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Abelization of join spaces of affine transformations of ordered field with proximity

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Abstract. Using groups of affine transformations of linearly ordered fields a certain construction of non-commutative join hypergroups is presented based on the criterion of reproducibility of semi-hypergroups which are determined by ordered semigroups. The aim of this paper is to construct the abelization of the non-commutative join space of affine transformations of ordered fields. A construction of commutative weakly associative hypergroup (H_ν-group) is made and a proximity is defined on this structure.

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Transformation groups which represent the classical and developing discipline are situated in the intersection of several parts of mathematical structures. Transformation groups in the discrete approach create in a very natural way commutative hyperstuctures, which are in addition join spaces. On the other hand, groups of affine transformations of the field create naturally a transformation group on the supporting set of this field. More precisely:

Let $(F, +, \cdot)$ be a field, $A(F)$ be a group of affine transformations of F of the form $\varphi_{a,b}(x) = ax + b$; $x \in F$, where the coefficient $a \in F$ is different from zero. Then we construct a discrete transformation group (X, T, π) by putting $X = F$; $T = A(F)$ and the action π we define by $\pi(x; \varphi_{a,b}) = \varphi_{a,b}(x)$. Clearly the identity axiom and homomorphism axiom are satisfied.

In this paper hyperstructures on groups of affine transformations of ordered fields are constructed. These affine transformations are represented by ordered pairs of elements of a given field. During the construction of final hyperstructures upper ends of the products of pairs of elements of a given field are used.

This idea is adopted from the functorial assignment of a commutative hypergroup to an arbitrary transformation (discrete) group. We will describe this construction in more details. Let $G = (X, T, \pi)$ be a transformation group, (i.e. X-phase set, T-phase group, π- action (projection): $X \times T \to X$). For any pair $x, y \in X$ we define

$$x *_G y = \pi(x, T) \cup \pi(y, T) = \{\pi(x, t); t \in T\} \cup \{\pi(y, t); t \in T\}.$$

It is easy to show that $(X, *_G)$ is an extensive commutative hypergroup—moreover a join space (see the definition below).

In the paper [1] the authors studied the non-commutative join hypergroups of affine transformations of ordered fields. The aim of this paper is to abelize this structure using the construction described in papers [7, 14].

Recall that an ordering on a field $(F, +, \,.\,)$ (with the zero element 0_F and the unit 1_F) is given by the choice of a set $P \subseteq F$ (called the positive cone of the ordering) which satisfies the following axioms:

(1) $P + P \subseteq P\,,$ (2) $P \,.\, P \subseteq P\,,$

(3) $P \cup (-P) = F\,,$ (4) $P \cap (-P) = \{0_F\}\,.$

As usual, we define a binary relation $\leqq_P$ on the set F by $x \leqq_P y$ (resp. $x <_P y$) if $y - x \in P$(resp. $y - x \in \dot{P}$, where $\dot{P} = P \setminus \{0_F\}$). The relation $\leqq_P$ results in a total ordering of the elements of F. In detail, the relation $\leqq_P$ on F is reflexive, anti symmetrical and transitive, compatible in the usual sense with commutative binary operations addition "$+$" and multiplication "$\,.\,$", and satisfies the law of trichotomy (for any pair $x, y \in F$ exactly one of the following possibilities occurs: $x = y$ or $x <_P y$ or $y <_P x$). Note that an ordered field $(F, +, \,.\,, \leqq_P)$ has necessarily the characteristic equal to zero. As usual, $\mathscr{P}^*(S)$ denotes the system of all nonempty subsets of S.

Let $(F, +, \,.\,)$ be a field of the characteristic zero. An affine transformation $f\colon F \to F$ of the form $f(x) = a \,.\, x + b$, $a, b \in F$ is uniquely represented by an ordered pair of its coefficients $[a, b] \in F \times F$. We shall consider non-constant transformations only, i.e., transformations $f(x) = a.x + b$ satisfying the condition $a \neq 0_F$. So, let us denote $A(F) = (F \setminus \{0_F\}) \times F$ and define a binary operation "$\cdot$" on the set $A(F)$ by the rule

$$[a, b] \cdot [c, d] = [a.c, a.d + b]\,,$$

which corresponds to the usual composition of affine transformations of F. It is easy to see that $\big(A(F), \cdot\big)$ is a non-commutative group with the identity $[1_F, 0_F]$ and inverse elements of the form $[a, b]^{-1} = [a^{-1}, -a^{-1}.b]$.

