Applied General Topology

Appl. Gen. Topol. 16, no. 2(2015), 127-139
doi:10.4995/agt.2015.3257

Rational criterion for testing the density of additive subgroups of $\mathbb{R}^n$ and $\mathbb{C}^n$

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Abstract

In this paper, we give an explicit criterion to decide the density of finitely generated additive subgroups of $\mathbb{R}^n$ and $\mathbb{C}^n$.

2010 MSC: *47A06; 47A99.*

Keywords: *dense; additive group; rationally independent; Kronecker.*

1. Introduction

It is a classical result of Kronecker that the additive group $\mathbb{Z}+\alpha\mathbb{Z}$, $\alpha\in\mathbb{R}$, is dense in $\mathbb{R}$ whenever α is irrational. In higher dimensions, this is generalized as follows: $\mathbb{Z}^n+\mathbb{Z}[\theta_1,\dots,\theta_n]^T$ is dense in $\mathbb{R}^n$ if and only if $1,\theta_1,\dots,\theta_n$ are rationally independent (see [6] and [5]). For general generated additive groups of the form $H=\sum_{k=1}^{p}\mathbb{Z}u_k$, where $p\geq 1$ and $u_k\in\mathbb{K}^n$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$), a criterion for the density was given by Waldschmidt ([7], see proposition 2.1 for the real case). However, the use of this theorem in higher dimension or with a large number of generators is more difficult. So the main aim of this paper is to give an explicit arithmetic way for checking the density of any finitely generated additive subgroup of $\mathbb{C}^n$ and $\mathbb{R}^n$, which may be used in a future algorithm. This criterion can be used as a tool to characterize the density of any orbit given by the natural action of any abelian linear or affine group on $\mathbb{K}^n$ (see for example, [1], [2], [3] and [4]).

Received 5 September 2014 – Accepted 27 July 2015

2. Preliminaries

First of all, let us introduce the following proposition which characterizes the density of additive subgroups $\mathbb{Z}u_1 + \cdots + \mathbb{Z}u_p$ of $\mathbb{R}^n$.

Proposition 2.1 ([7], Proposition 4.3, Chapter II)**.** *Let $H = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_p$ with $u_k \in \mathbb{R}^n$, $k = 1, \dots, p$. Then H is dense in $\mathbb{R}^n$ if and only if for every $(s_1, \dots, s_p) \in \mathbb{Z}^p \backslash \{0\}$:*

$$\operatorname{rank} \begin{bmatrix} u_1 & \dots & \dots & u_p \\ s_1 & \dots & \dots & s_p \end{bmatrix} = n + 1.$$

If $p = n + 1$ and $(u_1, \dots, u_n)$ is a basis of $\mathbb{R}^n$, the additive group $H = \sum_{i=1}^{p} \mathbb{Z}u_i$, where $u_{n+1} = \sum_{i=1}^{n} \theta_i u_i$ is isomorphic (by a linear map) to $\mathbb{Z}^n + \mathbb{Z}[\theta_1, \dots, \theta_n]^T$. In this case, proposition 2.1 becomes explicit and have the following form: H is dense in $\mathbb{R}^n$ if and only if $1, \theta_1, \dots, \theta_n$ are rationally independent.

Now, for the general case, if H is dense in $\mathbb{R}^n$ then $p \geq n+1$ and the vector space $\sum_{k=1}^{p} \mathbb{R}u_k$ is equal to $\mathbb{R}^n$ (proposition 2.1). The last condition means that a basis of $\mathbb{R}^n$ can be extracted from the set of vectors u_k, $k = 1, \dots, p$.

So let us assume here and after that this basis is $(u_1, \dots, u_n)$ and that $p \geq n + 1$.

With these assumptions, the rank condition in proposition 2.1 becomes:

$$\operatorname{rank} \begin{bmatrix} 1 & 0 & \dots & 0 & \alpha_{n+1,1} & \dots & \alpha_{p,1} \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & \alpha_{n+1,n} & \dots & \alpha_{p,n} \\ s_1 & \dots & \dots & s_n & s_{n+1} & \dots & s_p \end{bmatrix} = n + 1 \tag{2.1}$$

where the scalars $\alpha_{k,i}$ are the coordinates of u_k, $k = n+1, \dots, p$ in the basis $(u_1, \dots, u_n)$, *i.e.*

$$u_k = \sum_{i=1}^{n} \alpha_{k,i} u_i \qquad \text{for all } \; k = n+1, \dots, p$$

