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Pointwise convergence on the rings of functions which are discontinuous on a set of measure zero

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Abstract

Consider the ring $\mathcal{M}_\circ(X,\mu)$ of functions which are discontinuous on a set of measure zero which is introduced and studied extensively in [2]. In this paper, we introduce a ring $B_1(X,\mu)$ of functions which are pointwise limits of sequences of functions in $\mathcal{M}_\circ(X,\mu)$. We study various properties of zero sets, $B_1(X,\mu)$-separated and $B_1(X,\mu)$-embedded subsets of $B_1(X,\mu)$ and also establish an analogous version of Urysohn's extension theorem. We investigate a connection between ideals of $B_1(X,\mu)$ and $\mathcal{Z}_B$-filters on X. We study an analogue of Gelfand-Kolmogoroff theorem in our setting. We define real maximal ideals of $B_1(X,\mu)$ and establish the result $|\mathcal{R}Max(\mathcal{M}_\circ(X,\mu))| = |\mathcal{R}Max(B_1(X,\mu))|$, where $\mathcal{R}Max(\mathcal{M}_\circ(X,\mu))$ and $\mathcal{R}Max(B_1(X,\mu))$ are the sets of all real maximal ideals of $\mathcal{M}_\circ(X,\mu)$ and $B_1(X,\mu)$, respectively.

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Introduction

Let (X,τ) be a T_1 topological space. Let $\mathcal{A}$ be a σ-algebra containing τ, which is defined as follows: $\mathcal{A}$ is a collection of subsets of X satisfying (i) $X \in \mathcal{A}$, (ii) $\mathcal{A}$ is closed under complementation and (iii) $\mathcal{A}$ is closed under countable union. A mapping $\mu : \mathcal{A} \to [0,\infty)$ is called a measure on $(X,\mathcal{A})$ if $\mu(\varnothing) = 0$ and satisfies the countable additive property i.e., for any countable family $\{A_n : n \in \mathbb{N}\}$ of pairwise disjoint members of $\mathcal{A}$, $\mu(\bigcup\limits_{n=1}^{\infty} A_n) = \sum\limits_{n=1}^{\infty} \mu(A_n)$. The quadruplet $(X,\tau,\mathcal{A},\mu)$ is called a $\tau\mathcal{A}\mu$-space. The collection $\mathcal{M}_\circ(X,\mu) = \{f \in \mathbb{R}^X$: the measure of discontinuity-set D_f of f is zero$\}$ is a lattice ordered ring, discussed extensively in [2]. Now we define $B_1(X,\mu) = \{f \in \mathbb{R}^X$: there exists a sequence $\{f_n\}$ in $\mathcal{M}_\circ(X,\mu)$ such that $\{f_n\}$ converges to f pointwise$\}$. Then $B_1(X,\mu)$ is a commutative lattice ordered ring if the relevant operations are defined pointwise on X and moreover we have $\mathcal{M}_\circ(X,\mu) \subseteq B_1(X,\mu) \subseteq \mathcal{M}(X,\mathcal{A})$, where $\mathcal{M}(X,\mathcal{A})$ is the ring of measurable functions, discussed in [1].

It is shown in the paper [2] that the ring $C(X)$ of all real-valued continuous functions on X is a special case of the ring $\mathcal{M}_\circ(X,\mu)$ if we choose $\mathcal{A} = \mathcal{P}(X)$, the power set of X and μ is the counting measure on $\mathcal{P}(X)$. The ring $B_1(X)$ of all real-valued Baire class one functions on X, which lies between the rings $C(X)$ and $\mathcal{M}(X,\mathcal{A})$ has been investigated extensively in [3, 4, 5]. The goal of this article is to pursue research on the ring $B_1(X,\mu)$, a generalization of $B_1(X)$.

In Section 1, we show that $B_1(X,\mu)$ is a commutative lattice ordered ring which lies between $\mathcal{M}_\circ(X,\mu)$ and $\mathcal{M}(X,\mathcal{A})$. For $f \in B_1(X,\mu)$, $Z(f) = \{x \in X : f(x) = 0\}$ is called the zero-set of f. Let $Z[B_1(X,\mu)] = \{Z(f) : f \in B_1(X,\mu)\}$ be the collection of all zero-sets induced by elements of $B_1(X,\mu)$. It is easily verified that $Z[B_1(X,\mu)] = Z[B_1^*(X,\mu)]$, where $B_1^*(X,\mu) = \{f \in B_1(X,\mu) : f$ is bounded on $X\}$. In Theorem 1.7, we establish that $B_1(X,\mu)$ is closed under uniform limit and with help of this theorem, we prove Theorem 1.8 which states that $Z[B_1(X,\mu)]$ is closed under countable intersection. Using Theorem 2.10 of [2], it can be easily shown that for any $\tau\mathcal{A}\mu$-space $(X,\tau,\mathcal{A},\mu)$, there exists a quadruplet $(X,\tau,\mathcal{A}^*,\mu^*)$, where $\mathcal{A}^*$ is a σ-algebra containing τ and μ^* is a complete measure defined on $\mathcal{A}^*$ such that $B_1(X,\mu) = B_1(X,\mu^*)$ [Theorem 1.9]. With similar ideas, we establish $B_1(X,\mu) = B_1(X,\mu|_\beta)$, for any $\tau\mathcal{A}\mu$-space $(X,\tau,\mathcal{A},\mu)$ and for the quadruplet $(X,\tau,\beta,\mu|_\beta)$, where β is the Borel σ-algebra containing τ and $\mu|_\beta$ is the restriction of μ on β.

In the next section, we introduce and study the notions of $B_1(X,\mu)$-separated, $B_1(X,\mu)$-embedded and $B_1^*(X,\mu)$-embedded subsets of X. We establish an analogous version of Urysohn's extension theorem [see Theorem 2.5].

In Section 3, we introduce the notion of filter of zero sets in $Z[B_1(X,\mu)]$ and call it $\mathcal{Z}_B$-filter. We investigate the correspondence between ideals of $B_1(X,\mu)$ and $\mathcal{Z}_B$-filters. Also, we define $\mathcal{Z}_B$-ideals of $B_1(X,\mu)$ and in Theorem 3.6, we provide a characterization of prime $\mathcal{Z}_B$-ideals of $B_1(X,\mu)$. We establish an analogous version of Gelfnd-Kolmogoroff theorem in our setup (Theorem

3.10). In Example 3.11, we show that βX, $\beta_{\mathcal{M}_\circ}X$ and $\beta_{B_1}X$ are mutually not homeomorphic, where βX is the Stone-Čech compactification of X, $\beta_{\mathcal{M}_\circ}X$ is the index set for the family of all $\mathcal{Z}$-ultrafilters on X, defined in [2] and $\beta_{B_1}X$ is the index set for the family of all $\mathcal{Z}_B$-ultrafilters on X.

In Section 4, we define positive elements of the residue class of $B_1(X,\mu)$ modulo ideals and in Theorem 4.7, we study a complete description of non-negative elements of $B_1(X,\mu)/I$, when I is a $\mathcal{Z}_B$-ideal of $B_1(X,\mu)$. In this section, we also define and study real maximal ideals of $B_1(X,\mu)$. Theorem 4.14 is a characterization of infinitely large element of $B_1(X,\mu)/I$. We discuss the characterization of real maximal ideal in $B_1(X,\mu)$ in Theorem 4.17. Also, we define real compact spaces, analogous version of 8.1 [7] and provide a characterization of a real compact space via ring homomorphism from $B_1(X,\mu)$ to $\mathbb{R}$ (Theorem 4.20).

In the next section, we discuss relations between real maximal ideals of $\mathcal{M}_\circ(X,\mu)$ and maximal ideals of $B_1(X,\mu)$. In this section, we prove that a maximal ideal M of $\mathcal{M}_\circ(X,\mu)$ is real if and only if $M = M_B \cap \mathcal{M}_\circ(X,\mu)$, where $M_B = \{f \in \mathbb{R}^X$: there exists a sequence of functions $\{f_n\} \subseteq M$ such that $f_n \to f$ pointwise$\}$ (Theorem 5.2). We introduce the closed ideals of $B_1(X,\mu)$ and establish the result $|\mathcal{R}Max(\mathcal{M}_\circ(X,\mu))| = |\mathcal{R}Max(B_1(X,\mu))|$, where $\mathcal{R}Max(\mathcal{M}_\circ(X,\mu))$ and $\mathcal{R}Max(B_1(X,\mu))$ are sets of all real maximal ideals of $\mathcal{M}_\circ(X,\mu)$ and $B_1(X,\mu)$, respectively (Theorem 5.10).

Finally, we define a $B_1(X,\mu)$-compact space. Also, we show that every $B_1(X,\mu)$-compact space is $\tau\mathcal{A}\mu$-compact (see Theorem 6.3). The converse need not be true and it is established in Example 6.4. Lastly, we develop a result (see Theorem 6.8) which is an analogous version of the Stone Weierstrass Theorem ([12]).

1. Zero set in the ring $B_1(X,\mu)$

For any topological space X, we define $B_1(X,\mu) = \{f \in \mathbb{R}^X$: there exists a sequence $\{f_n\}$ in $\mathcal{M}_\circ(X,\mu)$ such that $\{f_n\}$ converges to f pointwise on X $\}$.

Let $\{f_n\}$ converge to f pointwise on X and $\{g_n\}$ converge to g pointwise on X. Then

(i) $\{f_n + g_n\}$ converges to $f + g$ pointwise on X.
(ii) $\{-f_n\}$ converges to $-f$ pointwise on X.
(iii) $\{f_n g_n\}$ converges to fg pointwise on X.
(iv) $\{|f_n|\}$ converges to $|f|$ pointwise on X.

Using the above results, and the fact that for any $f, g \in B_1(X,\mu)$, $f \vee g = \frac{1}{2}(f + g + |f - g|)$ and $f \wedge g = -(-f \vee -g)$ are in $B_1(X,\mu)$, it is easy to verify that $(B_1(X,\mu), +, \cdot)$ is a commutative lattice ordered ring if the relevant operations are defined pointwise on X. It is clear that $\mathcal{M}_\circ(X,\mu) \subseteq B_1(X,\mu)$ and the following example shows that $\mathcal{M}_\circ(X,\mu)$ is a proper subring of the ring $B_1(X,\mu)$.

Example 1.1. Let τ be the topology on $X = [0,1]$ inherited from the Euclidean topology on the set $\mathbb{R}$ of reals, $\mathcal{P}(X)$ be the power set of X. For any $A \in \mathcal{P}(X)$, define dirac measure δ_1 on $\mathcal{P}(X)$ as follows:

$$\delta_1(A) = \begin{cases} 1, & \text{if } 1 \in A \\ 0, & \text{if } 1 \notin A. \end{cases}$$

For each $n \in \mathbb{N}$, we define $f_n : X \to \mathbb{R}$ by $f_n(x) = x^n$ for all $x \in X$. Then each $f_n \in \mathcal{M}_\circ(X, \delta_1)$ and $f_n \to f$ pointwise on X, where

$$f(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } 0 \leq x < 1. \end{cases}$$

Clearly, $f \in B_1(X, \delta_1)$ and $f \notin \mathcal{M}_\circ(X, \delta_1)$. Therefore $\mathcal{M}_\circ(X, \delta_1) \subsetneqq B_1(X, \delta_1)$.

Let (X, τ) be a topological space and $\mathcal{A}$ be a σ-algebra on X containing τ. Then $(X, \mathcal{A})$ is called a measurable space. A function $f : X \to \mathbb{R}$ is called $\mathcal{A}$-measurable or a measurable function if $\{x \in X : f(x) > \alpha\} \in \mathcal{A}$, for any real number α. Then the set $\mathcal{M}(X, \mathcal{A})$ of all real valued measurable functions is a commutative lattice ordered ring with unity, discussed in [1]. Since $\mathcal{M}_\circ(X, \mu) \subsetneqq \mathcal{M}(X, \mathcal{A})$ and pointwise limit of measurable functions is again a measurable function, we have $B_1(X, \mu) \subseteq \mathcal{M}(X, \mathcal{A})$. Now we want to show that $B_1(X, \mu)$ is a proper subring of $\mathcal{M}(X, \mathcal{A})$. For this purpose we first state the following theorem.

