Applied General Topology

Appl. Gen. Topol. 15, no. 2(2014), 221-228
doi:10.4995/agt.2014.2139

Approximation in different smoothness spaces with the RAFU method

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Abstract

*The **RAFU** (radical functions) **method** is an original and unknown approximation procedure we can use in Approximation Theory. We know that the RAFU method provides a linear space uniformly dense in $C[a,b]$ by using some separation conditions. In this work, we will show we can employ the RAFU method to approximate functions of $C_0(\mathbb{R})$ and $C_{00}(\mathbb{R})$, Riemann integrable functions, Lebesgue integrable functions, functions of $L^p[a,b]$ and $L^p(\mathbb{R})$, $1 \leq p < \infty$ and measurable functions. Moreover, Riemann integrals can be approximated by the integrals of the functions that the RAFU method provides.*

2010 MSC: *41A30; 37L65.*

Keywords: *RAFU method; RAFU approximation; Approximation Theory.*

1. Introduction

Let f be an arbitrary function defined in $[a,b]$ and let $a = x_0 < x_1 < ... < x_n = b$ be a partition of $[a,b]$ for each natural n. The RAFU method on approximation is an approximation procedure to the function f by a sequence of radical functions $(C_n)_n$ defined by the formula

$$C_n(x) = f(x_1) + \sum_{p=2}^{n} [f(x_p) - f(x_{p-1})] \cdot F_n(x_{p-1}, x) \tag{1.1}$$

being $F_n(x_p, x) = \frac{\sqrt[2n+1]{x_p - a} + \sqrt[2n+1]{x - x_p}}{\sqrt[2n+1]{b - x_p} + \sqrt[2n+1]{x_p - a}}$, $p = 1, ..., n-1$.

Received 7 February 2014 – Accepted 15 May 2014

Blasco-Moltó [5], Tietze, Jameson, Mrowka and Garrido-Montalvo (see [6]), and recently Gassó-Hernández-Rojas [7] have studied the uniform density of a linear space of $C(K)$ where K is a compact Hausdorff space. In [2] we proved that the called RAFU linear space is uniformly dense in $C[a,b]$ by using a S-separation condition due to Blasco-Moltó [5] or its equivalent S'-separation condition due to Garrido-Montalvo [6]. This result was also obtained taking into account that the mentioned linear space S-separates Lebesgue-sets of $[a,b]$ [7]. This proof was not possible from well-known Kakutani-Stone's Theorem or Stone-Weierstras' Theorem because the RAFU linear space is not a lattice or an algebra. About basic properties of the RAFU method on $C[a,b]$, the reader can see [3].

The RAFU method can be used to uniformly approximate a continuous function f from average samples of the values $f(x_j)$, from linear combinations of $f(x_j)$ and $f(x_{j+1})$ and from local average samples given by $\left(\chi_{[-\frac{h}{2},\frac{h}{2}]} \star f\right)(x)$. Moreover, if the data $f(x_j)$ or average samples or local average samples are unknown, but approximate values of them are known, then it is also possible to obtain the uniform reconstruction of f. Such problems have been studied by many authors, for example, H. Behforooz, E.J.M. Delhez, F.G. Lang and X.P. Xu, T. Zhanlav and R. Mijiddorj, J. Huang and Y. Chen, J. Bustamante, R.C. Castillo and A.F. Collar. In [4] we solved all these problems with the only condition that $f \in C[a,b]$ and we gave error uniform bounds in each case.

The radical functions defined as (1.1) approach very well to the step functions [3]. On the other hand, it is well-known that the step functions are dense in many spaces of functions, so our aim in this work will be to prove that these radical functions can also be dense in all these spaces.

The paper is organized as follows. In Section 2 we will recall some basic results about the uniform approximation on $C[a,b]$. In Section 3 we will approach functions of the spaces $C_0(\mathbb{R})$ and $C_{00}(\mathbb{R})$. The Riemann integrable functions will be approximated with the RAFU method in Section 4. Moreover, the integral of a Riemann integrable function will be approximated by the sequence of integrals of the functions that the RAFU method provides. This approximation procedure will serve to approach the Lebesgue integrable functions in Section 5. Section 6 is devoted to approximate elements of $L^p[a,b]$ and $L^p(\mathbb{R})$, $1 \le p < \infty$. In Section 7 the RAFU method will be employed to approach measurable functions.

