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Strongly Lipschitz (ℓ_p, ℓ_q)-factorable mappings

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Abstract

In this paper we study the space of strongly Lipschitz (ℓ_p, ℓ_q)-factorable operators between metric spaces and a Banach spaces. In particular, a factorization of this class through ℓ_p and ℓ_q spaces is given. We show that this type of operators fits in the theory of composition α-Banach Lipschitz operator ideal. As a special case, we get a Lipschitz version of weakly p-nuclear operators.

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1. Introduction and background

The ideal of ℓ_p-factorable operators was introduced by Pietsch for $1 \leq p \leq \infty$ [19] (for $p = \infty$ see also Dazord [9]), and comprehensively examined recently by Kim [16] (called weakly p-nuclear operators). In [14], Jarchow introduced and studied the α-Banach ideal of (p, q)-operators $(1 \leq p \leq q \leq \infty)$, as a generalization of ℓ_p-factorable operators. Moreover, when $p = q$ we recover the ideal of ℓ_p-factorable operators. To avoid confusion, in this paper, we call it a (ℓ_p, ℓ_q)-factorable operators.

Farmer and Johnson [11] introduce the notion of p-summing operators for Lipschitz maps, prove their basic first properties and leave to interested readers a list of open problems. Since then, many works have appeared related to this

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class of Lipschitz mappings. In 2016 an axiomatic theory of Lipschitz operator ideals for Banach spaces-valued Lipschitz mappings was given by the first author, Rueda, Sánchez-Perez and Yahi [1] (see also [24]) and in [2] for Lipschitz operator ideals between pointed metric spaces). Very recently, several different operator ideals have been fruitfully generalized to the Lipschitz setting (see, for example [3, 4, 6, 8, 13, 17, 21, 22, 25] and the references therein). The present work is in this direction. We extend the class of (ℓ_p, ℓ_q)-factorable operators to Lipschitz operators.

The paper is organized as follows: after this introduction, where a preliminary discussion about the basics on Lipschitz mappings is given, in Section 2, we introduce and study the strongly Lipschitz (ℓ_p, ℓ_q)-factorable operator as a Lipschitz mapping between a pointed metric space and a Banach space by a representation of series and we prove a factorization theorem for these mappings through the spaces ℓ_p and ℓ_q. We show that the injective hull of strongly Lipschitz (ℓ_p, ℓ_q)-factorable operators can be expressed as a composition of the space of Lipschitz mappings and the class of quasi (ℓ_p, ℓ_q)-factorable linear operators. In particular, the Lipschitz α-Banach ideal (ℓ_p, ℓ_2)-factorable is injective for $1 \leq p \leq 2$. As an application, we compare our class with some well known Lipschitz operators. In Section 3, the notion of factorable Lipschitz quasi weakly (p, q)-nuclear operators is introduced. We present a characterization given by Lipschitz injective hull of strongly Lipschitz (ℓ_p, ℓ_q)-factorable operators. Also, we establish the relationship between our class and the class of Lipschitz (r, s)-summing operators. As a new result of this section, we characterize the adjoint operators of weakly p-compact operators.

Our notation is standard. X and Y will be pointed metric spaces with a base point denoted by 0 and metric will be denoted by d. We denote by B_X the closure of the ball centered at 0 with radius 1. Also, E and F will stand for Banach spaces over the same field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$) with dual spaces E^* and F^*. A Banach space E will be considered as pointed metric spaces with a base point 0 and distance $d(x, x') = \|x - x'\|$. With $Lip_0(X, Y)$ we denote the set of all Lipschitz mappings from X to Y such that maps 0 to 0 and we put

$$Lip(T) = \inf\{C > 0\colon d(T(x), T(x') \leq C d(x, x'));\ \forall x, x' \in X\}.$$

In particular, $Lip_0(X, E)$ is the Banach space of all Lipschitz mappings T from X to E that vanish at 0, under the Lipschitz norm $Lip(\cdot)$. When $E = \mathbb{K}$, $Lip_0(X, \mathbb{K})$ is denoted by $X^\#$ and it is called the Lipschitz dual of X. The space of all linear operators from E to F is denoted by $\mathcal{L}(E, F)$ and it is a Banach space with the usual supremum norm. It is clear that $\mathcal{L}(E, F)$ is a subspace of $Lip_0(E, F)$ and, in particular, E^* is a subspace of $E^\#$.

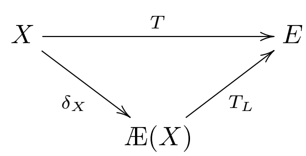
Let X be a metric space. A molecule on X is a scalar valued function m on X with finite support that satisfies $\sum\limits_{x \in X} m(x) = 0$. We denote by $\mathcal{M}(X)$ the linear space of all molecules on X. For $x, x' \in X$ the molecule $m_{xx'}$ is defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$, where χ_A is the characteristic function of the set A.

For $m \in \mathcal{M}(X)$ we can write $m = \sum_{j=1}^{n} \lambda_j m_{x_j x_j'}$ for some suitable scalars λ_j, and we write

$$\|m\|_{\mathcal{M}(X)} = \inf \left\{ \sum_{j=1}^{n} |\lambda_j| \, d(x_j, x_j'), \ m = \sum_{j=1}^{n} \lambda_j m_{x_j x_j'} \right\},$$

where the infimum is taken over all representations of the molecule m. Denote by Æ(X) the completion of the normed space $(\mathcal{M}(X), \|.\|_{\mathcal{M}(X)})$. This space was first introduced by Arens and Eells [5] in 1956. The terminology *Arens-Eells space* Æ(X) is due to Weaver [23]. The Arens-Eells space is also known as the Lipschitz-free Banach space of a metric space X.
A different approach to the preduality problem of $X^{\#}$ was taken with the next known result.