Theorem 1.2 ([14])**.** *Let X be a normal topological space and $B_1(X)$ denotes the set of all Baire class one functions from X to the real line $\mathbb{R}$. Then $f \in B_1(X)$ if and only if $f^{-1}(G)$ is an F_σ-set, for every open set $G \subseteq \mathbb{R}$.*

Example 1.3. Consider $(\mathbb{R}, \tau_u, \mathcal{L}, \mu_\infty)$, where τ_u is the usual topology on $\mathbb{R}$, $\mathcal{L}$ is the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}$ and μ_∞ is a measure on $\mathcal{L}$, defined as follows: for any $A \in \mathcal{L}$,

$$\mu_\infty(A) = \begin{cases} 0 & \text{if } A = \varnothing \\ \infty & \text{otherwise.} \end{cases}$$

Then $\mathcal{M}_\circ(\mathbb{R}, \mu_\infty) = C(\mathbb{R})$. Now consider a function $f : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, $f \in \mathcal{M}(\mathbb{R}, \mathcal{L})$, where the last set is the ring of all measurable functions from $\mathbb{R}$ to $\mathbb{R}$ with respect to above mentioned measure and $f^{-1}(\frac{1}{2}, \frac{3}{2}) = \mathbb{R} \setminus \mathbb{Q}$ is not a F_σ-set. Therefore $f \notin B_1(X, \mu_\infty)$ by Theorem 1.2.

Now we define zero set of $f \in B_1(X, \mu)$ by $Z(f) = \{x \in X : f(x) = 0\}$.

Theorem 1.4. *Let $f, g \in B_1(X, \mu)$ and $r \in \mathbb{R}$. Then*

(i) $Z(f^2 + g^2) = Z(f) \cap Z(g) = Z(|f| + |g|)$.
(ii) $Z(f \cdot g) = Z(f) \cup Z(g)$.
(iii) $\{x \in X : f(x) \geq r\}$ *and* $\{x \in X : f(x) \leq r\}$ *are zero sets in* X.
(iv) $Z(f) = Z(-\underline{1} \vee f \wedge \underline{1})$. *Thus* $B_1(X, \mu)$ *and* $B_1^*(X, \mu)$ *produce the same family of zero sets in* X, *where* $B_1^*(X, \mu) = \{f \in B_1(X, \mu) : f$ *is bounded* $\}$.

Here for $f, g \in B_1(X,\mu)$, the functions $f \vee g$ and $f \wedge g$ are in $B_1(X,\mu)$, defined in the most obvious manner: $(f \vee g)(x) = Max\{f(x), g(x)\}$ and $(f \wedge g)(x) = Min\{f(x), g(x)\}, x \in X$.

We denote $Z[B_1(X,\mu)]$ for the collection $\{Z(f) : f \in B_1(X,\mu)\} = \{Z(g) : g \in B_1^*(X,\mu)\}$ of all zero sets in X. It follows from the Theorem 1.4 that $Z[B_1(X,\mu)]$ is closed under finite union and also closed under finite intersection. Moreover, we will establish that $Z[B_1(X,\mu)]$ is closed under countable intersection too. For this, we first prove some results.

Lemma 1.5. *If $f \in B_1(X,\mu)$ and $|f| \leq M$ for some $M \in \mathbb{R}$, then there exists a sequence $\{g_n\} \subseteq \mathcal{M}_\circ(X,\mu)$ such that $g_n \to f$ pointwise and each g_n is bounded by M.*

Proof. Let $f \in B_1(X,\mu)$. Then there exists a sequence $\{f_n\}$ in $\mathcal{M}_\circ(X,\mu)$ such that $f_n \to f$ pointwise. Set $g_n = (-M \vee f_n) \wedge M$, then each $g_n \in \mathcal{M}_\circ(X,\mu)$ and $g_n \to f$ pointwise. This completes the proof. □

Lemma 1.6. *Let $\{f_k\} \subseteq B_1(X,\mu)$ and $|f_k(x)| \leq M_k$ for all $k \in \mathbb{N}$ $(M_k > 0)$ and for all $x \in X$. If $\sum\limits_{k=1}^{\infty} M_k < \infty$, then $f = \sum\limits_{k=1}^{\infty} f_k \in B_1(X,\mu)$.*

Proof. For each $f_k \in B_1(X,\mu)$, there exists a sequence $\{g_{ki}\}$ in $\mathcal{M}_\circ(X,\mu)$ such that $g_{ki} \to f_k$ pointwise. By Lemma 1.5, we can choose $\{g_{ki}\}$ such that $|g_{ki}| \leq M_k$ for all $i \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $h_n = g_{1n} + g_{2n} + \cdots + g_{nn}$, then $h_n \in \mathcal{M}_\circ(X,\mu)$. We will show that $h_n \to f$ pointwise. Since $\sum\limits_{k=1}^{\infty} M_k < \infty$, for any $\epsilon > 0$ there exists a $k' \in \mathbb{N}$ such that $\sum\limits_{k=k'+1}^{\infty} M_k < \epsilon$. Now we choose an integer $N > k'$ such that $|g_{ki}(x) - f_k(x)| < \frac{\epsilon}{k'}$ for $1 \leq k \leq k'$ and for all $i \geq N$. Again for any $n \geq N$, we have $|h_n(x) - f(x)| = |\sum\limits_{k=1}^{n} g_{kn}(x) - \sum\limits_{k=1}^{\infty} f_k(x)| \leq |\sum\limits_{k=1}^{n} (g_{kn}(x) - f_k(x))| + |\sum\limits_{k=n+1}^{\infty} f_k(x)| \leq |\sum\limits_{k=1}^{k'} ((g_{kn}(x) - f_k(x))) + \sum\limits_{k=k'+1}^{n} |g_{kn}(x)| + \sum\limits_{k=k'+1}^{\infty} |f_k(x)| \leq \sum\limits_{k=1}^{k'} \frac{\epsilon}{k'} + 2 \sum\limits_{k=k'+1}^{\infty} M_k \leq 3\epsilon$. It follows that $\{h_n\}$ converges pointwise to f. Thus $f(x) = \sum\limits_{k=1}^{\infty} f_k(x)$ belongs to $B_1(X,\mu)$ □

Theorem 1.7. *Let $\{f_n\}$ be a sequence of functions in $B_1(X,\mu)$ that converges to a function f uniformly on X. Then $f \in B_1(X,\mu)$ i.e., $B_1(X,\mu)$ is closed under uniform limit.*

Proof. Let $\{f_n\}$ be a sequence in $B_1(X,\mu)$ and $f_n \to f$ uniformly. We will show that $f \in B_1(X,\mu)$. By definition of uniform convergence, for each $k \in \mathbb{N}$, there exists a subsequence f_{n_k} such that $|f_{n_k}(x) - f(x)| < \frac{1}{2^k}$ for all $x \in X$. Consider the sequence $\{f_{n_{k+1}} - f_{n_k}\}$, then $|f_{n_{k+1}}(x) - f_{n_k}(x)| \leq |f_{n_{k+1}}(x) - f(x)| + |f_{n_k}(x) - f(x)| \leq \frac{1}{2^{k+1}} + \frac{1}{2^k} = \frac{3}{2}2^{-k}$. Set $M_k = \frac{3}{2}2^{-k}$, then $|f_{n_{k+1}}(x) - f_{n_k}(x)| \leq$

M_k for all $x \in X$ and $\sum_{k=1}^{\infty} M_k < \infty$. Then by Lemma 1.6, the sum $\sum_{k=1}^{\infty}[f_{n_{k+1}} - f_{n_k}]$ belongs to $B_1(X,\mu)$. Now $\sum_{k=1}^{\infty}[f_{n_{k+1}}(x) - f_{n_k}(x)] = \lim_{N\to\infty}\sum_{k=1}^{N}[f_{n_{k+1}}(x) - f_{n_k}(x)] = f(x) - f_{n_1}(x)$. Since $f_{n_1} \in B_1(X,\mu)$, $f \in B_1(X,\mu)$. □

Theorem 1.8. *$Z[B_1(X,\mu)]$ is closed under countable intersection.*

Proof. Let $f_n \in B_1(X,\mu)$ for each $n \in \mathbb{N}$. We have to show that $\bigcap_{n=1}^{\infty} Z(f_n) = Z(g)$ for some $g \in B_1(X,\mu)$. In fact, for each $x \in X$, we let $g(x) = \sum_{n=1}^{\infty}(\frac{1}{2^n} \bigwedge |f_n(x)|)$. Then by Weierstrass M-test, this series is uniformly convergent over X. Since for each $n \in \mathbb{N}$, $\frac{1}{2^n} \wedge |f_n| \in B_1(X,\mu)$, then by Theorem 1.7 we have $g \in B_1(X,\mu)$ and also it is clear that $Z(g) = \bigcap_{n=1}^{\infty} Z(f_n)$. □

The following theorem shows that to study the rings $B_1(X,\mu)$ and $B_1^*(X,\mu)$, we can take the measure μ on the σ-algebra containing τ being always complete.

Theorem 1.9. *Let $(X,\tau,\mathcal{A},\mu)$ be a $\tau\mathcal{A}\mu$-space. Then it is possible to construct another space $(X,\tau,\mathcal{A}^\star,\mu^\star)$ of the same type with the following properties: $\mathcal{A}^*$ is a σ-algebra on X containing $\mathcal{A}$; $\mu^* : \mathcal{A}^* \to [0,\infty]$ is a complete measure, extending the original measure $\mu : \mathcal{A} \to [0,\infty]$ and $B_1(X,\mu) = B_1(X,\mu^\star)$. Moreover $B_1^*(X,\mu) = B_1^*(X,\mu^\star)$.*

Proof. From the Theorem 2.10 [2], we have $\mathcal{M}_\circ(X,\mu) = \mathcal{M}_\circ(X,\mu^*)$. Therefore $B_1(X,\mu) = B_1(X,\mu^*)$ and also it is easy to see that $B_1^*(X,\mu) = B_1^*(X,\mu^\star)$. □

Now the notion of subspace of $(X,\tau,\mathcal{A},\mu)$ is defined as follows:

Definition 1.10 ([2]). Let $(X,\tau,\mathcal{A},\mu)$ be a $\tau\mathcal{A}\mu$-space. For any $E \in \mathcal{A}$, $\mathcal{A}|_E = \{E \cap A : A \in \mathcal{A}\}$ is a σ-algebra on the set E. Suppose that $(E,\tau|_E)$ is a subspace of (X,τ). Let $\mu|_E : (E,\mathcal{A}|_E) \to [0,\infty]$ be defined by $\mu|_E(F) = \mu(F)$ for any $F \in \mathcal{A}|_E$. Then $(X,\tau,\mathcal{A}|_E,\mu|_E)$ is called a subspace of the $\tau\mathcal{A}\mu$-space $(X,\tau,\mathcal{A},\mu)$.

Theorem 1.11. *Let $(X,\tau,\mathcal{A},\mu)$ be a $\tau\mathcal{A}\mu$-space. Take $(X,\tau,\beta,\mu|_\beta)$, where β is a Borel σ-algebra containing τ and $\mu|_\beta$ is the restriction of μ on $\beta(\subseteq \mathcal{A})$. Then $B_1(X,\mu) = B_1(X,\mu|_\beta)$.*

Proof. To prove this result, it is enough to show that $\mathcal{M}_\circ(X,\mu) = \mathcal{M}_\circ(X,\mu|_\beta)$. Let $f \in \mathcal{M}_\circ(X,\mu)$. Then $\mu(D_f) = 0$, where D_f is the discontinuity set of f. Since D_f is a F_σ-set (see [11]), $D_f \in \beta$. Thus $\mu|_\beta(D_f) = \mu(D_f) = 0$, so $f \in \mathcal{M}_\circ(X,\mu|_\beta)$. Therefore $\mathcal{M}_\circ(X,\mu) \subseteq \mathcal{M}_\circ(X,\mu|_\beta)$. Next, let $f \in \mathcal{M}_\circ(X,\mu|_\beta)$, then $\mu|_\beta(D_f) = 0$. Again D_f is a F_σ-set implies $D_f \in \beta \subseteq \mathcal{A}$. Thus $\mu|_\beta(D_f) = \mu(D_f) = 0$, so $f \in \mathcal{M}_\circ(X,\mu)$. Hence $\mathcal{M}_\circ(X,\mu|_\beta) \subseteq \mathcal{M}_\circ(X,\mu)$. Thus $\mathcal{M}_\circ(X,\mu) = \mathcal{M}_\circ(X,\mu|_\beta)$. Therefore $B_1(X,\mu) = B_1(X,\mu|_\beta)$. □

2. $B_1(X,\mu)$-separated and $B_1(X,\mu)$-embedded subsets of X

It is well known that two subsets A and B of a topological space X are said to be completely separated [see 1.15,[7]] if there exists a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

In an analogous manner we call two subsets A and B of a $\tau\mathcal{A}\mu$-space $(X,\tau,\mathcal{A},\mu)$, $B_1(X,\mu)$-separated ($\mathcal{M}_\circ$-separated [13]) if there is an element $f \in B_1(X,\mu)$ (respectively $f \in \mathcal{M}_\circ(X,\mu)$) such that $f(X) \subseteq [0,1]$ with $f(A) = \{1\}, f(B) = \{0\}$. Equivalently, for any two real numbers r,s with $r < s$, there exists $f : X \to [r,s]$, $f \in B_1(X,\mu)$(respectively $f \in \mathcal{M}_\circ(X,\mu)$) such that $f(A) = \{r\}$ and $f(B) = \{s\}$. Since $\mathcal{M}_\circ(X,\mu) \subseteq B_1(X,\mu)$, any two $\mathcal{M}_\circ$-separated subsets of X are also $B_1(X,\mu)$-separated. The following example shows that the converse need not be true.

