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On graph induced symbolic systems

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Abstract

In this paper, we investigate shift spaces arising from a multidimensional graph G. In particular, we investigate non-emptiness and existence of periodic points for a multidimensional shift space. We derive sufficient conditions under which these questions can be answered affirmatively. We investigate the structure of the shift space using the generating matrices. We prove that any d-dimensional shift of finite type is finite if and only if it is conjugate to a shift generated through permutation matrices. We prove that if any triangular pattern of the form $\,_{a}\,^{c}_{b}$ can be extended to a 1×1 square then the two dimensional shift space possesses periodic points. We introduce the notion of an E-pair for a two dimensional shift space. Using the notion of an E-pair, we derive sufficient conditions for non-emptiness of the two dimensional shift space under discussion.

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1. Introduction

Symbolic dynamics originated as a tool to investigate various natural and physical phenomena around us. The convenience of symbolic representation and easier computability of the system has attracted attention of several researchers around the globe and the topic has found applications in various branches of sciences and engineering. In particular, the area has found applications in areas like data storage, data transmission and communication systems

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to name a few [9, 14, 11]. The structure and dynamics of a symbolic system can be used to investigate the dynamics of a general dynamical system. In fact, it is known that every discrete dynamical system can be embodied in a symbolic dynamical system (with appropriate number of symbols) [4]. Consequently, it is sufficient to study the shift spaces and its subsystems to investigate the dynamics of a general discrete dynamical system.

Let $\mathcal{A} = \{a_i : i \in I\}$ be a finite set and let d be a positive integer. Let the set $\mathcal{A}$ be equipped with the discrete metric and let $\mathcal{A}^{\mathbb{Z}^d}$, the collection of all functions $c : \mathbb{Z}^d \to \mathcal{A}$ be equipped with the product topology. Any such function c is called a configuration over $\mathcal{A}$. Any configuration c is called periodic if there exists $u \in \mathbb{Z}^d$ $(u \neq 0)$ such that $c(v+u) = c(v)$ $\forall v \in \mathbb{Z}^d$. In particular, if c is periodic of period $(m, 0)$ $((0, m))$, then c is referred as a horizontally (vertically) periodic point. The set $\Gamma_c = \{w \in \mathbb{Z}^d : c(v+w) = c(v) \ \ \forall v \in \mathbb{Z}^d\}$ is called the lattice of periods for the configuration c. The function $\mathcal{D} : \mathcal{A}^{\mathbb{Z}^d} \times \mathcal{A}^{\mathbb{Z}^d} \to \mathbb{R}^+$ defined as $\mathcal{D}(x, y) = \frac{1}{n+1}$, where n is the least non-negative integer such that $x \neq y$ in $R_n = [-n, n]^d$, is a metric on $\mathcal{A}^{\mathbb{Z}^d}$ and generates the product topology. For any $a \in \mathbb{Z}^d$, the map $\sigma_a : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ defined as $(\sigma_a(x))(k) = x(k+a)$ is a d-dimensional shift and is a homeomorphism. For any $a, b \in \mathbb{Z}^d$, $\sigma_a \circ \sigma_b = \sigma_b \circ \sigma_a$ and hence $\mathbb{Z}^d$ acts on $\mathcal{A}^{\mathbb{Z}^d}$ through commuting homeomorphisms. For any nonempty $S \subset \mathbb{Z}^d$, any element of $\mathcal{A}^S$ is called a **pattern** over S. A pattern is said to be **finite** if it is defined over a finite subset of $\mathbb{Z}^d$. A pattern q over S is said to be **extension** of the pattern p over T if $T \subset S$ and $q|_T = p$. The extension q is said to be proper extension if $T \cap Bd(S) = \varnothing$, where $Bd(S)$ denotes the boundary of S. It may be noted that any d-dimensional pattern can be visualized as an adjacent placement of some $(d-1)$-dimensional patterns. For $(d-1)$-dimensional patterns $B_1, B_2, \ldots, B_r$, let $B = [B_1 B_2 \ldots B_r]_i$ denote the d-dimensional pattern obtained by placing $B_1, B_2, \ldots, B_r$ adjacently in the i-th direction. We say that a pattern $C = [C_1 C_2 \ldots C_r]_i$ **overlaps progressively** with $B = [B_1 B_2 \ldots B_r]_i$ in the i-th direction if $B_2 B_3 \ldots B_r = C_1 C_2 \ldots C_{r-1}$.

For any two dimensional shift space X defined over alphabet $\mathcal{A}$, let $\mathcal{B}_{(M,N)}(X)$ denote the collection of all $M \times N$ patterns allowed for the shift space X. Then, $\beta_{(M,N)} : X \to (\mathcal{B}_{(M,N)}(X))^{\mathbb{Z}^2}$ defined as $(\beta_{[(M,N)]}(x))_{[(i,j)]} = x_{[i,i+M-1]\times[j,j+N-1]}$ is called (M, N)-higher block code. It can be proved that $\beta_{(M,N)}(X)$ is a shift space. Further, it may be noted that for any configuration c in the shift space X, any rectangular patterns of size $M \times N$ appearing in $\beta_{(M,N)}(c)$ placed adjacently (in any direction) overlap progressively (in that direction). A two dimensional shift space of finite type $X_{\mathcal{F}}$ is said to be (m, n)**-step shift** if it can be described by a forbidden set consisting of rectangles of size $(m+1) \times (n+1)$. If the shift space can be described by a forbidden set consisting of blocks of size $1 \times (m+1)$ or $(m+1) \times 1$, then the shift space $X_{\mathcal{F}}$ is called a m**-step** shift. Analogously, for $P = (P_1, P_2, \ldots, P_k) \in \mathbb{N}^k$, one can define $\mathcal{B}_P(X)$ as the collection of all $P_1 \times P_2 \times \ldots \times P_k$ patterns allowed for a d-dimensional shift space X. Then, $\beta_P : X \to (\mathcal{B}_P(X))^{\mathbb{Z}^k}$ defined as $(\beta_P(x))_{[(i_1,i_2,\ldots,i_k)]} = x_{[i_1,i_1+P_1-1]\times[i_2,i_2+P_2-1]\times\ldots\times[i_k,i_k+P_k-1]}$ is called $(P_1, P_2, \ldots, P_k)$**-higher block** code(or P-higher block code). Once again, it can be proved that $\beta_P(X)$ is a shift space and the results made for the two dimensional case extend analogously for a d-dimensional shift space.

Let $\mathcal{F}$ be a given set of finite patterns (possibly over different subsets of $\mathbb{Z}^d$) and let $X = \overline{\{x \in \mathcal{A}^{\mathbb{Z}^d} : \text{any pattern from } \mathcal{F} \text{ does not appear in } x\}}$. The set X defines a subshift of $\mathbb{Z}^d$ generated by set of forbidden patterns $\mathcal{F}$. If the shift space X can be generated by a finite set of finite patterns, we say that the shift space X is a **shift of finite type**. We say that a pattern is **allowed** if it is not an extension of any forbidden pattern. We denote the shift space generated by the set of forbidden patterns $\mathcal{F}$ by $X_{\mathcal{F}}$. Two forbidden sets $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be **equivalent** if they generate the same shift space, i.e. $X_{\mathcal{F}_1} = X_{\mathcal{F}_2}$. Refer [9, 11, 12, 6, 16] for details.

