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Function lattices and compactifications

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Abstract

Let $\mathcal{F}$ be a lattice of real-valued functions on a non-empty set X such that $\mathcal{F}$ contains the constant functions. Using certain filters on X determined by $\mathcal{F}$, we construct a compact Hausdorff topological space δX with the property that every bounded member of $\mathcal{F}$ extends to δX and these extensions form a dense subspace of $C(\delta X)$. If $\mathcal{A}$ is any C^*-subalgebra of $\ell^\infty(X)$ containing the constant functions, then our construction gives a representation of the spectrum of $\mathcal{A}$ as a space of filters on X.

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1. Introduction

A widely used method to study topological compactifications and semigroup compactifications is to view these compactifications as the spectrums of some C^*-algebras of bounded, complex-valued functions. If X is a Tychonoff space, then every topological compactification of X can be realized as the spectrum of some C^*-algebra consisting of continuous functions on X and containing the constant functions. A similar statement holds for any semigroup compactification of a Hausdorff semitopological semigroup S (see [5]).

Some of these compactifications can be considered as spaces of filters. The most familiar example is the Stone-Čech compactification βX of a discrete topological space X, which may be regarded as the space of all ultrafilters on X (see [6] or [9]). If S is a discrete semigroup, then βS is actually a semigroup compactification of S, and the consideration of βS as the space of all ultrafilters on S is an extremely powerful approach while analyzing algebraic

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properties of βS (see [9]). For a general Tychonoff space X, the Stone-Čech compactification βX of X can also be considered as a space of filters on X, but this time one uses z-ultrafilters on X instead of ultrafilters (see [8] or [15]). (Ultrafilters and z-ultrafilters on X coincide if X is discrete.) The uniform compactification (or the Samuel compactification, see [10]) of a uniform space $(X,\mathcal{U})$ was represented as the space of all near ultrafilters on X by Koçak and Strauss in [12]. Near ultrafilters on X need not be filters in the ordinary sense of the word, since they need not be closed under finite intersections. Recently, a representation of the uniform compactification using filters was given by the author in [1]. In both [12] and [1], the given representation was used to study the $\mathcal{LUC}$-compactification of a topological group. The $\mathcal{LUC}$-compactification of a locally compact topological group G was also studied using filters by Budak and Pym in [3], where the $\mathcal{LUC}$-compactification of G was considered as a suitable quotient space of the Stone-Čech compactification of βG_d. Here, G_d denotes the group G endowed with the discrete topology. The $\mathcal{WAP}$-compactification of a discrete semigroup was studied using filters by Berglund and Hindman in [4] and a treatment of semigroup compactifications using equivalence classes of z-filters was given by Tootkaboni and Riazi in [14].

The original aim of this paper is to show that the spectrum of *any* C^*-algebra $\mathcal{F}$ of bounded, complex-valued functions on X, where X is *any* non-empty set and $\mathcal{F}$ contains the constant functions, can be considered as a space of filters on X. Since every topological compactification [semigroup compactification] is determined by the spectrum of some C^*-algebra of bounded functions, our development gives a unified treatment of all these compactifications as spaces of filters. As far as we are aware, for many C^*-algebras our approach is actually the first one using filters instead of equivalence classes of filters or quotient spaces of some other compactifications. If X is a discrete topological space, then our approach yields the usual representation of βX as the space of all ultrafilters on X. Independently of the C^*-algebra $\mathcal{F}$ in question, our approach has a number of similarities with the consideration of βX for a discrete topological space X as the space of all ultrafilters on X. For example, we obtain a bijective correspondence between non-empty, closed subsets of the spectrum of $\mathcal{F}$ and $\mathcal{F}$-filters on X. We believe that the method presented in this paper can serve as a valuable tool in the study of both topological compactifications and semigroup compactifications. This method was used by the author in [1] to study the smallest ideal of the $\mathcal{LUC}$-compactification of a topological group and in [2] to study the smallest ideal of any semigroup compactification of any semitopological semigroup.

For a large part of the theory developed in this paper, it is not necessary that we work with a C^*-algebra of bounded functions. Instead, it is the lattice structure of real-valued functions that is important for our development. Therefore, we work with a lattice of real-valued functions (which might contain unbounded functions) throughout Sections 3-8. In Section 3, we introduce the main object of this paper, namely $\mathcal{F}$-filters and $\mathcal{F}$-ultrafilters, and we study some of their basic properties. In Section 4, we define a topology on the set of all $\mathcal{F}$-ultrafilters and we show that the resulting space δX is a compact Hausdorff space. Furthermore, we show that the $\mathcal{F}$-filters describe the topology of δX in a similar way as filters describe the topology of the Stone-Čech compactification of a discrete topological space. Section 5 contains a study of continuous functions on δX. We show that every bounded member of $\mathcal{F}$ extends to δX and that these extensions form a dense subspace of the algebra of all continuous, real-valued functions on δX. It is remarkable that we do not need the Stone-Weierstrass Theorem to prove the density of these extensions. In Sections 6-8, our main results concern closed subalgebras of the algebra of all bounded, real-valued functions on X. In Section 7, we establish a correspondence between $\mathcal{F}$-filters and closed, proper ideals of $\mathcal{F}$. Section 8 contains a treatment of $\mathcal{F}$-filters on a Hausdorff topological space X in the case that every member of $\mathcal{F}$ is a continuous function on X. In the last section, we turn our attention to C^*-algebras of bounded, complex-valued functions. Here, we include a description how the developed theory so far can be used to produce an interpretation of the spectrum of such an algebra [compactification of X] as a space of filters on X.

Our construction of the space δX as the space of all $\mathcal{F}$-ultrafilters has some similarities with the consideration of the Smirnov compactification of a proximity space using maximal round filters (see [13]). If the function lattice $\mathcal{F}$ on a non-empty set X separates the points of X, then there is a bijective correspondence between $\mathcal{F}$-ultrafilters on X and maximal round filters on the proximity space (X, P), where P is the proximity on X generated by $\mathcal{F}$. An advantage of our construction is that it applies to any function lattice $\mathcal{F}$ on X, and so it applies also to those semigroup compactifications where the evaluation mapping is not necessarily injective. This includes, for example, the Bohr compactification of some topological groups.

2. Preliminaries

Throughout the paper, let X be any non-empty set. We denote by $F(X)$ the algebra of all real-valued functions on X. We denote by $\ell^\infty(X)$ the subalgebra of $F(X)$ consisting of all bounded members of $F(X)$. Recall that the space $\ell^\infty(X)$ is equipped with the norm of uniform convergence. A function $f \in F(X)$ is *positive* if and only if $f(x) \geq 0$ for every $x \in X$. For all $f, g \in F(X)$, the functions $(f \vee g) : X \to \mathbb{R}$ and $(f \wedge g) : X \to \mathbb{R}$ are defined by

$$(f \vee g)(x) = \max\{f(x), g(x)\} \text{ and } (f \wedge g)(x) = \min\{f(x), g(x)\}$$

for every $x \in X$, respectively. By a *function lattice* on X we mean a vector subspace $\mathcal{F}$ of $F(X)$ such that $\mathcal{F}$ contains the constant functions and $f \vee g \in \mathcal{F}$ and $f \wedge g \in \mathcal{F}$ for all $f, g \in \mathcal{F}$. Note that a vector subspace $\mathcal{F}$ of $F(X)$ is a function lattice on X if and only if $|f| \in \mathcal{F}$ for every $f \in \mathcal{F}$.

We denote by $\mathbb{N}$ the set of all positive integers, that is, $\mathbb{N} = \{1, 2, 3, \ldots\}$. We denote by $\mathcal{P}(X)$ the family of all subsets of X. A *filter* on X is a non-empty family φ of subsets of X with the following properties:

(i) If $A, B \in \varphi$, then $A \cap B \in \varphi$.
(ii) If $A \in \varphi$ and $A \subseteq B \subseteq X$, then $B \in \varphi$.
(iii) $\varnothing \notin \varphi$.

A *filter base* on X is a non-empty family $\mathcal{B}$ of subsets of X such that $\varnothing \notin \mathcal{B}$ and, for all sets $A, B \in \mathcal{B}$, there exists some $C \in \mathcal{B}$ such that $C \subseteq A \cap B$. If $\mathcal{B}$ is a filter base on X, then the filter φ on X *generated* by $\mathcal{B}$ is

$$\varphi = \{A \subseteq X : \text{there exists some } B \in \mathcal{B} \text{ such that } B \subseteq A\}.$$

Let φ be a filter on X. A family $\mathcal{B}$ of subsets of X is a *filter base* for φ if and only if $\mathcal{B} \subseteq \varphi$ and, for every $A \in \varphi$, there exists some $B \in \mathcal{B}$ such that $B \subseteq A$.

Let (Y, τ) be a (not necessarily Hausdorff) topological space. For every subset A of Y, we denote by $\mathrm{int}_{(Y,\tau)}(A)$ and $\mathrm{cl}_{(Y,\tau)}(A)$ the interior and the closure of A in (Y, τ), respectively, or simply by $\mathrm{int}_Y(A)$ and $\mathrm{cl}_Y(A)$ if τ is understood. We denote by $C(Y)$ the subalgebra of $\ell^\infty(X)$ consisting of all continuous members of $\ell^\infty(X)$. If Y is locally compact, then the subalgebra $C_0(X)$ of $C(X)$ consists of those members of $C(X)$ which vanish at infinity.