Example 2.1. Let $X = [0,1]$ with τ_u, the topology on it inherited from the usual topology on the set $\mathbb{R}$ of reals, $\mathcal{P}(X)$ be the power set of X and δ_1 be the dirac measure at 1. Now define $f_n : X \to \mathbb{R}$ by $f_n(x) = 1 - x^n$ for $n \in \mathbb{N}$, then each $f_n \in \mathcal{M}_\circ(X,\delta_1)$ and $f_n \to f$ pointwise, where

$$f(x) = \begin{cases} 0, & \text{if } \mathrm{x{=}1} \\ 1, & \text{if } 0 \leq \mathrm{x} < 1. \end{cases}$$

Thus $f \in B_1(X,\delta_1) \setminus \mathcal{M}_\circ(X,\delta_1)$ and it separates two sets $\{1\}$ and $[0,1)$. But there does not exist any function in $\mathcal{M}_\circ(X,\delta_1)$ which separates $\{1\}$ and $[0,1)$.

Theorem 2.2. *Two subsets P,Q of $(X,\tau,\mathcal{A},\mu)$ are $B_1(X,\mu)$-separated in X if and only if they are contained in two disjoint zero sets in $Z[B_1(X,\mu)]$.*

Proof. Let P and Q be two $B_1(X,\mu)$-separated subsets in X. Then there exists $f \in B_1(X,\mu), f : X \to [0,1]$ such that $f(P) = \{0\}$ and $f(Q) = \{1\}$. Let $Z_1 = \{x \in X : f(x) \leq \frac{1}{3}\}$ and $Z_2 = \{x \in X : f(x) \geq \frac{1}{2}\}$. Then Z_1, Z_2 are two disjoint zero sets in $Z[B_1(X,\mu)]$ with $P \subseteq Z_1, Q \subseteq Z_2$.
Conversely, let $P \subseteq Z(f), Q \subseteq Z(g)$, where $Z(f) \cap Z(g) = \phi, f,g \in B_1(X,\mu)$. Take $h = \frac{f^2}{f^2+g^2} : X \to [0,1]$. Then $Z(f) \cap Z(g) = Z(f^2+g^2) = \phi$ and so $h \in B_1(X,\mu)$. Again we have $h(P) = \{0\}$, $h(Q) = \{1\}$. Hence P,Q are $B_1(X,\mu)$-separated in X. □

Corollary 2.3. *Any two disjoint zero sets in $Z[B_1(X,\mu)]$ are $B_1(X,\mu)$-separated in X.*

We recall from [1.16, [7]] that a subset A of a topological space X is said to be C-embedded (C^*-embedded) in X if each function $f \in C(A)$(respectively $f \in C^*(A)$) can be extended to a function in $C(X)$. Urysohn's Extension Theorem [Theorem 1.17, [7]] in $C(X)$ tells that a subset A of X is $C^\star$-embedded in X if and only if any two completely separated sets in A are also completely separated in X.

Definition 2.4. A measurable subset E of X (i.e., $E \in \mathcal{A}$) is said to be $B_1(X,\mu)$-embedded ($B_1^*(X,\mu)$-embedded) in X if each $f \in B_1(E,\mu|_E)$ (respectively $f \in B_1^*(E,\mu|_E)$) has an extension to a $g \in B_1(X,\mu)$.

It is clear that if $E \in \mathcal{A}$ is $B_1^*(X,\mu)$-embedded in X, then each $f \in B_1^*(E,\mu|_E)$ has an extension to a $g \in B_1^*(X,\mu)$.

The following theorem is an analogous version of Urysohn's Extension Theorem in our setting.

Theorem 2.5. *A measurable subset E of X is $B_1^*(X,\mu)$-embedded in X if and only if any two members of $\mathcal{A}|_E$ which are $B_1(X,\mu)$-separated in E are also $B_1(X,\mu)$-separated in X.*

Proof. Let E be $B_1^*(X,\mu)$-embedded in X. Let $A, B \in \mathcal{A}|_E$ be two $B_1(X,\mu)$-separated sets in E. Then there exists $f \in B_1(X,\mu|_E), f : E \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Since E is $B_1^*(X,\mu)$-embedded in X, f has an extension $g \in B_1(X,\mu)$ such that $g|_E = f$. Consider $h = (\underline{0} \vee g) \wedge \underline{1}$. Then $h \in B_1^*(X,\mu), h(X) \subseteq [0,1]$, $h(A) = \{0\}$ and $h(B) = \{1\}$. Thus A and B are $B_1(X,\mu)$-separated in X.

Conversely, let each pair of $B_1(X,\mu)$-separated members of $\mathcal{A}|_E$ in E be $B_1(X,\mu)$-separated in E are also $B_1(X,\mu)$-separated in X. Let $f_1 \in B_1^\star(E,\mu|_E)$. Then $|f_1| \leq m$ for some $m \in \mathbb{N}$. Take $r_n = \frac{m}{2}(\frac{2}{3})^n$ for all $n \in \mathbb{N}$. Then we have $|f_1| \leq 3r_1$ and thus inductively given $f_n \in \mathcal{M}_\circ^\star(E,\mu|_E)$ we have $|f_n| \leq 3r_n$. Consider $A_n = \{x \in E : f_n(x) \leq -r_n\}$ and $B_n = \{x \in E : f_n(x) \geq r_n\}$. Then A_n, B_n are disjoint zero sets in $Z[B_1(E,\mu|_E)]$. Hence, by Theorem 2.2, A_n, B_n are $B_1(X,\mu)$-separated in E and so by hypothesis, A_n, B_n are $B_1(X,\mu)$-separated in X. Thus there exists $g_n \in B_1(X,\mu)$ such that $g_n(X) \subseteq [-r_n, r_n]$, $g_n(A_n) = \{-r_n\}, g_n(B_n) = \{r_n\}$. Now set $f_{n+1} = f_n - g_n|_E$. Then it is easy to check that $|f_{n+1}| \leq 2r_n = 3r_{n+1}$. Therefore the induction step is completed. For each $x \in X$, let $g(x) = \sum\limits_{n=1}^{\infty} g_n(x)$. Then by Weierstrass' formula the infinite series is uniformly convergent to g on X and hence by Theorem 1.7, $g \in B_1(X,\mu)$. Now for all $x \in E$, $g(x) = \lim\limits_{n\to\infty}\{g_1(x) + g_2(x) + \cdots + g_n(x)\} = \lim\limits_{n\to\infty}\{f_1(x) - f_2(x) + f_2(x) - f_3(x) + \cdots + f_n(x) - f_{n+1}(x)\} = f_1(x) - \lim\limits_{n\to\infty} f_{n+1}(x) = f_1(x)$. Hence, the proof is complete. $\square$

The following result decides when a $B_1^*(X,\mu)$-embedded subset of X become $B_1(X,\mu)$-embedded in X. The proof of this result can be figured out by closely adapting the arguments in the proof of Theorem 1.18 in [7] and thus, the proof is omitted.

Theorem 2.6. *Let $E \in \mathcal{A}$ be $B_1^*(X,\mu)$-embedded in X. Then E is $B_1(X,\mu)$-embedded in X if and only if it is $B_1(X,\mu)$-separated from any zero set in $Z[B_1(X,\mu)]$ disjoint from it.*

3. $\mathcal{Z}_B$-filters and $\mathcal{Z}_B$-ideals of $B_1(X,\mu)$

Throughout the article, an ideal will always be a proper ideal.

To develop a connection between ideals of $B_1(X,\mu)$ and $\mathcal{Z}_B$-filters on X, we first prove the following theorem which is a sufficient condition for an element f of $B_1(X,\mu)$ to be a unit in $B_1(X,\mu)$.

Theorem 3.1. *Let $f \in B_1(X,\mu)$ be such that either $f(x) > 0$ for all $x \in X$ or $f(x) < 0$ for all $x \in X$. Then $\frac{1}{f}$ exists and belongs to $B_1(X,\mu)$.*

Proof. We first take $f \in B_1(X,\mu)$ and $f(x) > 0$ for all $x \in X$. Then there exists a sequence of functions $\{f_n\} \subseteq \mathcal{M}_\circ(X,\mu)$ such that $f_n \to f$ pointwise on X. For each $x \in X$, let $g_n(x) = |f_n(x)| + \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $g_n(x) > 0$ for all $x \in X$ and for all $n \in \mathbb{N}$ and also $g_n \to f$ pointwise on X. Now consider the function $g : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by $g(x) = \frac{1}{x}$ on $\mathbb{R}\backslash\{0\}$. Then $g \circ g_n \in \mathcal{M}_\circ(X,\mu)$ for each $n \in \mathbb{N}$. We now show that $g \circ g_n \to g \circ f$ pointwise. Using the continuity of g, for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|g(g_n(x)) - g(f(x))| < \epsilon$ for $|g_n(x) - f(x)| < \delta$. Since $g_n \to f$ pointwise, there exists a $k \in \mathbb{N}$ such that $|g_n(x) - f(x)| < \delta$ for all $n \geq k$. Thus $|g(g_n(x)) - g(f(x))| < \epsilon$ for all $n \geq k$. Therefore $g \circ g_n \to g \circ f$ pointwise i.e., $g \circ f \in B_1(X,\mu)$ and $g \circ f(x) = \frac{1}{f(x)}$ shows $\frac{1}{f} \in B_1(X,\mu)$.

Similarly, we can prove the result when $f(x) < 0$ for all $x \in X$. □

Definition 3.2. A non-empty subfamily $\mathcal{F}$ of $Z[B_1(X,\mu)]$ is called a $\mathcal{Z}_B$-filter on X if it satisfies the following conditions:

(i) $\phi \notin \mathcal{F}$,
(ii) $Z_1, Z_2 \in \mathcal{F}$ implies $Z_1 \cap Z_2 \in \mathcal{F}$ and
(iii) If $Z \in \mathcal{F}$ and $Z' \in Z[B_1(X,\mu)]$ such that $Z \subseteq Z'$, then $Z' \in \mathcal{F}$.

A $\mathcal{Z}_B$-filter on X which is not properly contained in any $\mathcal{Z}_B$-filter on X is called $\mathcal{Z}_B$-ultrafilter. Using Zorn's lemma, it can be established that a $\mathcal{Z}_B$-filter on X can be extended to a $\mathcal{Z}_B$-ultrafilter on X. It is interesting to note that there is a duality between ideals (maximal ideals) in $B_1(X,\mu)$ and the $\mathcal{Z}_B$-filters (respectively $\mathcal{Z}_B$-ultrafilters) on X and this is emphasized by the following result.

