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A note on the fixed point theorem of F-contraction mappings in rectangular M-metric space

Mujahid Abbas [a], Rizwan Anjum [b] and Rabia Anwar [c]

[a] Department of Mathematics, Government College University Katchery Road, Lahore 54000, Pakistan and Department of Mathematics and Applied Mathematics, University of Pretoria Hatfield 002, Pretoria, South Africa (mujahid.abbas@up.ac.za)
[b] Department of Mathematics, Division of Science and Technology, University of Education, Lahore Pakistan. (rizwan.anjum@ue.edu.pk)
[c] Abdus Salam School of Mathematical Sciences, Government College University, Lahore 54600, Pakistan. (rabiaanwar@sms.edu.pk)

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Abstract

In this note, we show that the main result (Theorem 3.2) due to Asim et al. (Appl. Gen. Topol., 23(2), 363-376 (2022) https://doi.org/10.4995/agt.2022.17418) is still valid if we remove the assumption of continuity of the mapping.

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1. Introduction

One of the most important results used in nonlinear analysis is the well known Banach contraction principle [16] which states that any contraction mapping on a complete metric space has a unique fixed point. This principle [16] has played a key role in the development of metric fixed point theory in the recent past. The fruitfulness of this basic principle is evident from the fact

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that several researchers have obtained its several interesting extensions and generalizations in different directions (see [1, 2, 4, 8, 3]).

These generalization are done on the basis of two main aspects, one is the ambient space and the other is the contraction map. Motivated by this idea, several authors have studied various generalizations of this notation in different types of metric spaces.

In 1994, Matthews [19] introduced the notion of a partial metric space and proved the Banach contraction principle [16] in this new distance structure. Afterwards, many fixed point theorems in partial metric spaces were obtained by several mathematicians.

In 2014, Asadi et al.[13] extend the concept of partial metric spaces and presented some examples to show that their definition is a real generalization of partial metric space by introducing the concept of M -metric space. On the other hand, Altun et al. in [7] discussed on the topological structures of M-metric space. They emphasized that the sequential topology is stronger than the topology induced by open balls.

One of the interesting generalization of metric space was given by Branciari [17] by introducing the concept of generalized metric space. Moreover, Branciari proved the analogue of Banach fixed point theorem in the generalized metric spaces. Latter on, Özgür et al. [25] introduced the concept of a rectangular M-metric space, along with proving the analogue of Banach fixed point theorem in the rectangular M-metric spaces [25].

As mentioned earlier, another way to generalize the Banach fixed point theorem by extended the notion of contractive condition instead of distance structure. Based on this fact, in 2012, Wardowski [27] introduced the concept of F-contraction and proved a fixed point theorem which extends the Banach fixed point theorem in the setting of complete metric space. In 2018, employing the idea of Branciari and Wardowski, Zheng-ying [18] proved the analogue of Wardowski fixed point result in the generalized metric space. In 2019, Sahin et al. [26] proved two fixed point results for multivalued F-contraction on M-metric space. For more results in this direction, we refer to [11, 12, 14, 20, 21, 22, 23, 24, 26, 9, 10, 5, 6].

On the other hands, Asim et al. [15] introduced the concept of F-contraction in rectangular M-metric space as follows:
Let (X, m_r) be a rectangular M-metric space and $F \in \mathcal{F}$. A mapping $T : X \to X$ is called F-contraction if there exist $\tau > 0$ such that for all $x, y \in X$ with $m_r(Tx, Ty) > 0$, we have

$$\tau + F(m_r(Tx, Ty)) \leq F(m_r(x, y)), \tag{1.1}$$

where $\mathcal{F}$ is the set of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

$(F1)$ F is strictly increasing: $s < t \Rightarrow F(s) < F(t)$;

$(F2)$ For each sequence $\{s_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+$, $\lim_{n\to\infty} s_n = 0$ if and only if $\lim_{n\to\infty} F(s_n) = -\infty$;

$(F3)$ There exists $k \in (0,1)$ such that $\lim_{s\to 0^+} s^k F(s) = 0$.

Let us recall the statement of the Theorem 3.2 from [15].

Theorem 1.1 ([15]). *Let (X, m_r) be a rectangular M-metric space and $T : X \to X$ be a continuous F-contraction. Then, T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$, a sequence $\{T^n x_0 : n \in \mathbb{N}\}$ is convergent to x^*.*

In this paper, we show that the assumption of continuity considered in Theorem 1.1 can be removed. Moreover, we give an example of a mapping on a rectangular M-metric space where the result of Asim et al. [15] is not applicable.

2. Preliminaries

Let us recall some of the concepts given in [25].

