Applied General Topology
© Universidad Politécnica de Valencia
Volume 11, No. 1, 2010
pp. 21-27

One point compactification for generalized quotient spaces

V. Karunakaran and C. Ganesan *

Abstract. The concept of Generalized function spaces which were introduced and studied by Zemanian are further generalized as Boehmian spaces or as generalized quotient spaces in the recent literature. Their topological structure, notions of convergence in these spaces are also investigated. Some sufficient conditions for the metrizability are also obtained. In this paper we shall assume that a generalized quotient space is non-compact and realize its one point compactification as a quotient space.

2000 AMS Classification: 46F30, 46A19, 22A30.

Keywords: Generalized quotient space, compact, locally compact and Hausdorff, one point compactification.

1. Introduction

Schwartz distribution spaces are generalized in different ways in the literature. Some of these are "Generalized function spaces" (introduced and studied in detail by Zemanian see [8]), "Boehmian spaces" (motivated by the concept of regular operator introduced by Boehme (see [1]) and studied in [3, 4, 5]) and most recently "The generalized quotient spaces" (see [6])

In [2] the authors introduce the concepts of δ-convergence and Δ-convergence in these generalized quotient spaces and investigate the behavior of convergence sequences and the topological properties of these spaces under the quotient topology. Suitable conditions for metrizability of these spaces are also obtained. It turns out that these generalized function spaces, in general, are not compact. Further it is also difficult to find out suitable conditions under which these spaces (under the canonical quotient topology) are locally compact and

*The research of the second author is supported by a "University Grants Commission Research Fellowship in Sciences for Meritorious Students", India.

Hausdorff. Thus the problem of realizing the one point compactifications of these spaces assumes significance. In this paper we shall identify the one point compactification of a non-compact generalized quotient space as another quotient space. The results in this paper are also motivated by a desire to find an analogue of the following classical result which can be easily proved. Let X and Y be locally compact non-compact Hausdorff spaces and $p : X \to Y$ be a quotient map. Let $p^* : X^* \to Y^*$ be the natural extension of p to their respective one point compactifications. Then p^* is continuous and hence a quotient map if and only if $p^{-1}(K)$ is compact in X for every compact K in Y. Thus under certain conditions the one point compactification of a quotient space becomes another quotient space. The analogue of this result in the context of a generalized quotient space will be studied here. In Section 2 we shall develop the required preliminaries and in Section 3 we shall state and prove the main theorem. The conditions under which the one point compactifications of these generalized quotient spaces can be realized as generalized quotient spaces also guarantee that the original generalized quotient spaces are locally compact and Hausdorff.

2. Preliminaries

We shall briefly recall the concept of generalized quotient spaces as described in [6]. Let X be a non-empty set and let G be a commutative semi-group acting on X injectively. This means that to every $g \in G$ there corresponds an injective map $g : X \to X$ such that $(g_1g_2)(x) = g_1(g_2(x))$ for all $g_1, g_2 \in G$ and $x \in X$. For $g \in G$ and $x \in X$, $g(x)$ denotes the action of g on x in X.

Let $\mathcal{A} = X \times G$. For $(x, f), (y, g) \in \mathcal{A}$ we write $(x, f) \backsim (y, g)$ if $g(x) = f(y)$. Then $\backsim$ is an equivalence relation in $\mathcal{A}$. We define the space of generalized quotients as $\mathcal{B} = \mathcal{B}(X, G) = \mathcal{A}/\backsim$. We denote the equivalence class containing (x, f) by $\left[\frac{x}{f}\right]$.

Suppose G fails to act injectively on X then we proceed as follows: Let I be a non-empty index set and let $\Delta \subset G^I$ be a semi-group (this only means that Δ is closed for the canonical semi-group operation available in G^I which is defined as follows: If $\alpha, \beta \in G^I$ then $(\alpha\beta)(i) = \alpha(i)\beta(i)$ for all $i \in I$). For $\alpha \in \Delta$ and $x \in X$ define $\alpha x \in X^I$ by $(\alpha x)(i) = \alpha(i)(x)$, so that each α gives rise to a mapping from X in to X^I. We assume that these maps are injective. For $\alpha \in \Delta$ and $\psi \in X^I$ we also define $(\alpha\psi)(i) = \alpha(i)(\psi(i))$ so that $\alpha\psi$ defines an element of X^I.

Suppose $\chi \subset X^I$ satisfies the following conditions:

a: $\alpha x \in \chi$ for all $\alpha \in \Delta$ and all $x \in X$.

b: $\alpha\psi \in \chi$ for all $\alpha \in \Delta$ and all $\psi \in \chi$.

Let

$$\mathcal{A} = \{(\xi, \alpha)/\xi \in \chi, \alpha \in \Delta \text{ and } \alpha(i)(\xi(j)) = \alpha(j)(\xi(i)), i, j \in I\}.$$

For (f,ϕ), $(g,\psi)\in\mathcal{A}$ we write $(f,\phi)\backsim(g,\psi)$ if $\phi(i)(g(j))=\psi(j)(f(i))$, for all $i,j\in I$. Then $\backsim$ is an equivalence relation on $\mathcal{A}$. We define the space of generalized quotients as $\mathcal{B}=\mathcal{B}(\chi,\Delta)=\mathcal{A}/\backsim$ and we shall denote the equivalence class containing (f,ϕ) by $\left[\frac{f(i)}{\phi(i)}\right]$.