For multidimensional shifts of finite type, it is known that given a set of forbidden patterns, the non-emptiness problem for multidimensional shift spaces is undecidable [2]. In [6], the authors show that the sets of periods of multidimensional shifts of finite type are exactly the sets of integers of the complexity class NE. They also give characterizations for general sofic and effective subshifts. In [3], authors prove that a multidimensional shift of finite type has a power that can be realized as the same power of a tiling system. They show that the set of entropies of tiling systems equals the set of entropies of shifts of finite type. It is known that multidimensional shifts of finite type with positive topological entropy cannot be minimal[13]. In fact, if X is subshift of finite type with positive topological entropy, then X contains a subshift which is not of finite type, and hence contains infinitely many subshifts of finite type [13]. In [5], Hochman proved that $h \geq 0$ is the entropy of a $\mathbb{Z}^d$ effective dynamical system if and only if it is the lim inf of a recursive sequence of rational numbers. For two dimensional shifts, Lightwood proved that strongly irreducible shifts of finite type have dense set of periodic points [10]. In [15], the authors characterized a multidimensional shift of finite type using an infinite matrix. In [16], authors gave an algorithmic approach to address the non-emptiness problem for multidimensional shift space. They give an algorithm to generate the elements of the shift space using finite matrices. In the process, they prove that elements of d-dimensional shift of finite type can be characterized by a sequence of finite matrices.

Let G be a graph with finite set of vertices V and finite set of edges E. It can be seen that the set of bi-infinite walks over a graph is a 1-step one dimensional shift of finite type. Also, for any given shift of finite type X, there exists a higher block shift (conjugate to X) which can be generated by a finite graph G. Consequently, every one dimensional shift of finite type can be visualized as a shift generated from some graph [9, 11]. For a d-dimensional graph $G =$

$(G_1, G_2, \ldots, G_d)$ (where $G_i = (V, E_i)$), let X_G denote the d-dimensional shift space where i-th graph determines the compatibility of the vertices in the i-th direction.

In this paper, we investigate the relation between the structure of such a shift space and the structure of generating graphs G_i. In particular, we answer some of the questions relating the structure of the underlying graphs with the non-emptiness problem of the shift space and existence of periodic points. For example, can every shift of finite type X be generated by a finite set of graphs? When does a given collection $\{G_1, G_2, \ldots, G_d\}$ of graphs generate a non-empty shift space? When does a multidimensional shift generated by $\{G_1, G_2, \ldots, G_d\}$ exhibit periodic points? Does existence of periodicity in one direction ensure the periodicity in other directions? While the non-emptiness problem and existence of periodic points are known to be undecidable, we derive sufficient conditions under which these questions can be answered (in the affirmative or otherwise). While answering some of these questions, the authors were unaware of some of the existing works in this area and consequently ended up establishing some of the known results in the literature (Propositions 2.1, 2.4, 2.6, 2.12 [7, 12, 6, 8]). However, as the results were derived independently, we include the proofs in the article below. We now give answers to some of these questions relating the multidimensional shift space and the generating set of graphs.

2. Main Results

Throughout this paper we assume that none of the generating graphs have stranded vertices (equivalently, none of the generating matrices contain a zero row or zero column).

Proposition 2.1. *For any two dimensional one step shift of finite type X, there exists a two dimensional graph G such that $X = X_G$.*

Proof. Let X be a two dimensional one step shift of finite type over the finite alphabet set $\mathcal{A}$. As X is one step, X is generated by a forbidden set $\mathcal{F}$ such that any element of $\mathcal{F}$ is of the form $\substack{b\\a}$ or ab (where $a, b \in \mathcal{A}$). Define a graph H (V) with $\mathcal{A}$ as the set of vertices and $\exists$ a directed edge from vertex a to vertex b in H (V) if and only if ab $(\substack{b\\a})$ does not belong to $\mathcal{F}$. Then, as $G = (H, V)$ is a two dimensional graph that captures horizontal and vertical compatibility of the elements of $\mathcal{A}$, G generates any arbitrary element of X. Consequently, $X = X_G$ and the proof is complete. □

Remark 2.2. The above result establishes that any two dimensional one step shift of finite type can be generated by a two dimensional graph. It may be noted that for a d-dimensional one step shift of finite type X, if H_i is the graph that captures the compatibility of the symbols in the i-th direction, then similar arguments establish that $G = (H_1, H_2, \ldots, H_k)$ generates an arbitrary element of X (and conversely) and thus the above result holds for any higher dimensional one step shift. Further, note that as slicing any given configuration in patterns of size $M \times N$ at each $(r,s) \in \mathbb{Z}^2$ (and placing it at each $(r,s) \in \mathbb{Z}^2$) yields an element of $(\mathcal{B}_{(M,N)}(X))^{\mathbb{Z}^2}$. The correspondence is natural and defines a conjugacy between X and $X^{(M,N)}$. Further, if X is a d-dimensional shift space and $P \in \mathbb{N}^k$, then slicing any configuration in X in patterns of size P at each point in $\mathbb{Z}^k$ (and placing the slice at each point in $\mathbb{Z}^k$) extends the above result for a d-dimensional shift space and hence we get the following results.

Corollary 2.3. *For any d-dimensional one step shift of finite type X, there exists a d-dimensional graph G such that $X = X_G$.*

Proof. The proof follows from discussions in Remark 2.2. □

Proposition 2.4. *For any two dimensional shift space $X_{\mathcal{F}}$, $X^{(M,N)}$ is a shift space conjugate to $X_{\mathcal{F}}$.*

Proof. Let $X_{\mathcal{F}}$ be a shift space generated by the forbidden set $\mathcal{F}$ and let $(M,N) \in \mathbb{N}^2$. Let $\mathcal{F}^*$ be the set obtained by replacing any forbidden pattern P of size less than size $M \times N$ by all $M \times N$ extensions of P. Then, $X_{\mathcal{F}} = X_{\mathcal{F}^*}$ and hence we obtain a modified forbidden set generating $X_{\mathcal{F}}$ such that all the forbidden patterns in the generating forbidden set are bigger than a rectangle of size $M \times N$. Further, as all the forbidden patterns can be extended to rectangles of uniform size to generate the same space, we assume all the elements of the forbidden set to be rectangles of size $R \times S$ (for some integers $R, S \in \mathbb{N}$).

For any $P \in F$, define $P^{(M,N)}$ to be a pattern of size $(R-M+1)\times(S-N+1)$ over $(\mathcal{B}_{(M,N)}(X))$ defined as $P^{(M,N)}_{[(k,l)]} = P_{[k,k+M-1]\times[l,l+N-1]}$, i.e. the $M \times N$ rectangle with left bottom corner at (k,l) is placed at (k,l). Let $\mathcal{F}_1 = \{P^{(M,N)} : P \in \mathcal{F}\}$. Further, let $\mathcal{F}_2 = \{P_1P_2 : P_1, P_2 \in \mathcal{A}_X^{[M,N]}$ such that P_1 and P_2 do not overlap progressively horizontally$\}$ and let $\mathcal{F}_3 = \{{}^{P_2}_{P_1} : P_1, P_2 \in \mathcal{A}_X^{[M,N]}$ such that P_1 and P_2 do not overlap progressively vertically$\}$.