Denote by K the subset $\{[a, b]; a, b \in F, a >_P 0_F\}$ of $A(F) = (F \setminus \{0_F\}) \times F$. It is easy to see that $(K, \cdot)$ is a subgroup of the group $\big(A(F), \cdot\big)$. Define a binary relation "$\leqq$" on the set K by the rule:

$$[a, b] \leqq [c, d] \quad \text{for} \quad [a, b], [c, d] \in K$$

whenever $a = c$ and $b \leqq_P d$. Then evidently "$\leqq$" is an ordering on K and we have

Proposition 1 ([1])**.** *If $K = \{[a, b]; a, b \in F, a >_P 0_F\}$, then $(K, \cdot)$ is an ordered group such that it is a normal subgroup of the group $\big(A(F), \cdot\big)$.*

Recall that a *hypergroupoid* is a pair $(H,\cdot)$ where H is a (nonempty) set and $\cdot : H \times H \to \mathscr{P}^*(H)$ is a binary hyperoperation on the set H. If

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \text{for all} \quad a, b, c \in H\,, \quad (\text{associativity})\,,$$

then $(H,\cdot)$ is called a *semihypergroup.* A semihypergroup $(H,\cdot)$ is said to be a *hypergroup* if the following axiom

$$a \cdot H = H = H \cdot a \quad \text{for all} \quad a \in H\,, \quad (\text{reproduction axiom})\,,$$

is satisfied—see e.g. [3],[4].

Here, for $A, B \subseteq H$, $A \neq \varnothing \neq B$ we define $A \cdot B = \bigcup\{a.b; a \in A, b \in B\}$. Moreover, for subset A and B of H, it becomes convenient to use the relational notation $A \approx B$ (read A meets B) to assert that A and B have an element in common, that is, that $A \cap B \neq \varnothing$ ([16], p. 79).

A hypergroup $(H,\cdot)$ is called a *transposition hypergroup* if it satisfies the *transposition axiom*: For all $a, b, c, d \in H$ the relation

$$b\backslash a \approx c/d \quad \text{implies} \quad a \cdot d \approx b \cdot c\,,$$

where $b\backslash a = \{x \in H; a \in b \cdot x\}$, $c/d = \{x \in H; c \in x \cdot d\}$. A commutative transposition hypergroup $(H,*)$ is called a *join space* ([15]).

The hyperoperation $\star\colon H \times H \to \mathscr{P}^*(H)$ is called *weakly associative* hyperoperation if

$$\big(a \star (b \star c)\big) \cap \big((a \star b) \star c\big) \neq \varnothing$$

for any triad $a, b, c \in H$.

A *weak semihypergroup* (*H_ν-semigroup*) is a set H $(H \neq \varnothing)$ equipped with a weakly associative hyperoperation.

A H_ν-semigroup is called a *weak hypergroup* (*H_ν-group*) if moreover the reproduction axiom is satisfied for any $a \in H$ ([18]).

Proposition 2 ([5] Theorem 1)**.** *Let $(S, \cdot, \leqq)$ be an ordered semigroup. A binary operation $* : S \times S \to \mathscr{P}^*(S)$ defined by $x * y = [x\,.\,y)_{\leqq}$ $(= \{t \in S; x\,.\,y \leqq t\})$ for any pair $x, y \in S$ is associative. Then we have*

1° *The semi-hypergroup $(S,*)$ is commutative if and only if the semigroup $(S,\cdot)$ is commutative.*

2° *For the ordered semigroup $(S,\cdot,\leqq)$ the following conditions are equivalent:*

(i) *for any pair of elements $x, y \in S$ there exists a pair $z, z' \in S$ such that $y \cdot z \leqq x$, $z' \cdot y \leqq x$,*

(ii) *the semihypergroup $(S,*)$ satisfies the reproduction condition (i.e. $t*S = S = S*t$ for any $t \in S$), hence it is a hypergroup.*

Lemma 1 ([7, 14])**.** *Let $(H,\cdot)$ be a hypergroupoid. Define a hyperoperation "$\star$" on the diagonal Δ_H as follows: $[x,x] \star [y,y] = D(x \cdot y \cup y \cdot x) = \big\{[u,u]; u \in x \cdot y \cup y \cdot x\big\}$ for any pair $[x,x],[y,y] \in \Delta_H$. Then the following assertions hold:*