Theorem 1.1 ([23])**.** *Let X be a pointed metric space. Then there exist a Banach space Æ(X) and an isometric embedding $\delta_X : X \longrightarrow$ Æ$(X), \delta_X(x) = m_{x0}$ satisfying the following universal property: For each Banach space E and each map $T \in Lip_0(X, E)$, there is a unique operator $T_L \in \mathcal{L}($Æ$(X), E)$ such that $T_L \circ \delta_X = T$ that is, the diagram*



commutes, and $\|T_L\| = Lip(T)$. The correspondence $T \longleftrightarrow T_L$ establishes an isometric isomorphism between the Banach spaces $Lip_0(X, E)$ and $\mathcal{L}($Æ$(X), E)$. In particular, the spaces $X^{\#}$ and Æ$(X)^*$ are isometrically isomorphic.

Let $1 \leq p \leq \infty$, we write p^* the conjugate index of p, that is $1/p + 1/p^* = 1$. As usual, when $p = 1, p^* = \infty$.

Definition 1.2 ([6])**.** Let $(f_j)_j$ be a sequence in $X^{\#}$. The sequence $(f_j)_j$ is Lipschitz ω^*-p-summable if there is a constant C such that for all $n \in \mathbb{N}$ and for all $x, x' \in X$ we have

$$\left\| (f_j(x) - f_j(x'))_{j=1}^{n} \right\|_p \leq C d(x, x').$$

The smallest such constant C will be denoted by $\| (f_j)_j \|_p^{L,\omega^*}$ and $\ell_p^{L,\omega^*}(X^{\#})$ will denote the set of Lipschitz ω^*-p-summable sequences in $X^{\#}$. Clearly

$$\| (f_j)_j \|_p^{L,\omega^*} = \sup_{\substack{x, x' \in X \\ x \neq x'}} \frac{\left\| (f_j(x) - f_j(x'))_j \right\|_p}{d(x, x')}.$$

It is a well known result (see [6, Lemma 2.4 and Proposition 2.5]) that the canonical correspondence

$T \mapsto (\langle e_j^*, T(.)\rangle)_j$, provides an isometric isomorphism of $Lip_0(X,\ell_p)$ onto $\ell_p^{L,\omega^*}(X^\#)$ and

$$\left[\ell_p^{L,\omega^*}(X^\#), \|\cdot\|_p^{L,\omega^*}\right] = \left[\ell_p^{\omega}(X^\#), \|\cdot\|_p^{\omega}\right].$$

2. Strongly Lipschitz (ℓ_p,ℓ_q)-factorable mappings

Let $1 \le p \le q \le \infty$. Recall that a linear operator $T \in \mathcal{L}(E,F)$ is called (ℓ_p,ℓ_q)-factorable(see [14]) if T has a factorization $T = B \circ J_{p,q} \circ A$, where $A \in \mathcal{L}(E,\ell_p)$, $B \in \mathcal{L}(\ell_q, F)$ and $J_{p,q} : \ell_p \longrightarrow \ell_q$ is the canonical injection. The α-norm on the vector space $\mathcal{F}_{p,q}(E,F)$ of all (ℓ_p,ℓ_q)-factorable operators from E to F is defined by

$$\|T\|_{\mathcal{F}_{p,q}} := \inf\left\{\|A\|\|B\| : A \in \mathcal{L}(E,\ell_p), B \in \mathcal{L}(\ell_q,F) \text{ and } T = B\circ J_{p,q}\circ A\right\}.$$

It is shown in [14] that $(\mathcal{F}_{p,q}, \|\cdot\|_{\mathcal{F}_{p,q}})$ is a α-Banach operator ideal and that $T \in \mathcal{F}_{p,q}(E,F)$ if and only if T has a representation $T = \sum_{n=1}^{\infty} x_n^*(\cdot) y_n$, where $(x_n^*)_n \in \ell_p^{\omega}(E^*)$and $(y_n)_n \in \ell_{q^*}^{\omega}(F)$. Also, $\|T\|_{\mathcal{F}_{p,q}} = \inf\left\{\|(x_n^*)_n\|_p^{\omega}\|(y_n)_n\|_{q^*}^{\omega}\right\}$, with the infimum taken over all representations of T as above.
The notion of quasi (ℓ_p,ℓ_q)-factorable operators was introduced by Jarchow [14, p. 125]. For $1 \le p \le q \le \infty$, a linear map $T \in \mathcal{L}(E,F)$ is said to be quasi (ℓ_p,ℓ_q)-factorable if there exists a sequence $(x_n^*)_n \in \ell_p^{\omega}(E^*)$ such that

$$\|T(x)\| \le \|(x_n^*(x))_n\|_q \text{ for all } x \in E. \tag{2.1}$$

This class, which is denoted by $\mathcal{QF}_{p,q}$, endowed with the α-norm $\|\cdot\|_{\mathcal{QF}_{p,q}}$ defined as the infimum of $\|(x_n^*)_n\|_p^{\omega}$ taken over all the sequences $(x_n^*)_n$ satisfying the above inequality, becomes a α-Banach operator ideal.
Note that when $p = q$, we have the class of weakly p-nuclear operators (respectively, the class of quasi weakly p-nuclear) as defined in [16] (see also [19]). The ideal of weakly p-nuclear (respectively, quasi weakly p-nuclear) operators from E to F is represented by $\left(\mathcal{N}_{\omega,p}(E,F), \|\cdot\|_{\mathcal{N}_{\omega,p}}\right)$(respectively, $\mathcal{QN}_{w,p}(X,E), \|\cdot\|_{\mathcal{QN}_{w,p}}$).
Let us introduce the Lipschitz version of (ℓ_p,ℓ_q)-factorable operators.