Simplifying further (2.1) using elementary row operations, we get:

$$\operatorname{rank}\begin{bmatrix} 1 & 0 & \dots & 0 & \alpha_{n+1,1} & \dots & \alpha_{p,1} \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & \alpha_{n+1,n} & \dots & \alpha_{p,n} \\ 0 & \dots & \dots & 0 & s_{n+1}-\sum\limits_{i=1}^{n} s_i\alpha_{n+1,i} & \dots & s_p-\sum\limits_{i=1}^{n} s_i\alpha_{p,i} \end{bmatrix} = n+1$$

This condition is fulfilled if the last row is not null, which means that for every $(s_1,\dots,s_p)\in\mathbb{Z}^p\backslash\{0\}$, there is at least one integer $k_0\in\{n+1,\dots,p\}$ such that $s_{k_0}-\sum\limits_{i=1}^{n} s_i\alpha_{k_0,i}\neq 0$, which gives rise to the following proposition:

Proposition 2.2. *Let $H=\mathbb{Z}u_1+\dots+\mathbb{Z}u_p$, $p\geq n+1$ and such that $(u_1,\dots,u_n)$ is a basis of $\mathbb{R}^n$ with $u_k=\sum\limits_{i=1}^{n}\alpha_{k,i}u_i$, for every $k=n+1,\dots,p$. Then H is dense in $\mathbb{R}^n$ if and only if for every $(s_1,\dots,s_p)\in\mathbb{Z}^p\backslash\{0\}$, there is at least one integer $k_0\in\{n+1,\dots,p\}$ such that $s_{k_0}-\sum\limits_{i=1}^{n} s_i\alpha_{k_0,i}\neq 0$.*

Now, let us suppose that $1,\alpha_{k,i_1},\dots,\alpha_{k,i_{r_k}}$ is the longest sequence extracted from the list $\{1,\alpha_{k,1},\dots,\alpha_{k,n}\}$ which contains 1 and such that its elements are rationally independent.

Set $I_k:=\{i_1,\dots,i_{r_k}\}$.

• If $I_{k_0}=\{1,2,\dots,n\}$ for at least one integer $k_0\in\{n+1,\dots,p\}$ then H is dense in $\mathbb{R}^n$. Indeed, otherwise, by proposition 2.2, there exists $(s_1,\dots,s_p)\in\mathbb{Z}^p\backslash\{0\}$ such that for every $k=n+1,\dots,p$

$$s_k-\sum_{i=1}^{n} s_i\alpha_{k,i}=0 \tag{2.2}$$

As $I_{k_0}=\{1,2,\dots,n\}$ then using equation 2.2 with $k=k_0$, we get $s_{k_0}=0$ and $s_i=0$ for all $i=1,\dots,n$. Using again this equation for the other values of $k\in\{n+1,\dots,p\}$, we get $s_k=0$. Therefore $s_i=0$ for every $i=1,\dots,p$, which leads to a contradiction since $(s_1,\dots,s_p)\in\mathbb{Z}^p\backslash\{0\}$.

• If $I_k=\varnothing$ all the coordinates of the given vector u_k are rational. Actually if this condition is fulfilled for every $k=n+1,\dots,p$ then we have:

Proposition 2.3. *If $I_k=\varnothing$ for every $k=n+1,\dots,p$, then $\sum\limits_{j=1}^{p}\mathbb{Z}u_j$ is not dense in $\mathbb{R}^n$.*

We need the following lemma:

Lemma 2.4. *Let $(u_1, \dots, u_n)$ be a basis of $\mathbb{R}^n$, $n \geq 2$. Then:*

(i) The group $\sum\limits_{k=1}^{p} \mathbb{Z}u_k$ is closed in $\mathbb{R}^n$, for any $1 \leq p \leq n$.

(ii) The group $\sum\limits_{k=1}^{p} \mathbb{R}u_k + \sum\limits_{k=p+1}^{q} \mathbb{Z}u_k$ is closed in $\mathbb{R}^n$, for any $1 \leq p < q \leq n$.