3. $\mathcal{F}$-filters

Throughout this section, let $\mathcal{F}$ be a function lattice on X. We introduce the main object of the paper, namely $\mathcal{F}$-filters and $\mathcal{F}$-ultrafilters on X, and we describe some of their basic properties. For all $f \in \mathcal{F}$ and $r > 0$, we put

$$Z(f) = \{x \in X : f(x) = 0\} \quad \text{and} \quad X(f, r) = \{x \in X : |f(x)| \leq r\}.$$

Definition 3.1. An $\mathcal{F}$-*family* on X is a non-empty family $\mathcal{A}$ of non-empty subsets of X such that, for every $A \in \mathcal{A}$ with $A \neq X$, there exist some $B \in \mathcal{A}$ and a function $f \in \mathcal{F}$ such that $f(B) = \{0\}$ and $f(X \setminus A) = \{1\}$. An $\mathcal{F}$-*filter* on X is a filter φ on X which is also an $\mathcal{F}$-family on X.

Since $\mathcal{F}$ contains the constant functions, we may assume that the function $f \in \mathcal{F}$ in the previous definition satisfies $f(B) = \{1\}$ and $f(X \setminus A) = \{0\}$. Also, since $\mathcal{F}$ is closed under the lattice operations $\vee$ and $\wedge$, we may assume, if necessary, that $f(X) \subseteq [0, 1]$.

There exists at least one $\mathcal{F}$-filter on X, namely the filter $\varphi = \{X\}$. If $\mathcal{F}$ contains only the constant functions, then $\{X\}$ is the only $\mathcal{F}$-filter on X. On the other hand, if $\mathcal{F} = \ell^\infty(X)$, then every filter φ on X is an $\mathcal{F}$-filter on X.

Let φ be an $\mathcal{F}$-filter on X and suppose that $A \in \varphi$ satisfies $A \neq X$. Pick some $B \in \varphi$ and a function $f \in \mathcal{F}$ with $f(B) = \{0\}$ and $f(X \setminus A) = \{1\}$. Then $B \subseteq Z(f) \subseteq A$. Since $Z(f) \in \varphi$, the filter φ has a filter base consisting of zero sets (determined by $\mathcal{F}$) of X. However, not every zero set of X is contained in any $\mathcal{F}$-filter. For example, let $\mathcal{F} = C(\mathbb{R})$. Then $A = \{0\}$ is a zero set of $\mathbb{R}$ but there is no $\mathcal{F}$-filter φ on $\mathbb{R}$ satisfying $A \in \varphi$.

We shall apply the following remark frequently without any further notice.

Remark 3.2. Let $\mathcal{A}$ be a non-empty family of non-empty subsets of X. Suppose that, for every $A \in \mathcal{A}$ with $A \neq X$, there exist some $B \in \mathcal{A}$, real numbers s and r with $s < r$, and a function $f \in \mathcal{F}$ such that $f(x) \leq s$ for every $x \in B$

and $f(x) \geq r$ for every $x \in X \setminus A$. Then, using the lattice operations, it is easy to see that $\mathcal{A}$ is an $\mathcal{F}$-family on X.

Zorn's Lemma implies that every $\mathcal{F}$-filter on X is contained in some maximal (with respect to inclusion) $\mathcal{F}$-filter on X.

Definition 3.3. An $\mathcal{F}$-*ultrafilter* on X is an $\mathcal{F}$-filter on X which is not properly contained in any other $\mathcal{F}$-filter on X.

Note that if $\mathcal{F} = \ell^\infty(X)$, then a filter φ on X is an $\mathcal{F}$-ultrafilter if and only if φ is an ultrafilter on X. Also, the following fact about $\mathcal{F}$-ultrafilters is very useful: If p and q are $\mathcal{F}$-ultrafilters on X, then $p = q$ if and only if $p \subseteq q$.

Definition 3.4. Define

$$\mathcal{F}_0 = \{f \in \mathcal{F} : X(f, r) \neq \varnothing \text{ for every } r > 0\}.$$

For every non-empty subset A of X, define

$$\mathcal{Z}(A) = \{f \in \mathcal{F} : f(x) = 0 \text{ for every } x \in A\}.$$

The next statement follows from Remark 3.2.

Lemma 3.5. *The family $\mathcal{A} = \{X(f, r) : f \in \mathcal{F}', r > 0\}$ is an $\mathcal{F}$-family on X for every non-empty subset $\mathcal{F}'$ of $\mathcal{F}_0$.*

We will use the following lemma and its corollaries a number of times in this paper. Recall that a non-empty family $\mathcal{A}$ of subsets of X has the *finite intersection property* if and only if $\bigcap_{k=1}^n A_k \neq \varnothing$ whenever $A_1, \dots, A_n \in \mathcal{A}$ for some $n \in \mathbb{N}$.

Lemma 3.6. *If $\mathcal{A}$ is an $\mathcal{F}$-family on X such that $\mathcal{A}$ has the finite intersection property, then there exists an $\mathcal{F}$-ultrafilter p on X such that $\mathcal{A} \subseteq p$.*

Proof. We sketch the proof briefly. Let φ be the smallest filter on X containing the family $\mathcal{A}$. Let $n \in \mathbb{N}$ and suppose that $A_1, \dots A_n \in \mathcal{A}$ satisfy $A_k \neq X$ for every $k \in \{1, \dots, n\}$. If $k \in \{1, \dots, n\}$, then there exist some $B_k \in \mathcal{A}$ and a positive function $f_k \in \mathcal{F}$ with $f_k(B_k) = \{0\}$ and $f_k(X \setminus A_k) = \{1\}$. Put $B = \bigcap_{k=1}^n B_k$ and $f = \sum_{k=1}^n f_k$. Since $B \in \varphi$, $f \in \mathcal{F}$, $f(B) = \{0\}$, and $f(x) \geq 1$ for every $x \in X \setminus \bigcap_{k=1}^n A_k$, the filter φ is an $\mathcal{F}$-filter on X. □

The next two corollaries now follow from Lemma 3.5.

Corollary 3.7. *Let φ be an $\mathcal{F}$-filter on X and let $f \in \mathcal{F}$. If $X(f, r) \cap B \neq \varnothing$ for every $B \in \varphi$ and for every $r > 0$, then there exists an $\mathcal{F}$-ultrafilter p on X such that $\varphi \cup \{X(f, r) : r > 0\} \subseteq p$.*

Corollary 3.8. *Let φ be an $\mathcal{F}$-filter on X and let $A \subseteq X$. If $A \cap B \neq \varnothing$ for every $B \in \varphi$, then there exists an $\mathcal{F}$-ultrafilter p on X containing the family $\varphi \cup \{X(f, r) : f \in \mathcal{Z}(A), r > 0\}$.*

If $\mathcal{F} = \ell^\infty(X)$, then we may take the members of $\mathcal{F}$ in the next theorem to be characteristic functions of subsets of X. Then, except for statement (ii), the conclusion of the next theorem is the same as in [9, Theorem 3.6].

Theorem 3.9. *If $\varphi \subseteq \mathcal{P}(X)$, then the following statements are equivalent:*

(i) φ is an $\mathcal{F}$-ultrafilter on X.

(ii) φ is an $\mathcal{F}$-filter on X and, if $X(f,r) \notin \varphi$ for some $f \in \mathcal{F}$ and $r > 0$, then, for every real number t with $0 < t < r$, there exists some $A \in \varphi$ such that $X(f,t) \cap A = \varnothing$.

(iii) φ is a maximal $\mathcal{F}$-family on X such that φ has the finite intersection property.

(iv) φ is an $\mathcal{F}$-filter on X and, if $\bigcup_{k=1}^{n} A_k \in \varphi$ for some $n \in \mathbb{N}$ and for some $A_1, \ldots, A_n \subseteq X$, then there exists $k \in \{1, \ldots, n\}$ such that $X(f,r) \in \varphi$ for all $f \in \mathcal{Z}(A_k)$ and $r > 0$.

(v) φ is an $\mathcal{F}$-filter on X and, if $A \subseteq X$ satisfies $A \neq \varnothing$ and $A \neq X$, then, either $X(f,r) \in \varphi$ for every $f \in \mathcal{Z}(A)$ and for every $r > 0$, or $X(g,r) \in \varphi$ for every $g \in \mathcal{Z}(X \setminus A)$ and for every $r > 0$.

Proof. (i) $\Rightarrow$ (ii) This follows from Corollary 3.8 with $g = (|f| - t) \vee 0$. Note that $g \in \mathcal{Z}(X(f,t))$ and $X(g, r-t) \subseteq X(f,r)$.

(ii) $\Rightarrow$ (iii) This follows from the definition of an $\mathcal{F}$-family.

(iii) $\Rightarrow$ (iv) Suppose that (iii) holds. Let us first show that φ is a filter on X. Clearly, $X \in \varphi$, $\varnothing \notin \varphi$, and $B \in \varphi$ whenever $A \in \varphi$ and $A \subseteq B \subseteq X$. So, let $A, B \in \varphi$. Pick some $C, D \in \varphi$ and functions $f, g \in \mathcal{F}$ with $f(C) = g(D) = \{0\}$ and $f(X \setminus A) = g(X \setminus B) = \{1\}$. Since $C \cap D \subseteq X(|f| + |g|, r)$ for every $r > 0$, we have $X(|f| + |g|, r) \in \varphi$ for every $r > 0$ by Lemma 3.5. Since $X(|f| + |g|, 1/2) \subseteq A \cap B$, we have $A \cap B \in \varphi$, as required.