Note that as elements of $\mathcal{F}$ are forbidden for X, elements of $\mathcal{F}_1$ are not allowed for $X^{(M,N)}$. Also, as any two blocks placed adjacently for $X^{(M,N)}$ must overlap progressively, elements of $\mathcal{F}_2$ and $\mathcal{F}_3$ are also not allowed for $X^{(M,N)}$ and thus $X^{(M,N)} \subset \cap_{i=1}^3 X_{\mathcal{F}_i}$ or $X^{(M,N)} \subseteq X_{\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3}$. Conversely, for any element x in $X_{\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3}$, as adjacent placement of blocks not overlapping progressively is not allowed, any two adjacent blocks overlap progressively. Further, as elements of $\mathcal{F}_1$ are forbidden for $X_{\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3}$, any block forbidden for X does not appear in x. Consequently, $x \in X^{(M,N)}$ and the proof for $X^{(M,N)} = X_{\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3}$ is complete.

Further, for any $x \in X$ as $((\beta_{(M,N)})(x))_{(i,j)}$ is a $M \times N$ pattern of x with left corner at $x_{(i,j)}$, the map $\beta_{(M,N)}$ defines a conjugacy between shift space X and $X^{(M,N)}$. □

Corollary 2.5. *For any d-dimensional shift space $X_{\mathcal{F}}$ and $P \in \mathbb{N}^k$, X^P is a shift space conjugate to $X_{\mathcal{F}}$.*

Proof. The proof follows from discussions in Remark 2.9. □

Proposition 2.6. *For any two dimensional shift space of finite type $X_{\mathcal{F}}$, there exists a graph G such that X_G is conjugate to $X_{\mathcal{F}}$.*

Proof. Let $X_{\mathcal{F}}$ be a shift space of finite type generated by the forbidden set $\mathcal{F}$. If all the elements of $\mathcal{F}$ are of type $\{\alpha\beta\}$ or $\{\substack{\alpha \\ \beta}\}$, then $X_{\mathcal{F}}$ is one step shift of finite type. If not, let all the elements of $\mathcal{F}$ be rectangles of size $M \times N$. By previous proposition, since $X^{[M,N]}$ can be viewed as one step of finite type over alphabet $\mathcal{A}_X^{[(M,N)]}$, shift space $X_{\mathcal{F}}$ can be visualized as one step shift of finite type. For $\mathcal{V} = \mathcal{B}_{(M,N)}(X)$, define the graph $H_1 = (\mathcal{V}, E_1)$ as a graph with set of vertices $\mathcal{V}$ where any two elements of $\mathcal{V}$ are connected if they overlap progressively horizontally. Let $H_2 = (\mathcal{V}, E_2)$ be the graph with $\mathcal{V}$ as the set of vertices where any two elements of $\mathcal{V}$ are connected if they overlap progressively vertically. Then $G = (H_1, H_2)$ generates $X^{(M,N)}$ and the proof is complete. ☐

Corollary 2.7. *For every d-dimensional shift of finite type $X_{\mathcal{F}}$, there exists a d-dimensional graph G such that X_G is conjugate to $X_{\mathcal{F}}$.*

Proof. The proof follows from discussions in Remark 2.9. ☐

Example 2.8. Let X be two dimensional shift space with alphabet $\{e, f, g\}$ with forbidden pattern set $\mathcal{F} = \{ff, gg, fe, eg, \substack{f \\ f}, \substack{e \\ e}, \substack{g \\ g}, \substack{e \\ f}, \substack{g \\ e}\}$. Then, graph G for this shift space is given by Figure 1.

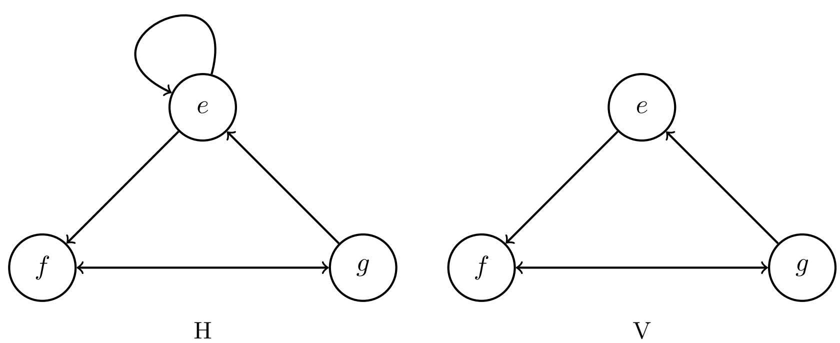

FIGURE 1

Then, as there exists 2×2 patterns (for example $\begin{matrix} g & e & f \\ f & g & e \\ e & f & g \end{matrix}$) whose infinite repetition (in both directions) tiles the plane in an allowed manner, the shift space is non-empty and exhibits periodic points. Further for any given configuration, as there exists arbitrarily large central blocks whose infinite repetition (in both directions) generates an element of X, the shift space exhibits a dense set of periodic points.

Remark 2.9. The above result establishes that any two dimensional shift is conjugate to its higher block code $X^{(M,N)}$ and hence any shift of finite type can be visualized as a one step shift of finite type. Further, as any one step shift of finite type can be generated through a graph, any two dimensional shift of finite type is conjugate to a shift generated by a two dimensional graph $G = (H, V)$. Also, as any d-dimensional one step shift of finite type can be generated through a d-dimensional graph (Corollary 2.3), any d-dimensional shift of finite type is conjugate to a shift generated by a d-dimensional graph. Consequently, an analogous extension of the above result is true and we get the following results.

Example 2.10. Let X be two dimensional Golden Mean shift space over alphabet $\{0, 1\}$ with forbidden pattern set $\mathcal{F} = \{11, \begin{smallmatrix}1\\1\end{smallmatrix}\}$. Then, $X = X_G$, where graph G is given by Figure 2.

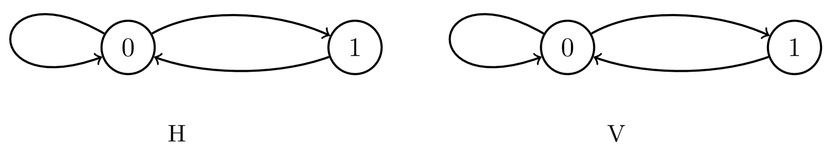

Figure 2

Then, appearance of two consecutive 1's is forbidden in horizontal and vertical directions. As the configuration comprising of all 0's is a valid element of X, the shift space X is indeed non-empty. Note that any allowed 2×2 pattern can be extended to a valid repetition of the shift space X. Once again, as there exists arbitrarily large central blocks whose infinite repetition (in both directions) generates an element of X, the shift space exhibits a dense set of periodic points (with finite orbits).

Remark 2.11. The above construction provides an example of a shift of finite type with dense set of periodic points. It may be noted that any (m, n) periodic point (with $m, n \neq 0$) can be realized as a vertical arrangement of shifts of an infinite horizontal strip of height n. As the number of blocks of a given finite size are finite, existence of periodic points is equivalent to existence of horizontally periodic points for a shift of finite type. As the periodic point generated is also vertically periodic, the proof establishes equivalence of existence of periodic points with existence of vertically periodic points for a shift of finite type. We now establish our claims below.