Theorem 3.3. *For the ring $B_1(X,\mu)$, the following statements are true.*

(i) *If I is an ideal(proper) of $B_1(X,\mu)$, then $Z[I] = \{Z(f) : f \in I\}$ is a $\mathcal{Z}_B$-filter on X. Dually for any $\mathcal{Z}_B$-filter $\mathcal{F}$ on X, $Z^{-1}[\mathcal{F}] = \{f \in B_1(X,\mu) : Z(f) \in \mathcal{F}\}$ is an ideal(proper) in $B_1(X,\mu)$.*
(ii) *If M is a maximal ideal of $B_1(X,\mu)$ then $Z[M]$ is a $\mathcal{Z}_B$-ultrafilter on X. If $\mathcal{U}$ is a $\mathcal{Z}_B$-ultrafilter on X, then $Z^{-1}[\mathcal{U}]$ is a maximal ideal of $B_1(X,\mu)$. Moreover the assignment: $M \to Z[M]$ defines a bijection on the set of all maximal ideals in $B_1(X,\mu)$ and the collection of all $\mathcal{Z}_B$-ultrafilters on X.*

Proof. (i) We first show that $\varnothing \notin Z[I]$. If possible, let $\varnothing \in Z[I]$, then there exists a $f \in I$ such that $Z(f) = \varnothing$. Then $f^2 \in I$ and it is a unit by Theorem 3.1, which contradicts our assumption that I is a proper ideal. Next let $Z(f)$ and $Z(g) \in Z[I]$. Then $Z(f) \cap Z(g) = Z(f^2 + g^2)$ in $Z[I]$, since $f, g \in I$ implies $f^2 + g^2 \in I$. Finally, let $Z(f) \in Z[I]$ and $Z(f) \subseteq Z(h)$ for some $h \in B_1(X,\mu)$, then $Z(h) = Z(f \cdot h) \in Z[I]$ as $f \cdot h \in I$. Therefore $Z[I]$ is a $\mathcal{Z}_B$-filter on X.

Since $\varnothing \notin \mathcal{F}$, $\underline{1} \notin Z^{-1}[\mathcal{F}]$ as $Z(\underline{1}) = \varnothing$. Thus $Z^{-1}[\mathcal{F}]$ is a proper subset of $B_1(X,\mu)$. Let $f, g \in Z^{-1}[\mathcal{F}]$. Then $Z(f), Z(g) \in \mathcal{F}$ and $Z(f) \cap Z(g) \in \mathcal{F}$ as

$\mathcal{F}$ is a $\mathcal{Z}_B$-filter. Now $Z(f) \cap Z(g) \subseteq Z(f-g)$ implies $Z(f-g) \in \mathcal{F}$ as $\mathcal{F}$ is a $\mathcal{Z}_B$-filter. This shows that $f - g \in Z^{-1}[\mathcal{F}]$. If $f \in Z^{-1}[\mathcal{F}]$ and $h \in B_1(X, \mu)$, then $Z(f) \cup Z(h) = Z(f \cdot h) \supset Z(f)$ implies $Z(f \cdot h) \in \mathcal{F}$ by the property of filter. Thus $f \cdot h \in Z^{-1}[\mathcal{F}]$. Therefore $Z^{-1}[\mathcal{F}]$ is an ideal of $B_1(X, \mu)$.

(ii) The proof of this part easily follows from the Theorem 2.5 [7]. ☐

Definition 3.4. An ideal I of $B_1(X, \mu)$ is called fixed if $\cap Z[I] \neq \varnothing$. Otherwise it is called a free ideal.

For any $p \in X$, $M_p = \{f \in B_1(X, \mu) : f(p) = 0\}$ is a fixed maximal ideal of $B_1(X, \mu)$ and each fixed maximal ideal of $B_1(X, \mu)$ is of this form. It follows from Theorem 3.3 that for any $p \in X$, $Z[M_p] = \mathcal{U}_p$, where $\mathcal{U}_p = \{Z \in Z[B_1(X, \mu)] : p \in Z\}$ is a typical fixed $\mathcal{Z}_B$-ultrafilter on X.

Definition 3.5. An ideal I of $B_1(X, \mu)$ is said to be $\mathcal{Z}_B$-ideal if $Z^{-1}Z[I] = I$ i.e., for $f, g \in B_1(X, \mu)$ with $Z(f) = Z(g)$, and $f \in I$ implies $g \in I$.

From the above definition and by Theorem 3.3, we can easily prove that every maximal ideal of $B_1(X, \mu)$ is a $\mathcal{Z}_B$-ideal. But if we take $(\mathbb{R}, \tau_u, \mathcal{L}, \mu)$ where τ_u is the usual topology on $\mathbb{R}$, $\mathcal{L}$ is the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}$ and μ is the Lebesgue measure on $\mathcal{L}$, then the ideal $I = \{f \in B_1(\mathbb{R}, \mu) : f(2) = f(3) = 0\}$ is a $\mathcal{Z}_B$-ideal that is not a maximal ideal.

The following theorem is a characterization of prime $\mathcal{Z}_B$-ideals of $B_1(X, \mu)$.

Theorem 3.6. *For a $\mathcal{Z}_B$-ideal I of $B_1(X, \mu)$, the following statements are equivalent:*

(i) *I is a prime ideal of $B_1(X, \mu)$.*
(ii) *I contains a prime ideal of $B_1(X, \mu)$.*
(iii) *If $fg = 0$ for $f, g \in B_1(X, \mu)$, then either $f \in I$ or $g \in I$.*
(iii) *For any $f \in B_1(X, \mu)$ there exists $Z \in Z[I]$ such that f does not change its sign on Z.*

Proof. The proof is analogous to the proof of Theorem 2.9 of [7] and thus, it is omitted. ☐

With the help of above theorem and the fact that the intersection of an arbitrary collection of $\mathcal{Z}_B$-deals of $B_1(X, \mu)$ is a $\mathcal{Z}_B$-ideal, we can state the following theorem which is an analogue version of Theorem 2.11 [7].

Theorem 3.7. *Every prime ideal of $B_1(X, \mu)$ can be extended to a unique maximal ideal of $B_1(X, \mu)$ and therefore $B_1(X, \mu)$ is a Gelfand ring.*

Let $Max(B_1(X, \mu))$ be the structure space of $B_1(X, \mu)$ i.e., $Max(B_1(X, \mu))$ is the set of all maximal ideals of $B_1(X, \mu)$ equipped with hull-kernel topology. Then $\{\mathcal{M}_f : f \in B_1(X, \mu)\}$ form a base for closed sets of this hull-kernel topology, 7M [7], where $\mathcal{M}_f = \{M \in Max(B_1(X, \mu)) : f \in M\}$. Using Theorem 1.2 of [10], we have $Max(B_1(X, \mu))$ is a Hausdorff compact space. It is checked that the structure space of $B_1(X, \mu)$ is the same with the set of all $\mathcal{Z}_B$-ultrafilters on X with Stone topology.

Let $\beta_{B_1}X$ be an index set for the family of all $\mathcal{Z}_B$-ultrafilters on X i.e., for each $p \in \beta_{B_1}X$, there exists a $\mathcal{Z}_B$-ultrafilter on X, which is denoted by $\mathcal{U}^p$. For any $p \in X$, we can find a fixed $\mathcal{Z}_B$-ultrafilter $\mathcal{U}_p$ and set $\mathcal{U}_p = \mathcal{U}^p$. Then we can think X as a subset of $\beta_{B_1}X$.

Now we want to define a topology on $\beta_{B_1}X$. Let $\beta = \{\overline{Z} : Z \in Z[B_1(X,\mu)]\}$, where $\overline{Z} = \{p \in \beta_{B_1}X : Z \in \mathcal{U}^p\}$. Then β is a base for closed sets for some topology on $\beta_{B_1}X$. Since X belongs to every $\mathcal{Z}_B$-ultrafilters on X, $\overline{X} = \beta_{B_1}X$. Again $p \in \overline{Z} \cap X \Leftrightarrow Z \in \mathcal{U}^p \Leftrightarrow p \in Z$. Therefore $\overline{Z} \cap X = Z$. It is easy to observe that if $Z_1, Z_2 \in Z[B_1(X,\mu)]$ with $Z_1 \subseteq Z_2$, then $\overline{Z_1} \subseteq \overline{Z_2}$. This leads to the following result.

Theorem 3.8. *For $Z \in Z[B_1(X,\mu)]$, $\overline{Z} = Cl_{\beta_{B_1}X}Z$.*

Proof. Let $Z \in Z[B_1(X,\mu)]$ and $\overline{Z_1} \in \beta$ be such that $Z \subseteq \overline{Z_1}$. Then $Z \subseteq \overline{Z_1} \cap X = Z_1$. This implies $\overline{Z} \subseteq \overline{Z_1}$. Therefore $\overline{Z}$ is the smallest basic closed set containing Z. Hence $\overline{Z} = Cl_{\beta_{B_1}X}Z$. □

Now, we want to show that $Max(B_1(X,\mu))$ and $\beta_{B_1}X$ are homeomorphic.

Theorem 3.9. *The map $\phi : Max(B_1(X,\mu)) \to \beta_{B_1}X$, defined by $\phi(M) = p$ is a homeomorphism, where $Z[M] = \mathcal{U}^p$.*

Proof. The map ϕ is bijective by Theorem 3.3 (ii). Basic closed set of $Max(B_1(X,\mu))$ is of the form $\mathcal{M}_f = \{M \in Max(B_1(X,\mu)) : f \in M\}$, for some $f \in B_1(X,\mu)$. Now $M \in \mathcal{M}_f \Leftrightarrow f \in M \Leftrightarrow Z(f) \in Z[M]$ (since maximal ideal is a $\mathcal{Z}_B$-ideal) $\Leftrightarrow Z(f) \in \mathcal{U}^p \Leftrightarrow p \in \overline{Z(f)}$. Thus $\phi(\mathcal{M}_f) = \overline{Z(f)}$. Therefore ϕ interchanges basic closed sets of $Max(B_1(X,\mu))$ and $\beta_{B_1}X$. Hence $Max(B_1(X,\mu))$ is homeomorphic to $\beta_{B_1}X$. □

Now we prove the following theorem which is an analogous version of the Gelfand-Kolmogoroff Theorem 7.3 [7].

Theorem 3.10. *Every maximal ideal of $B_1(X,\mu)$ is of the form $M^p = \{f \in B_1(X,\mu) : p \in Cl_{\beta_{B_1}X}Z(f)\}$, for some $p \in \beta_{B_1}X$.*

Proof. Let M be any maximal ideal of $B_1(X,\mu)$. Then $Z[M]$ is a $\mathcal{Z}_B$-ultrafilter on X. Thus $Z[M] = \mathcal{U}^p$, for some $p \in \beta_{B_1}X$. So, $f \in M \Leftrightarrow Z(f) \in Z[M]$ as M is a $\mathcal{Z}_B$-ideal $\Leftrightarrow Z(f) \in Z[M] = \mathcal{U}^p \Leftrightarrow p \in \overline{Z(f)} = Cl_{\beta_{B_1}X}Z(f)$. Hence $M = \{f \in B_1(X,\mu) : p \in Cl_{\beta_{B_1}X}Z(f)\}$ and so we can write $\{f \in B_1(X,\mu) : p \in Cl_{\beta_{B_1}X}Z(f)\} = M^p$, $p \in \beta_{B_1}X$. This completes the proof. □

It is interesting to note that the Stone-Čech compactification βX of X, $\beta_{\mathcal{M}_\circ}X$ (index set for the family of all $\mathcal{Z}$-ultrafilters on X, defined in [2]) and $\beta_{B_1}X$ (defined above) are the same if X is equipped with the discrete topology. The following example shows that these spaces may not be homeomorphic to each other.

Example 3.11. Let $X = (1,2) \cup \{3\}$. Consider $(X, \tau_u, \mathcal{P}(X), \delta_3^\infty)$, where τ_u is the subspace topology on X of the real line and $\mathcal{P}(X)$ be the power set of X

and for any $A \in \mathcal{P}(X)$, define a measure δ_3^∞ on $\mathcal{P}(X)$ as follows:

$$\delta_3^\infty(A) = \begin{cases} \infty, & \text{if } 3 \in A \\ 0, & \text{if } 3 \notin A. \end{cases}$$

Then we have $\mathcal{M}_\circ(X, \delta_3^\infty) = B_1(X, \delta_3^\infty) = \mathbb{R}^X$. So $\beta_{\mathcal{M}_\circ} X = \beta_{B_1} X$ is equal to the Stone-$\check{C}$ech compactification of X, if X is equipped with discrete topology. Now clearly $\beta_{\mathcal{M}_\circ} X = \beta_{B_1} X$ has uncountably many isolated points (in fact, each point of X). But βX has exactly one isolated point namely at 3. Hence $\beta_{\mathcal{M}_\circ} X = \beta_{B_1} X$ is not homeomorphic to βX.