Suppose now that $\bigcup_{k=1}^{n} A_k \in \varphi$ for some $n \in \mathbb{N}$ and for some non-empty subsets $A_1, \ldots, A_n$ of X. Suppose also that, for every $k \in \{1, \ldots, n\}$, there exist $r_k > 0$ and a function $f_k \in \mathcal{Z}(A_k)$ such that $X(f_k, r_k) \notin \varphi$. If $k \in \{1, \ldots, n\}$, then the family $\mathcal{A} = \varphi \cup \{X(f_k, t) : t > 0\}$ is an $\mathcal{F}$-family on X by Lemma 3.5. Since $\mathcal{A}$ contains φ properly, there exist some $B_k \in \varphi$ and $t_k > 0$ such that $B_k \cap X(f_k, t_k) = \varnothing$. Put $B = \bigcap_{k=1}^{n} B_k$. Then $B \in \varphi$ and $B \cap [\bigcup_{k=1}^{n} Z(f_k)] = \varnothing$, a contradiction.

(iv) $\Rightarrow$ (v) This is obvious.

(v) $\Rightarrow$ (i) Suppose that (v) holds. Suppose also that there exists an $\mathcal{F}$-filter ψ on X which properly contains φ. Pick some set $A \in \psi \setminus \varphi$. Pick some $B \in \psi$ and a function $f \in \mathcal{F}$ with $f(B) = \{0\}$ and $f(X \setminus A) = \{1\}$. Pick some $C \in \psi$ and a function $g \in \mathcal{F}$ with $g(C) = \{1\}$ and $g(X \setminus B) = \{0\}$. Since $X(f, 1/2) \subseteq A$, we have $X(f, 1/2) \notin \varphi$. Since $f \in \mathcal{Z}(B)$, we have $X(g, 1/2) \in \varphi$ by assumption. But now $X(g, 1/2) \cap C = \varnothing$, a contradiction. □

The two statements given in statement (v) of the previous theorem are not exclusive. Indeed, let $\mathcal{F} = C(\mathbb{R})$ and $A = \mathbb{Q}$. Then $\mathcal{Z}(A) = \mathcal{Z}(\mathbb{R} \setminus A) = \{0\}$, and so $X(f,r) = X(g,r) = X$ for all $f \in \mathcal{Z}(A)$, $g \in \mathcal{Z}(\mathbb{R} \setminus A)$, and $r > 0$.

4. The topological space δX

As in the previous section, we assume that $\mathcal{F}$ is a function lattice on X. Our next task is to define a topology on the set of all $\mathcal{F}$-ultrafilters on X

and establish some of the properties of the resulting space. In particular, we show that the resulting space is a compact Hausdorff space and that $\mathcal{F}$-filters describe its topology.

Definition 4.1. Define $\delta X = \{p : p \text{ is an } \mathcal{F}\text{-ultrafilter on } X\}$. For every subset A of X, put $\widehat{A} = \{p \in \delta X : A \in p\}$. For every $\mathcal{F}$-filter φ on X, put $\widehat{\varphi} = \{p \in \delta X : \varphi \subseteq p\}$.

To be precise, we should include the function lattice $\mathcal{F}$ in the notation above, such as $\delta_{\mathcal{F}}(X)$. Except in Section 6, we consider only one function lattice $\mathcal{F}$ in the same context, so we hope that the notation chosen above does not cause any misunderstandings.

Theorem 4.2. *If φ and ψ are $\mathcal{F}$-filters on X, then the following statements hold:*

(i) $\widehat{\varphi} = \bigcap_{A \in \varphi} \widehat{A}$.
(ii) $\varphi = \bigcap_{p \in \widehat{\varphi}} p$.
(iii) $\varphi \subseteq \psi$ *if and only if* $\widehat{\psi} \subseteq \widehat{\varphi}$.
(iv) $\varphi = \psi$ *if and only if* $\widehat{\varphi} = \widehat{\psi}$.

Proof. (i) This is obvious.

(ii) The inclusion $\varphi \subseteq \bigcap_{p \in \widehat{\varphi}} p$ is obvious, so suppose that A is a subset of X such that $A \notin \varphi$. By Corollary 3.8, there exists an element $p \in \widehat{\varphi}$ such that $\{X(f, r) : f \in \mathcal{Z}(X \setminus A), r > 0\} \subseteq p$. Now, it is enough to show that $A \notin p$. Suppose that $A \in p$. Pick some $B \in p$ and a function $f \in \mathcal{F}$ with $f(B) = \{1\}$ and $f(X \setminus A) = \{0\}$. Since $f \in \mathcal{Z}(X \setminus A)$, we have $X(f, 1/2) \in p$. But now $B \cap X(f, 1/2) = \varnothing$, a contradiction.

(iii) Necessity is obvious and sufficiency follows from statement (ii).

(iv) This follows from statement (iii). □

The family $\{\widehat{A} : A \subseteq X\}$ is a base for a topology on δX. We define the topology of δX to be the topology which has this family as its base. In particular, $\{\widehat{A} : A \in p\}$ is a neighborhood base of a point $p \in \delta X$. If $Y \subseteq \delta X$, then we denote $\mathrm{cl}_{\delta X}(Y)$ by $\overline{Y}$ with one exception: If $A \subseteq X$, then we use $\mathrm{cl}_{\delta X}(\widehat{A})$ instead of the cumbersome notation $\overline{\widehat{A}}$.

We denote by $\tau(\mathcal{F})$ the weakest topology τ on X such that every member of $\mathcal{F}$ is continuous with respect to τ. For every subset A of X, we denote $\mathrm{int}_{(X,\tau(\mathcal{F}))}(A)$ by A°. For every element $x \in X$, we denote by $\mathcal{N}_{\mathcal{F}}(x)$ the neighborhood filter of x in $(X, \tau(\mathcal{F}))$.

We shall apply the following remark frequently without any further notice.

Remark 4.3. Let φ be an $\mathcal{F}$-filter on X. Suppose that $A \in \varphi$ satisfies $A \neq X$. Pick some $B \in \varphi$ and a function $f \in \mathcal{F}$ with $f(B) = \{0\}$ and $f(X \setminus A) = \{1\}$. Then $B \subseteq \{x \in X : |f(x)| < 1\} \subseteq A$. In conclusion, if C is any subset of X, then $C \in \varphi$ if and only if $C^\circ \in \varphi$.

Theorem 4.4. *If $x \in X$, then the family*

$$\mathcal{A}_x = \{X(f,r) : f \in \mathcal{F},\ f(x) = 0,\ \text{and } r > 0\}$$

is a filter base on X. The filter on X generated by $\mathcal{A}_x$ is $\mathcal{N}_{\mathcal{F}}(x)$ and it is an $\mathcal{F}$-ultrafilter on X.

Proof. If $f, g \in \mathcal{F}$ and $r > 0$, then $X(|f|+|g|, r) \subseteq X(f,r) \cap X(g,r)$. This implies that $\mathcal{A}_x$ is a filter base on X. Clearly, $\mathcal{A}_x$ generates the filter $\mathcal{N}_{\mathcal{F}}(x)$, and so $\mathcal{N}_{\mathcal{F}}(x)$ is an $\mathcal{F}$-filter on X by Lemma 3.5. Then $\mathcal{N}_{\mathcal{F}}(x)$ is an $\mathcal{F}$-ultrafilter on X by Theorem 3.9 (iv). □

The following definition is reasonable by the previous theorem.

Definition 4.5. The function $e : X \to \delta X$ defined by $e(x) = \mathcal{N}_{\mathcal{F}}(x)$ for every $x \in X$ is the *canonical mapping*.

If $A \subseteq X$ and $x \in X$, then $e(x) \in \widehat{A}$ if and only if $x \in A^\circ$. Next, let $A, B \subseteq X$. In general, $\widehat{B} \cap e(A) = \varnothing$ does not imply $B \cap A = \varnothing$. However, this implication holds if B is a $\tau(\mathcal{F})$-open subset of X. We apply this fact repeatedly in what follows.

We gather some properties of the space δX in the following lemmas.

Lemma 4.6. *Let $A \subseteq X$ and let $p \in \delta X$. The following statements are equivalent:*

(i) $p \in \overline{e(A)}$.
(ii) $A \cap B \neq \varnothing$ *for every* $B \in p$.
(iii) $X(f,r) \in p$ *for every* $f \in \mathcal{Z}(A)$ *and for every* $r > 0$.

In particular, $p \in \overline{e(A)}$ *for every* $A \in p$.

Proof. (i) ⇒ (ii) If $A \cap B = \varnothing$ for some $B \in p$, then $A \cap B^\circ = \varnothing$, and so $e(A) \cap \widehat{B^\circ} = \varnothing$. Since $B^\circ \in p$, we have $p \notin \overline{e(A)}$.

(ii) ⇒ (iii) This follows from Corollary 3.8.

(iii) ⇒ (i) Suppose that $p \notin \overline{e(A)}$. Then there exists a $\tau(\mathcal{F})$-open subset B of X such that $B \in p$ and $\widehat{B} \cap e(A) = \varnothing$, and so $B \cap A = \varnothing$. Pick some $C \in p$ and a function $f \in \mathcal{F}$ with $f(C) = \{1\}$ and $f(X \setminus B) = \{0\}$. Then $f \in \mathcal{Z}(A)$. Since $X(f, 1/2) \cap C = \varnothing$, we have $X(f, 1/2) \notin p$, and so statement (iii) does not hold. □

Lemma 4.7. *If $A, B \subseteq X$, then the following statements hold:*

(i) $\widehat{X \setminus A} = \delta X \setminus \overline{e(A)}$.
(ii) If A is a $\tau(\mathcal{F})$-open subset of X, then $\overline{e(A)} = cl_{\delta X}(\widehat{A})$.
(iii) $\widehat{A} = \widehat{B}$ *if and only if* $A^\circ = B^\circ$.
(iv) $\widehat{A} = \varnothing$ *if and only if* $A^\circ = \varnothing$.
(v) $\widehat{A} = \delta X$ *if and only if* $A = X$.