Proposition 2.12. *For any 2-dimensional shift of finite type X, X has a horizontally periodic point if and only if X has a (m, n) periodic point (for some $m, n \in \mathbb{Z} \setminus \{0\}$).*

Proof. As any shift of finite type is conjugate to some one-step shift of finite type, we establish our result for any one step 2-dimensional shift space. Let

X be a one step shift of finite type and let $x \in X$ be a $(m, 0)$ periodic point. Then, note that x is a infinite horizontal repetition of an infinite vertical strip of width m (say $\mathbb{S}$). Further as $\mathbb{S}$ can be realized as a vertical arrangement of one dimensional strips of length m, there exists a $1 \times m$ block $a_1 a_2 \dots a_m$ which appears twice in $\mathbb{S}$ (say at heights u and v). Consequently, infinite repetition of the block $x_{[0,m-1]\times[u,v-1]}$ is an element of X and is periodic of period $(m, v-u)$.

Conversely, if X has a (m, n) periodic point then there exists an infinite (horizontal) strip $\mathbb{S}$ such that x is a vertical arrangement of shifts of $\mathbb{S}$ (where $\sigma^{(-m,0)}(\mathbb{S}), \mathbb{S}, \sigma^{(m,0)}(\mathbb{S}), \sigma^{(2m,0)}(\mathbb{S}), \dots$ are placed vertically one over the other to obtain x). As the blocks of size $m \times n$ are finite, there exists a block B_0 of size $m \times n$ that appears in x at $(u, 0)$ and $(v, 0)$. Consequently, if $B_0 B_1 \dots B_k B_0$ is a block appearing in X then the $k \times k$ rectangular arrangement of $B_0, B_1, \dots, B_k$ where $B_{(k-j+i+1)\text{mod}(k+1)}$ is placed at (i, j)-th position is an allowed rectangular block. Further, as infinite repetition of the block generated yields an allowed configuration of X, the shift space exhibits a horizontally periodic point and the proof is complete. □

Corollary 2.13. *For any 2-dimensional shift of finite type X, X has a vertically periodic point if and only if X has a (m, n) periodic point (for some $m, n \in \mathbb{Z} \setminus \{0\}$).*

Proof. The proof follows from discussions in Remark 2.11. □

Proposition 2.14. *A two dimensional shift of finite type is finite if and only if it is conjugate to a shift generated by a pair of permutation matrices.*

Proof. Firstly note that any finite shift space is a union of finitely many periodic points (with finite orbits). Also, if X itself is a single periodic orbit then X can be visualized as an infinite repetition (both horizontal and vertical) of an $m \times n$ rectangle. Then, if H and V are indexed with allowed rectangles of size $m \times n$ capturing horizontal and vertical compatibility of the indices then H and V are permutation matrices and the graph $G = (H, V)$ generates a shift conjugate to the shift space X. Finally if X is a union of periodic orbits, a similar argument applied to each periodic orbit (and collating the set of indices to generate H and V) generates a pair of permutation matrices that generate a shift conjugate to X and the proof of forward part is complete.

Conversely, let the shift be conjugate to a shift generated by a pair of permutation matrices. Note that for the shift generated by permutation matrices, fixing the entry at the origin fixes the entries in the immediate neighborhood and hence fixes all the entries at other coordinates. Consequently, the shift space X generated is finite and the proof is complete. □

Remark 2.15. The above result establishes that a two dimensional shift space is finite if and only if it is conjugate to a shift generated by a pair of permutation matrices. In [1], the authors proved that if all the elements of the shift space are periodic then the shift space must be finite. The above result not only provides an alternate view of the result established in [1] but also characterizes such spaces in terms of shift spaces generated through permutation matrices. However, it is worth mentioning that the finiteness of the shift space X_G does not enforce the generating matrices H and V to be permutation matrices. To establish our claim, let X be the shift space generated by the graph shown in Figure 3. Then, it can be seen that although the shift generated by the graph is finite, the associated adjacency matrices H and V are not permutation matrices and hence the claim is indeed true.

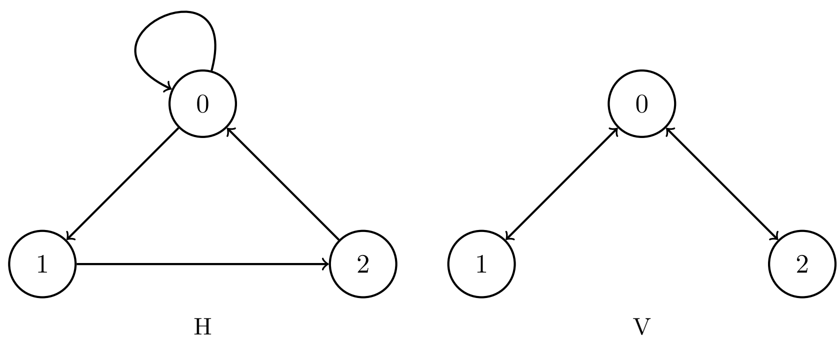

FIGURE 3

$$H = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{array} \qquad V = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

But X_G is finite as it is the orbit of a single periodic point (given below):

$$\begin{array}{cccccccccccccccccc}
\ldots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\
\ldots & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & \ldots \\
\ldots & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & \ldots \\
\ldots & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & \ldots \\
\ldots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots
\end{array}$$

Consequently, finiteness of the shift space X does not guarantee the generating matrices to be permutation matrices. However, it is worth mentioning that although a finite shift space may be generated by non-permutation matrices, such a space is always conjugate to a shift generated by permutation matrices.

Remark 2.16. We now discuss non-emptiness of shift spaces using adjacency matrices H and V. Note that while $(HV)_{ij}$ computes number of ways pattern $\begin{smallmatrix} & j \\ i & \end{smallmatrix}$ can be extended to triangular pattern of form $\begin{smallmatrix} & j \\ i & k \end{smallmatrix} \in \mathcal{B}(X_G)$, $(VH)_{ij}$ computes the number of ways pattern $\begin{smallmatrix} & j \\ i & \end{smallmatrix}$ can be extended to triangular pattern of form $\begin{smallmatrix} l & j \\ i & \end{smallmatrix} \in \mathcal{B}(X_G)$. As removing the vertices with no incoming (or outgoing) edge (both horizontally or vertically) does not alter the shift space generated, we assume that the generating matrices do not contain any zero row or zero column. It may be noted that for any shift generated by permutation matrices, as the immediate neighborhood of a symbol is uniquely determined, the shift space generated by permutation matrices is always finite (may be empty). It may be noted that a two dimensional shift space generated by a pair of irreducible permutation matrices is non-empty if and only if the generating matrices commute with each other. The proof follows from the fact that if $HV = VH$, any pattern of the form $\begin{smallmatrix} & c \\ a & b \end{smallmatrix}$ can be extended to a 2×2 square and hence shift space generated is non-empty. We now establish our claim below.

