contractible, [21]. Then the commutativity of the following diagram holds:

$$\begin{array}{ccc} X & \xrightarrow{h} & MSS'_6 \\ & \searrow f & \downarrow g \\ & & Y. \end{array}$$

Thus, we conclude that g is a digital fibrational substitute of f .

Lemma 3.3. Any digital map $f : (X, \kappa_1) \rightarrow (Y, \kappa_2)$ has a digital fibrational substitute.

Proof. Let $f : X \rightarrow Y$ be a digital map. For any $m \in \mathbb{N}$, set the digital image $Z = \{(x, \alpha) : x \in X, \alpha : [0, m]_{\mathbb{Z}} \rightarrow Y, \alpha(0) = f(x)\}$. Let α_* be the adjacency relation on the cartesian product digital image $X \times Y^{[0, m]_{\mathbb{Z}}}$. α_* is defined as follows: for any $(x_1, \alpha_1), (x_2, \alpha_2) \in X \times Y^{[0, m]_{\mathbb{Z}}}$, if x_1 and x_2 are adjacent points and α_1 and α_2 are adjacent paths, then (x_1, α_1) and (x_2, α_2) are adjacent pairs of the elements in the cartesian product image. Z is a subset of the cartesian product image so the adjacency relation on Z is the same with the cartesian product. Consider the digital map

$$g : Z \rightarrow Y$$

$$(x, \alpha) \mapsto \alpha(1).$$

We also define $h : X \rightarrow Z$ with $h(x) = (x, \alpha_x)$, where $\alpha_x(t) = f(x)$ for all $t \in [0, m]_{\mathbb{Z}}$. We first get

$$g \circ h(x) = g(x, \alpha_x) = g_x(1) = f(x).$$

In order to show that g is a digital fibration, we need to build a new digital map $\tilde{G} : X \times [0, m]_{\mathbb{Z}} \rightarrow Z$. Let $G : X \times [0, m]_{\mathbb{Z}} \rightarrow Y$ be a digital homotopy. If we take $\tilde{G}(x, t) = h(x)$ and $i(x) = (x, t)$ for $t \in [0, m]_{\mathbb{Z}}$, then we have that

$$\tilde{G} \circ i(x) = \tilde{G}(x, t) = h(x),$$

and

$$g \circ \tilde{G}(x, t) = g \circ h(x) = G \circ i(x) = G(x, t)$$

from $g \circ h = G \circ i$ using the definition of a digital fibration. This shows that g is a digital fibration. Finally, we say that $h : X \rightarrow Z$ is a digital homotopy equivalence with considering the digital map $k : Z \rightarrow X$, defined by $k(x, \alpha) = x$ for any $(x, \alpha) \in Z$. As a consequence, g is a digital fibrational substitute of f . \square

Definition 3.4. Let (E, λ_1) , (E', λ'_1) , (B, λ_2) and (B', λ'_2) be digital images. Then a digital map of digital fibrations $p : (E, \lambda_1) \rightarrow (B, \lambda_2)$ to $p' : (E', \lambda'_1) \rightarrow (B', \lambda'_2)$ is a commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ p \downarrow & & \downarrow p' \\ B & \xrightarrow{f} & B'. \end{array}$$

Definition 3.5. Let $p : (E, \lambda_1) \rightarrow (B, \lambda_2)$ be a digital fibration. Two digital maps $f_0, f_1 : (X, \lambda_3) \rightarrow (E, \lambda_1)$ are said to be digitally fiber homotopic provided that there is a digital homotopy $F : X \times [0, m]_{\mathbb{Z}} \rightarrow E$ between f_0 and f_1 such that $p \circ F(x, t) = p \circ f_0(x)$ for $x \in X$ and $t \in [0, m]_{\mathbb{Z}}$.

Definition 3.6. Let $p_1 : (E_1, \lambda_1) \rightarrow (B, \lambda)$ and $p_2 : (E_2, \lambda_2) \rightarrow (B, \lambda)$ be two digital fibrations. Then they are said to digital fiber homotopy equivalent if there exist digital maps $f : (E_1, \lambda_1) \rightarrow (E_2, \lambda_2)$ and $g : (E_2, \lambda_2) \rightarrow (E_1, \lambda_1)$ for which $g \circ f$ is digitally fiber homotopic to $1_{(E_1, \lambda_1)}$ and $f \circ g$ is digitally fiber homotopic to $1_{(E_2, \lambda_2)}$.

Lemma 3.7. Any two digital fibrational substitutes of a digital map $f : (X, \kappa_1) \rightarrow (Y, \kappa_2)$ are digitally fiber homotopy equivalent fibrations.

Proof. Let $f : (X, \kappa_1) \rightarrow (Y, \kappa_2)$ be a digital map and assume that \hat{f} and \hat{g} are two digital fibrational substitutes of it. Then we have the following diagram consisting of the union of two commutative diagrams;

$$\begin{array}{ccccc} & & Z' & \xleftarrow{k} & X & \xrightarrow{h} & Z & & \\ & & \searrow & & \downarrow f & & \swarrow & & \\ & & & & Y & & & & \end{array}$$

i.e. $\hat{f} \circ h = f$ and $\hat{g} \circ k = f$. Since both the digital maps h and k on (X, κ_1) are digitally homotopy equivalences, there exist two digitally continuous maps $h_1 : (Z, \kappa_3) \rightarrow (X, \kappa_1)$ and $k_1 : (Z', \kappa'_3) \rightarrow (X, \kappa_1)$ such that $h \circ h_1 \simeq_{(\kappa_3, \kappa_3)} 1_{(Z, \kappa_3)}$, $h_1 \circ h \simeq_{(\kappa_1, \kappa_1)} 1_{(X, \kappa_1)}$, $k_1 \circ k \simeq_{(\kappa_1, \kappa_1)} 1_{(X, \kappa_1)}$ and $k \circ k_1 \simeq_{(\kappa'_3, \kappa'_3)} 1_{(Z', \kappa'_3)}$.