Proof. (i) Suppose first that $p \in \widehat{X \setminus A}$. Since $\widehat{X \setminus A} \cap e(A) = \varnothing$, we have $p \notin \overline{e(A)}$. Suppose now that $p \in \delta X \setminus \overline{e(A)}$. Then there exists a $\tau(\mathcal{F})$-open subset C of X such that $C \in p$ and $\widehat{C} \cap e(A) = \varnothing$. Then $C \cap A = \varnothing$, that is, $C \subseteq X \setminus A$, and so $X \setminus A \in p$.

(ii) The inclusion $\mathrm{cl}_{\delta X}(\widehat{A}) \subseteq \overline{e(A)}$ holds for any subset A of X and follows from statement (i). Suppose now that A is $\tau(\mathcal{F})$-open and that $p \in \overline{e(A)}$. If $B \in p$, then $\widehat{B} \cap e(A) \neq \varnothing$, so $B^\circ \cap A \neq \varnothing$, and so $\widehat{B} \cap \widehat{A} \neq \varnothing$. Therefore, $p \in \mathrm{cl}_{\delta X}(\widehat{A})$.

Statement (iii) follows from Remark 4.3. Then statements (iv) and (v) follow from statement (iii). $\square$

Lemma 4.8. *Let $A, B \subseteq X$. Then $X(f,r) \cap X(g,r) \neq \varnothing$ for all $f \in \mathcal{Z}(A)$, $g \in \mathcal{Z}(B)$, and $r > 0$ if and only if $\overline{e(A)} \cap \overline{e(B)} \neq \varnothing$.*

Proof. Necessity follows from Lemma 4.6, so suppose that $X(f,r) \cap X(g,r) \neq \varnothing$ for all $f \in \mathcal{Z}(A)$, $g \in \mathcal{Z}(B)$, and $r > 0$. Put

$$\mathcal{A} = \{X(h,r) : h \in \mathcal{Z}(A) \cup \mathcal{Z}(B),\ r > 0\}.$$

Then $\mathcal{A}$ is an $\mathcal{F}$-family on X by Lemma 3.5. We claim that $\mathcal{A}$ has the finite intersection property. Let $f_1, \dots, f_n \in \mathcal{Z}(A)$ and let $g_1, \dots, g_m \in \mathcal{Z}(B)$ for some $n, m \in \mathbb{N}$. Then $f := \sum_{k=1}^{n} |f_k| \in \mathcal{Z}(A)$ and $g := \sum_{k=1}^{m} |g_k| \in \mathcal{Z}(B)$. If $r > 0$, then

$$X(f,r) \cap X(g,r) \subseteq \left(\bigcap_{k=1}^{n} X(f_k, r) \right) \cap \left(\bigcap_{k=1}^{m} X(g_k, r) \right),$$

thus verifying our claim. By Lemma 3.6, there exists an element $p \in \delta X$ such that $\mathcal{A} \subseteq p$. Then $p \in \overline{e(A)} \cap \overline{e(B)}$ by Lemma 4.6, as required. $\square$

Now, we are ready to prove first of the main theorems of this section.

Theorem 4.9. *The space δX is a compact Hausdorff space and $e(X)$ is dense in δX.*

Proof. First, $e(X)$ is dense in δX by Lemma 4.7 (iv). To see that δX is Hausdorff, let p and q be distinct points of δX. Pick some set $A \in p \setminus q$. Pick some $B \in p$ and a function $f \in \mathcal{F}$ with $f(B) = \{0\}$ and $f(X \setminus A) = \{1\}$. Since $X(f, 1/2) \notin q$, there exists some $C \in q$ such that $X(f, 1/3) \cap C = \varnothing$ by Theorem 3.9 (ii). Then $B \cap C = \varnothing$, and so $\widehat{B}$ and $\widehat{C}$ are disjoint neighborhoods of p and q, respectively.

Lemma 4.7 (i) implies that the family $\mathcal{B} = \{\overline{e(A)} : A \subseteq X\}$ is a base for the closed subsets of δX. Suppose that a subset $\mathcal{C}$ of $\mathcal{B}$ has the finite intersection property. To show that δX is compact, it is enough to show that $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$. Put $\mathcal{A}' = \{A \subseteq X : \overline{e(A)} \in \mathcal{C}\}$ and $\mathcal{A} = \{X(f,r) : A \in \mathcal{A}',\ f \in \mathcal{Z}(A),\ r > 0\}$. Then $\mathcal{A}$ is an $\mathcal{F}$-family on X by Lemma 3.5 and $\mathcal{A}$ has the finite intersection property by Lemma 4.8. By Lemma 3.6, there exists an element $p \in \delta X$ such that $\mathcal{A} \subseteq p$. Then $p \in \overline{e(A)}$ for every $A \in \mathcal{A}'$ by Lemma 4.6, and so $p \in \bigcap_{C \in \mathcal{C}} C$, thus finishing the proof. $\square$

We finish this section by showing that $\mathcal{F}$-filters describe the topology of δX. As with the Stone-Čech compactification of a discrete topological space, we have two natural candidates for the closure of an $\mathcal{F}$-filter in δX, namely $\widehat{\varphi}$ and the following.

Definition 4.10. Define $\overline{\varphi} = \bigcap_{A\in\varphi} \overline{e(A)}$ for every $\mathcal{F}$-filter φ on X.

Note that $\overline{\varphi}$ is a non-empty, closed subset of δX. The next statement follows from Lemma 4.6.

Theorem 4.11. *If φ is an $\mathcal{F}$-filter on X, then $\widehat{\varphi} = \overline{\varphi}$.*

Theorem 4.12. *If C is a non-empty, closed subset of δX, then there exists a unique $\mathcal{F}$-filter φ on X such that $\widehat{\varphi} = C$.*

Proof. Let C be a non-empty, closed subset of δX. Put $\varphi = \bigcap_{p\in C} p$. Clearly, φ is a filter on X. Let us show that φ is an $\mathcal{F}$-family on X, hence, an $\mathcal{F}$-filter on X. Suppose that $A \in \varphi$ satisfies $A \neq X$. If $p \in C$, then $A \in p$, and so there exist some $B_p \in p$ and a function $f_p \in \mathcal{F}$ with $f(X) \subseteq [0, 1]$, $f_p(B_p) = \{0\}$, and $f_p(X \setminus A) = \{1\}$. Now, $\{\widehat{B}_p : p \in C\}$ is an open cover of C, and so there exist points $p_1, \ldots, p_n \in C$ for some $n \in \mathbb{N}$ such that $C \subseteq \bigcup_{k=1}^{n} \widehat{B}_{p_k}$. Put $f = \sum_{k=1}^{n} f_{p_k}$ and $B = \bigcup_{k=1}^{n} B_{p_k}$. Then $B \in \varphi$. Since $f(x) \leq n - 1$ for every $x \in B$ and $f(x) = n$ for every $x \in X \setminus A$, the filter φ is an $\mathcal{F}$-family on X.

Let us verify the equality $\widehat{\varphi} = C$. The inclusion $C \subseteq \widehat{\varphi}$ is obvious, so suppose that $q \in \delta X \setminus C$. Then there exists a $\tau(\mathcal{F})$-open subset A of X such that $A \in q$ and $\widehat{A} \cap C = \varnothing$. For every $p \in C$, pick a $\tau(\mathcal{F})$-open subset B_p of X such that $B_p \in p$ and $\widehat{A} \cap \widehat{B}_p = \varnothing$. Then $A \cap B_p = \varnothing$ for every $p \in C$. As above, there exist $n \in \mathbb{N}$ and points $p_1, \ldots, p_n \in C$ such that $B := \bigcup_{k=1}^{n} B_{p_k} \in \varphi$. Since $A \cap B = \varnothing$, we have $q \notin \widehat{\varphi}$, as required.

Finally, the $\mathcal{F}$-filter φ on X satisfying $\widehat{\varphi} = C$ is unique by Theorem 4.2 (iv). ☐

5. Continuous functions on δX

Again, we assume that $\mathcal{F}$ is a function lattice on X. This section is devoted to a study of continuous, real-valued functions on the space δX. We show that every bounded member of $\mathcal{F}$ extends to δX and that these extensions form a dense subspace of $C(\delta X)$.

We leave the proof of the following lemma to the reader.

Lemma 5.1. *Let $p \in \delta X$, let $g \in C(\delta X)$, and let $r > 0$. Then*

$$\{x \in X : |g(p) - g(e(x))| \leq r\} \in p.$$

Theorem 5.2. *For every bounded function $f \in \mathcal{F}$, there exists a unique function $\widehat{f} \in C(\delta X)$ satisfying $f = \widehat{f} \circ e$.*

Proof. Let $p \in \delta X$ and put

$$C = \bigcap_{A \in p} \mathrm{cl}_{\mathbb{R}}(f(A)). \tag{5.1}$$

Then C is a non-empty subset of $\mathbb{R}$ by assumption. Choosing any element $\widehat{f}(p) \in C$, we obtain a function $\widehat{f} : \delta X \to \mathbb{R}$.