Definition 2.1 ([25]). Let X be a nonempty set. A mapping $m_r : X \times X \to [0, \infty)$ is said to be m_r-metric if for any $x, y \in X$, the following conditions hold:

(1) $m_r(x, y) = m_{r_{x,y}} = M_{r_{x,y}} \Leftrightarrow x = y$,
(2) $m_{r_{x,y}} \le m_r(x, y)$,
(3) $m_r(x, y) = m_r(y, x)$,
(4) $m_r(x, y) - m_{r_{x,y}} \le m_r(x, u) - m_{r_{x,u}} + m_r(u, v) - m_{r_{u,v}} + m_r(v, y) - m_{v,y}$ for all $u, v \in X \backslash \{x, y\}$,

where $m_{r_{x,y}} = \min\{m_r(x, x), m_r(y, y)\}$ and $M_{r_{x,y}} = \max\{m_r(x, x), m_r(y, y)\}$.

The pair (X, m_r) is called a rectangular M-metric space.

Definition 2.2 ([25]). A sequence $\{x_n\}$ in a rectangular M-metric space X is said to be:

(a): convergent to some $x \in X$ if and only if

$$\lim_{n\to\infty} \left(m_r\left(x_n, x\right) - m_{r_{x_n,x}}\right) = 0.$$

In this case we write $x_n \to x$ as $n \to \infty$;

(b): a m_r-Cauchy sequence if and only if

$$\lim_{n,m\to\infty} \left(m_r\left(x_n, x_m\right) - m_{r_{x_n,x_m}}\right) \text{ and } \lim_{n,m\to\infty} \left(M_{r_{x_n,x_m}} - m_{r_{x_n,x_m}}\right)$$

exist and finite.

A rectangular M-metric space X is said to be m_r-complete if every m_r-Cauchy sequence in X is convergent in X such that

$$\lim_{n\to\infty} \left(m_r\left(x_n, x\right) - m_{r_{x_n,x}}\right) = 0 \text{ and } \lim_{n,m\to\infty} \left(M_{r_{x_n,x}} - m_{r_{x_n,x}}\right) = 0.$$

Lemma 2.3 (). [25] *Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in a rectangular M-metric space. Then*

$$\lim_{n\to\infty}\left(m_r\left(x_n, y_n\right) - m_{r_{x_n,y_n}}\right) = m_r(x,y) - m_{r_{x,y}}.$$

Lemma 2.4 ([25]). *Assume that $x_n \to x$ as $n \to \infty$ in a rectangular M-metric space. Then*

$$\lim_{n\to\infty}\left(m_r\left(x_n, y\right) - m_{r_{x_n,y}}\right) = m_r(x,y) - m_{r_{x,y}}, \quad \forall y \in X.$$

3. Main Result

Before stating the main result, we first prove the following lemma for the class of F-contraction mappings on rectangular M-metric spaces.

Lemma 3.1. *Let T be a F-contraction on rectangular M-metric space (X, m_r). If Picard iteration defined by*

$$x_n = Tx_{n-1} \quad n \in \mathbb{N}, \tag{3.1}$$

where x_0 is an initial guess in domain of an mapping T, converges to $u^* \in X$. Then

$$\lim_{n\to\infty} Tx_n = Tu^*.$$

Proof. We divide the proof into follwing two cases.
Case 1 :
Suppose that

$$\lim_{m\to\infty} m_r(Tx_m, Tu^*) = 0. \tag{3.2}$$

Since

$$m_{r_{Tx_m,Tu^*}} = \min\{m_r(Tx_m, Tx_m), m_r(Tu^*, Tu^*)\} \leq m_r(Tx_m, Tu^*).$$

On taking limit as $m \to \infty$ on both sides of the above inequality, we have

$$\lim_{m\to\infty} m_{r_{Tx_m,Tu^*}} \leq \lim_{m\to\infty} m_r(Tx_m, Tu^*).$$

By (3.2), it follows that

$$\lim_{m\to\infty} m_{r_{Tx_m,Tu^*}} = 0. \tag{3.3}$$

From (3.2) and (3.3), we get

$$\lim_{m\to\infty}\left(m_r(Tx_m, Tu^*) - m_{r_{Tx_m,Tu^*}}\right) = 0.$$

Hence, $Tx_m \to Tu^*$ as $m \to \infty$.