We want to show that \widehat{f} and \widehat{g} are digitally fiber homotopy equivalent fibrations. So we have to guarantee the existence of two digital maps $Z \rightarrow Z'$ and $Z' \rightarrow Z$ such that their compositions are digitally fiber homotopic to identities on Z and Z' , respectively. Now, we consider the digital maps $k \circ h_1 : (Z, \kappa_3) \rightarrow (Z', \kappa'_3)$ and $h \circ k_1 : (Z', \kappa'_3) \rightarrow (Z, \kappa_3)$. It is easy to see that

$$(h \circ k_1) \circ (k \circ h_1) \simeq_{(\kappa_3, \kappa_3)} 1_{(Z, \kappa_3)}$$

and

$$(k \circ h_1) \circ (h \circ k_1) \simeq_{(\kappa'_3, \kappa'_3)} 1_{(Z, \kappa'_3)}.$$

We conclude that the two digital fibrational substitutes of f is digitally fiber homotopy equivalent because $\widehat{g} \circ (k \circ h_1) = \widehat{f}$ and $\widehat{f} \circ (h \circ k_1) = \widehat{g}$ hold by Definition 3.4. \square

Definition 3.8. The digital Schwarz genus of a digital fibration $p : (E, \lambda_1) \rightarrow (B, \lambda_2)$ is defined as a minimum number k such that

$$B = U_1 \cup U_2 \cup \dots \cup U_k$$

with the property that there is a digitally continuous map $s_i : (U_i, \lambda_1) \rightarrow (E, \lambda_2)$ for all $1 \leq i \leq k$, satisfies $p \circ s_i = 1_{U_i}$ over each $U_i \subset B$.

The digital Schwarz genus of a digital map f is defined as the digital Schwarz genus of the digital fibrational substitute of f .

Lemma 3.9. Let $f : (X, \kappa_1) \rightarrow (Y, \kappa_2)$ and $g : (Y, \kappa_2) \rightarrow (Z, \kappa_3)$ be two maps. Then the digital Schwarz genus of the map $g \circ f$ is not less than the digital Schwarz genus of the map g .

Proof. Let f and g be digital fibrations. Then $g \circ f$ is a digital fibration. Now assume that the digital Schwarz genus of the digital map $g \circ f$ is k . Then we have $Z = U_1 \cup U_2 \cup \dots \cup U_k$ such that there exists a digital map $s_j : (U_j, \kappa_3) \rightarrow (X, \kappa_1)$ with $(g \circ f) \circ s_j = 1_{U_j}$ over each $U_j \subset Z$, where $j = 1, 2, \dots, k$. For each U_j , we obtain a new digital map $t_j : (U_j, \kappa_3) \rightarrow (Y, \kappa_2)$ with $t_j = f \circ s_j$. This implies that $g \circ t_j = g \circ (f \circ s_j) = 1_{U_j}$. It shows that the digital Schwarz genus of the digital map g does not exceed k because the digital maps t_j are constructed by the digital maps s_j . If f and g are not digital fibrations, then we consider the digital fibrational substitutes of them. Similarly, the digital Schwarz genus of the digital map g is less than or equal to the the digital Schwarz genus of the map $g \circ f$. \square

Definition 3.10. Let (X, κ_1) and (Y, κ_2) be digital images. Then the digital map

$$E_{X,Y}^{\kappa_1, \kappa_2} : (Y^X \times X, \kappa_*) \rightarrow (Y, \kappa_2),$$

defined by $E_{X,Y}^{\kappa_1, \kappa_2}(f, x) = f(x)$, is called a digital evaluation map.

Proposition 3.11. The digital evaluation map is a digitally continuous map.

Proof. Let (f, x) and (g, x') be two points in the digital image $Y^X \times X$ such that they are κ_* -connected. Let λ be an adjacency relation on the set Y^X . By the definition of adjacency for the cartesian product, we have that f and g are λ -adjacent and x and x' are κ_1 -adjacent. These give us that $f(x)$ and $g(x')$ are κ_2 -adjacent. \square

Definition 3.12. Let (X, κ_1) , (Y, κ_2) and (Z, κ_3) be any three digital images such that $f : (X \times Y, \kappa_*) \rightarrow (Z, \kappa_3)$ is a digitally continuous map. Then the digital map

$$\bar{f} : (X, \kappa_1) \rightarrow (Z^Y, \lambda),$$

defined by $\bar{f}(x)(y) = f(x, y)$, is called a digital adjoint map.

Proposition 3.13. *If (X, κ_1) , (Y, κ_2) and (Z, κ_3) are any three digital images such that $f : (X \times Y, \kappa_*) \rightarrow (Z, \kappa_3)$ is a digitally continuous map, then the digital adjoint map*

$$\begin{aligned} \bar{f} : (X, \kappa_1) &\rightarrow (Z^Y, \lambda) \\ x &\mapsto \bar{f}(x) \end{aligned}$$

is also digitally continuous.

Proof. Let x and x' be two points in X such that x and x' are κ_1 -adjacent. Then (x, y) and (x', y) are κ_* -adjacent for any $y \in Y$. Since f is digitally continuous, we have that $f(x, y)$ and $f(x', y)$ are κ_3 -adjacent. Hence, $\bar{f}(x)(y)$ and $\bar{f}(x')(y)$ are κ_3 -adjacent. Consequently, $\bar{f}(x)$ and $\bar{f}(x')$ are λ -adjacent. \square

Proposition 3.14. *Let (X, κ_1) , (Y, κ_2) and (Z, κ_3) be digital images. Then the map*

$$\begin{aligned} \alpha : (Z^{X \times Y}, \kappa_*) &\rightarrow ((Z^Y)^X, \kappa) \\ f &\mapsto \alpha(f) = \bar{f} \end{aligned}$$

is digitally continuous, where κ_ and κ are adjacency relations on $Z^{X \times Y}$ and $(Z^Y)^X$, respectively.*

Proof. Let f and g be given in $Z^{X \times Y}$ such that f and g are κ_* -adjacent and let λ_* be an adjacency relation on $X \times Y$. We show that $\alpha(f)$ and $\alpha(g)$ are κ -adjacent with considering $\alpha(f) = \bar{f}$ and $\alpha(g) = \bar{g}$. Now let x and x' be two κ_1 -adjacent points in X . Then (x, y) and (x', y) are λ_* -adjacent in $X \times Y$. Moreover, we see that $f(x, y)$ and $g(x', y)$ are κ_3 -adjacent. This means that $\bar{f}(x)(y)$ and $\bar{g}(x')(y)$ are κ_3 -adjacent. Therefore, the result holds from Proposition 3.13. \square

