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Duality of locally quasi-convex convergence groups

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Abstract

In the realm of the convergence spaces, the generalisation of topological groups is the convergence groups, and the corresponding extension of the Pontryagin duality is the continuous duality. We prove that local quasi-convexity is a necessary condition for a convergence group to be c-reflexive. Further, we prove that every character group of a convergence group is locally quasi-convex.

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1. Introduction and Preliminaries

The character group $\hat{G}$ of an abelian topological group G is the group of all continuous characters equipped with the compact-open topology. Pontryagin duality theorem states that for any locally compact abelian (LCA) topological group, the canonical map $\alpha_G : G \to \hat{\hat{G}}$ (from the group to the double dual group) is a topological isomorphism. Martín-Peinador [10] proves that if G is reflexive and the evaluation map $e : \hat{G} \times G \to \mathbb{T}$ defined as $e(\chi, x) = \chi(x)$ is continuous, then the group G is locally compact. This result explains the role of local compactness in the Pontryagin duality theory. This result is further generalized in [11, Theorem 1.1] where the same statement with the condition "reflexive" replaced by the "quasi-convex compactness property" is proved.

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In the last seventy years, extending the Pontryagin duality theory beyond local compactness has gained the attention of several researchers [8, 13]. One of these approaches is the continuous duality theory [5]. The compact-open topology on the continuous character group is replaced with the continuous convergence structure, and as a consequence of this, the evaluation map is always continuous. A comparison of this approach with the Pontryagin duality is presented in [7].

On the other hand, there are several situations in analysis (like convergence in measure) where non-topological convergence originates, and the corresponding generalisation of a topological space is the convergence space. In the realm of the convergence spaces, the generalisation of topological groups is the convergence groups. For details regarding the continuous duality for convergence groups refer [4, 6]. Further, the notion of locally quasi-convex convergence groups is introduced in [12]. Here, we prove that local quasi-convexity is a necessary condition for a convergence group to be c-reflexive, and then we prove that every character group of a convergence group is locally quasi-convex.

Before proceeding further, we present certain terms and notations required for the rest of the article.

A filter $\mathcal{F}$ is a non-empty family of non-empty subsets of a set X which is closed under supersets and finite intersection. We denote the set of all filters on a set X by $\mathbb{F}X$. Further, a subset $\mathcal{H}$ of a filter $\mathcal{F}$ is called basis of the filter if each set in $\mathcal{F}$ contains a set in $\mathcal{H}$. If $\mathcal{F}$ and $\mathcal{G}$ are filters on a set X then, $\mathcal{F}$ is called coarser than $\mathcal{G}$ if $\mathcal{F} \subset \mathcal{G}$. Let λ be an arbitrary relation between X and $\mathbb{F}X$. The relation is called convergence (and the pair (X, λ) a convergence space) on X if for $\mathcal{F}_1, \mathcal{F}_2$ in $\mathbb{F}X$ and x in X the following conditions hold:
(i) **Centred:** $x^{\uparrow} \in \lambda(x)$,
(ii) **Isotone**: If $\mathcal{F}_1 \in \lambda(x)$ and $\mathcal{F}_1 \leq \mathcal{F}_2$, then $\mathcal{F}_2 \in \lambda(x)$, and
(iii) **Finitely deep**: If $\mathcal{F}_1, \mathcal{F}_2 \in \lambda(x)$, then $\mathcal{F}_1 \cap \mathcal{F}_2 \in \lambda(x)$.
This relation is also denoted by $\mathcal{F} \underset{\lambda}{\rightarrow} x$.

A convergence space X is Hausdorff if every filter on X converges to at most one point. Further, a map $f : (X, \lambda_1) \to (Y, \lambda_2)$ between two convergence spaces is said to be continuous if $(\mathcal{F} \underset{\lambda_1}{\longrightarrow} x \ \ \Rightarrow \ \ f(\mathcal{F}) \underset{\lambda_2}{\longrightarrow} f(x))$.

For any convergence spaces (X, λ_1) and (Y, λ_2) let $C(X, Y)$ denote the set of all continuous functions from X to Y. The evaluation mapping $e : C(X, Y) \times X \to Y$ is defined as

$$e(f, x) = f(x) \ \ \forall \ f \in C(X, Y) \text{ and } x \in X.$$

The continuous convergence structure on $C(X, Y)$ is defined as: a filter $\mathcal{G}$ converge to f in $C(X, Y)$ iff $e(\mathcal{G} \times \mathcal{F})$ converge to $f(x)$ in Y, whenever $\mathcal{F}$ converge to x in X. The space $C(X, Y)$ equipped with continuous convergence structure is denoted by $C_c(X, Y)$.

A convergence group is a group with a compatible (the group operations are continuous in the sense of convergence) convergence structure. The class of convergence groups contains the class of topological groups. Some other examples include the underlying groups of the convergence vector spaces.

For details about convergence spaces and convergence groups, we refer the reader to [3, 9].