Case 2 : Suppose that

$$\lim_{m\to\infty} m_r(Tx_m, Tu^*) > 0. \tag{3.4}$$

Since $m_r(Tx_m, Tu^*) \in [0, \infty)$, for all $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that

$$m_r(Tx_m, Tu^*) > 0, \quad \forall m \geq N. \tag{3.5}$$

By (1.1) and (3.5), we obtain that

$$F(m_r(Tx_m, Tu^*)) \leq F(m_r(x_m, u^*)) - \tau < F(m_r(x_m, u^*)), \quad \forall m \geq N.$$

From $(F1)$, we get

$$m_r(Tx_m, Tu^*) < m_r(x_m, u^*), \quad \forall m \geq N. \tag{3.6}$$

Further, we consider the following two subcases:
Case (a) : Suppose that

$$m_r(u^*, u^*) \leq \lim_{m\to\infty} m_r(x_m, x_m). \tag{3.7}$$

In this subcase, we show that $m_r(u^*, u^*) = 0$.
If $\lim_{m\to\infty} m_r(x_m, x_m) = 0$. Then it follows from (3.7) that

$$m_r(u^*, u^*) = 0.$$

On the other hand, if $\lim_{m\to\infty} m_r(x_m, x_m) > 0$. Then, there exists $\nu \in \mathbb{N}$ such that

$$m_r(x_m, x_m) > 0, \quad \forall m \geq \nu. \tag{3.8}$$

From (1.1), we have

$$F(m_r(x_m, x_m)) \leq F(m_r(x_{m-1}, x_{m-1})) - \tau, \quad \forall m \geq \nu.$$

Continuing this way, we can obtain

$$F(m_r(x_m, x_m)) \leq F(m_r(x_0, x_0)) - m\tau, \quad \forall m \geq \nu.$$

It follows that

$$\lim_{m\to\infty} F(m_r(x_m, x_m)) = -\infty. \tag{3.9}$$

By using $(F2)$ in (3.9), we get

$$\lim_{m\to\infty} m_r(x_m, x_m) = 0.$$

By (3.7), we obtain

$$m_r(u^*, u^*) \leq \lim_{m\to\infty} m_r(x_m, x_m) = 0.$$

That is,

$$m_r(u^*, u^*) = 0. \tag{3.10}$$

Since $m_r(Tu^*, Tu^*) \in [0, \infty)$. If $m_r(Tu^*, Tu^*) = 0$. Then from (3.10), we have

$$m_r(Tu^*, Tu^*) = 0 = m_r(u^*, u^*). \tag{3.11}$$

On the other hand, if $m_r(Tu^*, Tu^*) > 0$. Then from (1.1), we get

$$F(m_r(Tu^*, Tu^*)) \leq F(m_r(u^*, u^*)) - \tau < F(m_r(u^*, u^*)).$$

From $(F1)$, we obtain

$$m_r(Tu^*, Tu^*) < m_r(u^*, u^*).$$

By using (3.10), the above inequality becomes

$$m_r(Tu^*, Tu^*) = 0 = m_r(u^*, u^*). \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$m_{r_{x_m,u^*}} = \min\{m_r(x_m,x_m), m_r(u^*,u^*)\} = m_r(u^*,u^*) = 0, \quad \forall m \geq \nu \text{ and} \tag{3.13}$$

$$m_{r_{Tx_m,Tu^*}} = \min\{m_r(Tx_m,Tx_m), m_r(Tu^*,Tu^*)\} = m_r(Tu^*,Tu^*) = 0, \quad \forall m \geq \nu. \tag{3.14}$$

Since, $x_m \to u^*$ as $m \to \infty$. This implies that

$$m_r(x_m,u^*) - m_{r_{x_m,u^*}} \to 0, \quad m \to \infty.$$

By using (3.13) in the above inequality, we have

$$m_r(x_m,u^*) \to 0, \quad m \to \infty. \tag{3.15}$$

On taking limit as $m \to \infty$ in (3.6), we have

$$\lim_{m\to\infty} m_r(Tx_m,Tu^*) \leq \lim_{m\to\infty} m_r(x_m,u^*). \tag{3.16}$$

By using (3.15) in (3.16), we get

$$\lim_{m\to\infty} m_r(Tx_m,Tu^*) = 0. \tag{3.17}$$

Taking limit as $m \to \infty$ in (3.13), we have

$$m_{r_{Tx_m,Tu^*}} \to 0, \quad m \to \infty. \tag{3.18}$$

By combining (3.17) and (3.18), we obtain

$$m_r(Tx_m,Tu^*) - m_{r_{Tx_m,Tu^*}} \to 0, \quad m \to \infty.$$

Thus $Tx_m \to Tu^*$ as $m \to \infty$.
Case (b) : Now, suppose that

$$m_r(u^*,u^*) \geq \lim_{m\to\infty} m_r(x_m,x_m). \tag{3.19}$$

In this case, we show that

$$\lim_{m\to\infty} m_{r_{x_m,u^*}} = 0. \tag{3.20}$$

If $m_r(u^*,u^*) = 0$. Then from (3.19), we obtain (3.20).
If $m_r(u^*,u^*) > 0$. Assuming $\lim_{m\to\infty} m_r(x_m,x_m) = 0$, we have