Next, let us show that if $p = e(x)$ for some $x \in X$, then $C = \{f(x)\}$. This will establish the equality $f = \widehat{f} \circ e$. Clearly, $f(x) \in C$, so let $y \in \mathbb{R}$ be such that $y \neq f(x)$. Pick $r > 0$ such that $y \notin U := [f(x) - r, f(x) + r]$. Then $f^{-1}(U) \in e(x)$. Since $y \notin \mathrm{cl}_{\mathbb{R}}(f(f^{-1}(U)))$, we have $y \notin C$, as required.

The density of $e(X)$ in δX implies that the continuous function h on δX satisfying $f = h \circ e$ is unique. Therefore, it is enough to show that $\widehat{f}$ is continuous to finish the proof. To see that $\widehat{f}$ is continuous, let $p \in \delta X$ and put $g = f - \widehat{f}(p)$. First, we claim that $X(g, r) \in p$ for every $r > 0$. By Corollary 3.7, it is enough to show that $X(g, r) \cap B \neq \varnothing$ for every $B \in p$ and for every $r > 0$. So, let $B \in p$ and $r > 0$ be given. Since $\widehat{f}(p) \in \mathrm{cl}_{\mathbb{R}}(f(B))$, there exists a point $x \in B$ such that $|g(x)| = |f(x) - \widehat{f}(p)| \leq r$, and so $x \in X(g, r) \cap B$, as required. To finish the proof, let $r > 0$. If $q \in \widehat{X(g, r)}$, then $\widehat{f}(q) \in \mathrm{cl}_{\mathbb{R}}(f(X(g, r)))$, so $|\widehat{f}(q) - \widehat{f}(p)| \leq r$, and so $\widehat{f}$ is continuous at p. □

Although the canonical mapping need not be injective, we call the continuous function $\widehat{f}$ on δX satisfying $f = \widehat{f} \circ e$ an *extension* of f to δX.

We denote by $\mathbb{R}^*$ the one-point compactification of $\mathbb{R}$, that is, $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$.

Theorem 5.3. *For every function $f \in \mathcal{F}$, there exists a unique continuous function $\widehat{f} : \delta X \to \mathbb{R}^*$ satisfying $f = \widehat{f} \circ e$.*

Proof. Arguing as in the previous proof and using the compactness of $\mathbb{R}^*$, we need only to show that $\widehat{f}$ is continuous at a point $p \in \delta X$ with $\widehat{f}(p) = \infty$. Let $n \in \mathbb{N}$. By Lemma 4.6, it is enough to show that $A := \{x \in X : |f(x)| \geq n\} \in p$. Put $B = \{x \in X : |f(x)| \geq n + 1\}$ and $g = n + 1 - (|f| \wedge (n + 1))$. Then $g \in \mathcal{Z}(B)$ and $X(g, 1) \subseteq A$. Since $p \in \overline{e(B)}$, we have $A \in p$ by Lemma 4.6, as required. □

The points $p \in \delta X$ satisfying $\widehat{f}(p) = \infty$ have a simple characterization. Indeed, if $f \in \mathcal{F}$ is unbounded, then the sets $C_n = \{x \in X : |f(x)| \geq n\}$, where $n \in \mathbb{N}$, determine a filter base $\mathcal{B}$ on X. The filter φ on X generated by $\mathcal{B}$ is an $\mathcal{F}$-filter on X and satisfies $\widehat{\varphi} = \{p \in \delta X : \widehat{f}(p) = \infty\}$.

In the next theorem (and later), we put $\mathcal{F}_b = \mathcal{F} \cap \ell^\infty(X)$. Recall that the space $\mathcal{F}_b$ is equipped with the norm of uniform convergence. We could deduce the following theorem from the Stone-Weierstrass Theorem. However, we feel that the proof below is worth presenting, since it uses only properties of $\mathcal{F}$-filters instead of the Stone-Weierstrass Theorem. Also, in this way we obtain the Stone-Weierstrass Theorem as a corollary in Section 8.

Theorem 5.4. *The mapping* $\Gamma : \mathcal{F}_b \to C(\delta X)$ *defined by* $\Gamma(f) = \widehat{f}$ *is a linear isometry and* $\Gamma(\mathcal{F})$ *is dense in* $C(\delta X)$*. If* $\mathcal{F}_b$ *is an algebra, then* Γ *is also a homomorphism.*

Proof. Using the density of $e(X)$ in δX and the equality $f = \widehat{f} \circ e$, it is easy to verify that Γ is a linear isometry (homomorphism if $\mathcal{F}_b$ is an algebra) and we leave the details to the reader. Let us show that $\Gamma(\mathcal{F})$ is dense in $C(\delta X)$. To prove this, it is enough to show that, for every positive function $g \in C(\delta X)$ with $\|g\| = 1$ and for every $r > 0$, there exists a function $f \in \mathcal{F}_b$ such that $\|\widehat{f} - g\| \leq r$. So, let $g \in C(\delta X)$ be positive with $\|g\| = 1$ and let $r > 0$. Pick $n \in \mathbb{N}$ such that $1/n \leq r/3$. For every $k \in \{1, \ldots, n\}$, define the following subsets of $[0, 1]$, X, and δX, respectively:

$$I_k = [\frac{k-1}{n}, \frac{k}{n}], \quad A_k = \{x \in X : \frac{k-2}{n} < g(e(x)) < \frac{k+1}{n}\}, \quad C_k = g^{-1}(I_k).$$

Note that $A_k \cap A_j = \varnothing$ whenever $k, j \in \{1, \ldots, n\}$ and $k + 3 \leq j$.

Let $k \in \{1, \ldots, n\}$. If $p \in C_k$, then $A_k \in p$ by Lemma 5.1, and so there exist some $B_p \in p$ and $f_p \in \mathcal{F}_b$ with $f_p(B_p) = \{k/n\}$, $f_p(X \setminus A_k) = \{0\}$, and $f_p(X) \subseteq [0, k/n]$. Pick elements $p_1, \ldots, p_m \in C_k$ for some $m \in \mathbb{N}$ such that $C_k \subseteq \bigcup_{j=1}^m \widehat{B}_{p_j}$ and put $f_k = f_{p_1} \vee \ldots \vee f_{p_m}$. Note that $f_k(X \setminus A_k) = \{0\}$ and $f_k(x) = k/n$ for every $x \in X$ with $e(x) \in C_k$.

Put $f = f_1 \vee \ldots \vee f_n$. Then $f \in \mathcal{F}_b$ and we claim that $\|\widehat{f} - g\| \leq r$. To verify our claim, it is enough to show that $|f(x) - g(e(x))| \leq r$ for every $x \in X$. So, let $x \in X$. Suppose first that $g(e(x)) \geq (n-3)/n$. Then $e(x) \in C_k$ for some $k \in \mathbb{N}$ with $n - 2 \leq k \leq n$, so $f(x) \geq (n-2)/n$, and so $|f(x) - g(e(x))| \leq r$. Suppose now that $g(e(x)) < (n-3)/n$. Then there exists $k \in \{1, \ldots, n-3\}$ such that $(k-1)/n \leq g(e(x)) < k/n$. Then $x \in A_k$ and $e(x) \in C_k$, and so $f(x) \geq k/n$. Since $A_k \cap A_j = \varnothing$ for every $j \in \{1, \ldots, n\}$ with $j \geq k+3$, we have $f_j(x) = 0$ for every j with $k + 3 \leq j \leq n$, and so $f(x) \leq (k+2)/n$. Therefore, $|f(x) - g(e(x))| \leq 3/n \leq r$, thus finishing the proof. □

Any closed subalgebra of $\ell^\infty(X)$ containing the constant functions is a function lattice on X (see [16, p. 291] or [11, p. 265]). Therefore, we obtain the following corollary.

Corollary 5.5. *If* $\mathcal{F}$ *is a closed subalgebra of* $\ell^\infty(X)$ *containing the constant functions, then* $\Gamma : \mathcal{F} \to C(\delta X)$ *is an isometric isomorphism.*

Corollary 5.6. *If* $\mathcal{F}$ *is a function lattice on* X *such that* $\mathcal{F} \subseteq \ell^\infty(X)$*, then the closure of* $\mathcal{F}$ *in* $\ell^\infty(X)$ *is a closed subalgebra of* $\ell^\infty(X)$*.*

Proof. Denote by $\mathcal{F}'$ the closure of $\mathcal{F}$ in $\ell^\infty(X)$. Remark 3.2 implies that a filter φ on X is an $\mathcal{F}$-filter if and only if φ is an $\mathcal{F}'$-filter, and so the notation δX is unambiguous. Corollary 5.5 implies that the mapping $\Gamma : \mathcal{F}' \to C(\delta X)$ is an isometric isomorphism. Since $C(\delta X)$ is an algebra, the statement follows. □

Next, we show that $\mathcal{F}$-filters describe all dense images of X in compact Hausdorff spaces. Precise statement and details follow.

Theorem 5.7. *Let Y be a compact Hausdorff space and let $\varepsilon : X \to Y$ be a function such that $\varepsilon(X)$ is dense in Y. The following statements hold:*

(i) The set $\mathcal{F} = \{h \circ \varepsilon : h \in C(Y)\}$ is a closed subalgebra of $\ell^\infty(X)$ containing the constant functions.
(ii) $\mathcal{F}$ is isometrically isomorphic with $C(Y)$.
(iii) There exists a homeomorphism $F : \delta X \to Y$ such that $F \circ e = \varepsilon$.