For a convergence abelian group (G, λ) denote by ΓG the set of all continuous homomorphisms of G into the circle group $\mathbb{T}$. The set of all continuous homomorphisms with the structure of continuous convergence is defined as convergence dual of G and is denoted as $(\Gamma G, \lambda_c)$. The evaluation map $\kappa : G \to \Gamma\Gamma G$ defined as $\kappa(g)(\chi) = \chi(g)\ \forall\ g \in G,\ \chi \in \Gamma G$ is a continuous group homomorphism. The convergence group is c-reflexive if this evaluation map κ is a bicontinuous isomorphism.

As proved in [3, Example 8.5.14], there exists a locally compact non-reflexive convergence group. Therefore, the analog of Pontryagin duality theorem, valid in the context of abelian topological groups, cannot be directly extended to abelian convergence groups. Motivated from the notion of local quasi-convexity [2] in topological groups, the notion of local quasi-convexity for convergence groups is defined in [12].

For a convergence abelian group G we define the polar and the inverse polar of subsets of G and of ΓG as follows:

Definition 1.1 (Polar and inverse polar)**.** For any subset H of G and L of ΓG the polar and the inverse polar of H and L respectively are subsets defined as:

$$H^{\rhd} = \{\chi \in \Gamma G : \chi(H) \subset \mathbb{T}_+\}; \qquad L^{\lhd} = \{g \in G : \chi(g) \subset \mathbb{T}_+,\ \forall\ \chi \in L\},$$

here $\mathbb{T}_+ = \{z \in \mathbb{T} :\ \mathrm{Re}\ (z) \geq 0\}$.

The local quasi-convexity for convergence groups is defined as:

Definition 1.2 (Quasi-convex set [3])**.** A subset A of a convergence abelian group G is quasi-convex if for each point g in $G \backslash A$, there is a character χ in the polar set of A such that $Re\chi(g) < 0$, that is $A^{\rhd\lhd} = A$.

Proposition 1.3. *Let G be a convergence group, $H \subset G$ and $L \subset \Gamma G$. The polar $H^{\rhd}$ and the inverse polar $L^{\lhd}$ are quasi convex subsets of $\Gamma_c G$ and G respectively.*

Definition 1.4 (Locally quasi-convex convergence group [12])**.** A convergence group G is locally quasi-convex if for each filter $\mathcal{F} \underset{G}{\longrightarrow} 0$, there exists another filter $\mathcal{G}$ coarser than $\mathcal{F}$ such that $\mathcal{G} \underset{G}{\longrightarrow} 0$ and $\mathcal{G}$ has a filter base composed of quasi-convex sets.

2. Main results

Proposition 2.1. *For a filter $\mathcal{U}$ on a convergence group G and for all $U, V \in \mathcal{U}$ we have $(U \cap V)^{\rhd\rhd} \subseteq U^{\rhd\rhd} \cap V^{\rhd\rhd}$.*

Proof. The proof is trivial. □

Proposition 2.2. *$\{U^{\rhd\rhd} : U \in \mathcal{U}\}$ is a basis of a filter in $\Gamma_c\Gamma_c G$.*

Proof. The proof is similar to [3, Theorem 8.4.3]. □

We denote the filter generated by $\{U^{\rhd\rhd} : U \in \mathcal{U}\}$ as $\mathcal{U}^{\rhd\rhd}$.
Similarly, $\{U^{\rhd\lhd} : U \in \mathcal{U}\}$ is a basis of a filter in G which we denote by $\mathcal{U}^{\rhd\lhd}$.

Lemma 2.3. *For a convergence abelian group G, the following statements hold:*

(1) If a filter $\Phi \xrightarrow[\Gamma_c G]{} 0$, then $A^{\rhd} \in \Phi$ for every finite subset A of G.

(2) For every filter $\mathcal{F} \xrightarrow[G]{} 0$, there is $B \in \mathcal{F}$ such that $B^{\rhd} \in \Phi$.

Proof. The proof follows from [3, Proposition 8.1.8]. □

Theorem 2.4. *For a convergence abelian group G, if $\mathcal{U} \xrightarrow[G]{} 0$, then $\mathcal{U}^{\rhd\rhd} \xrightarrow[\Gamma_c\Gamma_c G]{} 0$.*

Proof. Applying Lemma 2.3 to $\Gamma_c G$ we need to prove that

a. $A^{\rhd} \in \mathcal{U}^{\rhd\rhd}$ for each finite set $A \subseteq \Gamma_c G$, and

b. for each filter Φ which converges to 0 in $\Gamma_c G$, there is some $P \in \Phi$ such that $P^{\rhd} \in \mathcal{U}^{\rhd\rhd}$.