Again take $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Consider $(X, \tau_u, \mathcal{P}(X), \mu_c)$, where τ_u is the subspace topology on X of real line and μ_c, the counting measure on $\mathcal{P}(X)$. Then $C(X) = \mathcal{M}_\circ(X, \mu_c)$. Thus $\beta X = \beta_{\mathcal{M}_\circ} X$. Since X is a perfectly normal space, by Theorem 2.6 in [9], $B_1(X) = B_1(X, \mu_c) \supseteq C(X)_F$ (ring of functions which are discontinuous on a finite set [6]). Since X contains only one non-isolated point, the cardinality of the discontinuity set of any $f \in \mathbb{R}^X$ is not more than 1 and hence $B_1(X, \mu_c) = \mathbb{R}^X = C(X, \tau_d)$, where $C(X, \tau_d)$ is rings of continuous functions with discrete topology τ_d. Hence $\beta_{B_1} X$ is the Stone-$\check{C}$ech compactification of X, if X is equipped with discrete topology and the cardinality of $\beta_{B_1} X$ is equal to $|\beta\mathbb{N}| = 2^c$, where $\beta\mathbb{N}$ is the Stone-$\check{C}$ech compactification of the set $\mathbb{N}$ of natural numbers. Since (X, τ_u) is a compact space, βX is homeomorphic to X. Now the cardinality of βX is $\aleph_\circ$ implies $\beta_{B_1} X$ is not homeomorphic to $\beta X = \beta_{\mathcal{M}_\circ} X$.

Therefore the spaces βX, $\beta_{\mathcal{M}_\circ} X$ and $\beta_{B_1} X$ are not homeomorphic to each other.

4. Residue class of $B_1(X, \mu)$ modulo ideals and real maximal ideal of $B_1(X, \mu)$

Definition 4.1. For a partial ordered ring R, an ideal I is called convex if $a, b, c \in R$ with $a \leq b \leq c$, and $a, c \in I$ implies $b \in I$.

Definition 4.2. For a lattice ordered ring R, an ideal I is called absolutely convex if $a, b \in R$ with $|a| \leq |b|$, and $b \in I$ implies $a \in I$.

Example 4.3. Let $\psi : B_1(X, \mu) \to B_1(Y, \mu')$ be a homomorphism. Then $Ker\psi$ is an absolute convex ideal of $B_1(X, \mu)$. Indeed, let $f, g \in B_1(X, \mu)$ with $|f| \leq |g|$ and $g \in Ker\psi$. Then $\psi(|g|) = |\psi(g)| = 0$. This implies $\psi(|f|) = |\psi(f)| = \psi(f) = 0$ as homomorphism preserves order. Thus $f \in Ker\psi$ and hence $Ker\psi$ is an absolute convex ideal of $B_1(X, \mu)$.

Example 4.4. Every $\mathcal{Z}_B$-ideal I of $B_1(X, \mu)$ is absolutely convex as $|f| \leq |g|$ and $g \in I$, implies $Z(g) \subseteq Z(f)$ and $Z(g) \in Z[I]$. Thus $Z(f) \in Z[I]$ and hence $f \in I$ as I is $\mathcal{Z}_B$-ideal. Thus every maximal ideal of $B_1(X, \mu)$ is absolutely convex.

The following theorem follows from the Theorems 5.2, 5.3 [7].

Theorem 4.5. *Let I be an absolute convex ideal of a lattice ordered ring R. Then*

(i) *R/I is a lattice ordered ring according to the definition: $I(a) \geq 0$ if there exists $x \in R$ such that $x \geq 0$ and $I(a) = I(x)$. Here $I(a)$ denote the residue class of a in R.*
(ii) *$I(a) \geq 0$ if and only if $I(a) = I(|a|)$.*
(iii) *$I(|a|) = |I(a)|$ for each $a \in R$.*

The following theorem is an immediate consequence of Example 4.4 and Theorem 4.5.

Theorem 4.6. *If I is a $\mathcal{Z}_B$-ideal of $B_1(X,\mu)$, then the quotient ring $B_1(X,\mu)/I$ is a lattice ordered ring.*

The following theorem gives a description of non-negative elements of $B_1(X,\mu)/I$, when I is a $\mathcal{Z}_B$-ideal of $B_1(X,\mu)$.

Theorem 4.7. *Let I be a $\mathcal{Z}_B$-ideal of $B_1(X,\mu)$ and $f \in B_1(X,\mu)$. Then $I(f) \geq 0$ in $B_1(X,\mu)/I$ if and only if there exists $Z \in Z[I]$ such that $f \geq 0$ on Z.*

Proof. First assume that $I(f) \geq 0$. Then by Theorem 4.5, $I(f) = I(|f|)$. This implies $f - |f| \in I$. Let $Z' = Z(f - |f|) \in Z[I]$. Then $f \geq 0$ on Z'. Conversely, assume that there exists $Z \in Z[I]$ such that $f \geq 0$ on Z. Then $f = |f|$ on $Z \implies Z \subseteq Z(f - |f|)$ and $Z \in Z[I] \implies Z(f - |f|) \in Z[I] \implies f - |f| \in I$, as I is $\mathcal{Z}_B$-ideal $\implies I(f) = I(|f|) \geq 0$. □

The following theorem is a description of the maximal ideal of $B_1(X,\mu)$ with the help of zero sets.

Theorem 4.8. *Let M be a maximal ideal of $B_1(X,\mu)$. Then for any $f \in B_1(X,\mu)$, there exists $Z \in Z[M]$ on which f does not change its sign.*

Proof. Let $f \in B_1(X,\mu)$ and M be a maximal ideal of $B_1(X,\mu)$. Since $(f \vee 0) \cdot (f \wedge 0) = 0$ and each maximal ideal is prime, $f \vee 0 \in M$ or $f \wedge 0 \in M$. This implies $Z(f \vee 0) \in Z[M]$ or $Z(f \wedge 0) \in Z[M]$. Also $f \geq 0$ on $Z(f \vee 0)$ and $f \leq 0$ on $Z(f \wedge 0)$. Thus there exists $Z \in Z[M]$ on which f does not change its sign. □

Corollary 4.9. *Let M be a maximal ideal of $B_1(X,\mu)$. Then the residue class ring $B_1(X,\mu)/M$ is totally ordered.*

Proof. Let $f \in B_1(X,\mu)$ and M be a maximal ideal of $B_1(X,\mu)$. Then by Theorem 4.8, there is a $Z \in Z[M]$ on which $f \geq 0$ or $f \leq 0$. Thus in view of Theorem 4.7, $M(f) \geq 0$ or $M(f) \leq 0$ in $B_1(X,\mu)/M$. Hence $B_1(X,\mu)/M$ is totally ordered. □

Definition 4.10. A maximal ideal M of $B_1(X,\mu)$(or $B_1^*(X,\mu)$) is called real if the canonical map $\psi : \mathbb{R} \to B_1(X,\mu)/M$(respectively $\psi : \mathbb{R} \to B_1^*(X,\mu)/M$) defined by $r \mapsto M(\underline{r})$ is onto. A maximal ideal M is called hyperreal if it is not real.

It is easy to check that ψ is an ordered preserving injective map.

By using Theorem 0.22 [7], we can show that a maximal ideal M of $B_1(X,\mu)$ is real if and only if $B_1(X,\mu)/M$ is isomorphic to $\mathbb{R}$.

A totally ordered field F is called archimedean if for any $a \in F$, there exists an $n \in \mathbb{N}$ such that $a \leq n$. So, a non-archimedean ordered field F contains an element $a \in F$ such that $a > n$ for all $n \in \mathbb{N}$. Such element a is called an infinitely large element in F.

The following theorem is noted in Theorem 0.21 [7].

Theorem 4.11. *An ordered field is archimedean if and only if it is isomorphic to a subfield of* $\mathbb{R}$.

Theorem 4.12. *Let M be a maximal ideal of $B_1(X,\mu)(B_1^*(X,\mu))$. Then M is real maximal of $B_1(X,\mu)(B_1^*(X,\mu))$ if and only if $B_1(X,\mu)/M$ (respectively $B_1^*(X,\mu)/M$) is archimedean.*

Proof. First we assume that M is real. Then $B_1(X,\mu)/M \cong \mathbb{R}$. Thus $B_1(X,\mu)/M$ is archimedean. Conversely, let $B_1(X,\mu)/M$ be archimedean. Then by Theorem 0.21 [7], there exists an isomorphism ϕ from $B_1(X,\mu)/M$ into $\mathbb{R}$. We claim that $\phi(B_1(X,\mu)/M) = \mathbb{R}$. If $\phi(B_1(X,\mu)/M) \subsetneqq \mathbb{R}$. Then $\phi \circ \psi$ is an isomorphism from $\mathbb{R}$ onto a proper subfield of $\mathbb{R}$, which contradicts the Theorem 0.22 [7]. Thus $\phi(B_1(X,\mu)/M) = \mathbb{R}$. Therefore $B_1(X,\mu)/M$ isomorphic to $\mathbb{R}$. Hence M is real.

By using the same arguments, we can show that the result is also true for $B_1^*(X,\mu)$. □

The following theorem characterizes all maximal ideals of $B_1^*(X,\mu)$.

Theorem 4.13. *Each maximal ideal M of $B_1^*(X,\mu)$ is always real.*

Proof. Choose $f \in B_1^*(X,\mu)$, then $|f| \leq n$ for some $n \in \mathbb{N}$. This implies $M(f) \leq M(\underline{n})$. Thus $B_1^*(X,\mu)/M$ contains no infinitely large element. Hence M is real by Theorem 4.12. □

The following result shows the relation between the infinitely large elements in the residue class field $B_1(X,\mu)/M$, where M is a maximal ideal of $B_1(X,\mu)$ and the unbounded functions of $B_1(X,\mu)$.

Theorem 4.14. *Let $f \in B_1(X,\mu)$ and M be a maximal ideal of $B_1(X,\mu)$. Then the following statements are equivalent.*

(i) *$|M(f)|$ is an infinitely large element of the residue class field $B_1(X,\mu)/M$.*
(ii) *For all $Z \in Z[M]$, f is unbounded on Z.*
(iii) *For all $n \in \mathbb{N}, Z_n = \{x \in X : |f(x)| \geq n\} \in Z[M]$.*

Proof. $(i) \Leftrightarrow (ii)$: Now $M(f) \leq M(\underline{n})$ for some $n \in \mathbb{N}$. $\Leftrightarrow$ $|f| \leq n$ on some $Z \in Z[M]$, as M is $\mathcal{Z}_B$-ideal. Thus f is bounded on some $Z \in Z[M]$. This proves $(i) \Leftrightarrow (ii)$.

$(ii) \implies (iii)$: Assume (ii) holds and choose $Z \in Z[M]$. So f is unbounded on Z. Hence $Z \cap Z_n \neq \varnothing$, for all $n \in \mathbb{N}$. Thus each Z_n meets each set Z in $Z[M]$. So $Z_n \in Z[M]$, for all $n \in \mathbb{N}$, as $Z[M]$ is a $\mathcal{Z}_B$-ultrafilter on X.

$(iii) \implies (ii)$: Suppose (ii) does not hold. Then there exists a $Z \in Z[M]$ on which f is bounded i.e., there exists $n \in \mathbb{N}$, for which $|f| \leq n$ on Z. Hence $Z_{n+1} \notin Z[M]$ as $Z_{n+1} \cap Z = \phi$. Hence, (iii) does not hold. □

Theorem 4.15. *A function $f \in B_1(X,\mu)$ is unbounded on X if and only if there exists a maximal ideal M in $B_1(X,\mu)$, for which $|M(f)|$ is an infinitely large element of $B_1(X,\mu)/M$.*

Proof. First we assume that $f \in B_1(X,\mu)$ is unbounded on X. Then $Z_n = \{x \in X : |f(x)| \geq n\}$ is non-empty for all $n \in \mathbb{N}$. In fact, $\{Z_n : n \in \mathbb{N}\}$ is a family of zero sets in X with finite intersection property. Hence there exists a $\mathcal{Z}_B$-ultrafilter $Z[M]$ on X for some maximal ideal M in $B_1(X,\mu)$ such that $Z_n \in Z[M]$, for all $n \in \mathbb{N}$. Then by Theorem 4.14, $|M(f)|$ becomes infinitely large in $B_1(X,\mu)/M$.
Conversely, suppose there exists a maximal ideal M in $B_1(X,\mu)$ for which $|M(f)|$ is infinitely large in $B_1(X,\mu)/M$. Then by Theorem 4.14, f becomes unbounded on each $Z \in Z[M]$. In particular f is unbounded on X. □

From the following theorem we can assert that each hyperreal maximal ideal must be a free ideal in $B_1(X,\mu)$.