Remark 3.15. A digitally continuous map $\varphi : X \rightarrow Z^Y$ induces a digital map

$$\underline{\varphi} : X \times Y \rightarrow Z,$$

defined by $\underline{\varphi} = E_{Y,Z}^{\kappa_2, \kappa_3} \circ (\varphi \times 1_Y)$;

$$X \times Y \xrightarrow{\varphi \times 1_Y} Z^Y \times Y \xrightarrow{E_{Y,Z}^{\kappa_2, \kappa_3}} Z.$$

The composite map $E_{Y,Z}^{\kappa_2, \kappa_3} \circ (\varphi \times 1_Y)$ is digitally continuous since $E_{Y,Z}^{\kappa_2, \kappa_3}$, φ and 1_Y are digitally continuous.

Proposition 3.16. *Let (X, κ_1) , (Y, κ_2) and (Z, κ_3) be any digital images. Then the map*

$$\begin{aligned} \beta : ((Z^Y)^X, \kappa) &\longrightarrow (Z^{X \times Y}, \lambda) \\ \varphi &\longmapsto \beta(\varphi) = \underline{\varphi} \end{aligned}$$

is digitally continuous.

Proof. Let φ and χ be two κ -adjacent digital maps in $(Z^Y)^X$, κ_* an adjacency relation on $X \times Y$, τ an adjacency relation on Z^Y and μ an adjacency relation on $Z^Y \times Y$. We show that $\underline{\varphi}$ and $\underline{\chi}$ are λ -adjacent in $Z^{X \times Y}$. Now assume that (x, y) and $(x', y') \in X \times Y$ are κ_* -adjacent. Then x and x' are κ_1 -adjacent in X , so $\varphi(x)$ and $\chi(x')$ are τ -adjacent in Z^Y . Since y and y' are κ_2 -adjacent, $(\varphi \times 1_Y)(x, y)$ and $(\chi \times 1_Y)(x', y')$ are μ -adjacent in $Z^Y \times Y$. By the digital continuity of the evaluation map $E_{Y,Z}^{\kappa_2, \kappa_3}$ and Remark 3.15, the result holds. \square

Proposition 3.17. *Let (X, κ_1) , (Y, κ_2) and (Z, κ_3) be any digital images. Then the maps $\alpha : (Z^{X \times Y}, \kappa_*) \rightarrow ((Z^Y)^X, \kappa)$ and $\beta : ((Z^Y)^X, \kappa) \rightarrow (Z^{X \times Y}, \lambda)$ are bijective.*

Proof. Consider the digital composition maps $\alpha \circ \beta : ((Z^Y)^X, \kappa) \rightarrow ((Z^Y)^X, \kappa)$ and $\beta \circ \alpha : (Z^{X \times Y}, \kappa_*) \rightarrow (Z^{X \times Y}, \kappa_*)$. Then we have that

$$\alpha \circ \beta(\varphi) = \alpha(\beta(\varphi)) = \alpha(\underline{\varphi}) = \overline{\underline{\varphi}} = \varphi$$

and

$$\beta \circ \alpha(f) = \beta(\alpha(f)) = \beta(\underline{f}) = \overline{\underline{f}} = f$$

for all digitally continuous maps $\varphi \in (Z^Y)^X$ and $f \in Z^{X \times Y}$. \square

We refer to [31] for more details on the topological setting.

Theorem 3.18. *If (X, κ_1) , (Y, κ_2) and (Z, κ_3) are three digital images, then the map $\alpha : (Z^{X \times Y}, \lambda) \rightarrow ((Z^Y)^X, \kappa)$, defined by $\alpha(g) = \overline{g}$, is a digital isomorphism, where λ is an adjacency relation on the digital image $Z^{X \times Y}$ and κ is an adjacency relation on the digital image $(Z^Y)^X$.*

Proof. From Proposition 3.11, $E_{X \times Y, Z}^{\lambda, \kappa_3}$ is digitally continuous, where λ_* is an adjacency relation on $X \times Y$. We can express another version of the digital image $E_{X \times Y, Z}^{\lambda, \kappa_3}$ as $E_{Y, Z}^{\kappa_2, \kappa_3} \circ (\alpha_1 \times 1_Y)$, where $\alpha_1 = E_{X, Z^Y}^{\kappa_1, \mu} \circ (\alpha \times 1_X)$ such that μ is an adjacency relation on Z^Y . As a result of Proposition 3.14, Proposition 3.11 and the digital continuity of the identity map, α_1 is digitally continuous. Finally, we conclude that α is digitally continuous.

Conversely, the digitally continuous map $E_{Y, Z}^{\kappa_2, \kappa_3} \circ (E_{X, Z^Y}^{\kappa_1, \mu} \times 1_Y)$ equals the digital map $E_{X \times Y, Z}^{\lambda, \kappa_3} \circ (\alpha^{-1} \times 1_{X \times Y})$. As a result of Proposition 3.17, Proposition 3.16,

Proposition 3.11 and the digital continuity of the identity map, α^{-1} is digitally continuous as well. \square

Corollary 3.19. *Let (Y, λ) be a digital image. Then*

$$\mu : (Y^{[0,n]_{\mathbb{Z}}})^{[0,m]_{\mathbb{Z}}} \longrightarrow (Y^{[0,m]_{\mathbb{Z}}})^{[0,n]_{\mathbb{Z}}},$$

for $m, n \in \mathbb{N}$, is a digital isomorphism.

Proof. Let κ_* and λ_* be adjacency relations on the digital images $[0, n]_{\mathbb{Z}} \times [0, m]_{\mathbb{Z}}$ and $[0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}$, respectively. First, we show that the digital map

$$\gamma : ([0, n]_{\mathbb{Z}} \times [0, m]_{\mathbb{Z}}, \kappa_*) \longrightarrow ([0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}, \lambda_*)$$

$$(s, t) \longmapsto \gamma(s, t) = (t, s)$$

is a digital isomorphism for any $s \in [0, n]_{\mathbb{Z}}$ and $t \in [0, m]_{\mathbb{Z}}$.

