Applied General Topology
Universidad Politécnica de Valencia
Volume 2, No. 1, 2001
pp. 1 - 7

On topological sequence entropy of circle maps

José S. Cánovas*

Abstract. We classify completely continuous circle maps from the point of view of topological sequence entropy. This improves a result of Roman Hric.

2000 AMS Classification: 37B40, 37E10.

Keywords: Topological sequence entropy, circle maps, chaos in the sense of Li–Yorke.

1. Introduction

Let (X,d) be a compact metric space and let $f : X \to X$ be a continuous map. Denote by $C(X,X)$ the set of continuous maps $f : X \to X$. (X,f) is called a *discrete dynamical system*. The map f is said *chaotic in the sense of Li–Yorke* (or simply *chaotic*) if there is an uncountable set $S \subset X$ such that for any $x, y \in S$, $x \neq y$, it holds that

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0, \tag{1.1}$$

and

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0. \tag{1.2}$$

S is said a *scrambled set* of f (see [10]). When f is chaotic we say that (X,f) is chaotic.

The notion of chaos plays an special role in the setting of discrete dynamical systems. So, some topological invariants have been porposed to give a charaterization of chaos. Maybe, the most important topological invariant in this setting is the topological entropy (see [1]). When $X = [a,b]$, $a, b \in \mathbb{R}$, it is well–known that positive topological entropy implies that f is chaotic, while the converse result is false (see [12]).

So, in order to characterize chaotic interval maps we need an extension of topological entropy called topological sequence entropy (see [7]). Given an

*This paper has been partially supported by the grant D.G.I.C.Y.T. PB98-0374-C03-01.

increasing sequence of positive integers $A = (a_i)_{i=0}^{\infty}$, a number $h_A(f)$ can be associated to each $f \in C(X, X)$. This number is also a topological invariant. Then, defining $h_\infty(f) = \sup_A h_A(f)$, chaotic interval maps can be characterized by the following result.

Theorem 1.1. *Let $f \in C([a, b], [a, b])$. Then*
- (a) *f is non–chaotic iff $h_\infty(f) = 0$.*
- (b) *f is chaotic with zero topological entropy iff $h_\infty(f) = \log 2$.*
- (c) *f is chaotic with positive topological entropy iff $h_\infty(f) = \infty$.*

Theorem 1.1 establishes a complete classification of maps from the point of view of topological sequence entropy. The part (a) was proved by Franzová and Smítal in [6]. (a) provides that any chaotic map holds $h_\infty(f) > 0$. In [3] was proved (b) and (c) in case of piecewise monotonic maps. This result was extended to the general case in [4].

Following [7], a map $f \in C(X, X)$ is said *null* if $h_\infty(f) = 0$. f is said *bounded* if $h_\infty(f) < \infty$ and *unbounded* if $h_\infty(f) = \infty$. In the general case, it is unknown when a continuous map is null, bounded or unbounded. It is easy to see that when f is stable in the Lyapunov sense (f has equicontinuous powers) the map is null (see [7]). Theorem 1.1 establishes a characterization of null, bounded and unbounded continuous interval maps.

The aim of this paper is to prove Theorem 1.1 in the setting of continuous circle maps. This will provide a classification of unbounded, bounded and null continuous circle maps. Before starting with this classification, let us point out that for any $f \in C(S^1, S^1)$, Hric proved in [8] that it is non–chaotic iff $h_\infty(f) = 0$, which classifies chaotic circle maps from the point of view of topological sequence entropy.

2. Preliminaries

Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous map. Denote by $C(X, X)$ the set of continuous maps $f : X \to X$. Let f^0 be the identity on X, $f^1 := f$ and $f^n = f \circ f^{n-1}$ for all $n \geq 1$. Consider an increasing sequence of positive integers $A = (a_i)_{i=1}^{\infty}$ and let $Y \subseteq X$ and $\varepsilon > 0$. We say that a subset $E \subset Y$ is $(A, \varepsilon, n, Y, f)$–separated if for any $x, y \in E$, $x \neq y$, there is an $i \in \{1, 2, ..., n\}$ such that $d(f^{a_i}(x), f^{a_i}(y)) > \varepsilon$. Denote by $s_n(A, \varepsilon, Y, f)$ the cardinality of any maximal $(A, \varepsilon, n, Y, f)$–separated set. Define

$$s(A, \varepsilon, Y, f) := \limsup_{n\to\infty} \frac{1}{n} \log s_n(A, \varepsilon, Y, f). \tag{2.3}$$

It is clear from the definition that if $Y_1 \subseteq Y_2 \subseteq X$, then

$$s(A, \varepsilon, Y_1, f) \leq s(A, \varepsilon, Y_2, f). \tag{2.4}$$

Let

$$h_A(f, Y) := \lim_{\varepsilon\to 0} s(A, \varepsilon, Y, f). \tag{2.5}$$

The *topological sequence entropy* of f respect to the sequence A is defined by

$$h_A(f) := h_A(f, X). \tag{2.6}$$

When $A = (i)_{i=0}^{\infty}$, we receive the classical definition of topological entropy (see [1]).