$$\lim_{m\to\infty} m_{r_{x_m,u^*}} = 0.$$

Now if

$$\lim_{m\to\infty} m_r(x_m,x_m) > 0.$$

Following the same procedure as in the case (a), we obtain that

$$\lim_{m\to\infty} m_r(x_m,x_m) = 0.$$

As $x_m \to u^*$ as $m \to \infty$, we have

$$m_r(x_m,u^*) - m_{r_{x_m,u^*}} \to 0 \qquad m \to \infty.$$

By using (3.20) into the above, it follows that

$$\lim_{m\to\infty} m_r(x_m,u^*) = 0. \tag{3.21}$$

By using (3.21) and (3.6), we have the following

$$\lim_{m\to\infty} m_r(Tx_m, Tu^*) = 0. \tag{3.22}$$

Moreover, we have

$$m_{r_{Tx_m,Tu^*}} \leq m_r(Tx_m, Tu^*). \tag{3.23}$$

On taking limit as $m \to \infty$, we obtain

$$\lim_{m\to\infty} m_{r_{Tx_m,Tu^*}} = 0. \tag{3.24}$$

By using (3.23) and (3.24), we have

$$Tx_m \to Tu^*, \quad m \to \infty.$$

□

We now need the following propositions to prove the main result.

Proposition 3.2. *Let (X, m_r) be a rectangular M-metric space, $T : X \to X$ a F-contraction mapping and $x_0 \in X$. If the Picard iteration defined by*

$$x_n = Tx_{n-1} \qquad n \geq 1$$

has the following property

$$m_r(x_n, x_n) = 0 \quad \textit{for some} \;\; n \in \mathbb{N}.$$

Then, we have

$$m_r(x_m, x_m) = 0, \qquad \forall m \geq n. \tag{3.25}$$

Proof. We will prove it by induction on M.
Suppose that the result is true for $m = k > n$. This can also be expressed as

$$m_r(x_k, x_k) = 0. \tag{3.26}$$

We now to prove that

$$m_r(x_{k+1}, x_{k+1}) = 0.$$

Assume on the contrary that $m_r(x_{k+1}, x_{k+1}) > 0$. By (1.1), we have

$$F(m_r(x_{k+1}, x_{k+1})) \leq F(m_r(x_k, x_k)) - \tau < F(m_r(x_k, x_k)).$$

It follows from $(F1)$ that

$$m_r(x_{k+1}, x_{k+1}) < m_r(x_k, x_k).$$

Using (3.26) into the above inequality, we obtain the desired result. □

Proposition 3.3. *Let (X, m_r) be a rectangular M-metric space, $x_0 \in X$ and $T : X \to X$ a F-contraction mapping. Suppose that a sequence $\{x_m\}$ is given by*

$$x_m = Tx_{m-1} \qquad m \geq 1.$$

Then, for each fixed $n \in \mathbb{N}$, we have

$$m_{r_{x_n,x_m}} = \min\{m_r(x_n, x_n), m_r(x_m, x_m)\} = m_r(x_m, x_m), \quad m > n. \tag{3.27}$$

Proof. On the contrary suppose that

$$m_{r_{x_n,x_m}} = m_r\left(x_n, x_n\right), \quad \forall m > n. \tag{3.28}$$

We now divide the proof into two the following cases.
Case 1 : If $m_r\left(x_n, x_n\right) = 0$. It follows from Proposition 3.2 that

$$m_r\left(x_m, x_m\right) = 0, \quad \forall m > n.$$

Note that

$$\begin{aligned} m_{r_{x_n,x_m}} &= \min\{m_r\left(x_n, x_n\right), m_r\left(x_m, x_m\right)\}, \quad \forall m > n. \\ &= \min\{0, 0\}, \quad \forall m > n. \\ &= 0 = m_r\left(x_m, x_m\right), \quad \forall m > n. \end{aligned}$$

Thus,

$$m_{r_{x_n,x_m}} = m_r\left(x_m, x_m\right), \quad \forall m > n.$$

Case 2 : If $m_r\left(x_n, x_n\right) > 0$.
It follows from (3.28) that

$$m_r\left(x_m, x_m\right) > 0 \quad \forall m > n. \tag{3.29}$$

By using (1.1) and (3.29), we have

$$\begin{aligned} F\left(m_r\left(x_m, x_m\right)\right) &= F\left(m_r\left(Tx_{m-1}, Tx_{m-1}\right)\right) \\ &\leq F\left(m_r\left(x_{m-1}, x_{m-1}\right)\right) - \tau \leq \cdots \leq F\left(m_r\left(x_n, x_n\right)\right) - (m-n)\tau \\ &< F\left(m_r\left(x_n, x_n\right)\right), \end{aligned}$$

that is,

$$F\left(m_r\left(x_m, x_m\right)\right) < F\left(m_r\left(x_n, x_n\right)\right).$$

By using $(F1)$, we obtain that

$$m_r\left(x_m, x_m\right) < m_r\left(x_n, x_n\right); \quad \forall m > n,$$

which is a contradiction to (3.28). □

Now, we will prove the main result.