1° *For any hypergroupoid $(H,\cdot)$ we have that $(\Delta_H,\star)$ is a commutative hypergroupoid.*

2° *If* $(H,\cdot)$ *is a weakly associative hypergroupoid, then the hypergroupoid* $(\Delta_H,\star)$ *is weakly associative, as well.*

3° *If* $(H,\cdot)$ *is a quasi-hypergroup, the hypergroupoid* $(\Delta_H,\star)$ *also satisfies the reproduction law, i.e., it is a quasi-hypergroup.*

Define a hyperoperation $*\colon K\times K\to\mathscr{P}^*(K)$ by

$$[a,b]*[c,d]=\big\{[x,y];[a,b]\cdot[c,d]\leqq[x,y]\big\}=\big\{[a.c,y];a.d+b\leqq_P y\big\}.$$

From Proposition 2 it is evident, that the hypergroupoid $(K,*)$ is non-commutative hypergroup. The hypergroup $(K,*)$ will be called to be determined by the ordered group $(K,\cdot,\leqq)$. The hypergroup $(K,*)$ is non-commutative, therefore we will abelize it.

Let us define the set $\Delta_K=\Big\{\big[[a,b],[a,b]\big];[a,b]\in K\Big\}$ and a hyperoperation $\star:\Delta_K\times\Delta_K\to\mathscr{P}^*(\Delta_K)$ by

(0.1)

$$\begin{aligned}
&\big[[a,b],[a,b]\big]\star\big[[c,d],[c,d]\big]=\\
&=\Big\{\big[[x,y],[x,y]\big];[x,y]\in([a,b]*[c,d])\cup([c,d]*[a,b])\Big\}\\
&=\Big\{\big[[x,y],[x,y]\big];[x,y]\in\big\{[a.c,u];a.d+b\leqq_P u\big\}\cup\big\{[c.a,v];c.b+d\leqq_P v\big\}\Big\}\\
&=\Big\{\big[[x,y],[x,y]\big];x=a.c\wedge z\leqq y)\Big\},
\end{aligned}$$

where $z=\min\{a.d+b;c.b+d\}$.

Theorem 1. *The hyperstructure* $(\Delta_K,\star)$ *is commutative weakly associative hypergroup, simply* H_ν*-group.*

Proof. Recall that the field F is commutative, so the multiplication is commutative, thus $a.c=c.a$.

It is evident, that $(\Delta_K,\star)\cong(M,\diamond)$, where $M=\big\{[a,b,a,b];[a,b]\in K\big\}$ and

$$[a_1,b_1,a_1,b_1]\diamond[a_2,b_2,a_2,b_2]=\big\{[a,b,a,b];a=a_1\cdot a_2,z\leqq_P b\big\},$$

$z=\min\{a_1\cdot b_2,+b_1,a_2,\cdot b_1+b_2,\}$. Due to Lemma 1 it is evident that $(\Delta_K,\star)$ is the commutative weakly associative hypergroup—simply abelian H_ν-group.

To investigate the associativity law in more detail let us prove the weak associativity law using the concrete form of the structure $(K,*)$. We want to verify that

$$\Big(\big[[a,b],[a,b]\big]\star\big[[c,d],[c,d]\big]\Big)\star\big[[p,q],[p,q]\big]\cap\\
\big[[a,b],[a,b]\big]\star\Big(\big[[c,d],[c,d]\big]\star\big[[p,q],[p,q]\big]\Big)\neq\varnothing.$$

for any $[a,b],[c,d],[p,q]\in K$. For the next computation let us denote

$$A=\Big(\big[[a,b],[a,b]\big]\star\big[[c,d],[c,d]\big]\Big)\star\big[[p,q],[p,q]\big],$$
$$B=\big[[a,b],[a,b]\big]\star\Big(\big[[c,d],[c,d]\big]\star\big[[p,q],[p,q]\big]\Big),$$
$$z_1=\min\{a.d+b;c.b+d\},z_2=\min\{c.q+d;p.d+q\}.$$

We get

$$A=\Big\{\big[[x,y],[x,y]\big];x=a.c\wedge z_1\leqq_P y\Big\}\star\big[[p,q],[p,q]\big]=$$
$$=\bigcup_{z_1\leqq_P y}\big[[a.c,y],[a.c,y]\big]\star\big[[p,q],[p,q]\big]$$
$$=\Big\{\big[[u,v],[u,v]\big];u=a.c.p\wedge w_1\leqq_P v\Big\},$$

where $w_1=\min\{a.c.q+y,p.y+q\}$.