Proof. We prove only statement (iii) and leave the verifications of statements (i) and (ii) to the reader. If $p \in \delta X$, then

$$C = \bigcap_{A \in p} \mathrm{cl}_Y(\varepsilon(A))$$

is a non-empty subset of Y, and we claim that C is a singleton. Suppose that C contains distinct elements x and y. By Urysohn's Lemma, there exists a function $h \in C(Y)$ such that $h(x) = 0$ and $h(y) = 1$. Put $f = h \circ \varepsilon$ and $A = \{x \in X : \widehat{f}(p) - 1/3 \leq f(x) \leq \widehat{f}(p) + 1/3\}$. Then $A \in p$ by Lemma 5.1, so $x, y \in \mathrm{cl}_Y(\varepsilon(A))$, and so $h(x), h(y) \in \mathrm{cl}_{\mathbb{R}}(f(A))$. Therefore, $|h(x) - h(y)| \leq 2/3$, a contradiction.

Since C is a singleton, we obtain a function $F : \delta X \to Y$. Clearly, $F \circ e = \varepsilon$. Since $e(X)$ and $\varepsilon(X)$ are dense subsets of δX and Y, respectively, it is enough to show that F is injective and continuous to finish the proof.

To see that F is injective, suppose that $p, q \in \delta X$ satisfy $p \neq q$. By Urysohn's Lemma, there exists a function $g \in C(\delta X)$ such that $g(p) = 0$ and $g(q) = 1$. Put $f = g \circ e$. Then $f \in \mathcal{F}$ by Corollary 5.5. Put $A = \{x \in X : f(x) \leq 1/3\}$ and $B = \{x \in X : f(x) \geq 2/3\}$. Then $A \in p$ and $B \in q$ by Lemma 5.1, and so $F(p) \in \mathrm{cl}_Y(\varepsilon(A))$ and $F(q) \in \mathrm{cl}_Y(\varepsilon(B))$. Statement (ii) implies that there exists a function $h \in C(Y)$ such that $f = h \circ \varepsilon$. Then $h(F(p)) \in \mathrm{cl}_{\mathbb{R}}(f(A))$ and $h(F(q)) \in \mathrm{cl}_{\mathbb{R}}(f(B))$, and so $F(p) \neq F(q)$, as required.

To show that F is continuous, let $p \in \delta X$ and let U be an open neighborhood of $F(p)$ in Y with $U \neq Y$. Again, there exists a continuous function $h \in C(Y)$ such that $h(F(p)) = 0$ and $h(Y \setminus U) = \{1\}$. Put $f = h \circ \varepsilon$. The continuity of h implies that $\widehat{f}(p) = 0$, and so $B = \{x \in X : -1/2 \leq f(x) \leq 1/2\} \in p$ by Lemma 5.1. If $q \in \widehat{B}$, then $h(F(q)) \in [-1/2, 1/2]$, and so $F(q) \in U$, thus finishing the proof. □

6. Some relationships between function lattices

Throughout this section, we assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are function lattices on X contained in $\ell^\infty(X)$. We denote by $\delta_1 X$ and $\delta_2 X$ the spaces of $\mathcal{F}_1$-ultrafilters on X and $\mathcal{F}_2$-ultrafilters on X, respectively. Also, we denote by e_1 and e_2 the canonical mappings from X to $\delta_1 X$ and $\delta_2 X$, respectively. If $A \subseteq X$, then the notation $\widehat{A}$ is ambiguous. However, we hope that it is clear from the notation used whether we consider $\widehat{A}$ as a subset of $\delta_1 X$ or $\delta_2 X$. If $f \in \mathcal{F}_1 \cap \mathcal{F}_2$, then f extends to both $\delta_1 X$ and $\delta_2 X$. We denote these extension by f^{δ_1} and f^{δ_2}, respectively.

Theorem 6.1. *If $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then the following statements are equivalent:*

(i) *$\mathcal{F}_1$ is dense in $\mathcal{F}_2$.*
(ii) *The set $\{f^{\delta_2} : f \in \mathcal{F}_1\}$ is dense in $C(\delta_2 X)$.*
(iii) *A filter φ on X is an $\mathcal{F}_1$-filter if and only if φ is an $\mathcal{F}_2$-filter.*
(iv) *$\delta_1 X = \delta_2 X$.*

Proof. The equivalence of statements (i) and (ii) and the implication (iv) $\Rightarrow$ (i) follow from Theorem 5.4, and the implication (iii) $\Rightarrow$ (iv) is obvious. To verify the implication (i) $\Rightarrow$ (iii), it is enough to show that a non-empty family $\mathcal{A}$ of non-empty subsets of X is an $\mathcal{F}_1$-family on X if and only if $\mathcal{A}$ is an $\mathcal{F}_2$-family on X. Since $\mathcal{F}_1 \subseteq \mathcal{F}_2$, necessity is obvious, and sufficiency follows from the density of $\mathcal{F}_1$ in $\mathcal{F}_2$ and Remark 3.2. □

For the rest of this section, we assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are closed subalgebras of $\ell^\infty(X)$ containing the constant functions.

Theorem 6.2. *The inclusion $\mathcal{F}_1 \subseteq \mathcal{F}_2$ holds if and only if there exists a continuous, surjective mapping $F : \delta_2 X \to \delta_1 X$ such that $e_1 = F \circ e_2$.*

Proof. Suppose first that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Let $p \in \delta_2 X$ and put

$$C = \bigcap_{A \in p} \mathrm{cl}_{\delta_1 X}(e_1(A)).$$

Similar arguments as used in the proof of Theorem 5.7 apply to show that C is a singleton, and so we obtain a function $F : \delta_2 X \to \delta_1 X$. Clearly, $e_1 = F \circ e_2$. Also, arguing as in the last part of the proof of Theorem 5.7, we see that F is continuous. Therefore, we need only to show that F is surjective.

If $q \in \delta_1 X$, then q is an $\mathcal{F}_2$-filter on X. Pick any $p \in \delta_2 X$ with $q \subseteq p$ and let $A \in q$. Since $\delta_1 X$ is a regular topological space, there exists a $\tau(\mathcal{F}_1)$-open subset B of X such that $B \in q$ and $\mathrm{cl}_{\delta_1 X}(\widehat{B}) \subseteq \widehat{A}$. Then $B \in p$, so $F(p) \in \mathrm{cl}_{\delta_1 X}(\widehat{B})$ by Lemma 4.7 (ii), and so $A \in F(p)$. Therefore, $q \subseteq F(p)$, and so $q = F(p)$, as required.

Suppose now that there exists a continuous mapping $F : \delta_2 X \to \delta_1 X$ with $e_1 = F \circ e_2$. Let $f \in \mathcal{F}_1$. By Theorem 5.2, there exists a function $g \in C(\delta_1 X)$ such that $f = g \circ e_1$. Since $g \circ F \in C(\delta_2 X)$ and $f = (g \circ F) \circ e_2$, we have $f \in \mathcal{F}_2$ by Corollary 5.5, thus finishing the proof. □

For the proof of the next theorem, recall the definition of $\widehat{f}$ from the proof of Theorem 5.2.

Theorem 6.3. *Suppose that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and let $F : \delta_2 X \to \delta_1 X$ be as in Theorem 6.2. If $p \in \delta_2 X$ and $q \in \delta_1 X$, then the following statements are equivalent:*

(i) *$q \subseteq p$.*
(ii) *$F(p) = q$.*
(iii) *$f^{\delta_2}(p) = f^{\delta_1}(q)$ for every $f \in \mathcal{F}_1$.*

Proof. (i) ⇒ (ii) This was proved already in the proof of Theorem 6.2.

(ii) ⇒ (iii) Suppose that $F(p) = q$. Let $f \in \mathcal{F}_1$. Since $e_1 = F \circ e_2$, the functions f^{δ_2} and $f^{\delta_1} \circ F$ agree on $e_2(X)$, hence, on $\delta_2 X$. Therefore, $f^{\delta_2}(p) = f^{\delta_1}(q)$.

(iii) ⇒ (i) Suppose that q is not contained in p. Pick some $A \in q \setminus p$. Pick $B \in q$ and a positive function $f \in \mathcal{F}_1$ with $f(B) = \{0\}$ and $f(X \setminus A) = \{1\}$. Then $f^{\delta_1}(q) = 0$. Since $X(f, 1/2) \notin p$, there exists some $C \in p$ such that $X(f, 1/3) \cap C = \varnothing$ by Theorem 3.9 (ii). Then $f^{\delta_2}(p) \geq 1/3$, thus finishing the proof. □

Define two closed equivalence relations $\sim$ and $\approx$ on $\delta_2 X$ as follows: $p \sim q$ if and only if $F(p) = F(q)$, and $p \approx q$ if and only if $f^{\delta_2}(p) = f^{\delta_2}(q)$ for every $f \in \mathcal{F}_1$. Theorem 6.3 shows that these relations are identical. Since F is a quotient mapping (see [16, pp. 60-61]), we obtain the following statement.

Corollary 6.4. *If $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then the quotient space $\delta_2 X/\approx$ is homeomorphic with $\delta_1 X$.*

7. $\mathcal{F}$-filters and ideals of $\mathcal{F}$

Throughout this section, we assume that $\mathcal{F}$ is a closed subalgebra of $\ell^\infty(X)$ containing the constant functions. We establish a correspondence between $\mathcal{F}$-filters on X and closed, proper ideals of $\mathcal{F}$. Roughly speaking, we show how the ideals of $\mathcal{F}$ can be used to generate $\mathcal{F}$-filters on X. We apply the following convention for the rest of this paper: *By an ideal of $\mathcal{F}$, we always mean a closed, proper ideal of $\mathcal{F}$.*

The next lemma follows from [7, (1.23) Proposition]. Since the proof of the cited proposition relies on the spectrums of single elements of C^*-algebras, we present the following short proof using only basic properties of Banach algebras.