Proof. let $\Phi : \mathbb{R}^p \longrightarrow \sum\limits_{k=1}^{p} \mathbb{R}u_k$ the natural isomorphism defined by

$$\Phi(x_1, \dots, x_p) = \sum_{k=1}^{p} x_k u_k.$$

Then Φ is a homeomorphism and we have $\Phi(\mathbb{Z}^p) = \sum\limits_{k=1}^{p} \mathbb{Z}u_k$. Since $\mathbb{Z}^p$ is closed in $\mathbb{R}^p$, so $\sum\limits_{k=1}^{p} \mathbb{Z}u_k$ is closed in $\sum\limits_{k=1}^{p} \mathbb{R}u_k$ and hence in $\mathbb{R}^n$. A similar argument can be used to the proof (ii) by considering $\mathbb{R}^p \times \mathbb{Z}^{q-p}$ which is closed in $\mathbb{R}^q$. □

Proof of proposition 2.3. If $I_k = \varnothing$ for every $k = n+1, \dots, p$, then the coordinates of every vector u_k are rational. So there exist $q_k \in \mathbb{N}^*$ and $p_{k,j} \in \mathbb{Z}$ such that $\alpha_{k,j} = \frac{p_{k,j}}{q_k}$. Therefore, $u_k = \frac{1}{q_k} \sum\limits_{j=1}^{n} p_{k,j} u_j$, for every $k = n+1, \dots, p$. Hence $\sum\limits_{j=1}^{p} \mathbb{Z}u_j \subset \frac{1}{q} \sum\limits_{j=1}^{n} \mathbb{Z}u_j$, where $q = q_{n+1} q_{n+2} \dots q_p$. By lemma 2.4, $\frac{1}{q} \sum\limits_{j=1}^{n} \mathbb{Z}u_j$ is closed in $\mathbb{R}^n$, therefore $\sum\limits_{j=1}^{p} \mathbb{Z}u_j$ is not dense in $\mathbb{R}^n$. □

• For a fixed $k = n+1, \dots, p$, assume that $I_k \neq \varnothing$ and $I_k \neq \{1, 2, \dots, n\}$. Then rewrite the scalars $\alpha_{k,j}$ for every $j \notin I_k$ as a function of 1 and the scalars $\{\alpha_{k,i}\ i \in I_k\}$. Thus there exist $\gamma^{(k)}_{j,i_1}, \dots, \gamma^{(k)}_{j,i_{r_k}}, t_{k,j} \in \mathbb{Q}$ such that

$$\alpha_{k,j} = t_{k,j} + \sum_{i \in I_k} \gamma^{(k)}_{j,i} \alpha_{k,i}$$

We obtain:

$$\begin{aligned} u_k &= \sum_{j=1}^{n} \alpha_{k,j} u_j \\ &= \sum_{j\in I_k} \alpha_{k,j} u_j + \sum_{j\notin I_k} \left(t_{k,j} + \sum_{i\in I_k} \gamma_{j,i}^{(k)} \alpha_{k,i} \right) u_j \\ &= \sum_{j\in I_k} \alpha_{k,j} u_j + \sum_{i\in I_k} \alpha_{k,i} \left(\sum_{j\notin I_k} \gamma_{j,i}^{(k)} u_j \right) + \sum_{j\notin I_k} t_{k,j} u_j \\ &= \sum_{j\in I_k} \alpha_{k,j} \left(u_j + \sum_{i\notin I_k} \gamma_{i,j}^{(k)} u_i \right) + \sum_{j\notin I_k} t_{k,j} u_j \end{aligned}$$

Let $q_k \in \mathbb{N}^*$ and $m_{i,j}^{(k)}, p_{k,j} \in \mathbb{Z}$ such that $t_{k,j} = \frac{p_{k,j}}{q_k}$ and $\gamma_{i,j}^{(k)} = \frac{m_{i,j}^{(k)}}{q_k}$. Therefore,

$$q_k u_k = \sum_{j\in I_k} \alpha_{k,j} \left(q_k u_j + \sum_{i\notin I_k} m_{i,j}^{(k)} u_i \right) + \sum_{j\notin I_k} p_{k,j} u_j \tag{2.3}$$

Notice that the choice of the scalar q_k is not unique as it can be replaced by a positive multiple of it.

Set

$$u'_{k,j} := q_k u_j + \sum_{i\notin I_k} m_{i,j}^{(k)} u_i$$

for every $k = n+1, \ldots, p$ and $j \in I_k$. So

$$q_k u_k = \sum_{j\in I_k} \alpha_{k,j} u'_{k,j} + \sum_{j\notin I_k} p_{k,j} u_j \tag{2.4}$$

For a fixed k, the family of vectors $\left(u'_{k,j},\ \ j \in I_k\right)$ and $(u_j,\ \ j \notin I_k)$ constitute all together a basis of $\mathbb{R}^n$ since the obtained set is a result of transforming the basis $(u_1, \ldots, u_n)$ using elementary operations.