2. Uniform approximation on $C[a,b]$ with the RAFU Method

Consider an arbitrary step function defined $[x_0, x_m]$ by

$$f(x) = k_1 \cdot \chi_{[x_0,x_1]} + \sum_{p=2}^{m} k_p \cdot \chi_{(x_{p-1},x_p]}, \qquad k_i \in \mathbb{R} \tag{2.1}$$

Then, the sequence of radical functions $(C_{m,n})_n$ given by the formula

$$C_{m,n}(x) = k_1 + \sum_{p=2}^{m} [k_p - k_{p-1}] \cdot F_n(x_{p-1}, m, x) \tag{2.2}$$

where

$$F_n(x_{p-1}, m, x) = \frac{\sqrt[2n+1]{x_{p-1} - x_0} + \sqrt[2n+1]{x - x_{p-1}}}{\sqrt[2n+1]{x_m - x_{p-1}} + \sqrt[2n+1]{x_{p-1} - x_0}}, \quad p = 2, ..., m$$

verifies the following results.

Proposition 2.1. *Let f be the function defined by (2.1). For any $\beta > 0$ such that $(x_i - \beta, x_i + \beta) \cap (x_j - \beta, x_j + \beta) = \varnothing$ where $i \neq j$ and $i, j \in \{1, ..., m-1\}$ the limit*

$$\lim_{n \to +\infty} C_{m,n} = f$$

is uniform on $[x_0,\, x_1 - \beta] \cup [x_1 + \beta,\, x_2 - \beta] \cup ... \cup [x_{m-1} + \beta,\, x_m]$.

Proposition 2.2. *Let $\beta > 0$ be such that $(x_i - \beta,\, x_i + \beta) \cap (x_j - \beta,\, x_j + \beta) = \varnothing$ where $i \neq j$ and $i, j \in \{1, ..., m-1\}$. Then, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$ it follows that*

1. $\mid C_{m,n}(x) - f(x) \mid < \mid k_{j+1} - k_j \mid + \varepsilon$
2. $|C_{m,n}(x) - (k_j \cdot (1 - \alpha) + k_{j+1} \cdot \alpha)| < \varepsilon$

where $x \in (x_j - \beta, x_j + \beta)$, $j = 1, ..., m-1$ and $\alpha \in (0, 1)$.

Let $a = x_0 < x_1 < ... < x_n = b$ be a partition of $[a, b]$ with $x_j = a + j \cdot \frac{b-a}{n}$, $j = 0, ..., n$. Define by C_n the subset of $C[a, b]$ formed by the functions $C_{m,n}$ with $m = n$. In this case, the functions $C_{n,n} = C_n$ have the form $C_n(x) = k_1 + \sum_{p=2}^{n} [k_p - k_{p-1}] \cdot F_n(x_{p-1}, x)$ where

$$F_n(x_{p-1}, x) = \frac{\sqrt[2n+1]{x_{p-1} - x_0} + \sqrt[2n+1]{x - x_{p-1}}}{\sqrt[2n+1]{x_n - x_{p-1}} + \sqrt[2n+1]{x_{p-1} - x_0}}, \quad p = 2, ..., n$$

Theorem 2.3. *Let C be the set defined by $\mathsf{C} = \cup_{n \in \mathbb{N}} \mathsf{C}_n$. Then C is a linear space uniformly dense on $C[a, b]$.*

Besides to prove the uniform density of C in $C[a, b]$, we also know the expression of each term C_n of the sequence $(C_n)_n$ which converges uniformly to f in $[a, b]$:

$$C_n(x) = f(a) + \sum_{j=2}^{n} [f(x_j) - f(x_{j-1})] \cdot \frac{\sqrt[2n+1]{x_{j-1} - a} + \sqrt[2n+1]{x - x_{j-1}}}{\sqrt[2n+1]{b - x_{j-1}} + \sqrt[2n+1]{x_{j-1} - a}} \tag{2.3}$$

with $x_j = a + j \cdot \frac{b-a}{n}$, $j = 0, 1, ..., n$.

Next result deals about the degree of uniform approximation of the RAFU method.

Theorem 2.4. *Let f be a continuous function defined in $[a, b]$. Then, there exists a sequence of radical functions $(C_n)_n$ defined in $[a, b]$ as (2.3) such that*

$$|C_n(x) - f(x)| \leq \frac{M - m}{\sqrt{n}} + \omega\left(f, \frac{b-a}{n}\right)$$

for all $n \geq 2$ being M and m the maximum and the minimum of f in $[a, b]$ respectively and $\omega\left(f, \frac{b-a}{n}\right)$ its modulus of continuity.