Finally, let

$$h_\infty(f, Y) := \sup_A h_A(f, Y) \tag{2.7}$$

and

$$h_\infty(f) := \sup_A h_A(f). \tag{2.8}$$

An $x \in X$ is *periodic* if there is an $n \in \mathbb{N}$ such that $f^n(x) = x$. The smallest positive integer holding this condition is called the *period* of x. The set of periods of f, $P(f)$, is defined by

$$P(f) := \{n \in \mathbb{N} : \exists x \in X \text{ periodic point of period} n\}.$$

3. Results on topological sequence entropy

In this section we prove some useful results concerning topological sequence entropy of continuous maps defined on arbitrary compact metric spaces.

Proposition 3.1. *Let $f \in C(X, X)$. For all $n \in \mathbb{N}$ it holds that $h_\infty(f^n) = h_\infty(f)$.*

Proof. First, we prove that $h_\infty(f^n) \le h_\infty(f)$. In order to see this, let $A = (a_i)_{i=1}^{\infty}$ be an increasing sequence of positive integers and define $nA = (na_i)_{i=1}^{\infty}$. Then it is straightforward to see that $h_A(f^n) = h_{nA}(f)$ and hence

$$\begin{aligned} h_\infty(f^n) &= \sup_A h_A(f^n) = \sup_A h_{nA}(f) \\ &\le \sup_B h_B(f) = h_\infty(f). \end{aligned}$$

Now, we prove the converse inequality. Let A be an increasing sequence of postive integers. By [8], there is another sequence $B = B(A)$ such that $h_A(f) \le h_B(f^n)$. Then

$$\begin{aligned} h_\infty(f) &= \sup_A h_A(f) \le \sup_A h_{B(A)}(f^n) \\ &\le \sup_A h_A(f^n) = h_\infty(f^n), \end{aligned}$$

which ends the proof. □

Corollary 3.2. *Under the conditions of Proposition 3.1, the following statements hold:*

(a) *f is null iff f^n is null for all $n \in \mathbb{N}$.*
(b) *f is bounded iff f^n is bounded for all $n \in \mathbb{N}$.*
(c) *f is unbounded iff f^n is unbounded for all $n \in \mathbb{N}$.*

Proposition 3.3. *Let $f \in C(X, X)$ have positive topological entropy. Then $h_\infty(f) = \infty$.*

Proof. Since $h(f) > 0$ it follows by [7] that for any increasing sequence of positive integers $A = (a_i)_{i=1}^{\infty}$, $h_A(f) = K(A)h(f)$, where

$$K(A) = \lim_{k\to\infty} \limsup_{n\to\infty} \frac{1}{n} \mathrm{Card}\{a_i, a_i + 1, ..., a_i + k : 1 \leq i \leq n\}. \tag{3.9}$$

Taking $A = (2^i)_{i=1}^{\infty}$ it holds that $K(A) = \infty$ and hence $h_A(f) = \infty$. □

Proposition 3.4. *Let (X, d) and (Y, e) be compact metric spaces and let $f : X \to X$ and $g : Y \to Y$ be continuous maps. Let $\pi : X \to Y$ be continuous and surjective such that $\pi \circ f = g \circ \pi$. Let A be an increasing sequence of positive integers A and let $Y_1 \subseteq Y$. Then, for any $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$s(A, \delta, \pi^{-1}(Y_1), f) \geq s(A, \varepsilon, Y_1, g). \tag{3.10}$$

In particular, $h_\infty(f) \geq h_\infty(g)$.