3. The main result: the real case

Now, assume that $I_k \neq \varnothing$ for at least one k and that $I_k \neq \{1, \ldots, n\}$ for every $k = n+1, \ldots, p$.

Definition 3.1. We define M_H to be the matrix of the coordinates of the vectors $u'_{k,j}$, $j \in I_k$ and $k = n+1, \ldots, p$.

The matrix M_H is actually defined up to scaling of its columns which are the vectors $\{u'_{k,j},\, j \in I_k\, ;\, k = n+1, \dots, p\}$. Indeed, for a fixed $k \in \{n+1, \dots, p\}$, the choice of q_k in the definition of $u'_{k,j}$ is not unique as it can be replaced by any positive multiple of it. However, the rank of the matrix M_H does not change with a particular choice of the set of vectors $u'_{k,j}$.

Example 3.2. Let $H = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_7$ with $u_1 = [1, 0, 0]^T$, $u_2 = [0, 1, 0]^T$, $u_3 = [0, 0, 1]^T$, $u_4 = [1, 3\sqrt{2}, 2]^T$, $u_5 = [0, \sqrt{2}, \sqrt{5}]^T$, $u_6 = [2\sqrt{2}, \sqrt{3}, 1]^T$, $u_7 = [3, \sqrt{2}, 2\sqrt{2}]^T$. So $n = 3$ and $p = 7$.

Let $k = 4$. Then $1, 3\sqrt{2}$ is the longest sequence which can be extracted from the set $\{1, 1, 3\sqrt{2}, 2\}$ (of the coordinates of u_4 along with 1) such that its elements are rationally independent. Since only $\alpha_{4,2}$ has been selected, so $I_4 = \{2\}$. Once $\alpha_{4,1}$ and $\alpha_{4,3}$ are written as a function of $1, 3\sqrt{2}$, we get:

$$t_{4,1} = 1, \quad t_{4,3} = 2, \quad \gamma^{(4)}_{1,2} = \gamma^{(4)}_{3,2} = 0$$

Using the same procedure for the remaining values of k, we obtain:

$$I_5 = \{2, 3\}, \quad t_{5,1} = 0, \quad \gamma^{(5)}_{1,2} = \gamma^{(5)}_{1,3} = 0$$

$$I_6 = \{1, 2\}, \quad t_{6,3} = 1, \quad \gamma^{(6)}_{3,1} = \gamma^{(6)}_{3,2} = 0$$

$$I_7 = \{2\}, \quad t_{7,1} = 3, \quad t_{7,3} = 0, \quad \gamma^{(7)}_{1,2} = 0, \quad \gamma^{(7)}_{3,2} = 2$$

We choose $q_4 = q_5 = q_6 = q_7 = 1$ so that $p_{k,j} = t_{k,j}$ and $m^{(k)}_{i,j} = \gamma^{(k)}_{i,j}$ for every $i \in I_k$, $j \notin I_k$ and $k = 4, 5, 6, 7$.

The vectors $u'_{k,j}$, $j \in I_k$, $k = 4, 5, 6, 7$ are:

$$\begin{aligned} u'_{4,2} &= u_2 \\ u'_{5,2} &= u_2 \\ u'_{5,3} &= u_3 \\ u'_{6,1} &= u_1 \\ u'_{6,2} &= u_2 \\ u'_{7,2} &= u_2 + 2u_3 \end{aligned}$$

So that M_H is given by:

$$M_H = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

Theorem 3.3. *Let $H = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_p$ with $u_k \in \mathbb{R}^n$ and M_H defined as above. Then H is dense in $\mathbb{R}^n$ if and only if $rank(M_H) = n$.*

We need the following lemmas for the proof of the theorem 3.3:

Lemma 3.4. *Let $u_1, \dots, u_{n+1} \in \mathbb{R}^n$ be such that $(u_1, \dots, u_n)$ is a basis of $\mathbb{R}^n$ and $u_{n+1} = \sum_{i=1}^{n} \alpha_i u_i$. Suppose that $1, \alpha_{k_1}, \dots, \alpha_{k_r}$ is the longest sequence extracted from the list $\{1,\ \alpha_1, \dots, \alpha_n\}$ which contains 1 and such that its elements are rationally independent. Then there exist $q \in \mathbb{N}^*$, $m_{k,1}, \dots, m_{k,r} \in \mathbb{Z}$ such that*

$$\sum_{j\in I} \mathbb{R} u'_j + \sum_{j\notin I} \mathbb{Z} u_j \subset \overline{\sum_{k=1}^{n+1} \mathbb{Z} u_k} \subset \sum_{j\in I} \mathbb{R} u'_j + \frac{1}{q} \sum_{j\notin I} \mathbb{Z} u_j$$

where $u'_j = q u_j + \sum_{k\notin I} m_{k,j} u_k$ for every $j \in I$ and $I = \{k_j,\ j = 1, \dots, r\}$.