Proofs of these results are in [2, 3].

3. Approximation on $C_0(\mathbb{R})$ and $C_{00}(\mathbb{R})$

Definition 3.1. $C_0(\mathbb{R})$ is the space of all continuous functions on $\mathbb{R}$ such that $\underset{|x|\to\infty}{lim} f(x)$ exists and equals 0.

$C_{00}(\mathbb{R})$is the space that consists of those functions on $\mathbb{R}$ with compact support.

Note that the functions C_n defined on $[a, b]$ as (2.3) can be defined on $\mathbb{R}$ by the same formula. In this case, it is easy to check that these functions verify $\underset{n\to\infty}{lim} C_n(x) = f(a)$ for all $x \in (-\infty, a]$ and $\underset{n\to\infty}{lim} C_n(x) = f(b)$ for all $x \in [b, +\infty)$.

If $f \in C_{00}(\mathbb{R})$ there exists $M > 0$ such that $f(x) = 0$ if $|x| \geq M$. Then the sequence $(C_n)_n$, defined on $\mathbb{R}$ as (2.3) being $a = -M$ and $b = M$, verifies that $\underset{n\to\infty}{lim} C_n = f$ uniformly on $[-M, M]$ and $\underset{n\to\infty}{lim} C_n(x) = f(x) = 0$ for all $|x| \geq M$.

Let f be an element of $C_0(\mathbb{R})$. Given $\epsilon > 0$, there exists N_ϵ such that $|f(x)| < \epsilon$ if $|x| > N_\epsilon$. For these $\epsilon > 0$ and N_ϵ, there is a function $C_{n,\epsilon}$, defined on $\mathbb{R}$ as (2.3) by considering $a = -N_\epsilon$, $b = N_\epsilon$ and by requiring $C_{n,\epsilon}$ to have the values 0 at the points $\pm N_\epsilon$, such that $|f - C_{n,\epsilon}| < \epsilon$ on $[-N_\epsilon, N_\epsilon]$. Thus, we can construct a sequence $(C_n)_n$ defined on $\mathbb{R}$ such that $\underset{n\to\infty}{lim} C_n(x) = f(x)$ for all $x \in \mathbb{R}$. Moreover, this limit becomes uniform on each interval $[-N_\epsilon, N_\epsilon]$.

4. Approximation to a Riemman integrable function

Proposition 4.1. *Let f be a bounded real-valued function on $[a, b]$. If f is Riemann integrable on $[a, b]$, then there is a sequence of radical functions $(C_n)_n$, defined as (2.3), such that $\underset{n\to\infty}{lim} C_n = f$ uniformly on $[a, b]$ except in a null set D.*

Proof. It is well-known that there is a sequence of step functions $(E_m)_m$ defined on $[a, b]$ which converges uniformly to f on $[a, b]$ except in a null set D that contains the points in which f is not continuous.

Given $\epsilon > 0$, $\exists n_1 \in \mathbb{N}$ such that if $m \geq n_1$, then $|E_m - f| < \frac{\epsilon}{3}$ on the compact $C = [a, b] - \bigcup_{n=1}^{\infty}(a_n, b_n)$ where $\sum_{n=1}^{\infty}(b_i - a_i) < \epsilon$. By Proposition 2.1, $\exists n_{2,m} \in \mathbb{N}$ such that $|C_n - E_m| < \frac{\epsilon}{3}$ on

$$\left(\left[x_0, x_1 - \frac{1}{K}\right] \cup \left[x_1 + \frac{1}{K}, x_2 - \frac{1}{K}\right] \cup ... \cup \left[x_{m-1} + \frac{1}{K}, x_m\right]\right) \cap C$$

if $n \geq n_{2,m}$, for some large $K = K_0$. Fixed this K_0 there exists $n_{3,m} \in \mathbb{N}$ such that $|C_n - E_m| < \left|f(x^{j+1}) - f(x^j)\right| + \frac{\epsilon}{3}$ on $\bigcup_{j=1}^{m} \left(x_j - \frac{1}{K_0}, x_j + \frac{1}{K_0}\right) \cap C$, if $n \geq n_{3,m}$. Since $\left|f(x^{j+1}) - f(x^j)\right| < \frac{\epsilon}{3}$ because $\left|x^{j+1} - x^j\right| < \delta$ then $|C_n - E_m| < \frac{2\epsilon}{3}$ on $\bigcup_{j=1}^{m} \left(x_j - \frac{1}{K_0}, x_j + \frac{1}{K_0}\right) \cap C$.