Theorem 3.4. *Let (X, m_r) be a complete rectangular M-metric space and $T : X \to X$ be an F-contraction mapping. Then T has a unique fixed point.*

Proof. We divide the proof into the following two cases.
Case 1 : If there exists a natural number n such that $x_{n+1} = x_n$. Then, x_n is a fixed point of T.
Case 2 : Suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$.
We divide this case into two further subcases.
Subcase 1 : Take

$$m_r\left(x_{n+1}, x_n\right) = 0, \quad \text{for some } n \in \mathbb{N}. \tag{3.30}$$

Note that

$$m_{r_{x_{n+1},x_n}} \leq m_r\left(x_{n+1}, x_n\right) = 0.$$

Thus

$$m_{r_{x_{n+1},x_n}} = 0. \tag{3.31}$$

From the Proposition 3.3, we have

$$m_{r_{x_{n+1},x_n}} = m_r\left(x_{n+1}, x_{n+1}\right). \tag{3.32}$$

Combining (3.31) and (3.32), we get

$$m_r\left(x_{n+1}, x_{n+1}\right) = 0. \tag{3.33}$$

Since $m_r\left(x_{n+1}, x_{n+1}\right) = 0$, it follows from the Proposition 3.2 that

$$m_r\left(x_{n+2}, x_{n+2}\right) = 0. \tag{3.34}$$

We now divide subcase 1 into further subcases:
Subcase 1_a : Suppose

$$m_r\left(x_{n+1}, x_{n+2}\right) = 0. \tag{3.35}$$

It follows from (3.33), (3.34) and (3.35) that

$$m_r\left(x_{n+1}, x_{n+1}\right) = m_r\left(x_{n+2}, x_{n+2}\right) = m_r\left(x_{n+1}, x_{n+2}\right) = 0.$$

By using the property of m_r, we have

$$x_{n+1} = x_{n+2},$$

that is,

$$x_{n+1} = Tx_{n+1}.$$

that is, x_{n+1} is the fixed point of T.
Subcase 1_b : Suppose that

$$m_r\left(x_{n+1}, x_{n+2}\right) > 0. \tag{3.36}$$

By using (1.1), we have

$$\begin{aligned} F\left(m_r\left(x_{n+1}, x_{n+2}\right)\right) &= F(m_r(Tx_n, Tx_{n+1})) \\ &\leq F\left(m_r\left(x_n, x_{n+1}\right)\right) - \tau \\ &< F\left(m_r\left(x_n, x_{n+1}\right)\right). \end{aligned}$$

From the condition $(F1)$, we obtain that

$$m_r(x_{n+1}, x_{n+2}) < m_r(x_n, x_{n+1}).$$

It follows from (3.30) that

$$m_r(x_{n+1}, x_{n+2}) < m_r(x_n, x_{n+1}) = 0.$$

So,

$$m_r(x_{n+1}, x_{n+2}) = 0,$$

a contradiction to (3.36).
Subcase 2 : Suppose that $m_r(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$.
Let

$$\beta_n = m_r\left(x_n, x_{n+1}\right), \quad \forall n \in \mathbb{N}.$$

By (1.1), we get

$$F\left(\beta_n\right) \leq F\left(\beta_{n-1}\right) - \tau \leq F\left(\beta_{n-2}\right) - 2\tau \leq \cdots \leq F\left(\beta_0\right) - n\tau, \quad \forall n \in \mathbb{N}.$$

Therefore,

$$F(\beta_n) \leq F(\beta_0) - n\tau, \quad \forall n \in \mathbb{N}. \tag{3.37}$$

On taking limit as $n \to \infty$ in (3.37), we have

$$\lim_{n\to\infty} F(\beta_n) = -\infty.$$

From $(F2)$, we have

$$\lim_{n\to\infty} \beta_n = 0. \tag{3.38}$$

By $(F3)$, there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} \beta_n^k F(\beta_n) = 0. \tag{3.39}$$