Definition 2.1. Let X be metric space and E be a Banach space. For $1 \le p \le q \le \infty$ and $T \in Lip_0(X,E)$, we say that $T : X \to E$ is strongly Lipschitz (ℓ_p,ℓ_q)-factorable operators if T can be written in the form

$$T(x) = \sum_{n=1}^{\infty} f_n(x) y_n, \forall x \in X, \tag{2.2}$$

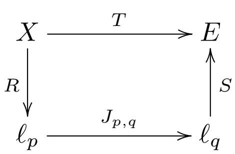
where $(f_n)_n \in \ell_p^{L,\omega^*}(X^\#)\,(c_0(X^\#)$ when $p = \infty)$ and $(y_n)_n \in \ell_{q^*}^{\omega}(E)$. Also, the set of all strongly Lipschitz (ℓ_p,ℓ_q)-factorable operators will be denoted by $\mathcal{SF}_{p,q}^L(X,E)$ and we set

$$\|T\|_{\mathcal{SF}_{p,q}^L} = \inf \|(f_n)_n\|_p^{L,\omega^*}\|(y_n)_n\|_{q^*}^{\omega},$$

with the infimum taken over all representations of T as in (2.2). In particular, whenever $p = q$, $\mathcal{SF}^L_{p,q}(X,E) = \mathcal{SN}^L_{\omega,p}(X,E)$ is the class of all Lipschitz weakly p-nuclear operators from X to E. With this notation, a Lipschitz version for the well-known notion of ℓ_p-factorable [19] (recently, called weakly p-nuclear operators [16]) is also given.

The first result concerning strongly Lipschitz (ℓ_p, ℓ_q)-factorable mappings is the factorization through the canonical injection $J_{p,q} : \ell_p \longrightarrow \ell_q$.

Theorem 2.2. *Let $1 \leq p \leq q \leq \infty$ and $T \in Lip_0(X,E)$. Then $T \in \mathcal{SF}^L_{p,q}(X,E)$ if and only if there exist $R \in Lip_0(X, \ell_p)$ ($R \in Lip_0(X, c_0)$ when $p = \infty$), $S \in \mathcal{L}(\ell_q, E)$ and $T = S \circ J_{p,q} \circ R$ such that the following diagram commutes:*



In this case;

$$\|T\|_{\mathcal{SF}^L_{p,q}} := \inf Lip(R).\|S\|,$$

wher the infimum is taken over all the above factorizations.

Proof. Let e_n^* and e_n, respectively, be the standard unit vector bases in ℓ_{p^*}(c_0 *when* $p =$ 1) and ℓ_q. Assume that T is strongly Lipschitz (ℓ_p, ℓ_q)-factorable operators. Let $(f_n)_n \in \ell_p^{L,\omega^*}(X^\#)$ and $(y_n)_n \in \ell_{q^*}^{\omega}(E)$, such that

$$T(x) = \sum_{n=1}^{\infty} f_n(x) y_n, \forall x \in X. \tag{2.3}$$

Define the mapping $R : X \longrightarrow \ell_p, x \mapsto (f_n(x))_n$. Then by [6, Lemma 2.4], we get that $R \in Lip_0(X, \ell_p)$ and $Lip(R) = \|(f_n)_n\|_p^{L,\omega^*}$. Consider the map $S : \ell_q \longrightarrow E, e_n \mapsto y_n$. It follows from [10, Proposition 2.2] that $S \in \mathcal{L}(\ell_q, E)$ and $\|S\| = \|(y_n)_n\|_{q^*}^{\omega}$.
Moreover, for each $x \in X$, we have

$$SJ_{p,q}R(x) = S\left(J_{p,q}(f_n(x))\right) = S\left(\sum_{n=1}^{\infty} f_n(x) e_n\right) = \sum_{n=1}^{\infty} f_n(x) y_n = T(x).$$

Thus, $T = S \circ J_{p,q} \circ R$ and $\inf Lip(R).\|S\| \leq \|T\|_{\mathcal{SF}^L_{p,q}}$.
Conversely, suppose that T has a factorization $T = S \circ J_{p,q} \circ R$, where $R \in Lip_0(X, \ell_p)$ and $S \in \mathcal{L}(\ell_q, E)$. Consider the sequences $f_n = \langle e_n^*, R(\cdot)\rangle$ and $y_n = S(e_n)$. Again by [6, Lemma 2.4] and [10, Proposition 2.2] we have $\|(e_n R)_n\|_p^{L,\omega} = \|R\|$ and $\|(Se_n)_n\|_{q^*}^{\omega} = \|S\|$. It follows that

$$T(x) = S \circ J_{p,q} \circ R(x) = S\left(\sum_{n=1}^{\infty} \langle e_n^*, R(x)\rangle e_n\right) = \sum_{n=1}^{\infty} \langle e_n^*, R(x)\rangle S(e_n) = \sum_{n=1}^{\infty} f_n(x) y_n.$$

Since the factorization of T was arbitrary,

$$\|T\|_{\mathcal{SF}^L_{p,q}} \leq \inf Lip(R).\|S\|.$$

□

The following facts are straightforward

Proposition 2.3.

1) *Every strongly Lipschitz (ℓ_p, ℓ_q)-factorable map T is bounded with $Lip(T) \leq \|T\|_{\mathcal{SF}^L_{p,q}}$.*
2) *Let $p \leq r$ and $q \leq s$. If T is strongly Lipschitz (ℓ_p, ℓ_q)-factorable, then T is strongly Lipschitz (ℓ_r, ℓ_s)-factorable with $\|T\|_{\mathcal{SF}^L_{r,s}} \leq \|T\|_{\mathcal{SF}^L_{p,q}}$.*

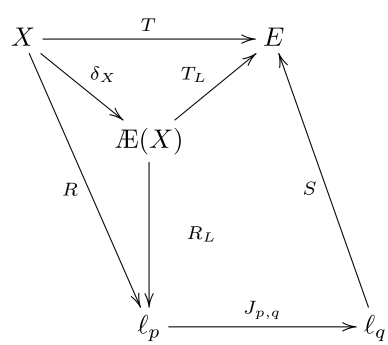
The next result give the relation of the strongly Lipschitz (ℓ_p, ℓ_q)-factorable mappings with the known linear theory using via their linearization.