Thus, for $n \geq max\{n_{2,m}, n_{3,m}\}$, it verifies that $|C_n - f| \leq |C_n - E_m| + |E_m - f| < \epsilon$ and the proof is complete. $\square$

Now we suggest the following approximation to the integral of a Riemann integrable function f by using the RAFU method.

Lemma 4.2. *Let E_m be a step function on $[a, b]$ defined as (2.1). Then, the sequence $(C_{m,n})_n$ defined from E_m verifies that $\lim\limits_{n\to\infty} \int_a^b C_{m,n} = \int_a^b E_m$.*

Proof.

$$\lim_{n\to\infty} \int_a^b C_{m,n}(x)dx = \lim_{n\to\infty} \int_a^b \left(k_1 + \sum_{p=2}^{m} [k_p - k_{p-1}] \cdot F_n\left(x_{p-1}, m, x\right)\right)(x)dx$$

$$= \lim_{n\to\infty} (k_1 \cdot (b-a) + \sum_{p=2}^{m} \left(\frac{(k_p - k_{p-1}) \cdot (b-a) \cdot \sqrt[2n+1]{x_{p-1} - a}}{\sqrt[2n+1]{b - x_{p-1}} + \sqrt[2n+1]{x_{p-1} - a}}\right) + \frac{2n+1}{2n+2}$$

$$\cdot \sum_{p=2}^{m} \left(\frac{(k_p - k_{p-1}) \cdot \left[\sqrt[2n+1]{(b - x_{p-1})^{2n+2}} - \sqrt[2n+1]{(a - x_{p-1})^{2n+2}}\right]}{\sqrt[2n+1]{b - x_{p-1}} + \sqrt[2n+1]{x_{p-1} - a}}\right))$$

$$= k_1 \cdot (b-a) + \sum_{p=2}^{m} \left(\frac{(k_p - k_{p-1}) \cdot (b-a)}{2}\right)$$

$$+ \sum_{p=2}^{m} \left(\frac{(k_p - k_{p-1}) \cdot (b - 2x_{p-1} + a)}{2}\right)$$

$$= \sum_{i=1}^{m} k_i \cdot (x_i - x_{i-1}) = \int_a^b E_m$$

$\square$

The expression

$$I_n\left(C_{m,n}\right) = \left[k_1 + \sum_{p=2}^{m} \left(\frac{(k_p - k_{p-1}) \cdot \sqrt[2n+1]{x_{p-1} - a}}{\sqrt[2n+1]{b - x_{p-1}} + \sqrt[2n+1]{x_{p-1} - a}}\right)\right] \cdot (b-a)$$

$$+ \frac{2n+1}{2n+2} \cdot \left(\sum_{p=2}^{m} \frac{(k_p - k_{p-1}) \cdot \left[\sqrt[2n+1]{(b - x_{p-1})^{2n+2}} - \sqrt[2n+1]{(a - x_{p-1})^{2n+2}}\right]}{\sqrt[2n+1]{b - x_{p-1}} + \sqrt[2n+1]{x_{p-1} - a}}\right)$$

can be considered a new formula of numerical approximation to the integral $\int_a^b E_m$. Thus, if f is a Riemann integrable function, we can approach $\int_a^b f$ taking into account that

$$\int_a^b f = \lim_{m\to\infty}\lim_{n\to\infty}\left[I_n\left(C_{m,n}\right)\right]$$

5. Approximation to a Lebesgue integrable function

Let I be an arbitrary interval on $\mathbb{R}$ and let $L\left(I\right)$ be the set of all Lebesgue integrable functions defined on I.