Theorem 4.16. *Every fixed maximal ideal in $B_1(X,\mu)$ is real.*

Proof. Any fixed maximal ideal of $B_1(X,\mu)$ is of the form $M_p = \{f \in B_1(X,\mu) : f(p) = 0\}$ for some $p \in X$. Consider the mapping $\psi : B_1(X,\mu) \to \mathbb{R}$ such that $f \mapsto f(p)$. Then ψ is an onto homomorphism. Therefore $B_1(X,\mu)/Ker\psi \cong \mathbb{R}$, by first isomorphism theorem. Now $Ker\psi = \{f \in B_1(X,\mu) : \psi(f) = 0\} = \{f \in B_1(X,\mu) : \psi(f) = f(p) = 0\} = M_p$. Therefore $B_1(X,\mu)/M_p \cong \mathbb{R}$. This shows that M_p is a real maximal ideal of $B_1(X,\mu)$. □

The following result gives a characterization of real maximal ideal of $B_1(X,\mu)$.

Theorem 4.17. *For a maximal ideal M of $B_1(X,\mu)$, the following statements are equivalent:*

(i) *M is a real maximal ideal of $B_1(X,\mu)$.*
(ii) *The $\mathcal{Z}_B$-ultrafilter $Z[M]$ is closed under countable intersection.*
(iii) *$Z[M]$ has countable intersection property.*

Proof. $(i) \implies (ii)$: Assume that (ii) is false. This means that there exists a sequence of functions $\{f_n\}$ in M such that $\bigcap\limits_{n=1}^{\infty} Z(f_n) \notin Z[M]$. Set $f = \sum\limits_{n=1}^{\infty}(|f_n| \wedge \frac{1}{2^n})$, then $f \in B_1(X,\mu)$ and $Z(f) = \bigcap\limits_{n=1}^{\infty} Z(f_n) \notin Z[M] \implies f \notin M \implies M(f) > 0$. For any $k \in \mathbb{N}$, set $Z = Z(f_1) \cap Z(f_2) \cap \cdots \cap Z(f_k)$. Now for all $x \in Z, f(x) = \sum\limits_{n=k+1}^{\infty}(|f_n(x)| \wedge \frac{1}{2^n}) \implies 0 \leq f(x) \leq \sum\limits_{n=k+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^k} \implies$

$0 \leq M(f) \leq M(\frac{1}{2^k})$, for all $k \in \mathbb{N}$. This shows that $M(f)$ is not an infinitely large element of $B_1(X,\mu)/M$. So, $B_1(X,\mu)/M$ becomes non archimedean i.e., M is hyperreal. Thus (i) becomes false.

$(ii) \implies (iii)$: Trivial.

$(iii) \implies (i)$: Assume (i) is false i.e., M is hyperreal. So, there exists $f \in B_1(X,\mu)$, for which $|M(f)|$ is an infinitely large in $B_1(X,\mu)/M$. Hence by Theorem 4.14, we can say that each $Z_n = \{x \in X : |f(x)| \geq n\} \in Z[M]$ for all $n \in \mathbb{N}$. We see that $\bigcap\limits_{n=1}^{\infty} Z_n = \phi$, which shows that the condition (iii) becomes false. This completes the proof. □

Definition 4.18. A $\tau\mathcal{A}\mu$-space X is called real compact if every real maximal ideal of $B_1(X,\mu)$ is fixed.

Example 4.19. Take $(\mathbb{R}, \tau_u, \mathcal{L}, \mu)$, where τ_u is the usual topology on $\mathbb{R}$, $\mathcal{L}$ is the set of all Lebesgue measurable subsets of $\mathbb{R}$ and μ is Lebesgue measure on $\mathcal{L}$. Let M be a real maximal ideal of $B_1(\mathbb{R},\mu)$. Then the identity map i on $\mathbb{R}$ is an element of $B_1(\mathbb{R},\mu)$. Since M is a real maximal ideal in $B_1(X,\mu)$, there exists $r \in \mathbb{R}$ such that $M(i) = M(\underline{r})$. Then $i - \underline{r} \in M$, and so $Z(i - \underline{r}) \in Z[M]$. Now $Z(i - \underline{r})$ is a singleton set. Thus M is fixed. Therefore it is a real compact space.

The following theorem characterizes real compact spaces with the help of ring homomorphisms from $B_1(X,\mu)$ into $\mathbb{R}$.

Theorem 4.20. *A $\tau\mathcal{A}\mu$-space X is real compact if and only if for each non-zero homomorphism $\psi : B_1(X,\mu) \to \mathbb{R}$, there exists a point $x \in X$ such that $\psi(f) = f(x)$ for all $f \in B_1(X,\mu)$.*

Proof. Let X be real compact. Let $\psi : B_1(X,\mu) \to \mathbb{R}$ be a non-zero homomorphism, then $\psi(\underline{r}) = r$ for all $r \in \mathbb{R}$ and $B_1(X,\mu)/Ker\psi \cong \mathbb{R}$. So $Ker\psi$ is of the form M_x for some $x \in X$. Now we define $\phi : B_1(X,\mu)/Ker\psi \to B_1(X,\mu)/Ker\psi$ by $\phi(f + Ker\psi) = \underline{f(x)} + Ker\psi$. Then ϕ is a homomorphism. Since the identity map is the only non-zero homomorphism from $B_1(X,\mu)/Ker\psi$ to $B_1(X,\mu)/Ker\psi$, thus $f + Ker\psi = \underline{f(x)} + Ker\psi$. This implies $\psi(f - \underline{f(x)}) = 0$. Hence $\psi(f) = f(x)$. Conversely, let M be a real maximal ideal of $B_1(X,\mu)$ and $\phi : B_1(X,\mu)/M \to \mathbb{R}$ be an isomorphism. Define a homomorphism $\psi : B_1(X,\mu) \to \mathbb{R}$ by $\psi(f) = \phi(f + M)$. Then by the given hypothesis $\psi(f) = f(x)$ for some $x \in X$ and for all $f \in B_1(X,\mu)$. Thus $\phi(f + M) = f(x)$, implies $f(x) = 0$ if and only if $f \in M$. Therefore $M = M_x$ is a fixed maximal ideal of $B_1(X,\mu)$. This completes the proof. □

5. Real maximal ideal of $\mathcal{M}_\circ(X,\mu)$ and $B_1(X,\mu)$

For an ideal I of $\mathcal{M}_\circ(X,\mu)$, we define $I_B = \{f \in \mathbb{R}^X$: there exists a sequence of functions $\{f_n\} \subseteq I$ such that $f_n \to f$ pointwise$\}$. We can easily prove that I_B is an ideal of $B_1(X,\mu)$ and $I \subseteq I_B \cap \mathcal{M}_\circ(X,\mu)$.

The next theorem states that for any fixed maximal ideal M of $\mathcal{M}_\circ(X,\mu)$, the ideal M_B of $B_1(X,\mu)$ is fixed.

Theorem 5.1. *For any $p \in X$, we have $(M_p)_B = \widetilde{M_p}$, where $M_p = \{f \in \mathcal{M}_\circ(X,\mu) : f(p) = 0\}$ and $\widetilde{M_p} = \{f \in B_1(X,\mu) : f(p) = 0\}$.*

Proof. Let $f \in (M_p)_B$. Then there exists a sequence $\{f_n\} \subseteq M_p$ such that $f_n \to f$ pointwise on X. Since each $f_n \in M_p$, $f_n(p) = 0$ for all $n \in \mathbb{N}$. Hence $f(p) = 0$ and thus $(M_p)_B \subseteq \widetilde{M_p}$. Next, let $f \in \widetilde{M_p}$. Then $f(p) = 0$. Since $f \in B_1(X,\mu)$, there exists $\{g_n\} \subseteq \mathcal{M}_\circ(X,\mu)$ such that $g_n \to f$ pointwise on X. Set $f_n = g_n - g_n(p)$, then $f_n(p) = 0$ for all $n \in \mathbb{N}$ and each $f_n \in \mathcal{M}_\circ(X,\mu)$. Also it is clear that $f_n \to f$ pointwise on X. Hence $f \in (M_p)_B$. Therefore $\widetilde{M_p} \subseteq (M_p)_B$. This completes the proof. □

A maximal ideal M of $\mathcal{M}_\circ(X,\mu)$ is called a $\tau\mathcal{A}\mu$-real maximal ideal (see Definition 9 in [13]) or simply a real maximal ideal of $\mathcal{M}_\circ(X,\mu)$ if $\mathcal{M}_\circ(X,\mu)/M$ is isomorphic to $\mathbb{R}$.

For any proper ideal I of $\mathcal{M}_\circ(X,\mu)$, we always have $I \subseteq I_B \cap \mathcal{M}_\circ(X,\mu)$. The following theorem shows when the equality holds.

Theorem 5.2. *A maximal ideal M of $\mathcal{M}_\circ(X,\mu)$ is real if and only if $M = M_B \cap \mathcal{M}_\circ(X,\mu)$.*

Proof. Let M be a real maximal ideal of $\mathcal{M}_\circ(X,\mu)$ and $f \in M_B \cap \mathcal{M}_\circ(X,\mu)$. Then there exists $\{f_n\} \subseteq M$ such that $f_n \to f$ pointwise. Since M is real, $Z[M]$ is closed under countable intersections (Theorem 18 in [13]). Thus $\bigcap\limits_{n=1}^{\infty} Z(f_n) \in Z[M]$. Also, $Z(f) \supseteq \bigcap\limits_{n=1}^{\infty} Z(f_n)$ and hence $Z(f) \in Z[M]$. By maximality of M, it follows that $f \in M$. Therefore $M = M_B \cap \mathcal{M}_\circ(X,\mu)$. Conversely, let M be a maximal ideal of $\mathcal{M}_\circ(X,\mu)$ and $M = M_B \cap \mathcal{M}_\circ(X,\mu)$. Consider a countable family of zero sets $\{Z(f_n) : n \in \mathbb{N}\}$ in $Z[M]$ and by maximality of M each $f_n \in M$. We construct a sequence $\{g_n\}$ as follows: $g_n = \sum\limits_{i=1}^{n} (\frac{1}{3^i} \wedge |f_i|)$, for each $n \in \mathbb{N}$. For each i, $Z(f_i) = Z(\frac{1}{3^i} \wedge |f_i|)$, this implies $\frac{1}{3^i} \wedge |f_i| \in M$. Thus $g_n \in M$ for all $n \in \mathbb{N}$. Then by Weierstrass test $g_n \to g$ and $g \in \mathcal{M}_\circ(X,\mu)$ as $\mathcal{M}_\circ(X,\mu)$ is closed under uniform limit (Theorem 2.2 [2]). Since each $g_n \in M$, $g \in M_B \cap \mathcal{M}_\circ(X,\mu) = M$. Thus $Z(g) = \bigcap\limits_{n=1}^{\infty} Z(f_n) \in Z[M]$. Therefore M is a real maximal ideal of $\mathcal{M}_\circ(X,\mu)$ by Theorem 18 in [13]. □

Next theorem states that if a maximal ideal M of $\mathcal{M}_\circ(X,\mu)$ is hyperreal then M_B is not a proper ideal of $B_1(X,\mu)$.

Theorem 5.3. *For a hyperreal maximal ideal M of $\mathcal{M}_\circ(X,\mu)$, $M_B = B_1(X,\mu)$.*

Proof. Since M is a hyperreal maximal ideal of $\mathcal{M}_\circ(X,\mu)$, by Theorem 5.2, we have $M \subsetneqq M_B \cap \mathcal{M}_\circ(X,\mu)$. Since M is maximal, $M_B \cap \mathcal{M}_\circ(X,\mu) = \mathcal{M}_\circ(X,\mu)$. This implies $\mathcal{M}_\circ(X,\mu) \subseteq M_B$ and $\underline{1} \in M_B$. Therefore $M_B = B_1(X,\mu)$, since M_B is an ideal of $B_1(X,\mu)$. □

For any proper ideal J of $B_1(X,\mu)$, $J \cap \mathcal{M}_\circ(X,\mu)$ is a proper ideal of $\mathcal{M}_\circ(X,\mu)$. Also $(J \cap \mathcal{M}_\circ(X,\mu))_B$ is an ideal of $B_1(X,\mu)$. Now we want to investigate when J and $(J \cap \mathcal{M}_\circ(X,\mu))_B$ coincide.