Theorem 2.4. *Let X ba a pointed metric space, E be a Banach space and let $1 \leq p \leq q \leq \infty$. An operator $T \in Lip_0(X, E)$ is strongly Lipschitz (ℓ_p, ℓ_q)-factorable operators if and only if its Linearization $T_L : \text{Æ}(X) \longrightarrow E$ is (ℓ_p, ℓ_q)-factorable linear operators. Moreover, we have*

$$\|T\|_{\mathcal{SF}^L_{p,q}} = \|T_L\|_{\mathcal{F}_{p,q}}.$$

In particular, whenever $p = q$; T is Lipschitz weakly p-nuclear operators if and only if its Linearization T_L is weakly p-nuclear linear operators.

Proof. Let $\varepsilon > 0$. Suppose that $T \in \mathcal{SF}^L_{p,q}(X, E)$, then $T = S \circ J_{p,q} \circ R$ with $R \in Lip_0(X, \ell_p)$, $S \in \mathcal{L}(\ell_q, E)$ and $\|R\|.\|S\| \leq (1 + \varepsilon)\|T\|_{\mathcal{SF}^L_{p,q}}$. Consider the following



Since S is linear, it follows frome uniqueness of linearization that $T_L = S \circ J_{p,q} \circ R_L$ with $R_L \in \mathcal{L}(\text{Æ}(X), \ell_p)$ and $S \in \mathcal{L}(\ell_q, E)$. By [14, (1.2)], $T_L \in \mathcal{F}_{p,q}(\text{Æ}(X), E)$ and

$$\begin{aligned} \|R_L\|.\|S\| &= Lip(R).\|S\| \\ &\leq (1 + \varepsilon)\|T\|_{\mathcal{SF}^L_{p,q}}. \end{aligned}$$

Letting $\varepsilon \longrightarrow 0$ we get
$$\|T_L\|_{\mathcal{F}_{p,q}} \leq \|T\|_{\mathcal{SF}^L_{p,q}}.$$
To show the other part, suppose that $T_L \in \mathcal{F}_{p,q}(Æ(X), E)$. By (1.2) in [14], there exist $A \in \mathcal{L}(Æ(X), \ell_p)$, $B \in \mathcal{L}(\ell_q, E)$ and $J_{p,q} : \ell_p \longrightarrow \ell_q$ such that $T_L = B \circ J_{p,q} \circ A$ and
$$\|A\|.\|B\| \leq (1+\varepsilon)\|T_L\|_{\mathcal{F}_{p,q}}.$$

Hence $T = T_L \circ \delta_X = B \circ J_{p,q} \circ A \circ \delta_X = S \circ J_{p,q} \circ R$ with $S = B \in \mathcal{L}(\ell_q, E)$ and $R = A \circ \delta_X \in Lip_0(X, \ell_p)$. Then $T \in \mathcal{SF}^L_{p,q}(X, E)$ and $\|T\|_{\mathcal{SF}^L_{p,q}} \leq (1+\varepsilon) \| T_L \|_{\mathcal{F}_{p,q}}$ and the proof follows. □

Lemma 2.5. *Let $1 < p \leq \infty$. Then $\mathcal{SN}^L_{\omega,p}(X, \ell_p)$ is isometrically equal to $Lip_0(X, \ell_p)$ (ℓ_p is replaced by c_0 when $p = \infty$).*

Proof. Let $T \in Lip_0(X, \ell_p)$. By Theorem 2.4 and [16, Lemma 2.3], $T \in \mathcal{SN}^L_{\omega,p}(X, \ell_p)$ if and only if $T_L \in \mathcal{L}(Æ(X), \ell_p)$. Again $T_L \in \mathcal{L}(Æ(X), \ell_p)$ if and only if $T \in Lip_0(X, \ell_p)$. The proof conclude. □

Remark 2.6.

1) Lemma 2.5 does not hold in general for the case $p = 1$. To clarify, if we were to assume that $\mathcal{SN}^L_{w,1}(\mathbb{N}, \ell_1)$ is equivalent to $Lip_0(\mathbb{N}, \ell_1)$, it would result in a contradiction. This stems from the fact that $\mathcal{SN}^L_{w,1}(\mathbb{N}, \ell_1) = \mathcal{N}_{w,1}(Æ(\mathbb{N}), \ell_1)$ isometrically and $Æ(\mathbb{N}) \equiv \ell_1$(see [12, Page 46]), by Remark 2.5 in [16], $\mathcal{N}_{w,1}(\ell_1, \ell_1)$ is not equal to $\mathcal{L}(\ell_1, \ell_1)$.
2) When $1 < p \leq \infty$, it becomes apparent that Lipschitz weakly p-nuclear operators are not necessarily compact. This observation is based on the equality between $\mathcal{SN}^L_{w,p}(\mathbb{N}, \ell_p)$, $\mathcal{N}_{w,p}(\ell_1, \ell_p)$ and Remark 2.6 in [16]. Additionally, it's important to emphasize that Lipschitz weakly 1-nuclear operators are typically non-compact.

From Theorem 2.2, Lemma 2.5 and [16, Lemma 2.3], we have

Corollary 2.7. *Let $1 < p \leq \infty$ and let $T \in Lip_0(X, E)$. Then $T \in \mathcal{SF}^L_{p,q}(X, E)$ if and only if there exist $R \in \mathcal{SN}^L_{\omega,p}(X, \ell_p)$, $S \in \mathcal{N}_{\omega,q}(\ell_q, E)$ and the canonical injection $J_{p,q} : \ell_p \longrightarrow \ell_q$ (ℓ_p is replaced by c_0 when $p = \infty$) such that $T = S \circ J_{p,q} \circ R$.*

Following [1] (see also [7]) an operator ideal between a pointed metric space and Banach space $\mathcal{A}_{Lip}$ is a subclass of Lip_0 such that for every pointed metric space X and every Banach space E the components
$$\mathcal{A}_{Lip}(X, E) := Lip_0(X, E) \cap \mathcal{A}_{Lip}$$
satisfy

i) $\mathcal{A}_{Lip}(X, E)$ is a linear subspace of $Lip_0(X, E)$.
ii) For any $f \in X^{\#}$ and $e \in E$, the map $fe\colon x \mapsto f(x)e \in \mathcal{A}_{Lip}(X, E)$.

iii) The ideal property: if $S \in Lip_0(Y,X)$, $T \in \mathcal{A}_{Lip}(X,E)$ and $w \in \mathcal{L}(E,F)$, then the composition $w \circ T \circ S$ is in $\mathcal{A}_{Lip}(Y,F)$.

