# Remarks on the finite derived set property

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ABSTRACT.   The finite derived set property asserts that any infinite subset of a space has an infinite subset with only finitely many accumulation points. Among other things, we study this property in the case of a function space with the topology of pointwise convergence.

Following [8], we say that a topological space $X$ has the *finite derived set* (briefly FDS) property if every infinite subset of $X$ contains an infinite subset with only finitely many accumulation points.

The class of spaces with the FDS property obviously contains the class of sequentially compact Hausdorff spaces. As sequential compactness, the validity of the FDS property involves in some cases the cardinal characteristic of the continuum.

An extensive investigation on the FDS property was initiated in [1]. In this note we will collect a few more results. In particular, we will establish some sufficient conditions for a function space to have the FDS property.

All undefined notions can be found in [6].

A space $X$ is said to be a SC space if every convergent sequence together with the limit point is a closed subset of $X$.

In a SC space a convergent sequence must have a unique accumulation point. Obviously, every SC space satisfies the separation axiom $T_1$.

As usual the formula $A \subseteq^* B$ means that $A \setminus B$ is finite. A set $A$ is a pseudointersection of a collection $\mathcal{S}$ provided that $A \subseteq^* S$ for every $S \in \mathcal{S}$.

The tower number $\mathfrak{t}$ is the smallest cardinality of a collection of infinite subsets of $\omega$ which is well-ordered by $\supseteq^*$ and has no infinite pseudointersection.

Two sets $A$ and $B$ are almost disjoint if $A \cap B$ is finite. A maximal family of infinite pairwise almost disjoint subsets of $\omega$ is briefly called a MAD family. A

collection $\Theta$ of MAD families is said to be splitting if for any infinite subset $A$ of $\omega$ there exists some $\mathcal{B} \in \Theta$ and $B_1, B_2 \in \mathcal{B}$ such that $|A \cap B_1| = |A \cap B_2| = \omega$.

$\mathfrak{h}$ denotes the smallest cardinality of a splitting collection $\Theta$ of MAD families.

$\mathfrak{s}$ denotes the smallest cardinality of a splitting collection $\Theta$ of MAD families each of which consists of two sets.

We have $\omega_1 \leq \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{s} \leq \mathfrak{c}$.

**Theorem 1.** *Every SC space of weight less than $\mathfrak{s}$ has the FDS property.*

*Proof.* Let $X$ be a SC space of weight $\kappa < \mathfrak{s}$ and let $A$ be a countable infinite subset of $X$. If $A$ contains an infinite subset without accumulation points in $X$ then we are done. On the contrary, the definition of $\mathfrak{s}$ and an argument analogous to the proof of Theorem 6.1 in [5] sufficie to construct a convergent sequence $S \subseteq A$. Since $X$ is a SC space, the set $S$ has only one accumulation point in $X$ and again we are done.                                    $\square$

As every Hausdorff space of net-weight $\kappa$ has a weaker Hausdorff topology of weight $\kappa$ and the FDS property is maintained by passing to a stronger topology, we have:

**Theorem 2.** *Every Hausdorff space of net-weight less than $\mathfrak{s}$ has the FDS property.*

**Corollary 1.** *Every Hausdorff space of cardinality less than $\mathfrak{s}$ has the FDS property.*

**Lemma 1.** *Let $X$ be a space with the FDS property and $A$ an infinite subset of $X$. If $A$ contains no infinite subset without accumulation points then every infinite subset of $A$ contains a convergent subsequence.*

*Proof.* Let $A$ be an infinite subset of $X$ containing no infinite subset without accumulation points and let $A'$ be any infinite subset of $A$. Since $X$ has the FDS property, there exists a countable infinite set $B \subseteq A'$ whose set of accumulation points is $\{x_0, \ldots, x_n\}$. If $B$ is a sequence converging to $x_0$, then we are done. If not, there exists an open neighbourhood $U_0$ of $x_0$ such that the set $B_1 = B \setminus U_0$ is infinite. If $B_1$ is a sequence converging to $x_1$, then we are done. If not we may choose an open neighbourhood $U_1$ of $x_1$ such that the set $B_2 = B_1 \setminus U_1$ is infinite. Continuing this process, it is clear that for some $k \leq n$ the corresponding set $B_k$ must converge to $x_k$.                                    $\square$

The previous Lemma immediately implies that any countably compact space with the FDS property is sequentially compact. Observe however that the same conclusion is no longer true for pseudocompact Tychonoff spaces, even in the class of spaces of countable tightness. A possible counterexample is the space constructed assuming [CH] in [4], Corollary 2.