Lemma 7.1. *If $f \in \mathcal{F} \setminus \mathcal{F}_0$, then $1/f \in \mathcal{F}$.*

Proof. Suppose first that $f \in \mathcal{F} \setminus \mathcal{F}_0$ is positive. Pick $r > 0$ such that $r \leq f(x)$ for every $x \in X$. Then

$$0 < \frac{r}{\|f\|} \leq \frac{f(x)}{\|f\|} \leq 1$$

for every $x \in X$. Put $g = f/\|f\|$. Then $\|1 - g\| < 1$ by the inequalities above, and so g is invertible in $\mathcal{F}$ (see [7, (1.3) Lemma]). Therefore, $1/f \in \mathcal{F}$.

If $f \in \mathcal{F} \setminus \mathcal{F}_0$ is any function, then $f^2 \in \mathcal{F} \setminus \mathcal{F}_0$ is positive. The equality $1/f = f/f^2$ and the first part of the proof imply that $1/f \in \mathcal{F}$. □

Corollary 7.2. *If I is an ideal of $\mathcal{F}$, then $I \subseteq \mathcal{F}_0$.*

Definition 7.3. For every ideal I of $\mathcal{F}$, define

$$\mathcal{B}(I) = \{X(f, r) : f \in I,\ r > 0\}. \tag{7.1}$$

Theorem 7.4. *If I is an ideal of $\mathcal{F}$, then $\mathcal{B}(I)$ is a filter base on X and the filter φ on X generated by $\mathcal{B}(I)$ is an $\mathcal{F}$-filter. Conversely, if φ is an $\mathcal{F}$-filter on X, then there exists an ideal I of $\mathcal{F}$ such that φ is generated by $\mathcal{B}(I)$.*

Proof. Suppose first that I is an ideal of $\mathcal{F}$. First, $X(f,r) \neq \varnothing$ for every $f \in I$ and for every $r > 0$ by Corollary 7.2, and so $\varnothing \notin \mathcal{B}(I)$. Next, let $f, g \in I$ and let $r > 0$. Since $f^2 + g^2 \in I$ and $X(f^2+g^2, r) \subseteq X(f,r) \cap X(g,r)$, the set $\mathcal{B}(I)$ is a filter base on X. Since $\mathcal{B}(I)$ is an $\mathcal{F}$-family on X by Lemma 3.5, the filter φ on X generated by $\mathcal{B}(I)$ is an $\mathcal{F}$-filter.

Suppose now that φ is an $\mathcal{F}$-filter on X. Put

$$I = \{f \in \mathcal{F} : X(f,r) \in \varphi \text{ for every } r > 0\}.$$

Clearly, $0 \in I$. Let $f_1, f_2 \in I$, let $h \in \mathcal{F}$ with $h \neq 0$, let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, let (g_n) be a sequence in I converging to some $g \in \mathcal{F}$, and let $r > 0$. The inclusions

$$\begin{aligned} X(f_1, r/2) \cap X(f_2, r/2) &\subseteq X(f_1 - f_2, r), \\ X(f_1, r/\|h\|) &\subseteq X(f_1 h, r), \\ X(f_1, r/|\alpha|) &\subseteq X(\alpha f_1, r), \\ X(g_n, r/2) &\subseteq X(g, r), \end{aligned}$$

where the last one holds if $\|g_n - g\| \leq r/2$, imply that I is an ideal of $\mathcal{F}$.

We claim that $\mathcal{B}(I)$ is a filter base for φ. Clearly, $\mathcal{B}(I) \subseteq \varphi$, so suppose that $A \in \varphi$ satisfies $A \neq X$. Pick some $B \in \varphi$ and a function $f \in \mathcal{F}$ with $f(B) = \{0\}$ and $f(X \setminus A) = \{1\}$. Since $B \subseteq X(f,r)$ for every $r > 0$ and $B \in \varphi$, we have $f \in I$. Since $X(f, 1/2) \subseteq A$, the claim follows. □

Let φ be an $\mathcal{F}$-filter on X. The previous theorem guarantees the existence of an ideal I of $\mathcal{F}$ such that φ is generated by $\mathcal{B}(I)$. The next theorem shows that this ideal I is unique.

Theorem 7.5. *Let I be an ideal of $\mathcal{F}$, let φ be the $\mathcal{F}$-filter on X generated by $\mathcal{B}(I)$, and let $f \in \mathcal{F}$. The following statements are equivalent:*

(i) $f \in I$.
(ii) $\widehat{f}(p) = 0$ *for every* $p \in \overline{\varphi}$.
(iii) $X(f,r) \in \varphi$ *for every* $r > 0$.

Proof. (i) ⇒ (ii) Suppose that $f \in I$. Let $p \in \overline{\varphi}$ and let $r > 0$. Since φ is generated by $\mathcal{B}(I)$, we have $X(f,r) \in p$ by Theorem 4.11, and so $|\widehat{f}(p)| \leq r$ by Lemma 4.6. Therefore, $\widehat{f}(p) = 0$.

(ii) ⇒ (iii) This follows from Lemma 5.1 and Theorem 4.2 (ii).

(iii) ⇒ (i) Suppose that (iii) holds. It is enough to show that $f \in \mathrm{cl}_{\mathcal{F}}(I)$, and so we may assume that $f \neq 0$. Let $0 < r < \|f\|$. Then $X(f,r) \neq X$. Since $\mathcal{B}(I)$ is a filter base for φ, there exist functions $h \in I$ and $g \in \mathcal{F}$ such that $g(X(h,1)) = \{0\}$ and $g(X \setminus X(f,r)) = \{1\}$. Now, $1/(|h| \vee 1)^2 \in \mathcal{F}$ by Lemma 7.1, so $k := h^2/(|h| \vee 1)^2 \in I$, and so $fk \in I$. The inclusion $X(h,1) \subseteq X(f,r)$ implies that f and fk agree on $X \setminus X(f,r)$. Therefore, $\|f - fk\| = \sup_{x \in X(f,r)} |f(x)(1 - k(x))| \leq r$, and so $f \in \mathrm{cl}_{\mathcal{F}}(I)$, thus finishing the proof. □

Let φ be an $\mathcal{F}$-filter on X. We say that a function $f \in \mathcal{F}$ *tends to zero* in the direction of φ if and only if, for every $r > 0$, there exists some $A \in \varphi$ with $|f(x)| \leq r$ for every $x \in A$. The previous theorem, then, says that a subset I of $\mathcal{F}$ is an ideal of $\mathcal{F}$ if and only if there exists an $\mathcal{F}$-filter φ on X such that I consists of those members of $\mathcal{F}$ which tend to zero in the direction of φ.

Let I be an ideal of $\mathcal{F}$. The equality $X(f, r) = X(|f|, r)$ for every $f \in \mathcal{F}$ and for every $r > 0$ implies that $|f| \in I$ for every $f \in I$.

For every ideal I of $\mathcal{F}$, we denote by $\varphi(I)$ the $\mathcal{F}$-filter on X generated by $\mathcal{B}(I)$. If I and J are ideals of $\mathcal{F}$, then the previous theorem implies that $I \subseteq J$ if and only if $\varphi(I) \subseteq \varphi(J)$. From this we conclude the following: An ideal I of $\mathcal{F}$ is a maximal ideal of $\mathcal{F}$ if and only if $\varphi(I)$ is an $\mathcal{F}$-ultrafilter on X.

We denote by $\mathcal{M}(\mathcal{F})$ the set of all maximal ideals of $\mathcal{F}$. If I is an ideal of $\mathcal{F}$, then the *hull* of I is the set $h(I) = \{J \in \mathcal{M}(\mathcal{F}) : I \subseteq J\}$. The *kernel* $k(\mathcal{J})$ of a non-empty subset $\mathcal{J}$ of $\mathcal{M}(\mathcal{F})$ is the set $k(\mathcal{J}) = \bigcap_{J \in \mathcal{J}} J$. Note that $k(\mathcal{J})$ is an ideal of $\mathcal{F}$. The *hull-kernel topology* on $\mathcal{M}(\mathcal{F})$ is defined by declaring a non-empty subset $\mathcal{J}$ of $\mathcal{M}(\mathcal{F})$ to be closed if and only if $\mathcal{J} = h(k(\mathcal{J}))$. In terms of $\mathcal{F}$-filters, this reads as follows: A non-empty subset $\mathcal{J}$ of $\mathcal{M}(\mathcal{F})$ is closed if and only if there exists an $\mathcal{F}$-filter φ on X such that $\mathcal{J} = \{J \in \mathcal{M}(\mathcal{F}) : \varphi \subseteq \varphi(J)\}$. Therefore, the mapping $J \mapsto \varphi(J)$ from $\mathcal{M}(\mathcal{F})$ to δX is a homeomorphism.

The following well-known property of $\mathcal{F}$ follows from Theorem 4.2 (ii).

Corollary 7.6. *If I is an ideal of $\mathcal{F}$, then $k(h(I)) = I$.*

8. $\mathcal{F}$-filters on topological spaces

In the previous sections, we made no assumption about algebraic or topological structure on the set X. In this section, we assume that (X, τ) is a Hausdorff topological space and that $\mathcal{F}$ is a function lattice on X such that $\mathcal{F} \subseteq C(X)$.