Proposition 5.1. *If $f \in L\left(I\right)$, then there exists a sequence of radical functions $\left(C_n\right)_n$ such that $\lim_{n\to\infty} C_n = f$ at almost every point of I.*

Proof. If $f \in L\left(I\right)$, then $f = u - v$ with $u, v \in U\left(I\right)$, where $U\left(I\right)$ is the set of all upper functions defined on I. In this case, there are sequences of step functions $\left(s_n\right)_n$ and $\left(t_n\right)_n$ such that $u = \lim_{n\to\infty} s_n$ at almost every point of I and $v = \lim_{n\to\infty} t_n$ at almost every point of I. So $f = u - v = \lim_{n\to\infty}\left(s_n - t_n\right)$ at almost every point of I. Note that s is a step function on I if there is a compact interval $[a,b] \subset I$ such that s is a step function on $[a,b]$ and $s(x) = 0$ on $I - [a,b]$.

Given $K \in \mathbb{N}$, there exists $n_K \in \mathbb{N}$ such that $\left|(s_p - t_p) - f\right| < \frac{1}{2K}$ at almost every point of I, if $p \geq n_K$. Since $s_p - t_p$ is a step function, there is $n_{K,p} \in \mathbb{N}$ such that $\left|C^p_{m_p,n} - (s_p - t_p)\right| < \frac{1}{2K}$ at almost every point of I, if $n \geq n_{K,p}$.

So, if $n_0 = max\left\{n_K, n_{K,p}\right\}$ and $n \geq n_0$, then

$$\left|C^p_{m_p,n} - f\right| < \left|C^p_{m_p,n} - (s_p - t_p)\right| + \left|(s_p - t_p) - f\right| < \frac{1}{K}$$

at almost every point of I. We take into account that the countable union of null sets is a null set. The sequence $\left(C_n\right)_n$ we want to obtain can be constructed by considering suitable $C_n = C^p_{m_p,n}$ for each p. □

Let I be a set defined by $I = I_1 \cup I_2$ where I_1 and I_2 are intervals such that $I_1 \cap I_2 = \varnothing$. Suppose that $f_1 \in L\left(I_1\right)$ and $f_2 \in L\left(I_2\right)$. Then, it is well-known that the function f defined by requiring to have value $f_1(x)$ at each point in I_1 and to have value $f_2(x)$ at each point in I_2 is a function that belongs to $L\left(I\right)$ and $\int_I f = \int_{I_1} f_1 + \int_{I_2} f_2$. So, the radical functions $C_{m,n}$ defined as (2.2) can approach this function $f \in L\left(I\right)$.

6. Approximation on $L^p\left[a,b\right]$ and $L^p\left(\mathbb{R}\right)$, $1 \leq p < \infty$

Let $\left(\left[a,b\right], \mathcal{A}, \mu\right)$ be a measure space, where $\mathcal{A}$ is the σ- algebra of Borel subsets of $[a,b]$ and $\mu$is the restriction of Lebesgue measure to $\mathcal{A}$. Let $\mathcal{L}^p\left(\left[a,b\right], \mathcal{A}, \mu\right)$

be the set of all $\mathcal{A}$-measurable functions $f : [a, b] \to \mathbb{R}$ such that $|f|^p$ is integrable. The function $\|\cdot\|_p : \mathcal{L}^p([a, b], \mathcal{A}, \mu) \to \mathbb{R}$ defined by

$$\|f\|_p = \left(\int_a^b |f|^p \, d\mu\right)^{\frac{1}{p}}$$

is a seminorm. Let $L^p([a, b], \mathcal{A}, \mu)$ be the set formed by identifying functions in $\mathcal{L}^p([a, b], \mathcal{A}, \mu)$ that agree almost everywhere.

We shall use $\mathcal{L}^p[a, b]$ and $L^p[a, b]$ as abbreviations for $\mathcal{L}^p([a, b], \mathcal{A}, \mu)$ and $L^p([a, b], \mathcal{A}, \mu)$ respectively.

Proposition 6.1. *The radical functions of the type (2.2) defined on $[a, b]$ are dense in $L^p[a, b]$.*

Proof. Of course, each continuous function defined on $[a, b]$ as (2.2) belongs to $L^p[a, b]$. Since the step functions on $[a, b]$ determine a dense subspace of $L^p[a, b]$, it is enough to prove that for each step function f and each positive number ϵ, there is a radical continuous function $C_{m,n}$ of the type (2.2) defined on $[a, b]$ that satisfies $\|f - C_{m,n}\|_p < \epsilon$. So, let f be a step function on $[a, b]$, then there are real numbers $a = x_0 < x_1 < ... < x_m = b$ such that f takes a constant value k_i on each interval (x_{i-1}, x_i). Define, for each $n \in \mathbb{N}$, the radical continuous function $C_{m,n}$ by the formula (2.2) at each point x on $[a, b]$. Denote $M = sup\{|f(x)| : x \in [a, b]\}$, $A_\beta = [x_0, x_1 - \beta] \cup [x_1 + \beta, x_2 - \beta] \cup ... \cup [x_{m-1} + \beta, x_m]$ and $B_\beta = \bigcup_{j=1}^{m-1} (x_j - \beta, x_j + \beta)$. In this case, we can put