- To show that γ is injective, let's define the digital map

$$\delta : ([0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}, \lambda_*) \longrightarrow ([0, n]_{\mathbb{Z}} \times [0, m]_{\mathbb{Z}}, \kappa_*)$$

with $\delta(t, s) = (s, t)$. Since

$$\gamma \circ \delta(t, s) = \gamma(s, t) = (t, s) = 1_{[0,m]_{\mathbb{Z}} \times [0,n]_{\mathbb{Z}}}$$

and

$$\delta \circ \gamma(s, t) = \delta(s, t) = (s, t) = 1_{[0,n]_{\mathbb{Z}} \times [0,m]_{\mathbb{Z}}},$$

γ has left and right inverses.

- Let (s, t) and (s', t') be two points in $[0, n]_{\mathbb{Z}} \times [0, m]_{\mathbb{Z}}$ such that they are κ_* -adjacent. Then s and s' are 2-adjacent and t and t' are 2-adjacent. Hence, we have that (t, s) and (t', s') are λ_* -adjacent. Since $\gamma(s, t) = (t, s)$ and $\gamma(s', t') = (t', s')$, γ is digitally continuous.
- Given two λ_* -adjacent points (t, s) and (t', s') in the digital image $[0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}$. This implies that t and t' are 2-adjacent and s and s' are 2-adjacent at the same time. Thus, we have that (s, t) and (s', t') are κ_* -adjacent. It shows that γ^{-1} is digitally continuous because of that $\gamma^{-1}(t, s) = (s, t)$ and $\gamma^{-1}(t', s') = (s', t')$.

After all, we now have that γ is a digital isomorphism and γ defines a digital map

$$\phi : Y^{[0,m]_{\mathbb{Z}} \times [0,n]_{\mathbb{Z}}} \longrightarrow Y^{[0,n]_{\mathbb{Z}} \times [0,m]_{\mathbb{Z}}}$$

$$f \longmapsto \phi(f) = f \circ \gamma$$

for any digitally continuous map f in the image $Y^{[0,n]_{\mathbb{Z}} \times [0,m]_{\mathbb{Z}}}$. Second, we show that ϕ is a digital isomorphism.

- To show that ϕ is bijective, we define a new digital map $\psi : Y^{[0,n]_{\mathbb{Z}} \times [0,m]_{\mathbb{Z}}} \longrightarrow Y^{[0,m]_{\mathbb{Z}} \times [0,n]_{\mathbb{Z}}}$ with $\psi(g) = g \circ \gamma^{-1}$ for any $g \in Y^{[0,n]_{\mathbb{Z}} \times [0,m]_{\mathbb{Z}}}$. It is easy to see that

$$\psi \circ \phi(f) = \psi(f \circ \gamma) = (f \circ \gamma) \circ \gamma^{-1} = f \circ 1_{Y^{[0,n]_{\mathbb{Z}} \times [0,m]_{\mathbb{Z}}}} = f$$

and

$$\phi \circ \psi(g) = \phi(g \circ \gamma^{-1}) = (g \circ \gamma^{-1}) \circ \gamma = g \circ 1_{Y^{[0,m]_{\mathbb{Z}} \times [0,n]_{\mathbb{Z}}}} = g.$$

So ϕ has left and right inverses.

- The digital continuity of f, g, γ and γ^{-1} shows that ϕ and ϕ^{-1} are digitally continuous maps.

By Theorem 3.18, the digital image $(Y^{[0,n]_{\mathbb{Z}}})^{[0,m]_{\mathbb{Z}}}$ is digitally isomorphic to $Y^{[0,m]_{\mathbb{Z}} \times [0,n]_{\mathbb{Z}}}$ and the digital image $(Y^{[0,m]_{\mathbb{Z}}})^{[0,n]_{\mathbb{Z}}}$ is digitally isomorphic to $Y^{[0,n]_{\mathbb{Z}} \times [0,m]_{\mathbb{Z}}}$. Finally, we conclude that

$$\mu : (Y^{[0,n]_{\mathbb{Z}}})^{[0,m]_{\mathbb{Z}}} \longrightarrow (Y^{[0,m]_{\mathbb{Z}}})^{[0,n]_{\mathbb{Z}}}$$

is a digital isomorphism because the digital isomorphism relation is an equivalence relation on digital images. \square

Definition 3.20. Let (A, τ) and (X, κ) be digital images such that $A \subset X$. We say that a pair of digital images (X, A) has a digital homotopy extension property with respect to the digital image (Y, χ) on condition that the digital map $g : X \longrightarrow Y$ and the digital homotopy $G : A \times [0, m]_{\mathbb{Z}} \longrightarrow Y$, where $m \in \mathbb{N}$, for which $g(x) = G(x, 0)$ for $x \in A$, satisfy that there exists a digital homotopy

$$F : X \times [0, m]_{\mathbb{Z}} \longrightarrow Y$$

with $F(x, 0) = g(x)$ and $F|_{A \times [0, m]_{\mathbb{Z}}} = G$ for $x \in X$.

Definition 3.21. Let (X', λ) and (X, κ) be two digital images. Then a digital map $f : X' \longrightarrow X$ is called a digital cofibration if given two digital maps $g : X \longrightarrow Y$ and $G : X' \times [0, n]_{\mathbb{Z}} \longrightarrow Y$ for arbitrary digital image Y such that $g \circ f(x') = G(x', 0)$ for $x' \in X'$ and $n \in \mathbb{N}$, then there exists a digital map $F : X \times [0, n]_{\mathbb{Z}} \longrightarrow Y$ satisfying $F(x, 0) = g(x)$ and $F(f(x'), t) = G(x', t)$ for $x \in X, x' \in X'$ and $t \in [0, n]_{\mathbb{Z}}$.

$$\begin{array}{ccc}
 X' \times 0 & \xrightarrow{\quad} & X' \times [0, n]_{\mathbb{Z}} \\
 \downarrow f \times 1_0 & \nearrow G & \downarrow f \times 1_{[0, n]_{\mathbb{Z}}} \\
 & Y & \\
 X \times 0 & \xrightarrow{\quad} & X \times [0, n]_{\mathbb{Z}} \\
 & \nearrow g & \searrow F \\
 & &
 \end{array}$$

Note that the digital cofibration $i : A \hookrightarrow X$ corresponds to that (X, A) has the digital homotopy extension property with respect to any digital image.