Recall that A° denotes the $\tau(\mathcal{F})$-interior of a subset A of X. If $A \subseteq X$, then $e^{-1}(\widehat{A}) = A^\circ$. Since $\mathcal{F} \subseteq C(X)$, the set A° is τ-open in X, and so the canonical mapping $e : X \to \delta X$ is continuous. For every element $x \in X$, we denote by $\mathcal{N}(x)$ the neighborhood filter of x in (X, τ). Since $\mathcal{F} \subseteq C(X)$, we have $\mathcal{N}_{\mathcal{F}}(x) \subseteq \mathcal{N}(x)$ for every $x \in X$.

The canonical mapping $e : X \to \delta X$ is an embedding if and only if the equality $\mathcal{N}(x) = \mathcal{N}_{\mathcal{F}}(x)$ holds for every $x \in X$. By Remark 3.5, this equality for every $x \in X$ is equivalent to statement (ii) below.

Lemma 8.1. *The following statements are equivalent:*

(i) The canonical mapping $e : X \to \delta X$ is an embedding.
(ii) For every $x \in X$ and for every neighborhood $U \in \mathcal{N}(x)$ with $U \neq X$, there exists a function $f \in \mathcal{F}$ with $f(x) = 1$ and $f(X \setminus U) = \{0\}$.

The next theorem follows from Theorem 5.7.

Theorem 8.2. *If Y is a compact Hausdorff space and $\varepsilon : X \to Y$ is a continuous mapping such that $\varepsilon(X)$ is dense in Y, then the following statements hold:*

(i) The set $\mathcal{F} = \{h \circ \varepsilon : h \in C(Y)\}$ is a closed subalgebra of $C(X)$ containing the constant functions.
(ii) $\mathcal{F}$ is isometrically isomorphic with $C(Y)$.
(iii) There exists a homeomorphism $F : \delta X \to Y$ such that $F \circ e = \varepsilon$.

Statements (ii) and (iii) of the next corollary constitute Stone-Weierstrass Theorem.

Corollary 8.3. *If X is compact, then the following statements are equivalent:*

(i) The canonical mapping $e : X \to \delta X$ is a homeomorphism.
(ii) $\mathcal{F}$ separates the points of X.
(iii) $\mathcal{F}$ is dense in $C(X)$.

Proof. Since X is compact, the canonical mapping $e : X \to \delta X$ is a continuous surjection. Therefore, e is a homeomorphism if and only if e is injective, and so (i) and (ii) are equivalent.

(i) $\Rightarrow$ (iii) Suppose that $e : X \to \delta X$ is a homeomorphism. Then it is easy to verify that the mapping $g \mapsto g \circ e$ from $C(\delta X)$ to $C(X)$ is an isometric isomorphism. Since $\{\widehat{f} : f \in \mathcal{F}\}$ is dense in $C(\delta X)$ by Theorem 5.4 and $\widehat{f} \circ e = f$ for every $f \in \mathcal{F}$, the statement follows.

(iii) $\Rightarrow$ (ii) This follows from Urysohn's Lemma. □

Next statement is a consequence of the Gelfand-Naimark Theorem. Here, it follows from Corollary 8.3. An isometric isomorphism $T : C(X) \to C(Y)$ induces a bijection between ideals of $C(X)$ and $C(Y)$. Then the maximal ideal spaces $\mathcal{M}(C(X))$ and $\mathcal{M}(C(Y))$ are homeomorphic (under their hull-kernel topologies).

Corollary 8.4. *If X and Y are compact Hausdorff spaces, then X and Y are homeomorphic if and only if $C(X)$ and $C(Y)$ are isometrically isomorphic.*

We finish this section with the following statement concerning locally compact topological spaces. If X is locally compact, then we denote by X_∞ the one-point compactification of X. Let $e_1 : X \to X_\infty$ denote the natural embedding. Then $\{h \circ e_1 : h \in C(X_\infty)\} = C_0(X) \oplus \mathbb{R}$, where $\mathbb{R}$ denotes the constant functions on X. Necessity of the following statement follows from Corollary 5.5 using the zero extension. Sufficiency follows from Theorem 6.2 and from the fact that X is embedded and open in X_∞.

Theorem 8.5. *Suppose that X is non-compact and locally compact and that $\mathcal{F}$ is a closed subalgebra of $C(X)$ containing the constant functions. The canonical mapping $e : X \to \delta X$ is an embedding and $e(X)$ is open in δX if and only if $C_0(X) \subseteq \mathcal{F}$.*

Remark 8.6. Let X and $\mathcal{F}$ be as above and suppose that X is embedded in δX. Then the family $\varphi_K = \{X \setminus K : K \subseteq X \text{ and } \mathrm{cl}_X(K) \text{ is compact}\}$ is an

$\mathcal{F}$-filter on X and $\delta X \setminus e(X) = \widehat{\varphi}_K$. In particular, if $\delta X = X_\infty$, then $\infty = \varphi_K$ as an $\mathcal{F}$-filter.

9. Spectrums of unital C^*-subalgebras of $\ell^\infty(X)$

In this last section, we change our notation from spaces of real-valued functions to spaces of complex-valued functions. We denote by $\ell^\infty(X)$ the C^*-algebra of all bounded, *complex-valued* functions on X. If X is a topological space, then we denote by $C(X)$ the C^*-subalgebra of $\ell^\infty(X)$ consisting of continuous members of $\ell^\infty(X)$. We explain briefly how the introduced filters can be used to represent the spectrum of any C^*-subalgebra of $\ell^\infty(X)$ as a space of filters on X.

Throughout this section, let $\mathcal{F}$ be a C^*-subalgebra of $\ell^\infty(X)$ such that $\mathcal{F}$ contains the constant functions. We consider the spectrum Δ of $\mathcal{F}$ as the space of all non-zero, multiplicative linear functionals on $\mathcal{F}$, that is,

$$\Delta = \{\mu \in \mathcal{F}^* : \mu \neq 0 \text{ and } \mu(fg) = \mu(f)\mu(g) \text{ for all } f, g \in \mathcal{F}\},$$

where $\mathcal{F}^*$ denotes the Banach dual of $\mathcal{F}$. The *evaluation mapping* $\varepsilon : X \to \Delta$ is defined by $[\varepsilon(x)](f) = f(x)$ for every $x \in X$ and for every $f \in \mathcal{F}$.

Under the relative weak* topology of $\mathcal{F}^*$, the space Δ is a compact Hausdorff space and $\varepsilon(X)$ is a dense subset of Δ. The characteristic property of the space Δ is the fact that $\mathcal{F}$ and $C(\Delta)$ are isometrically $*$-isomorphic. If $\mu \in \Delta$, then $\ker \mu = \{f \in \mathcal{F} : \mu(f) = 0\}$ is a maximal ideal of $\mathcal{F}$. Conversely, if I is a maximal ideal of $\mathcal{F}$, then there exists a unique element $\mu \in \Delta$ such that $I = \ker \mu$ (see [7]).

The space $\mathcal{F}_r$ of all real-valued members of $\mathcal{F}$ is a closed subalgebra of the space of all bounded, real-valued functions on X, and so $\mathcal{F}_r$ is a function lattice on X. We define δX to be the space of all $\mathcal{F}_r$-ultrafilters on X. If $f \in \mathcal{F}$, then Theorem 5.2 implies that the real and imaginary parts of f extend to δX, and so there exists a unique function $\widehat{f} \in C(\delta X)$ satisfying $f = \widehat{f} \circ e$. Then the mapping $f \mapsto \widehat{f}$ from $\mathcal{F}$ to $C(\delta X)$ is an isometric $*$-isomorphism by Theorem 5.4.

Small adjustments in Section 7 apply to show that, for every $\mathcal{F}$-filter φ on X, there exists a unique ideal I of $\mathcal{F}$ such that φ is generated by $\mathcal{B}(I)$. (Here, $\mathcal{B}(I)$ is defined as in (7.1).) In the second part of the proof of Lemma 7.1, we apply the equality $1/f = \overline{f}/|f|^2$. Here, $\overline{f}(x) = \overline{f(x)}$ for every $x \in X$ and $\overline{f(x)}$ denotes the complex-conjugate of $f(x)$. In the last part of the proof of Theorem 7.4, we apply the fact that $|f|^2 + |g|^2 \in I$. In the proof of implication (iii) $\Rightarrow$ (i) of Theorem 7.5, we define $k = |h|^2/(|h| \vee 1)^2$. Here, we apply the fact that $|f| \in \mathcal{F}$ for every $f \in \mathcal{F}$ (see [11, p. 265]). If I is an ideal of $\mathcal{F}$, then the equalities $X(f,r) = X(|f|,r) = X(\overline{f},r)$, which hold for every $f \in \mathcal{F}$ and for every $r > 0$, and Theorem 7.5 imply that $|f| \in I$ and $\overline{f} \in I$ for every $f \in I$.

Finally, the mapping $\mu \mapsto \ker \mu$ from Δ to the maximal ideal space $\mathcal{M}(\mathcal{F})$ of $\mathcal{F}$ is a bijection. Once $\mathcal{M}(\mathcal{F})$ is equipped with the hull-kernel topology, this mapping is a homeomorphism. Since the mapping $J \mapsto \varphi(J)$ from $\mathcal{M}(\mathcal{F})$ to

δX, where $\varphi(J)$ is the $\mathcal{F}$-ultrafilter on X generated by $\mathcal{B}(J)$, is a homeomorphism, we conclude that the mapping $\mu \mapsto p(\mu)$ from Δ to δX, where $p(\mu)$ is the $\mathcal{F}$-ultrafilter on X generated by $\mathcal{B}(\ker \mu)$, is a homeomorphism.

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