Multiplying β_n^kon the both sides of (3.37), we obtain that

$$\beta_n^k F(\beta_n) \leq \beta_n^k F(\beta_0) - \beta_n^k n\tau, \quad \forall n \in \mathbb{N},$$

that is,

$$\beta_n^k F(\beta_n) - \beta_n^k F(\beta_0) \leq -\beta_n^k n\tau, \quad \forall n \in \mathbb{N}.$$

On taking limit as $n \to \infty$ on both sides of the above inequality, we have

$$\lim_{n\to\infty} \beta_n^k F(\beta_n) - \lim_{n\to\infty} \beta_n^k F(\beta_0) \leq -\lim_{n\to\infty} \beta_n^k n\tau. \tag{3.40}$$

As $n \in \mathbb{N}$, $k \in (0,1), \tau > 0$ and $\beta_n \in [0,\infty)$,

$$\lim_{n\to\infty} \beta_n^k n\tau \geq 0. \tag{3.41}$$

Using (3.41) into (3.40), we obtain

$$\lim_{n\to\infty} \beta_n^k F(\beta_n) - \lim_{n\to\infty} \beta_n^k F(\beta_0) \leq -\lim_{n\to\infty} \beta_n^k n\tau \leq 0. \tag{3.42}$$

From (3.38) and (3.39), we have

$$0 \leq -\lim_{n\to\infty} \beta_n^k n\tau \leq 0, \quad \forall n \in \mathbb{N}.$$

Hence,

$$\lim_{n\to\infty} n\beta_n^k = 0.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that $n\beta_n^k \leq 1$ for all $n > n_0$ and so

$$\beta_n \leq \frac{1}{n^{1/k}} \quad \forall n > n_0. \tag{3.43}$$

We now prove that

$$\lim_{n\to\infty} m_r(x_n, x_{n+2}) = 0. \tag{3.44}$$

If $m_r(x_n, x_{n+2}) = 0$ for all $n \in \mathbb{N}$, then we have (3.44).
On the other hand, if $m_r(x_n, x_{n+2}) > 0$ for all $n \in \mathbb{N}$. By using (1.1), we get

$$F(m_r(x_n, x_{n+2})) \leq F(m_r(x_{n-1}, x_{n+1})) - \tau, \quad \forall n \in \mathbb{N}.$$

Continuing this way, we have

$$F(m_r(x_n, x_{n+2})) \leq F(m_r(x_{n-1}, x_{n+1})) - \tau \leq \cdots \leq F(m_r(x_0, x_2)) - n\tau, \quad \forall n \in \mathbb{N}. \tag{3.45}$$

On taking limit as $n \to \infty$ in (3.45), we obtain

$$\lim_{n\to\infty} F\left(m_r\left(x_n, x_{n+2}\right)\right) = -\infty.$$

From $(F2)$, we have

$$\lim_{n\to\infty} m_r\left(x_n, x_{n+2}\right) = 0.$$

We now prove that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is an m_r-Cauchy sequence. Let $m > n$ with $m = n + o$ where $o > 2$. We now consider two cases.
Case (i): Suppose that o is odd. Let $o = 2p + 1$, where $p \in \mathbb{N}$. Then

$$\begin{aligned}
m_r(x_n, x_m) - m_{r_{x_n,x_m}} &= m_r(x_n, x_{n+2p+1}) - m_{r_{x_n,x_{n+2p+1}}} \\
&\leq m_r(x_n, x_{n+1}) - m_{r_{x_n,x_{n+1}}} + \cdots + m_r(x_{n+2p}, x_{n+2p+1}) - m_{r_{x_{n+2p},x_{n+2p+1}}} \\
&< m_r(x_n, x_{n+1}) + \cdots + m_r(x_{n+2p}, x_{n+2p+1}) \\
&= \beta_n + \cdots + \beta_{n+2p} \\
&\leq \sum_{i=n}^{\infty} \beta_i \\
&\leq \sum_{n\geq n_0(\epsilon)} \frac{1}{n^{1/k}} < \epsilon.
\end{aligned}$$

Case (ii): Suppose that o is even. Let $o = 2p$, where $p \in \mathbb{N}$. Then

$$\begin{aligned}
m_r\left(x_n, x_m\right) - m_{r_{x_n,x_m}} &= m_r\left(x_n, x_{n+2p}\right) - m_{r_{x_n,x_{n+2p}}} \\
&\leq m_r(x_n, x_{n+2}) - m_{r_{x_n,x_{n+2}}} + m_r(x_{n+2}, x_{n+3}) - m_{r_{x_{n+2},x_{n+3}}} + \\
&\qquad \cdots + m_r(x_{n+2p-1}, x_{n+2p}) - m_{r_{x_{n+2p-1},x_{n+2p}}} \\
&< m_r(x_n, x_{n+2}) + m_r(x_{n+2}, x_{n+3}) + \cdots + m_r(x_{n+2p-1}, x_{n+2p}) \\
&\leq m_r(x_n, x_{n+2}) + \sum_{i=n+2}^{\infty} \beta_i \\
&\leq m_r(x_n, x_{n+2}) + \sum_{n\geq n_0(\epsilon)} \frac{1}{n^{1/k}} < \epsilon.
\end{aligned}$$