Proof. Let $E \subseteq Y_1$ be a maximal subset $(A, n, \varepsilon, Y_1, g)$–separated. Let $F \subseteq \pi^{-1}(Y_1)$ be a set containing exactly one element from $\pi^{-1}(y)$ for all $y \in E$. We claim that F is an $(A, n, \delta, \pi^{-1}(Y_1), f)$–separated subset for some $\delta > 0$. Assume the contrary. Since π is uniformly continuous, there is a $\delta = \delta(\varepsilon) > 0$ such that $d(x_1, x_2) < \delta$, $x_1, x_2 \in X$, implies $e(\pi(x_1), \pi(x_2)) < \varepsilon$. Now let $x_1, x_2 \in F$ be such that

$$d(f^{a_i}(x_1), f^{a_i}(x_2)) < \delta \tag{3.11}$$

for all $i \in \{1, 2, ..., n\}$. Let $y_1, y_2 \in E$ be such that $\pi(x_j) = y_j$ for $j = 1, 2$. Then, for all $i \in \{1, 2, ..., n\}$ we have that

$$\begin{aligned} e(g^{a_i}(y_1), g^{a_i}(y_2)) &= e(g^{a_i}(\pi(x_1)), g^{a_i}(\pi(y_2))) \\ &= e(\pi \circ f^{a_i}(x_1), \pi \circ f^{a_i}(y_2)) \leq \varepsilon, \end{aligned}$$

which leads us to a contradiction. Then $s_n(A, \delta, \pi^{-1}(Y_1), f) \geq s_n(A, \varepsilon, Y_1, f)$ and hence

$$\begin{aligned} s(A, \delta, \pi^{-1}(Y_1), f) &= \limsup_{n\to\infty} \frac{1}{n} \log s_n(A, \delta, \pi^{-1}(Y_1), f) \\ &\geq \limsup_{n\to\infty} \frac{1}{n} \log s_n(A, \varepsilon, Y_1, g) \\ &= s(A, \varepsilon, Y_1, g), \end{aligned}$$

which ends the proof. □

Under the conditions of Proposition 3.4, if π is an homemorphism, then f and g are said to be *conjugate*. Then

Corollary 3.5. *Under the conditions of Proposition 3.4, if f and g are conjugate, then $h_\infty(f) = h_\infty(g)$.*

4. Main results

In the sequel we will discuss the space of continuous circle maps denoted by $C(S^1, S^1)$. Let $f \in C(S^1, S^1)$ and let $l : \mathbb{R} \to S^1$ be defined by $l(x) = \exp(2\pi i x)$ for all $x \in \mathbb{R}$. Then, there are a countable number of continuous maps $F : \mathbb{R} \to \mathbb{R}$ such that $l \circ F = f \circ l$. An F holding this condition is called a *lifting* of f. If $\widetilde{F}$ is another lifting of f, then

$$\widetilde{F} - F = k \in \mathbb{N}. \tag{4.12}$$

By $|J|$ we denote the length of an interval $J \subseteq \mathbb{R}$.

Theorem 4.1. *Let $f \in C(S^1, S^1)$. Then*

(a) *f is non–chaotic iff $h_\infty(f) = 0$.*
(b) *f is chaotic with zero topological entropy iff $h_\infty(f) = \log 2$.*
(c) *f is chaotic with positive topological entropy iff $h_\infty(f) = \infty$.*

Proof. According to Chapter 3 from [2], $C(S^1, S^1)$ can be decomposed into the following classes:

$$C_1 = \{f \in C(S^1, S^1) : f \text{ has no periodic points}\}; \tag{4.13}$$

$$C_2 = \{f \in C(S^1, S^1) : P(f^n) = \{1\} \text{ or } P(f^n) = \{1, 2, 2^2, ...\} \text{ for some } n \in \mathbb{N}\}; \tag{4.14}$$

$$C_3 = \{f \in C(S^1, S^1) : P(f^n) = \mathbb{N} \text{ for some } n \in \mathbb{N}\}. \tag{4.15}$$

According to [8], any $f \in C_1$ is non–chaotic and holds that $h_\infty(f) = 0$. Let $f \in C_3$. Again by [8], it holds that f is chaotic and $h(f) > 0$. Then, by Proposition 3.3 we have that $h_\infty(f) = \infty$. So, we must consider only maps from C_2.

Let $f \in C_2$ and let $n \in \mathbb{N}$ be such that $P(f^n) = \{1\}$ or $P(f^n) = \{1, 2, 2^2, ...\}$. It is well–known that f is chaotic iff f^n is chaotic. So, applying Proposition 3.1, it is not restrictive to assume that $n = 1$. Since f has a fixed point, by Lemma 2.5 from [9], there is a lifting $F : \mathbb{R} \to \mathbb{R}$ and there is a compact interval J, with $|J| > 1$, such that $F(J) = J$. For the rest of the proof call $l = l|_J$. First assume that f is non–chaotic. Then by [8] it holds that $h_\infty(f) = 0$. Secondly, assume that f is chaotic. Hence F is also chaotic (see [8]) and has zero topological entropy (see [12]). By Proposition 3.4, for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that

$$s(A, \varepsilon, S^1, f) \le s(A, \delta, l^{-1}(S^1), F) = s(A, \delta, J, F). \tag{4.16}$$