Proposition 2.17. *Let G be graph with associated adjacency matrices H and V. If H and V are irreducible permutation matrices then $HV = VH$ iff $X_G \neq \varnothing$.*

Proof. Let X_G be the shift space generated by $G = (H, V)$ and let $HV = VH$. If H and V are permutation matrices then fixing an entry at origin uniquely determines the immediate neighbors of any symbol (in both horizontal and vertical directions). Further as HV and VH are permutation matrices (characterizing blocks of form $\begin{smallmatrix} & b \\ a & * \end{smallmatrix}$ and $\begin{smallmatrix} * & b \\ a & \end{smallmatrix}$ respectively), any block $\begin{smallmatrix} & b \\ a & * \end{smallmatrix}$ can be extended to a 2×2 square if and only if $HV = VH$. Consequently, for any element $x = \overline{x_1 x_2 \dots x_r}$ generated by H, as $x_1 x_2 \dots x_r x_1$ can be extended vertically (as $HV = VH$ holds), the element x can be extended (vertically) to a valid configuration in X_G. Thus, X_G is non-empty and the proof of forward part is complete.

Conversely, let X_G be non-empty. As H (and V) is an irreducible permutation matrix, the one dimensional shift space generated by H (V) comprises of a single periodic orbit. As the choice of extension (vertical) for any index is unique, if $HV \neq VH$, the periodic point generated by H cannot be extended vertically to a valid configuration and hence $X_G = \varnothing$. Consequently, $HV = VH$ and the proof of converse is complete. $\square$

Remark 2.18. Note that if $(HV)_{ij} \neq 0 \Leftrightarrow (VH)_{ij} \neq 0 \ \ \forall i, j$ then the shift space is non-empty and hence a more general form of the above result is true. In fact, note that if $(HV)_{ij} \neq 0 \implies (VH)_{ij} \neq 0$ (or $(VH)_{ij} \neq 0 \implies (HV)_{ij} \neq 0$), shift space generated does not contain any forbidden pattern of the form $\begin{smallmatrix} & c \\ a & b \end{smallmatrix}$ (or $\begin{smallmatrix} a & b \\ c & \end{smallmatrix}$). As the condition ensures that every pattern of the form $\begin{array}{cc} & c \\ a & b \end{array}$ is extendable to 2×2 square, the condition ensures existence of arbitrarily large valid patterns for the shift space and thus the shift space is non-empty under the imposed condition. However, once again the condition provides a sufficient criteria to establish non-emptiness of a shift space and the shift space can be non-empty in absence of the imposed condition. Further, as HV^T and V^TH characterizes allowed patterns of the form $\begin{array}{cc} a & b \\ & c \end{array}$ and $\begin{array}{cc} a & \\ b & c \end{array}$ respectively, the non-emptiness problem and existence of periodic points can be investigated using the matrices HV^T and V^TH. It is worth mentioning that the two conditions are indeed independent and hence can be used independently to investigate the shift space under discussion. We now establish our claims below.

Proposition 2.19. *Let X_G be a shift space generated by $G = (H, V)$. If $(HV)_{ij} \neq 0 \Rightarrow (VH)_{ij} \neq 0$. Then $X_G \neq \varnothing$.*

Proof. The proof follows from the fact that if $(HV)_{ij} \neq 0 \Rightarrow (VH)_{ij} \neq 0$, the shift space does not contain forbidden pattern of the form $\begin{array}{cc} & c \\ a & b \end{array}$. Consequently, any arbitrarily large $1 \times r$ pattern can be extended to an $r \times r$ square. As the shift space contains valid arbitrarily large squares, the shift space is non-empty and the proof is complete. □

Example 2.20. Let X be a shift space generated by the graph in Figure 4.

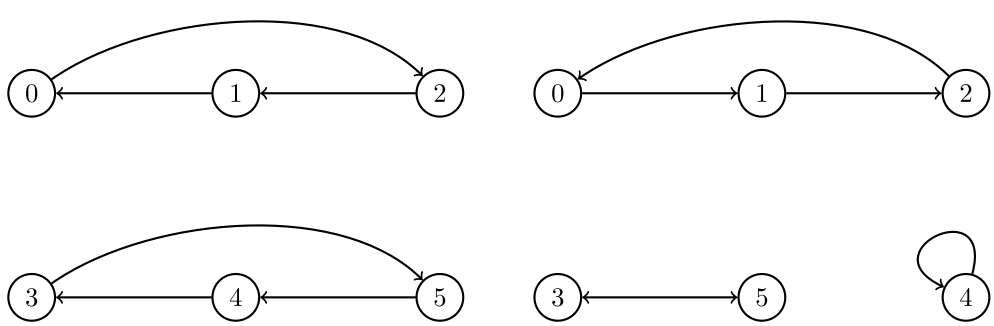

FIGURE 4

Then,

$$H = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \qquad V = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

Note that G can be written as a union of disjoint graphs G_1 and G_2 indexed by symbols $0, 1, 2$ and $3, 4, 5$ respectively. Further, while matrices capturing horizontal and vertical compatibility of G_1 commute, matrices capturing horizontal and vertical compatibility of G_2 do not commute and hence $X_{G_1} \neq \varnothing$ but $X_{G_2} = \varnothing$. Consequently, $X_G = X_{G_1}$ and the shift space is indeed non-empty. Thus, the shift space is generated by a non-commuting pair of permutation matrices.

Example 2.21. Let X be the shift space arising from graph in Figure 5 over symbol set $\{1, 2, 3\}$.

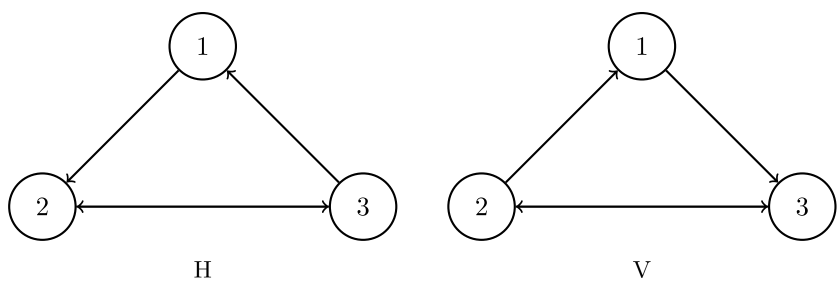

Figure 5

Then, generating matrices corresponding to given graph are:

$$H = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 1 & 0 \end{array} \qquad V = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & 0 \end{array}$$

Then,

$$HV = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 2 \end{array} \qquad VH = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 3 & 0 & 0 & 1 \end{array}$$

For the above example, one can find indices i, j such that of $(HV)_{ij} \neq 0$ but $(VH)_{ij} = 0$ (and indices k, l such that $(VH)_{kl} \neq 0$ but $(HV)_{kl} = 0$). Consequently, the condition $(HV)_{ij} \neq 0$ iff $(VH)_{ij} \neq 0$ does not hold and the derived results cannot be used to investigate the non-emptiness of the shift space. However, as $H = V^T$, we have $HV^T = V^T H$ and the shift space is indeed non-empty (and possesses periodic points)

Proposition 2.22. *Let X_G be a shift space generated by $G = (H, V)$. If $(HV)_{ij} \neq 0 \iff (VH)_{ij} \neq 0 \;\; \forall i, j$ then X_G possesses periodic points.*