Proposition 3.22. Let $\dot{I}_m = \{0, m\}$ and $[0, m]_{\mathbb{Z}}$ be two digital images for $m \in \mathbb{N}$. Then the pair $([0, m]_{\mathbb{Z}}, \dot{I}_m)$ has the digital homotopy extension property if and only if the digital image $\dot{I}_m \times [0, n]_{\mathbb{Z}} \cup [0, m]_{\mathbb{Z}} \times 0$ is a digital retract of $[0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}$ for $n \in \mathbb{N}$.

Proof. (\Rightarrow) : Suppose that $f : \dot{I}_m \rightarrow [0, m]_{\mathbb{Z}}$ is a digital cofibration and choose the arbitrary digital image Y as $\dot{I}_m \times [0, n]_{\mathbb{Z}} \cup [0, m]_{\mathbb{Z}} \times 0$ in the Definition 3.21. Then we have a digital map $F : [0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}} \rightarrow Y$. Hence, we can rewrite F as a retraction

$$r : [0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}} \rightarrow \dot{I}_m \times [0, n]_{\mathbb{Z}} \cup [0, m]_{\mathbb{Z}} \times 0.$$

(\Leftarrow) : Let $r : [0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}} \rightarrow \dot{I}_m \times [0, n]_{\mathbb{Z}} \cup [0, m]_{\mathbb{Z}} \times 0$ be a digital retraction. Assume that $g : [0, m]_{\mathbb{Z}} \rightarrow Y$ and $G : \dot{I}_m \times [0, n]_{\mathbb{Z}} \rightarrow Y$ are digital maps such that $g(a) = G(a, 0)$ for $a \in \dot{I}$. Define a digital map

$$k : \dot{I}_m \times [0, n]_{\mathbb{Z}} \cup [0, m]_{\mathbb{Z}} \times 0 \rightarrow Y$$

combining with the maps g and G . Then for all $a \in \dot{I}_m$, $s \in [0, m]_{\mathbb{Z}}$ and $t \in [0, n]_{\mathbb{Z}}$, the digital homotopy

$$\begin{aligned} F : [0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}} &\rightarrow Y \\ (s, t) &\mapsto F(s, t) = k(r(s, t)) \end{aligned}$$

gives us

$$\begin{aligned} F(s, 0) &= k(r(s, 0)) = k(s, 0) = g(s), \\ F(a, t) &= k(r(a, t)) = k(a, t) = G(a, t). \end{aligned}$$

Hence, F implies that $f : \dot{I}_m \rightarrow [0, m]_{\mathbb{Z}}$ is a digital cofibration. \square

Definition 3.23. Let $p : (E, \lambda_1) \rightarrow (B, \lambda_2)$ be a digital map. For $n \in \mathbb{Z}$, define a digital image $\overline{B} = \{(e, w) \in E \times B^{[0, n]_{\mathbb{Z}}} : w(0) = p(e)\}$. There is a digital map $p' : E^{[0, n]_{\mathbb{Z}}} \rightarrow \overline{B}$ defined by $p'(\overline{w}) = (\overline{w}(0), p \circ \overline{w})$ for any $\overline{w} : ([0, n]_{\mathbb{Z}}, 2) \rightarrow (E, \lambda_1)$. A digital lifting function of p is a digital map $\sigma : \overline{B} \rightarrow E^{[0, n]_{\mathbb{Z}}}$ for which $p' \circ \sigma = 1_{\overline{B}}$ holds.

The next three propositions and a theorem now guide us to reach our goal in this section. We first take the advantage of the relation between the digital lifting function and the digital fibration, and then we characterize the digital cofibration. Next, we obtain a new digital fibration using the characterization of the digital cofibration. As a consequence, we state the final theorem of this section with the help of these propositions.

Proposition 3.24. *Given two digital images (E, λ_1) and (B, λ_2) . Then a digital map $p : (E, \lambda_1) \rightarrow (B, \lambda_2)$ is a digital fibration if and only if there exists a digital lifting function σ of p .*

Proof. Suppose that p is a digital fibration. Let $F : \overline{B} \times [0, m]_{\mathbb{Z}} \rightarrow B$ and $f' : \overline{B} \rightarrow E$ be digital maps such that $F((e, w), t) = w(t)$ and $f'(e, w) = e$. Therefore, we have

$$F((e, w), 0) = w(0) = p(e) = p \circ f'(e, w).$$

There exists a digital homotopy $F' : \overline{B} \times [0, m]_{\mathbb{Z}} \rightarrow E$ because p is a digital fibration. The equalities $F'((e, w), 0) = f'(e, w)$ and $p \circ F' = F$ hold. By Theorem 3.18, the map $\sigma : \overline{B} \rightarrow E^{[0, m]_{\mathbb{Z}}}$, defined by $\sigma(e, w)(t) = F'((e, w), t)$,

is a digital lifting function for the map p . Conversely, let σ be a digital lifting function for the map p . We take the digital homotopy $F : X \times [0, m]_{\mathbb{Z}} \rightarrow E$ and the digital map $f' : X \rightarrow E$ such that $F(x, 0) = p \circ f'(x)$. We define $g : X \rightarrow B^{[0, m]_{\mathbb{Z}}}$ by $g(x)(t) = F(x, t)$. So we can construct a map

$$F' : X \times [0, m]_{\mathbb{Z}} \rightarrow E$$

$$(x, t) \mapsto F'(x, t) = \sigma(f'(x), g(x))(t).$$

Hence, the equalities $F'(x, 0) = f'(x)$ and $p \circ F' = F$ hold. As a conclusion, the map p has the digital homotopy lifting property. \square