A Lipschitz operator ideal $\mathcal{A}_{Lip}$ is a α-normed (α-Banach) Lipschitz operator ideal if there is $\|\cdot\|_{\mathcal{A}_{Lip}} \colon \mathcal{A}_{Lip} \longrightarrow [0,\infty[$ that satisfies

i') For every pointed metric space X and every Banach space E, the pair $(\mathcal{A}_{Lip}(X,E), \|\cdot\|_{\mathcal{A}_{Lip}})$ is a α-normed (α-Banach) space and $Lip(T) \leq \|T\|_{\mathcal{A}_{Lip}}$ for all $T \in \mathcal{A}_{Lip}(X,E)$.

ii') $\|Id_{\mathbb{K}} \colon \mathbb{K} \longrightarrow \mathbb{K}\|_{\mathcal{A}_{Lip}} = 1$, where $Id_{\mathbb{K}}$ is the identity map of $\mathbb{K}$.

iii') If $S \in Lip_0(Y,X)$, $T \in \mathcal{A}_{Lip}(X,E)$ and $w \in \mathcal{L}(E,F)$, the inequality $\|w \circ T \circ S\|_{\mathcal{A}_{Lip}} \leq Lip(S)\|T\|_{\mathcal{A}_{Lip}}\|w\|$ holds.

Clearly, the Lipschitz operator 1-Banach ideal is precisely the Lipschitz operator Banach ideal.

Let $\mathcal{A}$ be a linear α-Banach operator ideal. A Lipschitz mapping $T \in Lip_0(X,E)$ belongs to the *composition Lipschitz operator α-Banach ideal operator* $\mathcal{A} \circ Lip_0$ if its linearization T_L belongs to $\mathcal{A}(Æ(X), E)$. Moreover, $\mathcal{A} \circ Lip_0$ endowed with the α-norm $\|T\|_{\mathcal{A}\circ Lip_0} = \|T_L\|_{\mathcal{A}}$ is a α-Banach Lipschitz operator ideal. This way to obtain a α-Banach Lipschitz operator ideal from a α-Banach operator ideal is called composition method and the Lipschitz α-Banach operator ideals obtained in this way are called ideals of composition type.

In [1], the authors introduced the injective hull of a Banach Lipschitz operator ideal we can translate this for α-Banach Lipschitz operator ideal. The injective hull $\mathcal{A}_{Lip}^{inj}$ of an α-Banach Lipschitz operator ideal $\mathcal{A}_{Lip}$ is defined as follows;

$$\mathcal{A}_{Lip}^{inj}(X,E) = \{T \in Lip_0(X,E) : \iota_E \circ T \in \mathcal{A}_{Lip}(X, \ell_\infty(B_{E^*})\},$$

where $\iota_E \colon E \longrightarrow \ell_\infty(B_{E^*})$ is the natural isometry, and $\|T\|_{\mathcal{A}_{Lip}^{inj}} = \|i_E T\|_{\mathcal{A}_{Lip}}$ for $T \in \mathcal{A}_{Lip}^{inj}(X,E)$. If $\mathcal{A}_{Lip}$ is a α-Banach Lipschitz operator ideal, then $\mathcal{A}_{Lip}^{inj}$ is also a α-Banach Lipschitz operator ideal.

Mimicking the proof of [1, Corollary 3.3] and [2, Proposition 2.4], we have the following result.

Proposition 2.8. *If $\mathcal{A}$ is a α-Banach operator ideal then,*

a) *$\mathcal{A} \circ Lip_0$ is a α-Banach Lipschitz operator ideal.*

b) *$(\mathcal{A} \circ Lip_0)^{inj} = \mathcal{A}^{inj} \circ Lip_0$ isometrically.*
In particular, if $\mathcal{A}_{Lip}$ is a α-Banach Lipschitz operator ideal of composition type, then so is also $\mathcal{A}_{Lip}^{inj}$.

By Theorem 2.4 and the above criterion, we have the following.

Proposition 2.9. *The space $\left(\mathcal{SF}_{p,q}^L(X,E), \|\cdot\|_{\mathcal{SF}_{p,q}^L}\right)$ is α-Banach Lipschitz operator ideal of composition type. In other words*

$$\mathcal{SF}_{p,q}^L(X,E) = \mathcal{F}_{p,q} \circ Lip_0(X,E) \text{ isometrically}$$

for every pointed metric space X and every Banach space E.

As a consequence of the above, from Proposition 2.8 and [14, Page 125], we have the following.

Corollary 2.10. *Let X be pointed metric space and E be a Banach space. Then*

i) *For $1 \leq p \leq 2$*

$$\begin{aligned}(\mathcal{SF}^L_{p,2})^{inj} &= (\mathcal{F}_{p,2} \circ Lip_0)^{inj} \\ &= (\mathcal{F}_{p,2})^{inj} \circ Lip_0 \\ &= \mathcal{F}_{p,2} \circ Lip_0, \\ &= \mathcal{SF}^L_{p,2}\end{aligned}$$

ii) *For $1 \leq p \leq q \leq \infty$*

$$\begin{aligned}(\mathcal{SF}^L_{p,q})^{inj} &= (\mathcal{F}_{p,q} \circ Lip_0)^{inj} \\ &= (\mathcal{F}_{p,q})^{inj} \circ Lip_0 \\ &= \mathcal{QF}_{p,q} \circ Lip_0.\end{aligned}$$