It is easy to realize that $\mathfrak{s}$ is the smallest cardinal $\kappa$ such that any product of $\kappa$ non-trivial $T_1$ spaces does not have the FDS property.

More accurate estimate is in the following:

**Theorem 3.** $\mathfrak{h}$ *is the smallest cardinal $\kappa$ such that there is a family of $\kappa$ SC spaces with the FDS property whose product does not have the FDS property.*

*Proof.* Let $\kappa$ be the cardinal in the statement of the theorem. As there exists a family of $\mathfrak{h}$ many compact sequentially compact Hausdorff spaces whose product is not sequentially compact [7] and any compact Hausdorff space with the FDS property is sequentially compact, we immediately see that $\mathfrak{h} \le \kappa$. For the converse inequality, let $\lambda < \mathfrak{h}$ and let $\{X_\alpha : \alpha < \lambda\}$ be a family of SC spaces with the FDS property. Put $X = \prod\{X_\alpha : \alpha < \lambda\}$ and denote by $\pi_\alpha : X \to X_\alpha$ the usual projection mapping. Fix a countable infinite set $A \subseteq X$. If there exists some $\alpha$ such that the set $\pi_\alpha[A]$ contains an infinite subset $B$ without accumulation points in $X_\alpha$, then we are done by considering the set $A \cap \pi_\alpha^{-1}(B)$. In the other case, with the help of Lemma 1, we have that for each $\alpha$ every infinite subset of $\pi_\alpha[A]$ contains a convergent subsequence. Now, by arguing as in [7], we may construct a convergent sequence $S \subseteq A$. Since any product of SC spaces is still a SC space, the set $S$ has only one accumulation point and we are done. $\qquad\square$

**Remark 1.** *Corollary 1 provides a consistent negative answer to question 1.9 in [1]. For this, it is enough to consider a model where $\omega_1 = \mathfrak{p} < \mathfrak{s}$.*

The next assertion improves Theorem 1.6 in [1]. It should also be compared with Corollary 1.

**Theorem 4.** *Any Urysohn space of cardinality less than $\mathfrak{c}$ has the FDS property.*

*Proof.* Let $X$ be a Urysohn space without the FDS property and fix an infinite set $A$ such that every infinite subset of $A$ has infinitely many accumulation points. Let $A_\varnothing = A$. For any $s \in {}^{<\omega}2$, we may select an infinite subset $A_s$ of $A$ according to the following rule: if $t \in {}^n2$ then $A_{t\frown 0}$ and $A_{t\frown 1}$ are infinite subsets of $A_t$ such that $\overline{A_{t\frown 0}} \cap \overline{A_{t\frown 1}} = \varnothing$. For any $f \in {}^\omega 2$ we can fix an infinite set $B_f \subseteq A$ such that $|B_f \setminus A_{f\restriction n}| < \omega$ for each $n < \omega$. For any $f \in {}^\omega 2$ let $x_f$ be an accumulation point of the set $B_f$. It is easy to check that the map $f \to x_f$ is injective and consequently $|X| \ge \mathfrak{c}$. $\qquad\square$

The previous construction with some minor modifications provides a better result for Lindelöf spaces.

**Theorem 5.** *A Lindelöf Urysohn space of cardinality less than $2^{\mathfrak{t}}$ has the FDS property.*

*Proof.* Let $X$ be a Lindelöf Urysohn space and assume that $X$ does not have the FDS property. So, we may fix a countable infinite set $A \subseteq X$ such that every infinite subset of $A$ has infinitely many accumulation points. For any $\alpha \in \mathfrak{t}$ and any $f \in {}^\alpha 2$ we define an infinite set $A_f \subseteq A$ in such a way that:

(1) if $\beta < \alpha$ and $f \in {}^\alpha 2$ then $A_f \subseteq^* A_{f\restriction\beta}$;
(2) if $f, g \in {}^\alpha 2$ and $f \ne g$ then $\overline{A_f} \cap \overline{A_g}$ is finite.