Consider two digital images $\dot{I}_m = \{0, m\}$ and $[0, m]_{\mathbb{Z}}$. Let $f : \dot{I}_m \rightarrow [0, m]_{\mathbb{Z}}$ be a digital map defined by $f(0) = 0$ and $f(m) = m$. Define a digital image $\overline{[0, m]_{\mathbb{Z}}}$ which is the quotient space of the sum $(\dot{I}_m \times [0, n]_{\mathbb{Z}}) \vee ([0, m]_{\mathbb{Z}} \times 0)$ obtained by identifying $(x', 0) \in \dot{I}_m \times [0, n]_{\mathbb{Z}}$ with $(f(x'), 0) \in [0, m]_{\mathbb{Z}} \times 0$, for all $x' \in \dot{I}_m$. Then there exists a digital map

$$i' : (\overline{[0, m]_{\mathbb{Z}}}, \tau) \rightarrow ([0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}, \tau_*),$$

defined by $i'[x', t] = (f(x'), t)$, for all $t \in [0, n]_{\mathbb{Z}}$ and $x' \in \dot{I}_m$ and $i'[x, 0] = (x, 0)$ for all $x \in [0, m]_{\mathbb{Z}}$, where τ and τ_* are adjacency relations for the digital images $\overline{[0, m]_{\mathbb{Z}}}$ and $[0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}$, respectively.

Proposition 3.25. *Let $\dot{I}_m = \{0, m\}$ and $[0, m]_{\mathbb{Z}}$ be two digital images. Assume that $f : \dot{I}_m \rightarrow [0, m]_{\mathbb{Z}}$ is a digital map. Then f is a digital cofibration if and only if there exists a digital map*

$$\rho : ([0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}, \tau_*) \rightarrow (\overline{[0, m]_{\mathbb{Z}}}, \lambda)$$

such that ρ is a left inverse of the map i' .

Proof. (\Rightarrow): Suppose that the digital map f is a digital cofibration and consider the digital maps $g : [0, m]_{\mathbb{Z}} \rightarrow \overline{[0, m]_{\mathbb{Z}}}$ and $G : \dot{I}_m \times [0, n]_{\mathbb{Z}} \rightarrow \overline{[0, m]_{\mathbb{Z}}}$ with $g(x) = [x, 0]$ and $G(x', t) = [x', t]$. We now have that

$$G(x', 0) = [x', 0] = [f(x'), 0] = g \circ f(x').$$

Since f is a digital cofibration, we define a digital map

$$\rho : [0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}} \rightarrow \overline{[0, m]_{\mathbb{Z}}}$$

such that $\rho(x, 0) = g(x)$ and $\rho(f(x'), t) = G(x', t)$ hold for $x \in [0, m]_{\mathbb{Z}}$ and $x' \in \dot{I}_m$. Due to

$$\rho \circ i'[x, 0] = \rho(x, 0) = g(x) = [x, 0],$$

and

$$\rho \circ i'[x', t] = \rho(f(x'), t) = G(f(x'), t) = [x', t],$$

ρ is a left inverse of the map i' .

(\Leftarrow): Suppose that $\rho : ([0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}, \kappa_*) \rightarrow \overline{[0, m]_{\mathbb{Z}}}$ is a digital map with $\rho \circ i' = 1_{\overline{[0, m]_{\mathbb{Z}}}}$ and let $g : [0, m]_{\mathbb{Z}} \rightarrow Y$ and $G : \dot{I}_m \times [0, n]_{\mathbb{Z}} \rightarrow Y$ be two

digital maps such that $G(x', 0) = g \circ f(x')$ for $x' \in \dot{I}_m$. Then there is a digital map

$$\bar{G} : \overline{[0, m]}_{\mathbb{Z}} \longrightarrow Y$$

defined by $\bar{G}[x', t] = G(x', t)$ and $\bar{G}[x, 0] = g(x)$ for any $t \in [0, n]_{\mathbb{Z}}$. For all $x \in \overline{[0, m]}_{\mathbb{Z}}$ and $x' \in \dot{I}_m$, we have that

$$F(x, 0) = \bar{G} \circ \rho(x, 0) = \bar{G} \circ (\rho \circ i'([x, 0])) = \bar{G}[x, 0] = g(x),$$

and

$$F(f(x'), t) = \bar{G} \circ \rho(f(x'), t) = \bar{G} \circ (\rho \circ i'([x, t])) = \bar{G}[x', t] = G[x', t].$$

Thus, we obtain that F is a cofibration. □

Proposition 3.26. *If the map $f : \dot{I}_m \longrightarrow [0, m]_{\mathbb{Z}}$ is a digital cofibration, then the digital map*

$$\begin{aligned} p : Y^{[0, m]_{\mathbb{Z}}} &\longrightarrow Y^{\dot{I}_m} \\ g &\longmapsto p(g) = g \circ f \end{aligned}$$

is a digital fibration.

Proof. Assume that the map $f : \dot{I}_m \longrightarrow [0, m]_{\mathbb{Z}}$ is a digital cofibration. By Proposition 3.25, we have that the digital map $i' : \overline{[0, m]}_{\mathbb{Z}} \longrightarrow [0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}$ has a left inverse $\sigma : [0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}} \longrightarrow \overline{[0, m]}_{\mathbb{Z}}$. Define a new digital map $\sigma' : Y^{\overline{[0, m]}_{\mathbb{Z}}} \longrightarrow Y^{[0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}}$ with $\sigma'(g) = g \circ \sigma$, where $g \in Y^{\overline{[0, m]}_{\mathbb{Z}}}$. By using Corollary 3.19, we see that $Y^{[0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}}$ is digital isomorphic to the digital image $(Y^{[0, m]_{\mathbb{Z}}})^{[0, n]_{\mathbb{Z}}}$. Since $Y^{\overline{[0, m]}_{\mathbb{Z}}}$ is digital isomorphic to the digital image $\{(g, G) \in Y^{\overline{[0, m]}_{\mathbb{Z}}} \times (Y^{\dot{I}_m})^{[0, n]_{\mathbb{Z}}} : g \circ f = G(0)\}$, we conclude that σ' is a digital lifting function of p from Proposition 3.24. □

We now give a fundamental theorem of this section which is a key for defining the digital higher topological complexity in the next section.