Proof. Assume without loss of generality that $k_j = j,\ j = 1, \dots, r$. In the above discussion, we have introduced the vectors $u'_{k,j}$ when several vectors are added to the basis $(u_1, \dots, u_n)$. But in this case, only one vector has been added ($p = n+1$), so we drop the index k from the definition of $u'_{k,j}$, $m_{k,j}$, $p_{k,j}$ and I_k. Thus we have

$$q u_{n+1} = \sum_{j\in I} \alpha_j u'_j + \sum_{j\notin I} p_j u_j$$

where $u'_j = q u_j + \sum_{i\notin I} m_{i,j} u_i$. Moreover, let $H := \sum_{k=1}^{n+1} \mathbb{Z} u_k$ and

$$u'_{n+1} = q u_{n+1} - \sum_{j\notin I} p_j u_j = \sum_{j=1}^{r} \alpha_j u'_j.$$

Now, consider the vector space E of dimension r equipped with the basis $\mathcal{B}_1 = (u'_1, \dots, u'_r)$. The vector $u'_{n+1} \in E$ and its coordinates with respect to the basis $\mathcal{B}_1$ are $[\alpha_1, \dots, \alpha_r]^T$. Moreover, since $1, \alpha_1, \dots, \alpha_r$ are rationally independent, so for every $(s_1, \dots, s_{r+1}) \in \mathbb{Z}^{r+1} \backslash \{0\}$,

$$\det \begin{bmatrix} 1 & 0 & \dots & 0 & \alpha_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & \alpha_r \\ s_1 & \dots & \dots & s_r & s_{r+1} \end{bmatrix} = s_{r+1} - \sum_{i=1}^{r} s_i \alpha_i \neq 0.$$

By applying proposition 2.1 to $K' := \sum_{j=1}^{r} \mathbb{Z} u'_j + \mathbb{Z} u'_{n+1}$, we get:

$$\overline{K'} = E$$

On the other hand, $E \oplus \left(\sum\limits_{k=r+1}^{n} \mathbb{R}u_k\right) = \mathbb{R}^n$, so

$$\overline{\sum_{j=1}^{r}\mathbb{Z}u'_j + \mathbb{Z}u'_{n+1} + \sum_{k=r+1}^{n}\mathbb{Z}u_k} = \overline{\sum_{j=1}^{r}\mathbb{Z}u'_j + \mathbb{Z}u'_{n+1}} + \overline{\sum_{k=r+1}^{n}\mathbb{Z}u_k}$$

Using lemma 2.4, $\sum\limits_{k=r+1}^{n} \mathbb{Z}u_k$ is closed in $\mathbb{R}^n$, thus:

$$\overline{\sum_{j=1}^{r}\mathbb{Z}u'_j + \mathbb{Z}u'_{n+1} + \sum_{k=r+1}^{n}\mathbb{Z}u_k} = E + \sum_{k=r+1}^{n}\mathbb{Z}u_k$$

Finally, we have for every $1 \leq j \leq r$:

$$\begin{cases} u'_{n+1},\ u'_j \in \sum\limits_{k=1}^{n+1}\mathbb{Z}u_k, \\ u_{n+1},\ u_j \in E + \frac{1}{q}\sum\limits_{k=r+1}^{n}\mathbb{Z}u_k \end{cases}$$

So $K' \subset \sum\limits_{k=1}^{n+1}\mathbb{Z}u_k$. Then

$$E + \sum_{k=r+1}^{n}\mathbb{Z}u_k = \overline{K'} + \sum_{k=r+1}^{n}\mathbb{Z}u_k \subset \overline{\sum_{k=1}^{n+1}\mathbb{Z}u_k} \subset E + \frac{1}{q}\sum_{k=r+1}^{n}\mathbb{Z}u_k.$$

The proof is completed. □

Lemma 3.5. *Let* $\mathcal{B} = (u_1, \dots, u_n)$ *be a basis of* $\mathbb{R}^n$, $n \geq 2$, *and* $v_1, \dots, v_q \in \sum\limits_{i=1}^{n}\mathbb{Z}u_i$ *with* $1 \leq q < n$. *Then the group* $\sum\limits_{i=1}^{q}\mathbb{R}v_i + \sum\limits_{i=1}^{n}\mathbb{Z}u_i$ *is not dense in* $\mathbb{R}^n$.