$$\int_a^b |f - C_{m,n}| \, d\mu = \int_{A_\beta} |f - C_{m,n}| \, d\mu + \int_{B_\beta} |f - C_{m,n}| \, d\mu$$

Let δ be a positive number. It is possible to find $\beta > 0$ such that $\mu(B_\beta) < \delta$. Moreover we know that there exists $n_1 \in \mathbb{N}$ such that $|f - C_{m,n}|^p < (2M + \delta)^p$ on B_β for all $n \geq n_1$ by Proposition 2.2. On the other hand, for this fixed $\beta > 0$, there exist $n_2 \in \mathbb{N}$ such that $|f - C_{m,n}|^p < \delta^p$ on A_β for all $n \geq n_2$ by Proposition 2.1.

Then, for $n \geq max\{n_1, n_2\}$

$$\int_a^b |f - C_{m,n}| \, d\mu \leq \delta^p \cdot (b - a) + (2M + \delta)^p \cdot \delta$$

Since δ is arbitrary and M depends only on f, this proof is complete. □

Let us call a function on $\mathbb{R}$ a step function if for each interval $[a, b]$ its restriction to $[a, b]$ is a step function. Analogous of Proposition 6.1 holds for $L^p(\mathbb{R}, \mathcal{A}, \mu)$ where $\mathcal{A}$ and is the σ- algebra of Borel subsets of $\mathbb{R}$ and μ is the Lebesgue measure on $\mathbb{R}$ if we replace the set of step functions on $[a, b]$ with the set of step functions on $\mathbb{R}$ that vanish outside some bounded interval, and if we replace the set of radical continuous functions on $[a, b]$ with the set of radical continuous functions on $\mathbb{R}$ that vanish outside some bounded interval.

7. Approximation to a measurable function

The function $f = 1$ is the limit of step functions on $\mathbb{R}$, however f is not a Lebesgue integrable function, $f \notin L(\mathbb{R})$. So, the set of functions that are limit of step functions, namely $\mathcal{M}(\mathbb{R})$, contains the set of Lebesgue integrable functions $L(\mathbb{R})$.

Theorem 7.1. *Let I be an arbitrary interval on $\mathbb{R}$ and let $f \in \mathcal{M}(I)$. Then there exists a sequence of radical continuous functions $(C_n)_n$ defined as (2.3) such that $\lim_{n\to\infty} C_n(x) = f(x)$ at almost every point of I.*

Proof. If $f \in \mathcal{M}(I)$, then there exists a sequence of step functions $(s_n)_n$ on I such that $\lim_{n\to\infty} s_n(x) = f(x)$ at almost every point of I.

Given $K \in \mathbb{N}$, there is $n_K \in \mathbb{N}$ such that $|s_p - f| < \frac{1}{2K}$ at almost every point of I, if $p \geq n_K$. Since s_p is a step function, there exists $n_{K,p} \in \mathbb{N}$ such that $\left|C^p_{m_p,n} - s_p\right| < \frac{1}{2K}$ at almost every point of I, if $n \geq n_{K,p}$.

So, if $n_0 = max\{n_K, n_{K,p}\}$ and $n \geq n_0$ then

$$\left|C^p_{m_p,n} - f\right| < \left|C^p_{m_p,n} - s_p\right| + |s_p - f| < \frac{1}{K}$$

at almost every point of I. We take into account that the countable union of null sets is a null set. The sequence $(C_n)_n$ we want to obtain can be constructed by considering suitable $C_n = C^p_{m_p,n}$ for each p. □

Let f be a function defined on an arbitrary interval I and suppose that $(f_n)_n$is a sequence of measurable functions on I such that $\lim_{n\to\infty} f_n(x) = f(x)$ at almost every point of I. Then, it is well-known that f is measurable on I. There are no measurable functions but this result shows us that it is not easy to construct examples of them. So, by means of the RAFU method on approximation we can approach almost all functions defined on an arbitrary interval I.

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