Proof. Let X_G be a shift space generated by $G = (H, V)$ and let $m \in \mathbb{N}$. Let u be a block of size $1 \times m$. As $(HV)_{ij} \neq 0 \iff (VH)_{ij} \neq 0 \;\; \forall i, j$, u can extended to a pattern of size $k \times m$ (for any $k \in \mathbb{N}$). Without loss of generality, let u be extended to a rectangle v of size $s \times m$ such that $v_{00} = v_{ms}$. As $v_{00} = v_{ms}$, the block v can be further extended to the block $\begin{matrix} & v \\ v & \end{matrix}$ (along the line $sx - my = 0$) to obtain a valid pattern of X. Further, as $(HV)_{ij} \neq 0 \iff (VH)_{ij} \neq 0 \;\; \forall i, j$, the pattern can be extended to valid $2m \times 2s$ pattern for the shift space. Finally, note that the infinite such repetition of v (along the line $sx - my = 0$) and extending the pattern with the same choices (as in the previous step) yields a valid periodic point for the shift space. As the proof holds for any $m \in \mathbb{N}$, the shift space contains periodic points and the proof is complete. □

Remark 2.23. The above result establishes the existence of periodic points under the condition $(HV)_{ij} \neq 0 \iff (VH)_{ij} \neq 0 \;\; \forall i, j$. The proof uses the condition to extend a horizontal pattern to a valid $2m \times 2s$ pattern. As such a repetition can be made infinitely often, filling the choices in a unique manner at each step yields a periodic point for the shift space. Note that as such an extension is possible under $(HV)_{ij} \neq 0 \implies (VH)_{ij} \neq 0 \;\; \forall i, j$, the result holds under a weaker condition. Further, as similar arguments establish the result under the condition $(VH)_{ij} \neq 0 \implies (HV)_{ij} \neq 0 \;\; \forall i, j$, we get the following corollary.

Corollary 2.24. *Let X_G be a shift space generated by $G = (H, V)$. If $(HV)_{ij} \neq 0 \implies (VH)_{ij} \neq 0 \;\; \forall i, j$ (or $(VH)_{ij} \neq 0 \implies (HV)_{ij} \neq 0 \;\; \forall i, j$) then X_G possesses periodic points.*

Proof. The proof follows from discussions in Remark 2.23. □

Remark 2.25. Let X be a shift space generated by a graph G. It may be noted that if $(HV)_{ij} = 0$ then any block of the form $\begin{matrix} & j \\ i & * \end{matrix}$ is forbidden for the shift space X. Consequently, the set $\{(i, j) : (HV)_{ij} = 0 \text{ but } (VH)_{ij} \neq 0\}$ characterizes all patterns of the form $\begin{matrix} * & j \\ i & \end{matrix}$ which cannot be extended to a 2×2 square. Similarly, the set $\{(i, j) : (VH)_{ij} = 0 \text{ but } (HV)_{ij} \neq 0\}$ characterizes all patterns of the form $\begin{smallmatrix} & j \\ i & * \end{smallmatrix}$ which cannot be extended to a 2×2 square. As such patterns do not contribute towards generation of points of X, one may ignore such patterns (by simply considering them as invalid) and generate configurations in X_G using matrices of reduced complexity.

Let $A_1 = \{ \begin{smallmatrix} & c \\ a & b \end{smallmatrix} : \exists \ d \ \in \mathcal{V}(G) \ such \ that \begin{smallmatrix} d & c \\ a & b \end{smallmatrix} \in \mathcal{B}(X_G)\}$ and $A_2 = \{\begin{smallmatrix} y & z \\ x & \end{smallmatrix} : \exists \ w \in \mathcal{V}(G) \ such \ that \begin{smallmatrix} y & z \\ x & w \end{smallmatrix} \in \mathcal{B}(X_G)\}$. Let M and N be matrices indexed by elements of A_1 and A_2 in the following manner:

For $I = \begin{smallmatrix} & a_3 \\ a_1 & a_2 \end{smallmatrix}$, $J = \begin{smallmatrix} & a_5 \\ a_3 & a_4 \end{smallmatrix}$, $R = \begin{smallmatrix} b_2 & b_3 \\ b_1 & \end{smallmatrix}$ and $S = \begin{smallmatrix} b_4 & b_5 \\ b_3 & \end{smallmatrix}$

$$M_{IJ} = \begin{cases} 1, & if \ \begin{smallmatrix} a_3 & a_4 \\ a_2 & \end{smallmatrix} \in A_2 \\ 0, & otherwise \end{cases}$$

and $$N_{RS} = \begin{cases} 1, & if \ \begin{smallmatrix} & b_4 \\ b_2 & b_3 \end{smallmatrix} \in A_1 \\ 0, & otherwise \end{cases}$$

Remark 2.26. It may be noted that M and N generate valid "staircase patterns" for the shift space X_G. Further, as extension of a staircase pattern using complementary staircase patterns enables construction of arbitrarily large valid patterns for X_G, the same can be used to verify the non-emptiness of the shift space under discussion. We refer to the pair of indices (I, J) ($I = \begin{smallmatrix} & a_3 \\ a_1 & a_2 \end{smallmatrix}$ and $J = \begin{smallmatrix} a_4 & a_3 \\ a_1 & \end{smallmatrix}$) as an E-pair. It may be noted that if $I = \begin{smallmatrix} & a_3 \\ a_1 & a_2 \end{smallmatrix}$ and $J = \begin{smallmatrix} a_4 & a_3 \\ a_1 & \end{smallmatrix}$ form an E-pair then $\begin{smallmatrix} d & c \\ a & b \end{smallmatrix}$ is a valid pattern for the shift space. In such a case we refer $J \quad (I)$ as an E-partner of the pattern $I \quad (J)$. It may be noted that if for every $M_{ij} \neq 0$ and for every E- partner "i_1" of i, $\exists$ an E-partner "j_1" of j such that $N_{i_1 j_1} \neq 0$ then the shift space is non-empty. The proof follows from the fact that the imposed condition ensures the extension of compatible E-partners into a 3×3 square (and hence to a valid rectangular pattern of arbitrarily large size) and consequently ensures the non-emptiness of the shift space under discussion. It is worth mentioning that any shift space satisfying the conditions imposed in the above proposition must possess periodic points. A similar argument proves that if or every $N_{kl} \neq 0$ and for every E- partner "k_1" of k, $\exists$ an E-partner "l_1" of l such that $M_{k_1 l_1} \neq 0$ then the shift space is non-empty and hence we get the following results.

Proposition 2.27. *Let X_G be a two dimensional shift generated by a graph $G = (H, V)$ and let the sequence space generated by M and N be non-empty. If for every $M_{ij} \neq 0$ and for every E-partner "i_1" of i, $\exists$ an E-partner "j_1" of j such that $N_{i_1 j_1} \neq 0$ then, $X_G \neq \varnothing$.*

Proof. Let X_G be a shift of finite type such that the sequence spaces generated by M and N are non-empty. Let $M_{ij} \neq 0$ and let i_1 be an E-partner of i. If there exists an E-partner "j_1" of j such that $N_{i_1 j_1} \neq 0$ then the pattern $\begin{smallmatrix} & a_3 & a_5 \\ a_1 & a_2 & a_4 \end{smallmatrix}$ can be extended to a 3×3 pattern. As the shift spaces generated by M

and N respectively (generating valid staircase patterns for X_G) are non-empty, any finite pattern generated by M can be extended to an allowed rectangle of arbitrarily large size and hence can be extended to a point in the shift space X_G. Consequently, X_G is non-empty and the proof is complete. □

Remark 2.28. The above result establishes the non-emptiness of the shift space using the notion of an E-pair. In particular, the proof establishes that if for every $M_{ij} \neq 0$ and for every E- partner "i_1" of i, ∃ an E-partner "j_1" of j such that $N_{i_1 j_1} \neq 0$ then the shift space is non-empty. It may be noted that the proof ensures the extension of compatible E-partners into a 3×3 square and hence the condition " for every $M_{ij} \neq 0$ and for every E-pair "i_1" of i, ∃ an E-pair "j_1" of j such that $N_{i_1 j_1} \neq 0$" is sufficient (but not necessary) to ensure non-emptiness of the shift space. It is worth mentioning that any shift space satisfying the conditions imposed in the above proposition must possess periodic points. A similar argument proves that if for every $N_{kl} \neq 0$ and for every E- partner "k_1" of k, ∃ an E-partner "l_1" of l such that $M_{k_1 l_1} \neq 0$ then the shift space is non-empty and hence we get the following corollary.