For $1+\frac{1}{r} \geq \frac{1}{s}+\frac{1}{t}$, recall the operator ideal $[\mathcal{SN}^L_{(r,s,t)}, \|\cdot\|_{\mathcal{SN}^L_{(r,s,t)}}]$ of strongly Lipschitz (r,s,t)-nuclear (see [6]). The ideal $\mathcal{SN}^L_{(r,s,t)}$ is defined as all operators T which have a representation

$$T(x) = \sum_{n=1}^{\infty} \lambda_n f_n(x) y_n,$$

where $(\lambda_n)_n \in \ell_r$ if $1 \leq r < \infty$ or $(\lambda_n)_n \in c_0$ if $r = \infty$, $(f_n)_n \in \ell^{L,w^*}_{t^*}(X^\#)$ and $(y_n)_n \in \ell^w_{s^*} \in (E)$.
and we set

$$\|T\|_{\mathcal{SN}^L_{(r,s,t)}} := \inf \|(\lambda_n)_n\|_r . \|(f_n)_n\|^{L,\omega^*}_{t^*} . \|(y_n)_n\|^{\omega}_{s^*},$$

where the infimum is taken over all the strongly Lipschitz (r,s,t)-nuclear representations of T. Strongly Lipschitz $(p,p,1)$-nuclear are simply called strongly Lipschitz p-nuclear operators [8, Corollary 2.9]. The ideal of all strongly Lipschitz p-nuclear operators is denoted by $\left[\mathcal{SN}^L_p(X,E), \|\cdot\|_{\mathcal{SN}^L_p}\right]$.

Proposition 2.11. *Let X be a pointed metric space, E be a Banach space and let $1 \leq p \leq q \leq \infty$ satisfying:*

$$\left(\frac{1}{r}+\frac{1}{t^*}\right)^{-1} \leq p \leq \min\left\{t^*, \left(\frac{1}{r}+\frac{1}{t^*}+\frac{1}{s^*}-1\right)^{-1}\right\} \text{ and } \frac{1}{q} = 1+\frac{1}{p}-\frac{1}{r}-\frac{1}{t^*}-\frac{1}{s^*}.$$

Then, $\mathcal{SN}^L_{(r,s,t)}(X,E) \subset \mathcal{SF}^L_{p,q}(X,E)$ and $\|T\|_{\mathcal{SF}^L_{p,q}} \leq \|T\|_{\mathcal{SN}^L_{(r,s,t)}}$.

Proof. By [6, Proposition 2.12], [14, Page 127] and Proposition 2.9, we get

$$\begin{aligned}\mathcal{SN}^L_{(r,s,t)}(X,E) &= \mathcal{N}_{r,s,t} \circ Lip_0(X,E) \\ &\subset \mathcal{F}_{p,q} \circ Lip_0(X,E) \\ &= \mathcal{SF}^L_{p,q}(X,E),\end{aligned}$$

□

As a consequence of above Proposition, whenever $r = p, s = p$ and $t = 1$ we obtain the following corollary.

Corollary 2.12. *Let* $1 \leq p \leq \infty$. *Then*

$$\left[\mathcal{SN}_p^L(X,E), \|.\|_{\mathcal{SN}_p^L}\right] \subset \left[\mathcal{SN}_{\omega,p}^L(X,E), \|.\|_{\mathcal{SN}_{\omega,p}^L}\right].$$

3. Factorable Lipschitz quasi weakly (p,q)-nuclear operator

Motivated by the Corollary 2.10 and the ideas from [18, Proposition 4.8], we have the following definition.

Definition 3.1. Let X be pointed metric space, E be a Banach space and $1 \leq p \leq q \leq \infty$. A mapping $T \in Lip_0(X,E)$ is called factorable Lipschitz quasi weakly (p,q)-nuclear operator if there exists a sequence $(f_n)_n \in \ell_p^{L,\omega^*}(X^{\#})$ such that

$$\left\|\sum_{j=1}^{N} \lambda_j (T(x_j) - T(x_j'))\right\| \leq \left(\sum_{n=1}^{\infty} \left|\sum_{j=1}^{N} \lambda_j (f_n(x_j) - f_n(x_j'))\right|^q\right)^{\frac{1}{q}}, \tag{3.1}$$

for all $x_j, x_j' \in X, \lambda_j \in \mathbb{K}$, $(1 \leq j \leq N)$.
We denote by $\mathcal{FN}_{\omega,(p,q)}^{LQ}(X,E)$ the space of all factorable quasi weakly (p,q)-nuclear Lipschitz mappings between pointed metric spaces X and Banach space E. In such case, we put

$$\|T\|_{\mathcal{FN}_{\omega,(p,q)}^{LQ}} = \inf\left\{\|(f_n)_n\|_p^{L,\omega^*} : (f_n)_n \text{ satisfying (3.1)}\right\}.$$

In particular, whenever $p = q$, $\mathcal{FN}_{\omega,(p,q)}^{L}(X,E) = \mathcal{FN}_{\omega,p}^{L}(X,E)$ is the class of all factorable Lipschitz quasi weakly p-nuclear operators from X to E. With this notation, a Lipschitz version for the well-known notion of quasi weakly p-nuclear [16] is also given.

Theorem 3.2. *Let X be pointed metric space, E be a Banach space and $1 \leq p \leq q \leq \infty$. Then $T \in \mathcal{FN}_{\omega,(p,q)}^{LQ}(X,E)$ if and only if $T_L \in \mathcal{QF}_{p,q}(Æ(X), E)$.*

Proof. Let $m = \sum_{j=1}^{N} \lambda_j m_{x_j x_j'}$ for all $x_j, x_j' \in X$, and $\lambda_j \in \mathbb{K}, (1 \leq j \leq N)$, we have

$$\|T_L(m)\| = \|\sum_{j=1}^{N} \lambda_j (T(x_j) - T(x_j'))\|. \tag{3.2}$$

Suppose that $T \in \mathcal{FN}_{\omega,(p,q)}^{LQ}(X,E)$, then there exist a sequence $(f_n)_n \in \ell_p^{L,\omega^*}(X^{\#})$ such that

$$\|T_L(m)\| \leq \left(\sum_{n=1}^{\infty} \left|\sum_{j=1}^{N} \lambda_j (f_n(x_j) - f_n(x_j'))\right|^q\right)^{\frac{1}{q}}.$$