On the other hand, by [4], there is a compact interval $J_i \subseteq J$, holding that $F^{2^i}(J_i) = J_i$ such that

$$s(A, \delta, J, F) \le s(A, \delta/6, \cup_{j=1}^{2^i} F^j(J_i), F) \le \log 2. \tag{4.17}$$

By (4.16) and (4.17) we conclude that

$$s(A, \varepsilon, S^1, f) \le \log 2. \tag{4.18}$$

Since ε was arbitrary chosen, we obtain

$$h_A(f) \le \log 2, \tag{4.19}$$

and

$$h_\infty(f) = \sup_A h_A(f) \leq \log 2. \tag{4.20}$$

Now, we prove the converse inequality. By [12], there is a compact interval J_i, with $|J_i| < 1$ and $F^{2^i}(J_i) = J_i$ such that $F^{2^i}|_{J_i}$ is chaotic. By [6], there is an increasing sequence of positive integers B such that $s(B, \varepsilon, J_i, F^{2^i}) \geq \log 2$ for a suitable $\varepsilon > 0$. Since $l|_{J_i} : J_i \to l(J_i)$ is an homemorphism, we can apply Proposition 3.4 to $l|_{J_i} = l$ to obtain a $\delta > 0$ such that

$$s(A, \delta, l(J_i), f^{2^i}) \geq s(A, \varepsilon, J_i, F^{2^i}) \geq \log 2. \tag{4.21}$$

Hence

$$h_\infty(f) \geq h_{2^iA}(f) = h_A(f^{2^i}) \geq s(A, \delta, l(J_i), f^{2^i}) \geq \log 2, \tag{4.22}$$

which concludes the proof. $\square$

Remark 4.2. *When two–dimensional maps are concerned, Theorems 1.1 and 4.1 are false in general. More precisely, in* [11] *and* [5] *a chaotic map* $F \in C([0,1]^2, [0,1]^2)$ *with* $h_\infty(F) = 0$ *and a non chaotic map* $G \in C([0,1]^2, [0,1]^2)$ *holding* $h_\infty(G) > 0$ *have been constructed. It seems that the dimension of the space* X *plays a special role in Theorems 1.1 and 4.1. We conjecture that Theorem 1.1 remains true for continuous maps defined on finite graphs, that is, in the special setting of one–dimensional dynamics.*

References

[1] R. L. Adler, A. G. Konheim and M. H. McAndrew, *Topological entropy,* Trans. Amer. Math. Soc. **114** (1965), 309–319.

[2] Ll. Alsedá, J. Llibre and M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, World Scientific (1993).

[3] J. S. Cánovas, *On topological sequence entropy of piecewise monotonic mappings,* Bull. Austral. Math. Soc. **62** (2000), 21–28.

[4] J. S. Cánovas, *Topological sequence entropy of interval maps,* Prepublicación 6/00 del Departamento de Matemática Aplicada y Estadística. Universidad Politécnica de Cartagena.

[5] G. L. Forti, L. Paganoni and J. Smítal, *Strange triangular maps of the square,* Bull. Austral. Math. Soc. **51** (1995), 395–415.

[6] N. Franzová and J. Smítal, *Positive sequence topological entropy characterizes chaotic maps,* Proc. Amer. Math. Soc. **112** (1991), 1083–1086.

[7] T. N. T. Goodman, *Topological sequence entropy,* Proc. London Math. Soc. **29** (1974), 331–350.

[8] R. Hric, *Topological sequence entropy for maps of the circle,* Comment. Math. Univ. Carolin. **41** (2000), 53–59.

[9] M. Kuchta, *Characterization of chaos for continuous maps of the circle,* Comment. Math. Univ. Carolinae **31** (1990), 383–390.

[10] T. Y. Li and J. A. Yorke, *Period three implies chaos,* Amer. Math. Monthly **82** (1975), 985–992.

[11] L. Paganoni and P. Santambrogio, *Chaos and sequence topological entropy for triangular maps,* Grazer. Math. Ber. **339** (1999), 279–290.

[12] J. Smítal, *Chaotic functions with zero topological entropy,* Trans. Amer. Math. Soc. **297** (1986), 269–282.

Received July 2000

Revised version October 2000

José S. Cánovas
Dep. de Matemática Aplicada y Estadística
Universidad Politécnica de Cartagena
30203 Cartagena
Spain
E-mail address: Jose.Canovas@upct.es