Corollary 2.29. *Let X_G be a two dimensional shift generated by a graph $G = (H, V)$ and let the sequence space generated by M and N be non-empty. If for every $N_{kl} \neq 0$ and for every E- partner "k_1" of k, ∃ an E-partner "l_1" of l such that $M_{k_1 l_1} \neq 0$ then $X_G \neq \varnothing$.*

Proof. The proof follows from discussions in Remark 2.28 □

Proposition 2.30. *A shift space X_G is finite if it follows two conditions:*

(1) *M and N are permutation matrices.*
(2) *Every pattern in A_1 and A_2 has unique E-partner.*

Proof. Let X be a shift space and let (1) and (2) hold. Firstly, it may be noted that as M and N are permutation matrices, the shift spaces generated by M and N (respectively) are finite (union of periodic orbits). Further, as every triangular pattern is uniquely extendable to a 2×2 pattern, every infinite pattern generated by M is uniquely extendable to an element of the shift space. Consequently, the shift space is finite and the proof is complete. □

Remark 2.31. The above proposition establishes a sufficient condition for the shift space X_G to be finite. In particular, the result establishes that if M and N are permutation matrices and every pattern in A_1 (A_2) has a unique E-partner then the shift space X_G must be finite. However, the conditions once again provide a sufficient criteria to establish the finiteness of a shift space and are not necessary for the shift space to be finite. We now give an example in support of our claim.

Example 2.32. Let X_G be a shift space over the alphabet set $\mathcal{A} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ generated by the adjacency matrices:

$$H = \begin{array}{c|cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \quad \text{and} \quad V = \begin{array}{c|cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

It can be seen that X_G contains two disjoint periodic orbits generated by the following configurations:

$$\begin{array}{ccccccccccccccccc} \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & \dots \\ \dots & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & \dots \\ \dots & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & \dots \\ \dots & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & \dots \\ \dots & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

and

$$\begin{array}{cccccccccccccccccc} \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & 4 & 5 & 1 & 2 & 4 & 5 & 1 & 2 & 4 & 5 & 1 & 2 & 4 & 5 & 1 & 2 & \dots \\ \dots & 6 & 7 & 8 & 3 & 6 & 7 & 8 & 3 & 6 & 7 & 8 & 3 & 6 & 7 & 8 & 3 & \dots \\ \dots & 4 & 5 & 1 & 2 & 4 & 5 & 1 & 2 & 4 & 5 & 1 & 2 & 4 & 5 & 1 & 2 & \dots \\ \dots & 6 & 7 & 8 & 3 & 6 & 7 & 8 & 3 & 6 & 7 & 8 & 3 & 6 & 7 & 8 & 3 & \dots \\ \dots & 4 & 5 & 1 & 2 & 4 & 5 & 1 & 2 & 4 & 5 & 1 & 2 & 4 & 5 & 1 & 2 & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

Further, as any $1 \times n$ patterns containing the blocks 4512 and 123 (6783 and 123) cannot be extended to a valid configuration (cannot be extended vertically to height more than 2), X_G is the union of two disjoint periodic trajectories and hence is finite. However, as ${}_{1}{}^{3}_{2}$ can be extended to 2×2 square in a non-unique manner and ${}_{1}{}^{3}_{2}$ can be extended diagonally in non-unique manner, none of the conditions stated in proposition 2.30 hold. Consequently, the shift space may be finite even when none of the conditions in Proposition 2.30 hold.

We now give examples to show that shift space may not be finite if any of the above two conditions are dropped.

Example 2.33. Let X be the shift space arising from Figure 6. Then, the adjacency matrices associated with graph are:

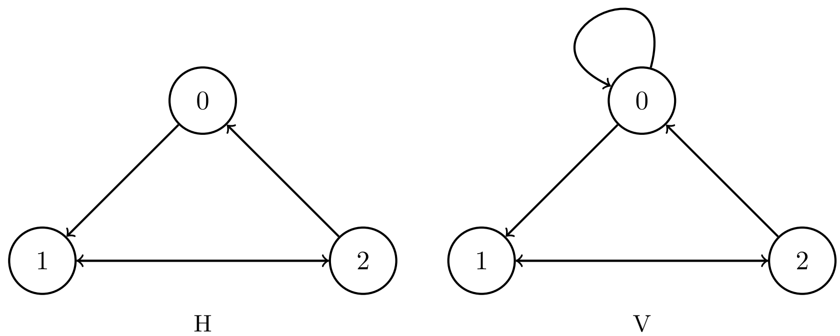

Figure 6

$$H = \begin{array}{cc} & \begin{matrix}0&1&2\end{matrix} \\ \begin{matrix}0\\1\\2\end{matrix} & \begin{pmatrix}0&1&0\\0&0&1\\1&1&0\end{pmatrix}\end{array} \qquad V = \begin{array}{cc} & \begin{matrix}0&1&2\end{matrix} \\ \begin{matrix}0\\1\\2\end{matrix} & \begin{pmatrix}1&1&0\\0&0&1\\1&1&0\end{pmatrix}\end{array}$$

Then,

$$HV = \begin{array}{cc} & \begin{matrix}0&1&2\end{matrix} \\ \begin{matrix}0\\1\\2\end{matrix} & \begin{pmatrix}0&0&1\\1&1&0\\1&1&1\end{pmatrix}\end{array} \qquad VH = \begin{array}{cc} & \begin{matrix}0&1&2\end{matrix} \\ \begin{matrix}0\\1\\2\end{matrix} & \begin{pmatrix}0&1&1\\1&1&0\\0&1&1\end{pmatrix}\end{array}$$

Note that there exists indices i, j such that $(HV)_{ij} \neq 0$ but $(VH)_{ij} = 0$ (and there exists k, l such that $(VH)_{kl} \neq 0$ but $(HV)_{kl} = 0$). Updating the matrices HV and VH we obtain

$$HV = \begin{array}{cc} & \begin{matrix}0&1&2\end{matrix} \\ \begin{matrix}0\\1\\2\end{matrix} & \begin{pmatrix}0&0&1\\1&1&0\\0&1&1\end{pmatrix}\end{array} \qquad VH = \begin{array}{cc} & \begin{matrix}0&1&2\end{matrix} \\ \begin{matrix}0\\1\\2\end{matrix} & \begin{pmatrix}0&0&1\\1&1&0\\0&1&1\end{pmatrix}\end{array}$$