Put $A_\varnothing = A$ and assume to have defined everything for each $\beta < \alpha$. If $\alpha$ is a limit ordinal and $f \in {}^\alpha 2$, then take as $A_f$ any infinite pseudointersection of the family $\{A_{f \restriction \beta} : \beta < \alpha\}$. If $\alpha = \gamma + 1$ denote by $f'$ and $f''$ the only two elements in ${}^\alpha 2$ such that $f' \restriction \gamma = f'' \restriction \gamma = g$ for some $g \in {}^\gamma 2$ and define $A_{f'}$ and $A_{f''}$ by selecting two infinite subsets of $A_g$ having disjoint closures. By the Lindelöfness of $X$, for any $f \in {}^\mathfrak{t} 2$ we may pick a point $x_f \in \bigcap\{A'_{f \restriction \alpha} : \alpha \in \mathfrak{t}\}$ (here $A'$ is the derived set of $A$). As the mapping $f \to x_f$ is injective, we see that $|X| \geq 2^\mathfrak{t}$. $\qquad \square$

The next result is an attempt to weaken the separation axiom in the previous theorem. Its proof mimics Theorem 2.5 in [2].

**Theorem 6.** *Any Lindelöf SC space of cardinality not exceeding $\mathfrak{t}$ has the FDS property.*

*Proof.* Let $X$ be a Lindelöf SC space and assume that $X = \{x_\alpha : \alpha \in \mathfrak{t}\}$. Let $A$ be a countable infinite subset of $X$. If $A$ contains some non-trivial convergent sequence, then thanks to the SC property we obtain a subset of $A$ with only one accumulation point. In the opposite case, for any $\alpha \in \mathfrak{t}$ we may select an open set $U_\alpha$ and an infinite set $A_\alpha \subseteq A$ in such a way that:

(1) $x_\alpha \in U_\alpha$;
(2) $A_\alpha \cap U_\alpha = \varnothing$;
(3) $A_\alpha \subseteq^* A_\beta$ whenever $\beta \leq \alpha$.

At the first step, since $A$ does not converge to $x_0$ there exists an open set $U_0$ such that $A \setminus U_0$ is infinite and we may put $A_0 = A \setminus U_0$. At step $\alpha$ let $B$ be an infinite pseudointersection of the family $\{A_\beta : \beta < \alpha\}$ and select an open set $U_\alpha$ such that $x_\alpha \in U_\alpha$ and $A_\alpha = B \setminus U_\alpha$ is infinite. Since $X$ is a Lindelöf space, the family $\{U_\alpha : \alpha \in \mathfrak{t}\}$ has a countable subcover $\mathcal{V}$. Since $\mathfrak{t}$ is a regular cardinal, we have $\mathcal{V} \subseteq \{U_\alpha : \alpha \leq \gamma\}$ for some $\gamma \in \mathfrak{t}$. To finish, observe that the set $A_\gamma$ has a finite intersection with each member of $\mathcal{V}$ and therefore it does not have any accumulation point in $X$. $\qquad \square$

**Corollary 2.** *Every countable SC space has the FDS property.*

Recall that a space $X$ is said to be $\omega$-monolithic if any separable subspace of $X$ has a countable net-work.