Since $Æ(X)^*$ and $X^\#$ are isometrically isomorphic, then by Proposition 2.5 in [6], there is a sequence $(m_n^*)_n \in \ell_p^\omega(Æ(X)^*)$ such that

$$m_n^*(m_{x_j x_j'}) = (f_n)_L(m_{x_j x_j'}) = f_n(x_j) - f_n(x_j'), \tag{3.3}$$

for all $x_j, x_j' \in X$, $(1 \leq j \leq N)$, we obtain

$$\begin{aligned} \|T_L(m)\| &\leq (\sum_{n=1}^{\infty} |\sum_{j=1}^{N} \lambda_j m_n^*(m_{x_j x_j'})|^q)^{\frac{1}{q}} \\ &= (\sum_{n=1}^{\infty} |m_n^*(\sum_{j=1}^{N} \lambda_j m_{x_j x_j'})|^q)^{\frac{1}{q}} \\ &= (\sum_{n=1}^{\infty} |m_n^*(m)|^q)^{\frac{1}{q}}. \end{aligned}$$

Therefore, by (2.1) $T_L \in \mathcal{QF}_{p,q}(Æ(X), E)$. The converse follows from (3.2) and (3.3). □

From Proposition 2.9, Corollary 2.10 and the above theorem we obtain the following corollary.

Corollary 3.3. *Let X be pointed metric space, E be a Banach space and $1 \leq p \leq q \leq \infty$. Then $T \in (\mathcal{SF}_{p,q}^L)^{inj}$ if and only if $T \in \mathcal{FN}_{\omega,(p,q)}^{LQ}(X, E)$.*

From Proposition 2.3 and the above corollary, we have

Corollary 3.4. *Let $1 \leq p \leq q \leq r \leq s \leq \infty$. Let X be a pointed metric and E be a Banach space. Then, $\mathcal{FN}_{\omega,(p,s)}^{LQ}(X, E) \subseteq \mathcal{FN}_{\omega,(q,r)}^{LQ}(X, E)$ and $\|T\|_{\mathcal{FN}_{\omega,(q,r)}^{LQ}} \leq \|T\|_{\mathcal{FN}_{\omega,(p,s)}^{LQ}}$, for every $T \in \mathcal{FN}_{\omega,(p,s)}^{LQ}(X, E)$.*

Johnson and Schechtman in [15] introduced the concept of Lipschitz (r, s)-summing operators. For $r, s \geq 1$, a Lipschitz map T is called Lipschitz (r, s)-summing, if there exists a constant $C > 0$ such that regardless of the choice of points $x_1, \ldots, x_n, x_1', \ldots, x_n'$ in X and for all $\lambda_1, ..., \lambda_n \in \mathbb{R}_+$ we have

$$\left(\sum_{i=1}^{n} \lambda_i \|T(x_i) - T(x_i')\|_E^r\right)^{\frac{1}{r}} \leq C \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^{n} \lambda_i |f(x_i) - f(x_i')|^s\right)^{\frac{1}{s}}. \tag{3.4}$$

In this case we put $\pi_{(r,s)}^L(T) = \inf\{C : \text{satisfying } (3.4)\}$. The space of all Lipschitz (r, s)-summing operators from X to E is denoted by $\Pi_{(r,s)}^L(X, E)$. If we take $r = s$ we find the definition of Lipschitz r-summing operators of Farmer and Johnson [11]. Notice that the definition is the same if we restrict to $\lambda_i = 1$. From the definition, it follows that T is Lipschitz (r, s)-summing and $\pi_{(r,s)}^L(T) \leq \pi_{(r,s)}(T_L)$ whenever T_L is (r, s)-summing.

We can establish the following comparison between the classes of factorable Lipschitz quasi weakly (p, q)-nuclear, Lipschitz weakly p-nuclear and Lipschitz (r, s)-summing.

Proposition 3.5. *Let X be a pointed metric space, E be a Banach space and $1 \leq p \leq q \leq \infty, p < \infty$. Put $r = p$ for $2 \leq q$ or $r = (\frac{1}{2} + \frac{1}{p} - \frac{1}{q})^{-1}$ for $q < 2$. Then we have, with $\alpha \geq \pi_{r,1}(J_{p,q}) : \mathcal{FN}^{LQ}_{\omega,(p,q)}(X, E) \subset \Pi^L_{(r,1)}(X, E)$; $\pi^L_{(r,1)}(\cdot) \leq \alpha \| \cdot \|_{\mathcal{FN}^{LQ}_{\omega,(p,q)}}$.*

Proof. Let $T \in Lip_0(X, E)$, then

$$\begin{aligned} T \in \mathcal{FN}^{LQ}_{\omega,(p,q)}(X, E) &\Leftrightarrow T_L \in \mathcal{QF}_{p,q}(Æ(X), E) \\ &\Rightarrow T_L \in \Pi_{(r,1)}(Æ(X), E) \\ &\Rightarrow T \in \Pi^L_{(r,1)}(X, E), \end{aligned}$$

where the first equivalence follows Theorem 3.2 and the second implication from (1.7) in [14]. □

Proposition 3.6. *Let X be a pointed metric space, E be a Banach space and let $1 \leq p \leq \infty$. If T is Lipschitz weakly p-nuclear, then T is Lipschitz p-summing with $\pi^L_p(T) \leq \|T\|_{\mathcal{SN}^L_{\omega,p}}$.*

Proof. Let $T \in \mathcal{SN}^L_{\omega,p}(X, E)$ then, $T(x) = \sum_{n=1}^{\infty} f_n(x) y_n$ where $(f_n)_n \in \ell^{L,\omega^*}_p(X^\#)$, $(y_n)_n \in \ell^{\omega}_{p^*}(E)$ and

$$\|T\|_{\mathcal{SN}^L_{\omega,p}} := \inf \|(f_n)_n\|^{L,\omega^*}_p . \|(y_n)_n\|^{\omega}_{p^*}.$$