As $p\in F,p>_P 0_F$, we have

$$\begin{aligned}w_1&=\min\big\{a.c.q+y,p.\min\{a.d+b,c.b+d\}+q\big\}=\\&=\min\big\{a.c.q+\min\{a.d+b,c.b+d\},\min\{p.a.d+p.b,p.c.b+p.d\}+q\big\}=\\&=\min\big\{\min\{a.c.q+a.d+b,a.c.q+c.b+d\},\\&\qquad\min\{p.a.d+p.b+q,p.c.b+p.d+q\}\big\}=\\&=\min\{a.c.q+a.d+b,a.c.q+c.b+d,p.a.d+p.b+q,p.c.b+p.d+q\}.\end{aligned}$$

So, the set $A=\Big\{\big[[u,v],[u,v]\big];u=a.c.p\wedge w_1\leqq_P v\Big\}$. On the other hand

$$B=\big[[a,b],[a,b]\big]\star\Big\{\big[[x,y],[x,y]\big];x=c.p\wedge z_2\leqq_P y\Big\}$$
$$=\bigcup_{z_2\leqq_P y}\big[[a,b],[a,b]\big]\star\big[[x,y],[x,y]\big]=\Big\{\big[[u,v],[u,v]\big];u=a.c.p\wedge w_2\leqq_P v\Big\},$$

where $w_2=\min\{a.y+b,x.b+y\}$.

Similarly, because $a\in F,a>_P 0_F$

$$\begin{aligned}w_2&=\min\big\{a.\min\{c.q+d,p.d+q\}+b,x.b+\min\{c.q+d;p.d+q\}\big\}=\\&=\min\big\{\min\{a.c.q+a.d,a.p.d+a.q\}+b,\\&\qquad\min\{x.b+c.q+d,x.b+p.d+q\}\big\}=\\&=\min\big\{\min\{a.c.q+a.d+b,a.p.d+a.q+b\},\\&\qquad\min\{c.p.b+c.q+d,c.p.b+p.d+q\}\big\}=\\&=\min\{a.c.q+a.d+b,a.p.d+a.q+b,c.p.b+c.q+d,c.p.b+p.d+q\}.\end{aligned}$$

So, the set $B=\Big\{\big[[u,v],[u,v]\big];u=a.c.p\wedge w_2\leqq_P v\Big\}$.

Choose $\big[[u_0,v_0],[u_0,v_0]\big]\in\Delta_K$, such that $u_0=a.c.p$ and $v_0\geqq_P\max\{w_1,w_2\}$. Evidently, this pair of pairs belongs to $A\cap B$. $\square$

Remark 1. *Using the previous calculations it is easy to check that the structure* $(\Delta_K, \star)$ *is never associative. To see it let us first consider* $F = \mathbb{Q}$. *Necessarily,* $\mathbb{Q}^+ \subset P$. *Choose, for example* $a = 1$, $b = 2$, $c = 3$, $d = -4$, $p = 5$, $q = -7$. *From the above notations we have* $w_1 = \min\{-23, -19, -17, 3\} = -23$, $w_2 = \min\{-23, -25, 5, 3\} = -25$. *Thus the triple* $[a, b], [c, d], [p, q]$ *fulfils only the weak associativity law.*

As it was mentioned earlier each ordered field F *contains in itself a copy of the set* $\mathbb{Q}$. *Thus each time it is possible to find a triple* $[a, b], [c, d], [p, q] \in \mathbb{Q}^+ \times \mathbb{Q}$ *such that* $w_1 \neq w_2$ *and therefore the structure* $(\Delta_K, \star)$ *is only weakly associative.*

Remark 2. *In the sense of the paper* [14] *it is possible to define a proximity on this structure, for example, in this way. Let* $A, B \subset K$, *then*

$$A\, p\, B \text{ if and only if } [A)_{\leqq} \cap [B)_{\leqq} \neq \varnothing.$$

where for $M \subset K$ *we define* $[M)_{\leqq} = \bigcup_{m \in M} [m)_{\leqq}$. *We mean the proximity in the sense of the Čech monograph* [2]*:*

A relation p *on the family of all subsets of the set* H *is called a* proximity *on the set* H *if* p *satisfies the following conditions:*

P1. $\varnothing\ non\, p\, H$

P2. *The relation* p *is symmetric, i.e.,* $A, B \subset H$, $A\, p\, B$ *implies* $B\, p\, A$.