**Lemma 2.** *Every Hausdorff $\omega$-monolithic space has the FDS property.*

*Proof.* Let $X$ be a Hausdorff $\omega$-monolithic space and fix a countably infinite set $A \subseteq X$. Assume that $A$ is not closed and discrete and let $x$ be an accumulation point of $A$. Since the subspace $Y = \overline{A}$ has a countable net-work, there exists a decreasing family $\{V_n : n \in \omega\}$ of open neighbourhoods of $x$ in $Y$ such that $\bigcap\{\overline{V_n} : n \in \omega\} = \{x\}$. For any $n$ select a point $a_n \in V_n \cap (A \setminus \{x\})$ and let $B$ be the set so obtained. All accumulation points of $B$ must lye in $\overline{V_n}$ for each $n$ and therefore the set $B$ has at most only $x$ has accumulation point. $\qquad \square$

Now, we look at the FDS property in function spaces in the topology of pointwise convergence (see [3]). For this reason, in the sequel all spaces are assumed to be Tychonoff.

To begin, observe that a space of the form $C_p(X)$ may fail to have the FDS property: it suffices to take as $X$ any discrete space of cardinality at least $\mathfrak{s}$.

A space $X$ is said to be $\omega$-stable if all continuous images of $X$ that can be injectively mapped into a second countable space have a countable net-work.

The class of $\omega$-stable spaces contains all pseudocompact ([3], 2.6.2c), all Lindelöf $\Sigma$ ([3], 2.6.21) and all Lindelöf P spaces ([3], 2.6.28). Thus, we have:

**Corollary 3.** *If $X$ is either a pseudocompact of a Lindelöf $\Sigma$ or a Lindelöf P space then $C_p(X)$ has the FDS property.*

*Proof.* The result follows from Lemma 2 and the fact that a space $X$ is $\omega$-stable if and only if $C_p(X)$ is $\omega$-monolithic ([3], 2.6.8). $\qquad\square$

**Corollary 4.** *Every normed linear space in the weak topology has the FDS property.*

*Proof.* Let $X$ be a normed linear space and $U^*$ be the unit ball in the dual space $X^*$. The well known theorem of Alaoglu says that $U^*$ is compact in the weak* topology on $X^*$. The result follows by observing that the space $X$ in the weak topology is just a subspace of $C_p(U^*)$. $\qquad\square$

**Theorem 7.** *The space $C_p(X)$ has the FDS property in the following cases:*

(1) *$X$ has density less than $\mathfrak{s}$;*
(2) *$X$ has a dense set which is the union of less than $\mathfrak{h}$ pseudocompact (or Lindelöf $\Sigma$, or Lindelöf P) subspaces.*

*Proof.* Observe that if $Y$ is dense in $X$ then there is an injective continuous mapping from $C_p(X)$ into $C_p(Y)$. Therefore, if $C_p(Y)$ has the FDS property then the same happens to $C_p(X)$. Item 1 follows from Theorem 1. If $Y = \bigcup\{Y_\alpha : \alpha < \kappa\}$ then $C_p(Y)$ is homeomorphic to a subspace of the space $\prod\{C_p(Y_\alpha) : \alpha < \kappa\}$. Taking into account Theorem 3, item 2 follows from Corollary 3. $\qquad\square$

**Corollary 5.** *If $\xi$ is an ordinal then $C_p(\xi)$ has the FDS property.*

Looking again at Corollary 3, one may wonder whether the second iteration of a function space, namely $C_p(C_p(X))$, on a compact space $X$ has yet the FDS property. This is not the case in general because the FDS property of $C_p(C_p(X))$ implies that the compact space $X$ must be sequentially compact. We do not know if the converse always holds, although it happens for sure in the class of compact $\omega$-monolithic spaces. However, the FDS property of $C_p(C_p(X))$ is far to imply either $\omega$-monolithicity or $\omega$-stability of the base space $X$. For instance, if $X$ is a subspace of the Sorgenfrey line of cardinality $\omega_1$ then $X$ is neither $\omega$-monolithic nor $\omega$-stable, but by assuming $\omega_1 < \mathfrak{s}$ $C_p(C_p(X))$ has the FDS property.

Of course, Corollary 3 and Theorem 7 leave open the general problem to characterize the spaces $X$ for which $C_p(X)$ has the FDS property.

Some specific questions are:

1. Is it true that any Hausdorff space of cardinality less than $\mathfrak{c}$ has the FDS property?

2. Find a compact sequentially compact space $X$ such that $C_p(C_p(X))$ does not have the FDS property.

## References

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