Indeed, the series $\sum_{n=i}^{\infty} \frac{1}{n^{1/k}}$ converges and $\lim_{n\to\infty} m_r(x_n, x_{n+2}) = 0$. Thus,

$$\lim_{n,m\to\infty} \left(m_r\left(x_n, x_m\right) - m_{r_{x_n,x_m}}\right),$$

exist and finite.

Now, if $M_{r_{x_n,x_m}} = 0$ for all $m > n$, then $m_{r_{x_n,x_m}} = 0$ for all $m > n$ which implies that

$$M_{r_{x_n,x_m}} - m_{r_{x_n,x_m}} = 0, \quad \forall m > n,$$

and hence

$$\lim_{n,m\to\infty} (M_{r_{x_n,x_m}} - m_{r_{x_n,x_m}}) = 0.$$

Assume that

$$M_{r_{x_n,x_m}} = \max\{m_r\left(x_n, x_n\right), m_r\left(x_m, x_m\right)\} > 0, \quad \forall m > n.$$

It follows from Proposition 3.3 that

$$M_{r_{x_n,x_m}} = \max\{m_r(x_n,x_n), m_r(x_m,x_m)\} = m_r(x_n,x_n) > 0, \quad \forall m > n.$$

Suppose $\mu_n = m_r(x_n,x_n)$ for all $n \in \mathbb{N}$. Then by (1.1), we have

$$F(\mu_n) \leq F(\mu_{n-1}) - \tau \leq F(\mu_{n-2}) - 2\tau \leq \cdots \leq F(\mu_0) - n\tau. \tag{3.46}$$

On taking limit as $n \to \infty$ on both sides of the inequality (3.46), we get

$$\lim_{n\to\infty} F(\mu_n) = -\infty.$$

By $(F2)$, we have

$$\lim_{n\to\infty} \mu_n = 0. \tag{3.47}$$

Therefore, we obtain that

$$\mu_n \leq \frac{1}{n^{1/h}}, \quad \forall n > n_1.$$

Note that

$$\begin{aligned} M_{r_{x_n,x_m}} - m_{r_{x_n,x_m}} &= m_r(x_n,x_n) - m_r(x_m,x_m) \\ &< m_r(x_n,x_n) \\ &\leq m_r(x_n,x_n) + m_r(x_{n+1},x_{n+1}) + \cdots + m_r(x_m,x_m) \\ &\leq \mu_n + \mu_{n+1} + \cdots + \mu_m \\ &\leq \sum_{i=n}^{\infty} \mu_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/h}} < \epsilon. \end{aligned}$$

Indeed, the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/h}}$ converges. Thus,

$$\lim_{n,m\to\infty} (M_{r_{x_n,x_m}} - m_{r_{x_n,x_m}}),$$

exist and finite.
Thus $\{x_n\}_{n\in\mathbb{N}}$ is an m_r-Cauchy sequence. We now take a point $u^* \in X$ such that $\{x_n\}$ converges to u^*.
Since $m_r(x_n,x_{n+1}) > 0$, by using (1.1) and $(F2)$, we conclude that

$$\lim_{n\to\infty} m_r(x_n,Tx_n) = 0. \tag{3.48}$$

Using $m_{r_{x_n,Tx_n}} \leq m_r(x_n,Tx_n)$, we have

$$\lim_{n\to\infty} m_{r_{x_n,Tx_n}} = 0. \tag{3.49}$$

From (3.48) and (3.49), we have

$$\lim_{n\to\infty} (m_r(x_n,Tx_n) - m_{r_{x_n,Tx_n}}) = 0. \tag{3.50}$$

Since

$$x_n \to u^*, \quad n \to \infty. \tag{3.51}$$

Therefore, it follows from Lemma 3.1 that

$$Tx_n \to Tu^*, \quad n \to \infty. \tag{3.52}$$

Using (3.51) and (3.52) into the Lemma 2.3, (3.50) becomes

$$m_r\left(u^*, Tu^*\right) = m_{r_{u^*,Tu^*}}. \tag{3.53}$$

By Proposition 3.3, we have

$$m_{r_{x_n,Tx_n}} = m_r\left(Tx_n, Tx_n\right), \quad \forall n \in \mathbb{N}.$$