Proof. Without loss of generality, we can assume that the vectors $v_1, \dots, v_q$ are linearly independent. So they can be completed to the basis $\mathcal{B}' = (v_1, \dots, v_q, v_{q+1}, \dots, v_n)$ using the basis $\mathcal{B}$. We may also assume that $v_i = u_i$ for every $i = q+1, \dots, n$. Since $v_i \in \sum\limits_{j=1}^{n}\mathbb{Z}u_j$ for every $i = 1, \dots, n$, it follows that, through a change of basis, we have $u_i \in \sum\limits_{j=1}^{n}\mathbb{Q}v_j$. So there exist $p \in \mathbb{N}^*$ and $n_{i,j} \in \mathbb{Z}$ such that

$$u_i = \sum_{j=1}^{n}\frac{n_{i,j}}{p}v_j$$

Hence $u_i \in \frac{1}{p}\sum_{j=1}^{n}\mathbb{Z}v_j$ for every $i = 1, \ldots, n$. Therefore,

$$\begin{aligned}\sum_{i=1}^{q}\mathbb{R}v_i + \sum_{i=1}^{n}\mathbb{Z}u_i &\subset \sum_{i=1}^{q}\mathbb{R}v_i + \frac{1}{p}\sum_{i=1}^{n}\mathbb{Z}v_i \\ &= \sum_{i=1}^{q}\mathbb{R}v_i + \frac{1}{p}\sum_{i=q+1}^{n}\mathbb{Z}v_i\end{aligned}$$

By lemma 2.4, the group $\sum_{i=1}^{q}\mathbb{R}v_i + \frac{1}{p}\sum_{i=q+1}^{n}\mathbb{Z}v_i$ is closed in $\mathbb{R}^n$, so the group $\sum_{i=1}^{q}\mathbb{R}v_i + \sum_{i=1}^{n}\mathbb{Z}u_i$ is not dense in $\mathbb{R}^n$. □

Proof of theorem 3.3. Let $H := \sum_{i=1}^{p}\mathbb{Z}u_i$. Suppose that $\overline{H} = \mathbb{R}^n$, and define $H_k := \overline{\sum_{i=1}^{n}\mathbb{Z}u_i + \mathbb{Z}u_k}$, for every $k = n+1, \ldots, p$. As

$$H \subset \sum_{k=n+1}^{p} H_k$$

So, we have

$$\overline{\sum_{k=n+1}^{p} H_k} = \mathbb{R}^n \tag{3.1}$$

On the other hand, by lemma 3.4, we have

$$\sum_{j\in I_k}\mathbb{R}u'_{k,j} + \sum_{j\notin I_k}\mathbb{Z}u_j \subset H_k \subset \sum_{j\in I_k}\mathbb{R}u'_{j,k} + \frac{1}{q_k}\sum_{j\notin I_k}\mathbb{Z}u_j$$

where for every $j \in I_k$, $u'_{k,j} = q_k u_j + \sum_{i\notin I_k} m_{i,j}^{(k)} u_i$, with $m_{i,j}^{(k)} \in \mathbb{Z}$ and $q_k \in \mathbb{N}^*$. It follows that

$$\begin{aligned}\sum_{k=n+1}^{p} H_k &\subset \sum_{k=n+1}^{p}\left(\sum_{j\in I_k}\mathbb{R}u'_{k,j} + \sum_{j\notin I_k}\frac{1}{q_k}\mathbb{Z}u_j\right) \\ &\subset \sum_{k=n+1}^{p}\left(\sum_{j\in I_k}\mathbb{R}u'_{k,j}\right) + \sum_{k=n+1}^{p}\left(\sum_{j\notin I_k}\frac{1}{q_k}\mathbb{Z}u_j\right)\end{aligned}$$

Let $q = q_{n+1} \dots q_p$, the last formula then simplifies to

$$\begin{aligned}\sum_{k=n+1}^{p} H_k &\subset \sum_{k=n+1}^{p} \left(\sum_{j\in I_k} \mathbb{R}u'_{k,j} \right) + \frac{1}{q} \sum_{k=n+1}^{p} \left(\sum_{j\notin I_k} \mathbb{Z}u_j \right) \\ &\subset \sum_{k=n+1}^{p} \left(\sum_{j\in I_k} \mathbb{R}u'_{k,j} \right) + \frac{1}{q} \sum_{j=1}^{n} \mathbb{Z}u_j\end{aligned}$$

and by equation 3.1, we have

$$\overline{\sum_{k=n+1}^{p} \left(\sum_{j\in I_k} \mathbb{R}u'_{k,j} \right) + \frac{1}{q} \sum_{j=1}^{n} \mathbb{Z}u_j} = \mathbb{R}^n \tag{3.2}$$