Definition 5.4. A proper ideal J of $B_1(X,\mu)$ is called closed if $J = (J \cap \mathcal{M}_\circ(X,\mu))_B$.

Using Theorem 5.1, it can be easily shown that every fixed maximal ideal $\widetilde{M_p}$ of $B_1(X,\mu)$ is closed.

Theorem 5.5. *If J is any closed ideal of $B_1(X,\mu)$ containing an ideal I of $\mathcal{M}_\circ(X,\mu)$, then $I_B \subseteq J$.*

Proof. Since $I \subseteq J \cap \mathcal{M}_\circ(X,\mu)$, $I_B \subseteq (J \cap \mathcal{M}_\circ(X,\mu))_B = J$ as J is closed. $\square$

Let $\mathcal{R}Max(\mathcal{M}_\circ(X,\mu))$ be the set of all real maximal ideals of $\mathcal{M}_\circ(X,\mu)$, $\mathcal{R}Max(B_1(X,\mu))$ be the set of all real maximal ideals of $B_1(X,\mu)$ and we denote $\mathcal{C}(B_1(X,\mu)) = \{\widetilde{M} \in Max(B_1(X,\mu)) : (\widetilde{M} \cap \mathcal{M}_\circ(X,\mu))_B = \widetilde{M}$ and $\widetilde{M} \cap \mathcal{M}_\circ(X,\mu) \subseteq M$ for some $M \in \mathcal{R}Max(\mathcal{M}_\circ(X,\mu))\}$.

Now we want to discuss the relation between $\mathcal{R}Max(B_1(X,\mu))$ and $\mathcal{C}(B_1(X,\mu))$ and finally show that $|\mathcal{R}Max(\mathcal{M}_\circ(X,\mu))| = |\mathcal{R}Max(B_1(X,\mu))|$, where $|P|$ stands for the cardinality of P.

Theorem 5.6. *If $M \in \mathcal{R}Max(\mathcal{M}_\circ(X,\mu))$, then $M_B \in \mathcal{C}(B_1(X,\mu))$.*

Proof. Since $M \in \mathcal{R}Max(\mathcal{M}_\circ(X,\mu))$, by Theorem 5.2, $M = M_B \cap \mathcal{M}_\circ(X,\mu)$. This implies $M_B = (M_B \cap \mathcal{M}_\circ(X,\mu))_B$. Now M_B is a proper ideal of $B_1(X,\mu)$, otherwise $M = M_B \cap \mathcal{M}_\circ(X,\mu) = B_1(X,\mu) \cap \mathcal{M}_\circ(X,\mu) = \mathcal{M}_\circ(X,\mu)$, a contradiction.
We claim that M_B is maximal among all closed maximal ideals of $B_1(X,\mu)$. Let J be a closed maximal ideal of $B_1(X,\mu)$ such that $M_B \subseteq J$. Then $M = M_B \cap \mathcal{M}_\circ(X,\mu) \subseteq J \cap \mathcal{M}_\circ(X,\mu)$. Since M is a maximal ideal of $\mathcal{M}_\circ(X,\mu)$, $M = J \cap \mathcal{M}_\circ(X,\mu)$ and $M_B = (J \cap \mathcal{M}_\circ(X,\mu))_B = J$ as J is closed.
Now we show that M_B is a maximal ideal in $B_1(X,\mu)$. If possible, let $\widetilde{M}$ be an ideal of $B_1(X,\mu)$ such that $M_B \subsetneqq \widetilde{M}$. Since M_B is maximal among closed ideals in $B_1(X,\mu)$, $\widetilde{M}$ is not closed in $B_1(X,\mu)$. So, $\widetilde{M}$ must be free. Now $M = M_B \cap \mathcal{M}_\circ(X,\mu) \subseteq \widetilde{M} \cap \mathcal{M}_\circ(X,\mu)$ and by maximality of M, $M = \widetilde{M} \cap \mathcal{M}_\circ(X,\mu)$. Thus M is free, otherwise $\widetilde{M} \cap \mathcal{M}_\circ(X,\mu) = M_p$ for some $p \in X$ implies $M_B = (\widetilde{M} \cap \mathcal{M}_\circ(X,\mu))_B = (M_p)_B = \widetilde{M_p}$, which contradicts that M_B is not a maximal ideal. Since M is any real maximal ideal of $\mathcal{M}_\circ(X,\mu)$ and every fixed maximal ideal of $\mathcal{M}_\circ(X,\mu)$ is real ([13]), M cannot be always free which contradicts that M must be free. Hence M_B is a maximal ideal of $B_1(X,\mu)$. $\square$

Theorem 5.7. *If $\widetilde{M} \in \mathcal{R}Max(B_1(X,\mu))$ then $\widetilde{M} \cap \mathcal{M}_\circ(X,\mu)$ is a member of $\mathcal{R}Max(\mathcal{M}_\circ(X,\mu))$.*

Proof. Let $\widetilde{M}$ be a real maximal ideal of $B_1(X,\mu)$. Then for the canonical map $\psi : B_1(X,\mu) \to B_1(X,\mu)/\widetilde{M}$ defined by $\psi(f) = \widetilde{M}(f)$, there exists $r \in \mathbb{R}$ such that $\widetilde{M}(f) = \widetilde{M}(\underline{r})$ and therefore $\psi|_{\mathcal{M}_\circ(X,\mu)} : \mathcal{M}_\circ(X,\mu) \to B_1(X,\mu)/\widetilde{M}$ such that $f \mapsto \widetilde{M}(f)$ is an onto homomorphism. Now $\widetilde{M}$ is real implies $B_1(X,\mu)/\widetilde{M} \cong \mathbb{R}$. Then by 1st isomorphism theorem, $\mathcal{M}_\circ(X,\mu)/Ker(\psi|_{\mathcal{M}_\circ(X,\mu)}) \cong \mathbb{R}$. Thus $Ker(\psi|_{\mathcal{M}_\circ(X,\mu)})$ is a real maximal ideal. Now $Ker(\psi|_{\mathcal{M}_\circ(X,\mu)}) = \{f \in \mathcal{M}_\circ(X,\mu) : \widetilde{M}(f) = 0\} = \{f \in \mathcal{M}_\circ(X,\mu) : f \in \widetilde{M}\} = \mathcal{M}_\circ(X,\mu) \cap \widetilde{M}$. Therefore $\mathcal{M}_\circ(X,\mu) \cap \widetilde{M}$ is a real maximal ideal of $\mathcal{M}_\circ(X,\mu)$. This completes the proof. □

Theorem 5.8. *If $\widetilde{M} \in \mathcal{C}(B_1(X,\mu))$, then there exists a unique $M \in \mathcal{R}Max(\mathcal{M}_\circ(X,\mu))$ such that $\widetilde{M} = M_B$.*

Proof. Since $\widetilde{M} \cap \mathcal{M}_\circ(X,\mu)$ is a prime ideal of $B_1(X,\mu)$ and $\mathcal{M}_\circ(X,\mu)$ is a Gelfand ring (Theorem 4.6 [2]), there exists a unique maximal ideal M of $\mathcal{M}_\circ(X,\mu)$ such that $\widetilde{M} \cap \mathcal{M}_\circ(X,\mu) \subseteq M$. Since $\widetilde{M} \in \mathcal{C}(B_1(X,\mu))$, $M \in \mathcal{R}Max(\mathcal{M}_\circ(X,\mu))$. So, $(\widetilde{M} \cap \mathcal{M}_\circ(X,\mu))_B \subseteq M_B$. But $\widetilde{M}$ is closed implies $\widetilde{M} = (\widetilde{M} \cap \mathcal{M}_\circ(X,\mu))_B \subseteq M_B$. By maximality of $\widetilde{M}$, we obtain $\widetilde{M} = M_B$, for some $M \in \mathcal{R}Max(\mathcal{M}_\circ(X,\mu))$. □

Theorem 5.9. *For any $\tau\mathcal{A}\mu$-space X, $\mathcal{R}Max(B_1(X,\mu)) = \mathcal{C}(B_1(X,\mu))$.*

Proof. Let $\widetilde{M}$ be any real maximal ideal of $B_1(X,\mu)$ and $g \in (\widetilde{M} \cap \mathcal{M}_\circ(X,\mu))_B$. Then there exists $\{g_n\} \subseteq \widetilde{M} \cap \mathcal{M}_\circ(X,\mu)$ such that $g_n \to g$ pointwise. Since $\widetilde{M}$ is real, $\bigcap_{n=1}^{\infty} Z(g_n) \in Z[\widetilde{M}]$. This implies $Z(g) \in Z[\widetilde{M}]$ as $Z(g) \supseteq \bigcap_{n=1}^{\infty} Z(g_n)$. Again $\widetilde{M}$ is a $\mathcal{Z}_B$-ideal, implies $g \in \widetilde{M}$. Thus $(\widetilde{M} \cap \mathcal{M}_\circ(X,\mu))_B \subseteq \widetilde{M}$. By Maximality of $\widetilde{M}$, $(\widetilde{M} \cap \mathcal{M}_\circ(X,\mu))_B = \widetilde{M}$. Using Theorem 5.7, we have $\widetilde{M} \cap \mathcal{M}_\circ(X,\mu)$ is a real maximal ideal of $\mathcal{M}_\circ(X,\mu)$ and so $(\widetilde{M} \cap \mathcal{M}_\circ(X,\mu))_B \in \mathcal{C}(B_1(X,\mu))$ by Theorem 5.6. Thus $\widetilde{M} \in \mathcal{C}(B_1(X,\mu))$. Hence $\mathcal{R}Max(B_1(X,\mu)) \subseteq \mathcal{C}(B_1(X,\mu))$.

Now, let $\widetilde{M} \in \mathcal{C}(B_1(X,\mu))$. Then by Theorem 5.8, there exists a unique $M \in \mathcal{R}Max(\mathcal{M}_\circ(X,\mu))$ such that $\widetilde{M} = M_B$. Let $\{f_n\}$ be a countable subset of M_B. Then each $f_n \in M_B$ and so there exists $\{f_{n_i}\} \subseteq M$ such that $f_{n_i} \to f_n$ pointwise. Since M is a real maximal ideal of $\mathcal{M}_\circ(X,\mu)$, for each $n \in \mathbb{N}$, $\bigcap_{i=1}^{\infty} Z(f_{n_i}) \in Z[M]$. Thus $\bigcap_{n=1}^{\infty}\bigcap_{i=1}^{\infty} Z(f_{n_i}) \in Z[M] \subseteq Z[M_B]$. Again, $\bigcap_{n=1}^{\infty} Z(f_n) \supseteq \bigcap_{n=1}^{\infty}\bigcap_{i=1}^{\infty} Z(f_{n_i})$ and $Z[M_B]$ is a $\mathcal{Z}_B$-filter implies $\bigcap_{n=1}^{\infty} Z(f_n) \in Z[M_B]$. Thus $Z[M_B]$ is closed under countable intersection. Hence $\mathcal{C}(B_1(X,\mu)) \subseteq \mathcal{R}Max(B_1(X,\mu))$. This completes the proof. □

Theorem 5.10. $|\mathcal{R}Max(\mathcal{M}_\circ(X,\mu))| = |\mathcal{R}Max(B_1(X,\mu))|$.

Proof. In view of Theorem 5.6 and Theorem 5.9, we define a function $\phi : \mathcal{R}Max(\mathcal{M}_\circ(X,\mu)) \to \mathcal{R}Max(B_1(X,\mu))$ by $\phi(M) = M_B$. By Theorem 5.8, for each $\widetilde{M} \in \mathcal{R}Max(B_1(X,\mu)) = \mathcal{C}(B_1(X,\mu))$, there exists $M \in \mathcal{R}Max(\mathcal{M}_\circ(X,\mu))$ such that $\widetilde{M} = M_B$ i.e., ϕ maps $\mathcal{R}Max(\mathcal{M}_\circ(X,\mu))$ onto $\mathcal{R}Max(B_1(X,\mu))$. Now, for any $M, S \in \mathcal{R}Max(\mathcal{M}_\circ(X,\mu))$ and $M_B = S_B$ implies that $M_B \cap \mathcal{M}_\circ(X,\mu) = S_B \cap \mathcal{M}_\circ(X,\mu)$. Then by Theorem 5.2, $M = S$. Therefore ϕ is one-one. Hence $|\mathcal{R}Max(\mathcal{M}_\circ(X,\mu))| = |\mathcal{R}Max(B_1(X,\mu))|$. □

6. $B_1(X,\mu)$-Compact spaces

Definition 6.1. A quadruplet $(X,\tau,\mathcal{A},\mu)$ or a $\tau\mathcal{A}\mu$-space is called $B_1(X,\mu)$-compact if every family of zero sets in $Z[B_1(X,\mu)]$ with finite intersection property has non-empty intersection. In short, we shall say X is $B_1(X,\mu)$-compact.