P3. *For any pair of subset* $A, B \subset H$, $A \cap B \neq \varnothing$ *implies* $A\, p\, B$.

P4. *If* A, B, C *are subsets of* H *then* $(A \cup B)\, p\, C$ *if and only if either* $A\, p\, C$ *or* $B\, p\, C$.

Recall that a triad $(H, \cdot, p_H)$ *such that* $(H, \cdot)$ *is a hypergroupoid and* (H, p_H) *is a proximity space will be called a* hypergroupoid with a proximity. *If for any triad of elements* $x, y, z \in H$

$$\big(x \cdot (y \cdot z)\big)\, p_H\, \big((x \cdot y) \cdot z\big)$$

is valid, then the hyperoperation "$\cdot$" is called proximally weakly associative*–see e.g.* [9],[14].

In fact, axioms P 1, P 2 and P 3 from the definition of proximity are obvious. It remains to show that the axiom P 4 is satisfied too.

First let us prove the equality $[A \cup B)_{\leqq} = [A)_{\leqq} \cup [B)_{\leqq}$, which will be helpful in the next calculations.

1. $x \in L \Rightarrow \exists\, u \in A \cup B\colon\ u \leqq x \Rightarrow \exists\, u \in A\colon u \leqq x$ or $\exists\, u \in B\colon u \leqq x \Rightarrow x \in [A)_{\leqq}$ or $x \in [B)_{\leqq}$
2. $x \in P \Rightarrow x \in [A)_{\leqq}$ or $x \in [B)_{\leqq} \Rightarrow x \in [A \cup B)_{\leqq}$

(By L we mean the left hand side of the equation and by P the right hand side.)

Now we can prove the axiom P 4: If A, B, C are subsets of K, then $(A \cup B)\, p\, C$ if and only if either $A\, p\, C$ or $B\, p\, C$.

"⇒"
$(A \cup B)\, p\, C \Rightarrow [A \cup B)_{\leqq} \cap [C)_{\leqq} \neq \varnothing \Rightarrow \big([A)_{\leqq} \cup [B)_{\leqq}\big) \cap [C)_{\leqq} \neq \varnothing \Rightarrow A\, p\, C$ or $B\, p\, C$
"⇐"
$A\, p\, C$ or $B\, p\, C \Rightarrow [A)_{\leqq} \cap [C)_{\leqq} \neq \varnothing$ or $[B)_{\leqq} \cap [C)_{\leqq} \neq \varnothing$. Since $[A)_{\leqq} \subseteq [A \cup B)_{\leqq}$ and $[B)_{\leqq} \subseteq [A \cup B)_{\leqq}$, we have $[A \cup B)_{\leqq} \cap [C)_{\leqq} \neq \varnothing$, i.e., $(A \cap B)\, p\, C$.

Thus we obtain

Theorem 2. *The hypergroup $(K, *, p)$ is a hypergroup with proximity.*

Theorem 3. *The hypergroupoid $(\Delta_K, \star)$ is a weakly associative and commutative transposition hypergroup, i.e., a weakly associative join space.*

Proof. First we will show that the reproduction axiom is fulfilled. Due to Theorem 2 the structure $(K, *)$ is the hypergroup, thus $[a,b] * K = K = K * [a,b]$ for any $[a,b] \in K$. If we define

$$\begin{aligned} [a,b] \circledast [c,d] &= \big\{[x,y]; [x,y] \in ([a,b] * [c,d]) \cup ([c,d] * [a,b])\big\} \\ &= \big\{[x,y]; [x,y] \in \{[a.c,u]; a.d + b \leqq_P u\} \cup \{[c.a,v]; c.b + d \leqq_P v\}\big\} \\ &= \big\{[x,y]; x = a.c \wedge z \leqq y)\big\}, \end{aligned}$$

where $z = \min\{a.d + b; c.b + d\}$, evidently $[a,b] \circledast [c,d] \supset [a,b] * [c,d]$. Therefore

$$K \supset [a,b] \circledast K \supset [a,b] * K = K,$$

which implies that $K \circledast [a,b] = K$ and similarly $[a,b] \circledast K = K$. From this we obtain that reproduction axiom holds in $(\Delta_K, \star)$.