Suppose that $\sum_{k=n+1}^{p} \left(\sum_{j\in I_k} \mathbb{R}u'_{k,j} \right) \neq \mathbb{R}^n$, then we can extract a maximal set of independent vectors $\{v_1, \dots, v_m\}$ with $m < n$ from the set of vectors $\{u'_{k,j},\ j \in I_k,\ k = n+1, \dots, p\}$. As $u'_{k,j} \in \frac{1}{q} \sum_{j=1}^{n} \mathbb{Z}u_j$ so $v_i \in \frac{1}{q} \sum_{j=1}^{n} \mathbb{Z}u_j$ for every $i = 1, \dots, m$. Using lemma 3.5, $\sum_{k=n+1}^{p} \left(\sum_{j\in I_k} \mathbb{R}u'_{k,j} \right) + \frac{1}{q} \sum_{j=1}^{n} \mathbb{Z}u_j = \sum_{j=1}^{m} \mathbb{R}v_j + \frac{1}{q} \sum_{j=1}^{n} \mathbb{Z}u_j$ is not dense in $\mathbb{R}^n$, this leads to a contradiction with equation 3.2. So $\sum_{k=n+1}^{p} \left(\sum_{j\in I_k} \mathbb{R}u'_{k,j} \right) = \mathbb{R}^n$.

Since $\sum_{k=n+1}^{p} \left(\sum_{j\in I_k} \mathbb{R}u'_{k,j} \right)$ is the span of the columns of the matrix M_H so $\text{rank}(M_H) = n$.

Conversely, suppose $\text{rank}(M_H) = n$, *i.e.* $\sum_{k=n+1}^{p} \left(\sum_{j\in I_k} \mathbb{R}u'_{k,j} \right) = \mathbb{R}^n$. By lemma 3.4, we have: for every $k = n+1, \dots, p$

$$\sum_{j\in I_k} \mathbb{R}u'_{k,j} \ \subset \sum_{j\in I_k} \mathbb{R}u'_{k,j} + \sum_{j\notin I_k} \mathbb{Z}u_j \ \subset H_k$$

So

$$\sum_{k=n+1}^{p} \left(\sum_{j\in I_k} \mathbb{R}u'_{k,j} \right) \ \subset \sum_{k=n+1}^{p} H_k$$

As $H_k \subset \overline{H}$, so $\sum_{k=n+1}^{p} H_k \subset \overline{H}$. Thus

$$\mathbb{R}^n = \sum_{k=n+1}^{p} \left(\sum_{j \in I_k} \mathbb{R} u'_{k,j} \right) \subset \overline{H}$$

It follows that $\overline{H} = \mathbb{R}^n$. $\square$

Example 3.6. Let $H = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_7$, where $u_1 = [1,0,0]^T$, $u_2 = [0,1,0]^T$, $u_3 = [0,0,1]^T$, $u_4 = [1,\sqrt{2},1]^T$, $u_5 = [0,1,\sqrt{3}]^T$, $u_6 = [\sqrt{2},\sqrt{3},1]^T$, $u_7 = [1,\sqrt{2},\sqrt{2}]^T$. So $n = 3$ and $p = 7$. The sets I_k, $k = 4, \ldots, 7$ are:
$I_4 = \{2\}$, $I_5 = \{3\}$, $I_6 = \{1,2\}$, $I_7 = \{2\}$.

We obtain:

$$\begin{aligned} u'_{4,2} &= u_2 \\ u'_{5,3} &= u_3 \\ u'_{6,1} &= u_1 \\ u'_{6,2} &= u_2 \\ u'_{7,2} &= u_2 + u_3 \end{aligned}$$

So that:

$$M_H = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Since $\operatorname{rank}(M_H) = 3$ then H is dense in $\mathbb{R}^3$.

Now, let us summarize the approach to follow in order to test the density of a given additive group $H = \sum_{k=1}^{p} \mathbb{Z}u_k$ of $\mathbb{R}^n$:

(1) If $p \leq n$ or $\sum_{k=1}^{p} \mathbb{R}u_k \neq \mathbb{R}^n$, then H is not dense in $\mathbb{R}^n$.