Thies implies $\|(f_n)_n\|^{L,\omega^*}_p . \|(y_n)_n\|^{\omega}_{p^*} \leq (1 + \varepsilon)\|T\|_{\mathcal{SN}^L_{\omega,p}}$ for every $\varepsilon > 0$.
We may assume that $\|(f_n)_n\|^{L,\omega^*}_p = 1$, then

$$\|(y_n)_n\|^{\omega}_{p^*} \leq (1 + \varepsilon)\|T\|_{\mathcal{SN}^L_{\omega,p}}.$$

Hence, for each $f_n \in B_{X^\#}, n \in \mathbb{N}$ $x, x^{'} \in X$ and by Hölder inequality, we have

$$\begin{aligned} \|T(x) - T(x^{'})\| &= \| \sum_{n=1}^{\infty} (f_n(x) - f_n(x^{'})) y_n \|_E \\ &= \sup_{y^* \in Y^*} | \sum_{n=1}^{\infty} (f_n(x) - f_n(x^{'}) \langle y_n, y^* \rangle | \\ &\leq \left[\sum_{n=1}^{\infty} |f_n(x) - f_n(x^{'})|^p \right]^{\frac{1}{p}} \sup_{y^* \in Y^*} \left[\sum_{n=1}^{\infty} |\langle y_n, y^* \rangle|^{p^*} \right]^{\frac{1}{p^*}} \end{aligned}$$

$$= \left[\sum_{n=1}^{\infty} |f_n(x) - f_n(x^{'})|^p\right]^{\frac{1}{p}} .\|(y_n)_n\|_{p^*}^{\omega}$$

$$\leq \left[\sum_{n=1}^{\infty} |f_n(x) - f_n(x^{'})|^p\right]^{\frac{1}{p}} .(1+\varepsilon)\|T\|_{\mathcal{SN}_{\omega,p}^{L}}$$

$$= (1+\varepsilon)\|T\|_{\mathcal{SN}_{\omega,p}^{L}} \left(\int_{B_{X^\#}} |f(x) - f(x^{'})|^p d\mu(f)\right)^{\frac{1}{p}}$$

where $\mu = \sum_{n=1}^{\infty} \delta_{f_n}$ is a probability on $B_{X^\#}$.
Consequently, by the Pietsch domination theorem for the class Π_p^L (see [11, Theorem 1]), T is Lipschitz p-summing and $\pi_p^L(T) \leq (1+\varepsilon)\|T\|_{\mathcal{SN}_{\omega,p}^{L}}$. □

Sinha and Karn [20] introduced an ideal of weakly p-compact. Let $1 \leq p \leq \infty$. A subset $K$of a Banach space E is called weakly p-compact if there exists $(x_n)_n \in \ell_p^{\omega}(E)$ ($c_0(E)$ when $p = \infty$) such that

$$K \subset p\text{-}co(x_n)_n := \left\{\sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n) \in B_{\ell_{p^*}}\right\}.$$

We denote the unit ball of E by B_E and replace $B_{\ell_{p^*}}$ with B_{c_0} if $p = 1$. For a linear map $T : F \longrightarrow E, T \in \mathcal{W}_p(F, E)$ if $T(B_F)$ is a weakly p-compact subset of E. In such case, we put

$$\|T\|_{\mathcal{W}_p} := \inf\{\|(x_n)_n\|_p : (x_n)_n \in \ell_p^{\omega}(E) \textit{ and } T(B_F) \subset p - co(x_n)_n\}.$$

For each $T \in Lip_0(X, E)$, the linear operator $T^t : E^* \longrightarrow X^\#$, given by $T^t(y^*) = y^* \circ T$ for all $y^* \in E^*$, is called the Lipschitz transpose map of T. The norm $\|T^t\|$ is given by $\|T^t\| = Lip(T)$. If $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a Banach ideal, then the Lipschitz dual of $\mathcal{A}$ is defined by Achour et al. [1, Definition 3.8] as follows

$$\mathcal{A}^{Lip-dual}(X, E) = \left\{T \in Lip_0(X, E) : T^t \in \mathcal{A}(E^*, X^\#)\right\},$$

for a pointed metric space X and a Banach space E.
We define, $\|T\|_{\mathcal{A}^{Lip-dual}} = \|T^t\|_{\mathcal{A}}$. Then $(\mathcal{A}^{Lip-dual}, \|.\|_{\mathcal{A}^{Lip-dual}})$ becomes a Banach Lipschitz ideal and $\mathcal{A}^{Lip-dual}(X, E) = \mathcal{A}^{dual} \circ Lip_0(X, E)$ (see [1, Theorem 3.9]).
In the next, we will show that in fact the factorable Lipschitz quasi p-nuclear operators are the adjoints of weakly p-compact linear operators.

Proposition 3.7. *Let X be a pointed metric space, E be a Banach space, and $1 \leq p < \infty$. For an operator $T \in Lip_0(X, E)$, $T^t : E^* \longrightarrow X^\#$ is weackly p-compact operator if and only if $T \in \mathcal{FN}_{\omega,p}^{LQ}(X, E)$ and $\|T\|_{\mathcal{FN}_{\omega,p}^{LQ}} = \|T^t\|_{\mathcal{W}_p}$. In other words, $\mathcal{FN}_{\omega,p}^{LQ}(X, E) = \mathcal{W}_p^{Lip-dual}(X, E)$ isometrically.*

Proof. By Theorem 3.2 and Theorem 3.7 in [16], we have

$$\begin{aligned}\mathcal{FN}^{LQ}_{\omega,p}(X,E) &= \mathcal{QN}_{\omega,p}\circ Lip_0(X,E)\\ &= (\mathcal{W}_p)^{dual}\circ Lip_0(X,E)\\ &= \mathcal{W}_p^{Lip-dual}(X,E).\end{aligned}$$

The result follows. □

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