Second we will verify the transposition axiom. With respect to the definition of join space and (0.1) we get

$$\begin{aligned} A &= \big[[b_1,b_2],[b_1,b_2]\big] \big\backslash \big[[a_1,a_2],[a_1,a_2]\big] \\ &= \big\{\big[[x,y],[x,y]\big]; [a_1,a_2] \in [b_1,b_2] \circledast [x,y]\big\} \\ &= \big\{\big[[x,y],[x,y]\big]; a_1 = b_1, .x \wedge a_2 \geqq_P \min\{b_1.y + b_2 b_2.x + y\}\big\} = \\ &= \big\{\big[[x,y],[x,y]\big]; x = a_1.b_1^{-1} \wedge y \leqq_P \max\{(a_2 - b_2).b_1^{-1}, a_2 - b_2.a_1.b_1^{-1}\}\big\}, \end{aligned}$$

$$\begin{aligned} B &= \big[[c_1,c_2],[c_1,c_2]\big] \big/ \big[[d_1,d_2],[d_1,d_2]\big] \\ &= \big\{\big[[x,y],[x,y]\big]; [c_1,c_2] \in [x,y] \circledast [d_1,d_2]\big\} \\ &= \big\{\big[[x,y],[x,y]\big]; c_1 = d_1.x \wedge c_2 \geqq_P \min\{d_1.y + d_2, d_2.x + y\}\big\} = \\ &= \big\{\big[[x,y],[x,y]\big]; x = c_1.d_1^{-1} \wedge y \leqq_P \max\{(c_2 - d_2).d_1^{-1}, c_2 - d_2.c_1.d_1^{-1}\}\big\}, \end{aligned}$$

$$\begin{aligned} C &= \big[[a_1,a_2],[a_1,a_2]\big] \star \big[[d_1,d_2],[d_1,d_2]\big] = \\ &= \big\{\big[[u,v],[u,v]\big]; u = a_1.d_1 \wedge v \geqq_P \min\{a_1.d_2 + a_2, a_2.d_1 + d_2\}\big\}, \end{aligned}$$

$$\begin{aligned} D &= \big[[b_1,b_2],[b_1,b_2]\big] \star \big[[c_1,c_2],[c_1,c_2]\big] = \\ &= \big\{\big[[u,v],[u,v]\big]; u = b_1.c_1 \wedge v \geqq_P \min\{b_1.c_2 + b_2, b_2.c_1 + c_2\}\big\}. \end{aligned}$$

We have $\big[[x,y],[x,y]\big] \in A \cap B$, i.e., $A \approx B$, if and only if $x = a_1.b_1^{-1} = c_1.d_1^{-1}$ and

$$y \leqq_P \min\big\{\max\{(a_2 - b_2).b_1^{-1}, a_2 - b_2.a_1.b_1^{-1}\},\\ \max\{(c_2 - d_2).d_1^{-1}, c_2 - d_2.c_1.d_1^{-1}\}\big\}.$$

If $A \approx B$, then necessarily $a_1.d_1 = b_1.c_1$. Let us denote $u_0 = a_1.d_1$. For any v_0 such that

$$v_0 \geqq_P \max\big\{\min\{a_1.d_2 + a_2, a_2.d_1 + d_2\}, \min\{b_1.c_2 + b_2, b_2.c_1 + c_2\}\big\}$$

we obtain $\big[[u_0,v_0],[u_0,v_0]\big] \in C \cap D$ which proves the transposition axiom. □

Remark 3. *It is easy to verify that under the assumption of the previous theorem even the following equivalence holds:*

$$b\backslash a \approx c/d \quad \textit{if and only if} \quad a\,.\,d \approx b\,.\,c\,.$$

In fact, one implication follows from Theorem 2. To obtain the converse one (using the notation from the proof of the mentioned theorem) suppose $C \approx D$. *Thus* $a_1.d_1 = b_1.c_1$. *Let us denote* $x_0 = a_1.b_1^{-1} = c_1.d_1^{-1}$. *Then for an arbitrary* y_0 *such that*

$$y_0 \leqq_P \min\big\{\max\{(a_2 - b_2).b_1^{-1}, a_2 - b_2.a_1.b_1^{-1}\},\\ \max\{(c_2 - d_2).d_1^{-1}, c_2 - d_2.c_1.d_1^{-1}\}\big\}.$$

we have $\big[[x_0,y_0],[x_0,y_0]\big] \in A \cap B$, *i.e.,* $A \approx B$.

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