(2) Otherwise, compute the sets I_k, $k = n+1, \ldots, p$:
- If $I_k = \varnothing$ for every $k = n+1, \ldots, p$, then H is not dense in $\mathbb{R}^n$.
- If there is an integer $k_0 \in \{n+1, \ldots, p\}$ such that $I_{k_0} = \{1, \ldots, n\}$, then H is dense in $\mathbb{R}^n$.
- If $I_k \neq \varnothing$ for at least one k and $I_k \neq \{1, \ldots, n\}$ for every $k = n+1, \ldots, p$, then compute the vectors $u'_{k,j}$, $j \in I_k$, $k = n+1, \ldots, p$.

(3) Determine the matrix M_H and its rank.
Then H is dense in $\mathbb{R}^n$ iff $\operatorname{rank}(M_H) = n$.

4. The complex case: Density of additive subgroups of $\mathbb{C}^n$

Let $u_1, \dots, u_p \in \mathbb{C}^n$, $p \geq 2n+1$. Suppose that $(u_1, \dots, u_{2n})$ is a basis of $\mathbb{C}^n$ over $\mathbb{R}$. Denote by $\widetilde{u_k} = [\Re(u_k),\ \Im(u_k)]^T$ for every $k = 1, \dots, p$, where $\Re(w)$ and $\Im(w)$ are respectively the real and the imaginary part of a vector $w \in \mathbb{C}^n$. We let $\widetilde{H} = \mathbb{Z}\widetilde{u_1} + \cdots + \widetilde{u_p}$, so $\widetilde{H} \subset \mathbb{R}^{2n}$.

Theorem 4.1. *Let $H = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_p$ with $u_k \in \mathbb{C}^n$. Then H is dense in $\mathbb{C}^n$ if and only if $rank(M_{\widetilde{H}}) = 2n$.*

Proof. The proof results directly from theorem 3.3 and the fact that $\overline{H} = \mathbb{C}^n$ if and only if $\overline{\widetilde{H}} = \mathbb{R}^{2n}$. □

Example 4.2. Let $H = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_8$ with $u_1 = [1,\ 0]^T$, $u_2 = [0,\ 1]^T$, $u_3 = [i,\ 0]^T$,$u_4 = [0,\ i]^T$, $u_5 = [1+2i,\ 5\sqrt{3}]^T$, $u_6 = [i\sqrt{2}, \sqrt{3}+i]^T$, $u_7 = [2\sqrt{3}+i,\ \sqrt{2}+2i]^T$, $u_8 = [1+4i\sqrt{2},\ 5i\sqrt{3}]^T$.

Then $\widetilde{H} = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_8$, where $\widetilde{u}_1 = [1,0,0,0]^T$, $\widetilde{u}_2 = [0,1,0,0]^T$, $\widetilde{u}_3 = [0,0,1,0]^T$, $\widetilde{u}_4 = [0,0,0,1]^T$, $\widetilde{u}_5 = [1, 5\sqrt{3}, 2, 0]^T$, $\widetilde{u}_6 = [0, \sqrt{3}, \sqrt{2}, 1]^T$, $\widetilde{u}_7 = [2\sqrt{3}, \sqrt{2}, 1, 2]^T$, $\widetilde{u}_8 = [1, 0, 4\sqrt{2}, 5\sqrt{3}]^T$.

The sets $I_k,\ k = 5, \dots, 8$ are:
$I_5 = \{2\}$, $I_6 = \{2,3\}$, $I_7 = \{1,2\}$ and $I_8 = \{3,4\}$.

We obtain:

$$\begin{aligned}
\widetilde{u}'_{5,2} &= \widetilde{u}_2\\
\widetilde{u}'_{6,2} &= \widetilde{u}_2\\
\widetilde{u}'_{6,3} &= \widetilde{u}_3\\
\widetilde{u}'_{7,1} &= \widetilde{u}_1\\
\widetilde{u}'_{7,2} &= \widetilde{u}_2\\
\widetilde{u}'_{8,3} &= \widetilde{u}_3\\
\widetilde{u}'_{8,4} &= \widetilde{u}_4
\end{aligned}$$

Then

$$M_{\widetilde{H}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $\text{rank}(M_{\tilde{H}}) = 4$ then $\tilde{H}$ is dense and so is H.

ACKNOWLEDGEMENTS. *This work is supported by the research unit: systèmes dynamiques et combinatoire: 99UR15-15*

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