Theorem 3.27. *For a digital image (X, κ_1) , the digital map*

$$\begin{aligned} p : X^{[0, m]_{\mathbb{Z}}} &\longrightarrow X \times X \\ w &\longmapsto p(w) = (w(0), w(m)) \end{aligned}$$

is a digital fibration.

Proof. For $m, n \in \mathbb{Z}$, it is easy to see that $\{0, m\} \times [0, n]_{\mathbb{Z}} \cup [0, m]_{\mathbb{Z}} \times 0$ is a digital retract of $[0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}$. By Proposition 3.22, the pair of digital images $([0, m]_{\mathbb{Z}}, \{0, m\})$ has the digital homotopy extension property, i.e. $i : \{0, m\} \longrightarrow [0, m]_{\mathbb{Z}}$ is a digital cofibration for $m \in \mathbb{N}$. Proposition 3.26 also implies that

$$\begin{aligned} X^{[0, m]_{\mathbb{Z}}} &\longrightarrow X^{\{0, m\}} \\ w &\longmapsto w \circ i \end{aligned}$$

is a digital fibration. On the other hand, $X^{\{0, m\}}$ is digital isomorphic to $X \times X$

via $g \mapsto (g(0), g(m))$ for any digital map $g : \{0, m\} \rightarrow X$. Therefore, the digital map

$$p : X^{[0,m]_{\mathbb{Z}}} \rightarrow X \times X$$

$$w \mapsto p(w) = (w(0), w(m))$$

is a digital fibration for any digital image (X, κ_1) . □

4. THE DIGITAL HIGHER TOPOLOGICAL COMPLEXITY

In this section we will define higher digital topological complexity and introduce its properties. Digital topological complexity can be equivalently introduced as follows.

Definition 4.1. Let (X, κ_1) be a connected digital image and $p : X^{[0,m]_{\mathbb{Z}}} \rightarrow X \times X$ a digital fibration, defined by $p(w) = (w(0), w(1))$ for any $w \in X^{[0,m]_{\mathbb{Z}}}$. The digital Schwarz genus of p is called the digital topological complexity of (X, κ_1) .

It is clear that Definition 4.1 is equivalent to Definition 2.8.

Definition 4.2. Let X be any κ -connected digital image. Let J_n be the wedge of n -digital intervals $[0, m_1]_{\mathbb{Z}}, \dots, [0, m_n]_{\mathbb{Z}}$ for a positive integer n , where $0_i \in [0, m_i]$, $i = 1, \dots, n$, are identified. Then the digital higher topological complexity $TC_n(X, \kappa)$ is defined by the digital Schwarz genus of the digital fibration

$$e_n : X^{J_n} \rightarrow X^n$$

$$f \mapsto (f(m_1)_1, \dots, f(m_n)_n),$$

where $(m_i)_k$, $k = 1, \dots, n$, denotes the endpoints of the i -th interval for each i .

Theorem 4.3. Let (X, κ) be a digital κ -connected image and n be a positive integer. Then the following hold:

- (1) $TC_1(X, \kappa) = 1$.
- (2) $TC_n(X, \kappa)$ is a homotopy invariant in digital images.
- (3) $TC_2(X, \kappa)$ coincides with $TC(X, \kappa)$.
- (4) $TC_n(X, \kappa) \leq TC_{n+1}(X, \kappa)$.

Proof. (1) Consider a digital map $e_1 : X^{J_1} \rightarrow X$ defined by $e_1(f) = f(1_1)$. Since there is a digitally continuous map $s_1 : X \rightarrow X^{J_1}$ such that $e_1 \circ s_1 = 1_X$, $TC(X, \kappa) = 1$.

(2) The proof is a repeated use of the proof of Theorem 4.1 in [23]. Let X and Y be digitally homotopy equivalent images with the maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$. Let $f \circ g$ be digitally homotopic to 1_Y . We first assume that U is a subset of $X \times X$ for which $s : U \rightarrow PX$ is a digitally continuous motion planning. Then we define $V = (g \times g)^{-1}(U)$ and find a digitally continuous motion planning $\alpha : V \rightarrow PY$. For the TC_n version of the proof, we have to define V as n -copies of $(g \times g \times \dots \times g)^{-1}(U)$, where U belongs to $X \times X \times \dots \times X$ because the range of the digital map e_n is n -copies of X . We also take digitally continuous maps $s : U \rightarrow X^{J_n}$ and $\alpha : V \rightarrow Y^{J_n}$ since the domain of the

digital map e_n includes the digitally continuous maps defined from any digitally connected space to J_n .

(3) We first note that $e^n : X^{J_n} \rightarrow X^n$ is a digital fibrational substitute of the diagonal map of digital images $\Delta_n : X \rightarrow X^n$. By Definition 3.1, we must show that $e_n \circ h = 1_{X^n} \circ \Delta_n$. Here we define $h : X \rightarrow X^{J_n}$ as a digital map taking any point x in (X, κ) and turning it into the wedge of digital loops starting and finishing at x . To say h is a digital homotopy equivalence, we consider that there exists a digital map $g : X^{J_n} \rightarrow X$, defined by $g(\beta) = \beta(0)$ for any $\beta \in X^{J_n}$, such that $g \circ h$ is digitally homotopy equivalent to 1_X and $h \circ g$ is digitally homotopy equivalent to $1_{X^{J_n}}$. Hence, we have $e_n \circ h = 1_{X^n} \circ \Delta_n$. On the other hand, the digital fibration e_n is digitally homotopy equivalent to the digital fibration

$$f_n : X^{[0,m]z} \rightarrow X^n$$

$$\alpha \mapsto (\alpha(0), \alpha(1), \alpha(2), \dots, \alpha(n-1)),$$

where m is a positive integer such that $n \leq m - 1$. Since we take the digital homotopy equivalence h as the digital constant map at x in the digital image (X, κ) , we have $f_n \circ h = \Delta_n$. As a conclusion, $\text{TC}_n(X, \kappa)$ is the digital Schwarz genus of f_n and $\text{TC}(X, \kappa)$ is the digital Schwarz genus of p in Definition 4.1. Moreover, for $n = 2$, the digital fibration f_2 coincides with p . It shows that $\text{TC}_2(X, \kappa) = \text{TC}(X, \kappa)$.