Clearly, every finite T_1-space is a $B_1(X,\mu)$-compact space.

The following theorem provides various equivalent conditions of a $B_1(X,\mu)$-compact space.

Theorem 6.2. *Consider a $\tau\mathcal{A}\mu$-space X. Then the following are equivalent.*

(i) *X is $B_1(X,\mu)$-compact.*
(ii) *Every ideal of $B_1(X,\mu)$ is fixed.*
(iii) *Every maximal ideal of $B_1(X,\mu)$ is fixed.*
(iv) *Every $\mathcal{Z}_B$-filter on X is fixed.*
(v) *Every $\mathcal{Z}_B$-ultrafilter on X is fixed.*

Proof. $(i) \Rightarrow (ii)$: Assume (i) holds and let I be an ideal of $B_1(X,\mu)$. Then $Z[I]$ is a family of zero sets having finite intersection property. Then by definition of $B_1(X,\mu)$-compact space, $\cap Z[I] \neq \varnothing$. Hence I is fixed.

$(ii) \Rightarrow (iii)$: Trivial.

$(iii) \Rightarrow (i)$: Let $\mathcal{B}$ be a family of zero sets in $Z[B_1(X,\mu)]$ having finite intersection property. By a straightforward use of Zorn's lemma, $\mathcal{B}$ can be extended to a $\mathcal{Z}_B$-ultrafilter $\mathcal{U}^p$ for some $p \in \beta_{B_1}X$. Then $\mathcal{U}^p = Z[M^p]$, where M^p is a maximal ideal of $B_1(X,\mu)$ and so, by given hypothesis, $\cap\mathcal{U}^p = Z[M^p] \neq \varnothing$. This implies $\cap\mathcal{B} \neq \varnothing$ as $\mathcal{B} \subseteq \mathcal{U}^p$.

$(ii) \Rightarrow (iv)$: Let $\mathcal{U}$ be a $\mathcal{Z}_B$-filter on X. Then $Z^{-1}[\mathcal{U}]$ is an ideal I of $B_1(X,\mu)$. This implies $\cap Z[I] = \cap\mathcal{U} \neq \varnothing$ by (ii). Thus $\mathcal{U}$ is fixed.

$(iv) \Rightarrow (v)$: Trivial.

$(v) \Rightarrow (iii)$: Let M be a maximal ideal of $B_1(X,\mu)$. Then $Z[M]$ is a $\mathcal{Z}_B$-ultrafilter on X. Thus by the given hypothesis, M is fixed. □

We recall that a $\tau\mathcal{A}\mu$-space X is called $\tau\mathcal{A}\mu$-compact [13] if every family of zero sets in $Z[\mathcal{M}_\circ(X,\mu)]$ with finite intersection property has non-empty intersection or equivalently, if every maximal ideal of $\mathcal{M}_\circ(X,\mu)$ is fixed. The following theorem gives a relation between $\tau\mathcal{A}\mu$-compact and $B_1(X,\mu)$-compact spaces.

Theorem 6.3. *If X is $B_1(X,\mu)$-compact, then X is $\tau\mathcal{A}\mu$-compact.*

Proof. Let X be $B_1(X,\mu)$-compact. Let M be a maximal ideal of $\mathcal{M}_\circ(X,\mu)$. Then the $\mathcal{Z}$-ultrafilter $Z[M]$ (defined in [2]) has finite intersection property.

Now $B_1(X,\mu) \supseteq \mathcal{M}_\circ(X,\mu)$ implies $Z[M] \subseteq Z[B_1(X,\mu)]$. Therefore $Z[M]$ is a family of zero sets having finite intersection property and so $\cap Z[M] \neq \varnothing$ as X is $B_1(X,\mu)$-compact. This shows that X is $\tau\mathcal{A}\mu$-compact. □

But the converse of the above theorem is false which is shown by the following example.

Example 6.4. Let $X = [0,1]$. Consider $(X, \tau_u, \mathcal{P}(X), \mu_c)$, where τ_u is the subspace topology of the usual topology of $\mathbb{R}$ on X, and μ_c is the counting measure on $\mathcal{P}(X)$. Clearly $C(X) = \mathcal{M}_\circ(X,\mu)$ and hence every maximal ideal of $\mathcal{M}_\circ(X,\mu)$ is fixed. Thus X is a $\tau\mathcal{A}\mu$-compact space. Since X is a perfectly normal space (A topological space is called perfectly normal if it is normal and every closed subset of it is a G_δ-subset), by Theorem 3.7 ([4]), each characteristic function $\{\chi_{\{x\}} : x \in X\}$ belongs to $B_1(X,\mu_c)$ and the ideal generated by the family $\mathcal{B} = \{\chi_{\{x\}} : x \in X\}$ is free. Thus by Theorem 6.2, X is not $B_1(X,\mu)$-compact.

Now we can establish the following theorem which is a characterization of $B_1(X,\mu)$-compact spaces in terms of co-zero sets.

Theorem 6.5. *A space X is $B_1(X,\mu)$-compact if and only if every family of co-zero sets, which covers X, has a finite sub-cover.*

Proof. Let X be $B_1(X,\mu)$-compact and $\{G_\alpha\}_{\alpha\in\Lambda}$ be a family of co-zero sets such that $\bigcup\limits_{\alpha\in\Lambda} G_\alpha = X$. Thus $X \setminus \bigcup\limits_{\alpha\in\Lambda} G_\alpha = \varnothing \Rightarrow \bigcap\limits_{\alpha\in\Lambda}(X\setminus G_\alpha) = \varnothing$, where each $X\setminus G_\alpha$ is a zero set of $Z[B_1(X,\mu)]$. Since X is $B_1(X,\mu)$-compact, there exists a finite sub-collection $\{G_1, G_2, \cdots, G_n\}$ such that $\bigcap\limits_{i=1}^{n}(X\setminus G_i) = \varnothing$, which means that $X = \bigcup\limits_{i=1}^{n} G_i$. Therefore $\{G_\alpha\}_{\alpha\in\Lambda}$ has a finite sub-cover.

Conversely, let $F = \{Z_\alpha : \alpha \in \Lambda\}$ be a family of zero sets having finite intersection property. If possible, let $\bigcap\limits_{\alpha\in\Lambda} Z_\alpha = \varnothing$. Then $X = X \setminus \bigcap\limits_{\alpha\in\Lambda} Z_\alpha = \bigcup\limits_{\alpha\in\Lambda}(X \setminus Z_\alpha)$. By our assumption, there exists a finite sub-collection $\{Z_1, Z_2, \cdots, Z_n\}$ of F such that $X = \bigcup\limits_{i=1}^{n}(X\setminus Z_i) = X \setminus \bigcap\limits_{i=1}^{n} Z_i$. This implies $\bigcap\limits_{i=1}^{n} Z_i = \varnothing$, a contradiction. This completes the proof. □

Now we want to develop a theorem like Stone Weierstrass theorem [12], in our set up. For this purpose we first prove the two following lemmas.

Lemma 6.6. *Let X be a $B_1(X,\mu)$-compact space with more than one point and let L be a closed sub-lattice of $B_1(X,\mu)$ with the property: if x and y are two distinct points of X and a, b are any two real numbers, then there exists a real valued function f in L such that $f(x) = a$ and $f(y) = b$. Then $L = B_1(X,\mu)$.*

Proof. Let f be an arbitrary function in $B_1(X,\mu)$. We want to show that $f \in L$. Choose an arbitrary small real number $\epsilon > 0$. Since L is closed, it is sufficient to construct a function $g \in L$ such that $f(z) - \epsilon < g(z) < f(z) + \epsilon$ for all $z \in X$.

Let x be a fixed point of X and $y \in X$ be any point different from x. By our assumption, there exists a function $f_y \in L$ such that $f_y(x) = f(x)$ and $f_y(y) = f(y)$. Now consider the co-zero set $G_y = \{z : f_y(z) < f(z) + \epsilon\}$. It is clear that both $x, y \in G_y$. So the class G_y's for all points y different from x is a cover of X. Since X is $B_1(X, \mu)$-compact, by Theorem 6.5, there exists a finite family of co-zero sets $\{G_{y_1}, G_{y_2}, \cdots, G_{y_n}\}$ that covers X. If the corresponding functions in L are denoted by $f_{y_1}, f_{y_2}, \cdots, f_{y_n}$ then $g_x = f_{y_1} \wedge f_{y_2} \wedge \cdots \wedge f_{y_n} \in L$ such that $g_x(x) = f(x)$ and $g_x(z) < f(z) + \epsilon$ for all $z \in X$.
Now consider the co-zero set $H_x = \{z : g_x(z) > f(z) - \epsilon\}$. Since $x \in H_x$, the class H_x's for all $x \in X$ is a cover of X. Again since the space X is $B_1(X, \mu)$-compact, by Theorem 6.5, there exists a finite subfamily of co-zero sets $\{H_{x_1}, H_{x_2}, \cdots, H_{x_m}\}$ that covers X. We denote the corresponding functions in L by $g_{x_1}, g_{x_2}, \cdots, g_{x_m}$ and we define g as $g = g_{x_1} \vee g_{x_2} \vee \cdots \vee g_{x_m}$. It is clear that $g \in L$ with the property that $f(z) - \epsilon < g(z) < f(z) + \epsilon$ for all $z \in X$. This completes the proof. ☐

It is routine check to see that $B_1(X, \mu)$ is a normed algebra if we define the norm as $||f|| = \sup\limits_{x \in X} |f(x)|$ for $f \in B_1(X, \mu)$ and we have the following lemma.

Lemma 6.7. *Let X be an arbitrary topological space. Then every closed sub-algebra of $B_1(X, \mu)$ is also a closed sub-lattice of $B_1(X, \mu)$.*

Proof. Let A be a closed sub-algebra of $B_1(X, \mu)$. To show that A is a sub-lattice, it is sufficient to show that if $f \in A$ then $|f| \in A$. Let $\epsilon > 0$ be any arbitrary real number. Since $|t|$ is a continuous function of real variable t, by Weierstrass approximation theorem, there exists a polynomial p' with the property that $||t| - p'(t)| < \frac{\epsilon}{2}$ for every t on the closed interval $[-||f||, ||f||]$. Set $p(t) = p'(t) - p'(0)$, then p is a polynomial with 0 as its constant term which has the property that $||t| - p(t)| < \epsilon$ for every t in $[-||f||, ||f||]$. Since A is an algebra, $p(f) \in A$. Also $||f(x)| - p(f(x))| < \epsilon$ for every x in X. This implies that $||f| - p(f)| < \epsilon$. Since A is a closed sub-algebra and the fact that $|f|$ is approximated by the function $p(f)$ in A , we have $|f| \in A$. ☐

Now we can easily prove the following theorem by adopting the proof of Stone Weierstrass theorem.

Theorem 6.8. *Let X be a $B_1(X, \mu)$-compact space and let A be a closed sub-algebra of $B_1(X, \mu)$, which separates points and contains a non-zero constant function. Then $A = B_1(X, \mu)$.*

Proof. If X has only one point, then $B_1(X, \mu)$ contains only constant functions. Since A contains a non-zero constant function and it is an algebra, it contains all constant functions and thus $A = B_1(X, \mu)$. We may assume that X has more than one point. Let x, y be two distinct points of X and a, b two real numbers. Since A separates points, there exists $g \in A$ such that $g(x) \neq g(y)$. Now we define f by $f(z) = a\frac{g(z)-g(y)}{g(x)-g(y)} + b\frac{g(z)-g(x)}{g(y)-g(x)}$. Then $f \in A$ and $f(x) = a, f(y) = b$. Then by Lemmas 6.6 and 6.7, we have $A = B_1(X, \mu)$. □

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