On taking limit as $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} m_{r_{x_n,Tx_n}} = \lim_{n\to\infty} m_r\left(Tx_n, Tx_n\right),$$

that is,

$$\lim_{n\to\infty} \left(m_{r_{x_n,Tx_n}} - m_r\left(Tx_n, Tx_n\right)\right) = 0. \tag{3.54}$$

Using (3.51) and (3.52) into the Lemma 2.3, (3.54) becomes

$$m_{r_{u^*,Tu^*}} - m_r\left(Tu^*, Tu^*\right) = 0. \tag{3.55}$$

By using (3.53) and (3.55), we have

$$m_r\left(u^*, Tu^*\right) = m_{r_{u^*,Tu^*}} = m_r\left(Tu^*, Tu^*\right). \tag{3.56}$$

From (3.50), we obtain

$$\lim_{n\to\infty} \left(m_r\left(x_n, x_{n-1}\right) - m_{r_{x_n,Tx_n}}\right) = 0. \tag{3.57}$$

By using (3.51) and (3.52) into the Lemma 2.3, then (3.57) becomes

$$m_r\left(u^*, u^*\right) = m_{r_{u^*,Tu^*}}. \tag{3.58}$$

By using (3.56) and (3.58), we get

$$m_r\left(u^*, u^*\right) = m_r\left(Tu^*, u^*\right) = m_r\left(Tu^*, Tu^*\right).$$

This implies that $Tu^* = u^*$. Suppose that there exist two elements $x, y \in X$ such that $x = Tx$ and $y = Ty$ with $x \neq y$.

Let us consider the following cases:

Case A : If $m_r(Tx, Ty) = m_r(x, y) = 0$. Without loss of generality, suppose that

$$m_{r_{x,y}} = m_r(x, x).$$

Notice that

$$m_r(x, x) = m_{r_{x,y}} \leq m_r(x, y) = 0.$$

It follows that

$$m_r(x, x) = 0.$$

Further, we divide the case A into two subcases.

Subcase A_1 : If $m_r(y, y) = 0$. Then, clearly $x = y$.

Subcase A_2 : Suppose that $m_r(y, y) > 0$.

By using (1.1), we have

$$F(m_r(y, y)) = F(m_r(Ty, Ty)) \leq F(m_r(y, y)) - \tau < F(m_r(y, y)),$$

It follows that

$$F(m_r(y, y)) < F(m_r(y, y)).$$

By $(F1)$, we have

$$m_r(y, y) < m_r(y, y),$$

a contradiction.
Case B : If $m_r(Tx, Ty) = m_r(x, y) > 0$. Then, by (1.1), we deduce that

$$F(m_r(x, y)) = F(m_r(Tx, Ty)) \leq F(m_r(x, y)) - \tau < F(m_r(x, y)),$$

that is,

$$F(m_r(x, y)) < F(m_r(x, y)).$$

By $(F1)$, we have

$$m_r(x, y) < m_r(x, y),$$

a contradiction. □

Example 3.5. Let $X = \{1, 2, 3, 4\}$. Define $m_r : X \times X \to [0, \infty)$ by

$$\begin{aligned}
&m_r(1,1) = 1 \text{ and } m_r(2,2) = m_r(3,3) = m_r(4,4) = 0\\
&m_r(1,2) = m_r(2,1) = 4\\
&m_r(1,3) = m_r(3,1) = 4\\
&m_r(1,4) = m_r(4,1) = 4\\
&m_r(2,3) = m_r(3,2) = 3\\
&m_r(2,4) = m_r(4,2) = 6\\
&m_r(3,4) = m_r(4,3) = 9.
\end{aligned}$$

Note that (X, m_r) is a complete rectangular M-metric space. On the other hand, (X, m_r) is not a M-metric space. Indeed,

$$9 = m_r(3, 4) \geq m_r(3, 1) + m_r(1, 4) = 4 + 4 = 8.$$

Define $T : X \to X$ as

$$T(x) = \begin{cases} 2, & x = 1, 2, 3 \\ 3, & x = 4 \end{cases}$$

For $x \in \{1, 2, 3\}$ and $y = 4$, we have $m_r(Tx, Ty) = m_r(2, 3) = 3 > 0$.
Therefore,

$$\ln(m_r(Tx, Ty)) + m_r(Tx, Ty) \leq \ln(m_r(x, y)) + m_r(x, y) - 0.5.$$

If we take $F(t) = \ln(t) + t$ and $\tau = 0.5$. Then, T is a F-contraction.
Hence, all the conditions of Theorem 3.4 are satisfied. Moreover, $x = 2$ is the fixed point of T.

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