(4) Assume that $\text{TC}_{n+1}(X, \kappa) = r$ and examine the digital higher topological complexity $\text{TC}_n(X, \kappa)$. Then the digital Schwarz genus of the digital map e_{n+1} equals r . So we have that $X^{n+1} = U_1 \cup U_2 \cup \dots \cup U_r$ and there exists $s_i : U_i \rightarrow X^{J_{n+1}}$ such that $e_{n+1} \circ s_i = 1_{U_i}$ for each $i \in [1, r]$. Consider the digital map $a_n : X^{J_{n+1}} \rightarrow X^{J_n}$ that takes the set of $n + 1$ paths and deletes the last one into the set of first n paths. Now choose a point $a \in X$ and establish a new digital map $b_n : X^n \rightarrow X^{n+1}$ that assigns n points of X to the $n + 1$ points of it by adding a to the end of the n points. Last, we set

$$V_i = \{(x_1, x_2, \dots, x_n) \in X^n : (x_1, x_2, \dots, x_n, a) \in U_i\} \subset X^n.$$

Therefore, we obtain $t_i = a_n \circ s_i \circ b_n : V_i \rightarrow X^{J_n}$ with the help of the digital map s_i such that $e_n \circ t_i = 1_{V_i}$ for $i = 1, 2, \dots, r$. It shows that the digital Schwarz genus of e_n can be at most r and so $\text{TC}_n(X, \kappa) \leq r$. \square

We currently finalize this section with an example about frequently used the digital surface MSS_{18} and the digital curve MSC_8 in digital topology, [21].

Example 4.4. Consider the minimal simple surface $MSS_{18} = \{a_i\}_{i=0}^9$ in \mathbb{Z}^3 , where

$$a_0 = \{0, 0, 0\}, a_1 = \{1, 1, 0\}, a_2 = \{0, 1, -1\},$$

$$a_3 = \{0, 2, -1\}, a_4 = \{1, 2, 0\}, a_5 = \{0, 3, 0\},$$

$$a_6 = \{-1, 2, 0\}, a_7 = \{0, 2, 1\}, a_8 = \{0, 1, 1\}, a_9 = \{-1, 1, 0\}.$$

It is shown that any loop is 18-contractible in MSS_{18} , [9] and $\text{cat}_{18}(MSS_{18}) = 1$, [2]. By [23, Theorem 5.1], we have $\text{TC}(MSS_{18}, 18) \geq 1$

but [23, Corollary 3.3]] says that $TC(MSS_{18}, 18)$ cannot be greater than 1. Therefore, we conclude that $TC_1(MSS_{18}, 18) = 1$, $TC_2(MSS_{18}, 18) = 1$ and $TC_n(MSS_{18}) \geq 1$, for $n \geq 3$, from Theorem 4.3.

On the other hand, if we consider the minimal simple closed curve $MSC_8 = \{b_i\}_{i=0}^5$ in \mathbb{Z}^2 , where

$$b_0 = \{0, 0\}, b_1 = \{-1, 1\}, b_2 = \{-1, 2\}, \\ b_3 = \{0, 3\}, b_4 = \{1, 2\}, b_5 = \{1, 1\},$$

then we have $TC(MSC_8, 8) = 2$ by [23, Theorem 3.5]]. It shows that $TC_1(MSC_8, 8) = 1$, $TC_2(MSC_8, 8) = 2$ and $TC_n(MSC_8, 8) \geq 2$, for $n \geq 3$, from Theorem 4.3.

5. A DIFFERENT RESULT FOR THE DIGITAL TOPOLOGICAL COMPLEXITY

Cohomological lower bounds are not valid for the digital topological complexity $TC(X, \kappa)$, where (X, κ) is a digital image. Let us explain it with an example.

Example 5.1. Let F be a field. Consider the two same digital images $X = Y = [0, m]_{\mathbb{Z}}$ with 2-adjacency for any $m \in \mathbb{N}$ and assume that cohomological lower bounds are holds in digital images. We know

$$H^{*,2}(X; F) = H^{*,2}(Y; F) = \begin{cases} F & , n = 0 \\ 0 & , n \neq 0, \end{cases}$$

and

$$H^{*,4}(X \times Y; F) = \begin{cases} F & , n = 0, 1 \\ 0 & , n \neq 0, 1, \end{cases}$$

from [14] and [1], respectively. Since X and Y are 2-contractible, we have that $TC(X, 2) = 1 = TC(Y, 2)$. So the zero-divisor-cup-lengths of $H^{*,2}(X; F)$ and $H^{*,2}(Y; F)$ must be equal to zero. Moreover, $TC(X \times Y, 4) = 1$ because $X \times Y$ is 4-contractible. This implies that the zero-divisor-cup-length of $H^{*,4}(X \times Y; F)$ is equal to zero as well. On the other hand, the zero-divisor-cup-lengths of $H^{1,2}(X; F)$ and $H^{1,2}(Y; F)$ are equal to zero by $H^{1,2}(X; F) = H^{1,2}(Y; F) = 0$. In addition to this, the zero-divisor-cup-length of $H^{1,4}(X \times Y; F)$ is greater than or equal to 1. We can choose a nonzero fundamental class $u \in H^{1,4}(X \times Y; F)$ such that $\bar{u} = 1 \otimes u - u \otimes 1$ is a zero-divisor by $H^{1,4}(X \times Y; F) = F$. We focus on the first dimension of digital cohomologies with considering that the minimum adjacency is 4 for $X \times Y$ because it is a digital image in \mathbb{Z}^2 . Then we get a contradiction!

6. CONCLUSION

This study introduces the theory of higher topological complexity in digital images. We first start with establishing the homotopy background for this. As a continuation, we deal with the number of digital topological complexities.

Furthermore, we answer the open problem given in the conclusion section of [23]. We offer a clear solution in digital setting to the question of what relationship between the digital cohomological cup-product and the digital topological complexity $TC(X, \kappa)$ will have. Our answer is negative, i.e. it is not suitable to build such a cohomological structure in digital topology. We can study on the various areas with different examples. One open problem is to find a relation between TC_n and digital Lusternik-Schnirelmann category cat which is introduced in [2].

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