Using above matrices, we obtain:
$\mathcal{A}_1 = \{ \begin{smallmatrix} & 2 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} & 0 \\ 1 & 2 \end{smallmatrix}, \begin{smallmatrix} & 1 \\ 1 & 2 \end{smallmatrix}, \begin{smallmatrix} & 1 \\ 2 & 0 \end{smallmatrix}, \begin{smallmatrix} & 2 \\ 2 & 1 \end{smallmatrix} \}$ and $\mathcal{A}_2 = \{ \begin{smallmatrix} 1 & 2 \\ 0 & \end{smallmatrix}, \begin{smallmatrix} 2 & 0 \\ 1 & \end{smallmatrix}, \begin{smallmatrix} 2 & 1 \\ 1 & \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 2 & \end{smallmatrix}, \begin{smallmatrix} 1 & 2 \\ 2 & \end{smallmatrix} \}$

It can be verified that every element of $\mathcal{A}_1$ and $\mathcal{A}_2$ can be extended to 1×1 square uniquely and hence condition (2) holds. The matrices M and N are :

$$M = \begin{array}{c} \\ \begin{smallmatrix} & 2 \\ 0 & 1 \end{smallmatrix} \\ \begin{smallmatrix} & 0 \\ 1 & 2 \end{smallmatrix} \\ \begin{smallmatrix} & 1 \\ 1 & 2 \end{smallmatrix} \\ \begin{smallmatrix} & 1 \\ 2 & 0 \end{smallmatrix} \\ \begin{smallmatrix} & 2 \\ 2 & 1 \end{smallmatrix} \end{array} \begin{array}{c} \begin{array}{ccccc} \begin{smallmatrix} & 2 \\ 0 & 1 \end{smallmatrix} & \begin{smallmatrix} & 0 \\ 1 & 2 \end{smallmatrix} & \begin{smallmatrix} & 1 \\ 1 & 2 \end{smallmatrix} & \begin{smallmatrix} & 1 \\ 2 & 0 \end{smallmatrix} & \begin{smallmatrix} & 2 \\ 2 & 1 \end{smallmatrix} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

$$N = \begin{array}{c} \\ \begin{smallmatrix} 1 & 2 \\ 0 & \end{smallmatrix} \\ \begin{smallmatrix} 2 & 0 \\ 1 & \end{smallmatrix} \\ \begin{smallmatrix} 2 & 1 \\ 1 & \end{smallmatrix} \\ \begin{smallmatrix} 0 & 1 \\ 2 & \end{smallmatrix} \\ \begin{smallmatrix} 1 & 2 \\ 2 & \end{smallmatrix} \end{array} \begin{array}{c} \begin{array}{ccccc} \begin{smallmatrix} 1 & 2 \\ 0 & \end{smallmatrix} & \begin{smallmatrix} 2 & 0 \\ 1 & \end{smallmatrix} & \begin{smallmatrix} 2 & 1 \\ 1 & \end{smallmatrix} & \begin{smallmatrix} 0 & 1 \\ 2 & \end{smallmatrix} & \begin{smallmatrix} 1 & 2 \\ 2 & \end{smallmatrix} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

It can be seen that M and N are not permutation matrices and the shift space X is not finite. Consequently, the Proposition 2.30 does not hold if M and N are not ensured to be permutation matrices.

Example 2.34. Let X be the shift space arising from graph G in Figure 7. Then, the adjacency matrices corresponding to the graph G are:

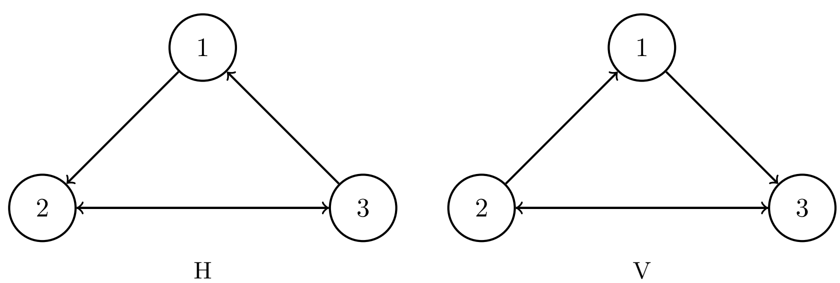

Figure 7

$$H = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array} \qquad V = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

Further,

$$HV = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \end{array} \qquad VH = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Once again, note that there exists indices i, j such that $(HV)_{ij} \neq 0$ but $(VH)_{ij} = 0$ (and there exists k, l such that $(VH)_{kl} \neq 0$ but $(HV)_{kl} = 0$). Updating the matrices HV and VH we obtain

$$HV = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{array} \qquad VH = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Consequently,
$\mathcal{A}_1 = \{ \begin{smallmatrix} & 1 \\ 1 & 2 \end{smallmatrix}, \begin{smallmatrix} & 2 \\ 2 & 3 \end{smallmatrix}, \begin{smallmatrix} & 3 \\ 3 & 1 \end{smallmatrix}, \begin{smallmatrix} & 3 \\ 3 & 2 \end{smallmatrix} \}$ and $\mathcal{A}_2 = \{ \begin{smallmatrix} 3 & 1 \\ 1 & \end{smallmatrix}, \begin{smallmatrix} 3 & 2 \\ 2 & \end{smallmatrix}, \begin{smallmatrix} 1 & 2 \\ 2 & \end{smallmatrix}, \begin{smallmatrix} 2 & 3 \\ 3 & \end{smallmatrix} \}$

and

$$M = \begin{array}{c c} & \begin{array}{cccc} \begin{smallmatrix} & 1 \\ 1 & 2 \end{smallmatrix} & \begin{smallmatrix} & 2 \\ 2 & 3 \end{smallmatrix} & \begin{smallmatrix} & 3 \\ 3 & 1 \end{smallmatrix} & \begin{smallmatrix} & 3 \\ 3 & 2 \end{smallmatrix} \end{array} \\ \begin{array}{c} \begin{smallmatrix} & 1 \\ 1 & 2 \end{smallmatrix} \\ \begin{smallmatrix} & 2 \\ 2 & 3 \end{smallmatrix} \\ \begin{smallmatrix} & 3 \\ 3 & 1 \end{smallmatrix} \\ \begin{smallmatrix} & 3 \\ 3 & 2 \end{smallmatrix} \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$N = \begin{array}{c c} & \begin{array}{cccc} \begin{smallmatrix} 3 & 1 \\ 1 & \end{smallmatrix} & \begin{smallmatrix} 3 & 2 \\ 2 & \end{smallmatrix} & \begin{smallmatrix} 1 & 2 \\ 2 & \end{smallmatrix} & \begin{smallmatrix} 2 & 3 \\ 3 & \end{smallmatrix} \end{array} \\ \begin{array}{c} \begin{smallmatrix} 3 & 1 \\ 1 & \end{smallmatrix} \\ \begin{smallmatrix} 3 & 2 \\ 2 & \end{smallmatrix} \\ \begin{smallmatrix} 1 & 2 \\ 2 & \end{smallmatrix} \\ \begin{smallmatrix} 2 & 3 \\ 3 & \end{smallmatrix} \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Clearly, M and N are permutation matrices but not every triangular pattern is getting extended uniquely to 2×2 pattern. It can be seen that the shift space generated is not finite and hence the shift space need not be finite if any of the two conditions are dropped.

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