a) Let $A = \{\chi_1, ..., \chi_n\}$, then $\chi_i(\mathcal{U}) \to 0$ for all $i = 1, \dots, n$. Further, there are $U_i \in \mathcal{U}$ such that $\chi_i(U_i) \subseteq \mathbb{T}_+$. If $U = U_1 \cap ... \cap U_n$, then $\chi_i(U) \subseteq \mathbb{T}_+$ for all i which implies $A \subseteq U^{\rhd}$. Finally we have, $U^{\rhd\rhd} \subseteq A^{\rhd}$ and hence, $A^{\rhd} \in \mathcal{U}^{\rhd\rhd}$.

b) If $\Phi \to 0$ in $\Gamma_c G$, then $\Phi(\mathcal{U}) \to 0$ in $\mathbb{T}$. Further, there are $P \in \Phi$ and $U \in \mathcal{U}$ such that $P(U) \subseteq \mathbb{T}_+$. This gives $P \subseteq U^{\rhd}$ which implies $U^{\rhd\rhd} \subseteq P^{\rhd}$ and hence, $P^{\rhd} \in \mathcal{U}^{\rhd\rhd}$.

□

Theorem 2.5. *If a convergence group G is c-reflexive then it must be locally quasi-convex.*

Proof. To prove this result it is sufficient to prove that if a convergence group is embedded then it must be locally quasi-convex.
In a convergence abelian group G let,

$$\kappa_G : G \to \Gamma_c\Gamma_c G$$

be an embedding. If $\mathcal{U} \to 0$ in G then by Theorem 2.4, we have, $\mathcal{U}^{\rhd\rhd} \to 0$ in $\Gamma_c\Gamma_c G$ and so $\kappa^{-1}(\mathcal{U}^{\rhd\rhd}) \to 0$ in G. Now we have

$$\kappa^{-1}(\mathcal{U}^{\rhd\rhd}) = \mathcal{U}^{\rhd\lhd} \subseteq \mathcal{U}.$$

In view of Proposition 1.3 the proof follows. □

Next we prove that the continuous character group of a convergence group is locally quasi-convex.

Lemma 2.6. *Let X be a convergence space and A is a subset of X. Furthermore, let G be a convergence group and $M \subseteq G$ a quasi-convex set. Then*

$$P(A, M) = \{g \in C_c(X, G) : g(A) \subset M\}$$

is quasi-convex in the group $C_c(X, G)$.

Proof. Take any $g_0 \notin P(A, M)$. Then there is a point $x_0 \in A$ such that $g_0(x_0) \notin M$. Since M is quasi-convex there is some $\chi \in \Gamma(G)$ such that $\chi(M) \subset \mathbb{T}_+$ while $\chi(g_0(x_0)) \notin \mathbb{T}_+$. Define $\phi : C_c(X, G) \to \mathbb{T}$ by

$$\phi(g) = \chi(g(x_0)).$$

Then ϕ is a continuous character, $\phi(g_0) = \chi(g_0(x_0)) \notin \mathbb{T}_+$ while $\phi(g) = \chi(g(x_0)) \in \chi(M) \subseteq M$ for all $g \in P(A, M)$. □

The next lemma is an extension of [1, Proposition 6.2, (ii)] to the realm of convergence groups.

Lemma 2.7. *Let G be a convergence group and $\mathcal{B}$ be a family of quasi-convex subsets of G. Then*

$$B_0 = \bigcap\{B : B \in \mathcal{B}\}$$

is a quasi-convex subset of G.

Proof. A direct proof is easy, so omitted. □

Theorem 2.8. *Let X be a convergence space and G be a locally quasi-convex topological group. Then $C_c(X, G)$ is locally quasi-convex.*

Proof. Choose a zero neighbourhood basis $\mathcal{B}$ in G consisting of quasi-convex sets and take a filter $\mathcal{G}$ on $C_c(X, G)$ which converges to 0. Then $\mathcal{G}(\Phi) \to 0 \in G$ for each $x \in X$, and each filter Φ on X which converges to x. So for each $B \in \mathcal{B}$ there are $G_{x,\Phi,B} \in \mathcal{G}$ and $F_{x,\Phi,B} \in \Phi$ such that

$$G_{x,\Phi,B}(F_{x,\Phi,B}) \subseteq B.$$

Set

$$H_{x,\Phi,B} = P(F_{x,\Phi,B}, B)$$

then $\{H_{x,\Phi,B} : \Phi \to x, B \in \mathcal{B}\}$ is the subbasis of a filter $\mathcal{H}$ which converges to 0 in $C_c(X, G)$. By the Lemma 2.6 and Lemma 2.7, $\mathcal{H}$ has a base of quasi-convex sets and $\mathcal{H} \subseteq \mathcal{G}$ since $G_{x,\Phi,B} \subseteq H_{x,\Phi,B}$ for all x, Φ, B. □

Corollary 2.9. *For each convergence group G, $\Gamma_c G$ is locally quasi-convex.*

Proof. Since $\mathbb{T}$ and $C_c(G, \mathbb{T})$ are locally quasi-convex so is $\Gamma_c(G)$ as a subgroup. □

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