We shall assume that the reader is familiar with the above construction of generalized quotient spaces. Further we shall assume the following:

(1) X is a non-compact locally compact Hausdorff space, G is a commutative semi group acting continuously on X (but not necessarily injectively) equipped with a Hausdorff topology.
(2) The mapping $\Lambda : X\times G\to X$ defined by $\Lambda(x,g)=g(x)$ is continuous and that $\Lambda^{-1}(K)$ is compact in $X\times G$ for each compact K in X.
(3) $\chi\subset X^I$ is closed in X^I.
(4) $\Delta\subset G^I$ is compact.

With these assumptions, the constructed generalized quotient space will be denoted by $\mathcal{B}$. We explicitly assume that such a $\mathcal{B}$ is non-compact.

We shall now give an example to show that the above conditions are realizable. Let $X=(-\infty,-2]\cup[2,\infty)$ considered as a subspace of the real line under the usual topology. Let $G=\mathbb{Z}\setminus\{0\}$ (the set of all non-zero integers) with discrete topology. Note that G is a commutative semi-group under usual multiplication. Let $I=\mathbb{N}$ (the set of natural numbers). We shall allow G to act continuously on X by $g(x)=x^{|n|}$ where $g=n\in G$. Note that even though G acts continuously on X the action is not injective for any even integer n. It is now easy to prove the following points.

(1) $\Lambda : X\times G\to X$ defined by $\Lambda(x,n)=x^{|n|}$ is continuous. (Note that $x_j\to x_0$ and $n_j\to n_0$ as $j\to\infty$ imply that sequence n_j is eventually a constant ($=n_0$, say) and hence $\Lambda(x_j,n_j)\to\Lambda(x_0,n_0)$ as $j\to\infty$).
(2) For any compact (and hence bounded) set K in X, $\Lambda^{-1}(K)\subset A\times B$ with A compact in X and B is a finite subset of G and hence $\Lambda^{-1}(K)$ is compact.

We shall now take $\chi=X^I$ so that χ is closed in X^I. We shall also take

$$\Delta=\left\{(\alpha_n)\in G^{\mathbb{N}}/\ \alpha_n=1 \text{ for odd } n \text{ and } \alpha_n=\pm1 \text{ for even } n\right\}\subset S^{\mathbb{N}}$$

where $S=\{-1,1\}$.

It is now easy to see that Δ is a semi-group, is closed in $S^{\mathbb{N}}$ and that it is compact because $S^{\mathbb{N}}$ is compact. Further each $\beta\in\Delta$ induces an injective map from X to $X^{\mathbb{N}}$ given by $(\beta x)(i)=\beta(i)(x)$ as required for the construction of a generalized quotient space.

We shall need the following lemmas.

Lemma 2.1. *Let X be locally compact non-compact Hausdorff space and G a Hausdorff space. If $\Lambda : X\times G\to X$ is continuous and $\Lambda^{-1}(K)$ is compact in $X\times G$ for each K compact in X then $g : X\to X$ is continuous and $g^{-1}(K)$ is compact in X for each K compact in X.*

Proof. Fix $g \in G$. Then $g(x) = \Lambda(x, g) = \Lambda_g(x)$ is continuous in the variable x. Let K be any compact set in X. Define a mapping $F : X \to X \times G$ by $F(x) = (x, g)$. Then we have $(\Lambda \circ F)(x) = g(x)\ \forall\ x \in X$. Now

$$\begin{aligned} g^{-1}(K) &= \{x \in X/g(x) \in K\} \\ &= (\Lambda \circ F)^{-1}(K) \\ &= F^{-1}(\Lambda^{-1}(K)) = F^{-1}(H) \text{ (where } H = \Lambda^{-1}(K) \subset X \times G \text{ is compact)} \\ &= \{x \in X/F(x) \in H\} \\ &= \{x \in X/(x, g) \in H\} \\ &= \pi(H \cap (X \times \{g\})) \text{ where } \pi : X \times G \to X \text{ is defined by } \pi(x, h) = x \end{aligned}$$

Since $X \times G$ is Hausdorff and H is compact subset of $X \times G$, H is closed in $X \times G$. It is clear that $X \times \{g\}$ is closed in $X \times G$. Hence $H \cap (X \times \{g\})$ is closed in $X \times G$. But $H \cap (X \times \{g\}) \subset H$ and H is closed in $X \times G$ implies that $H \cap (X \times \{g\})$ is closed in H and hence $H \cap (X \times \{g\})$ is compact in H. Thus $H \cap (X \times \{g\})$ is compact in $X \times G$. Now the continuity of π will show that $g^{-1}(K) = \pi(H \cap (X \times \{g\}))$ is compact in X. □

Note that the condition on Λ already implies that G acts continuously on X, a fact which we have explicitly assumed.

Lemma 2.2. *Let X, G, Λ and g be as in Lemma* 2.1. *Let X^* be the one point compactification of X and let $g^* : X^* \to X^*$ be the natural extension of g to X^* ie., $g^*|_X = g$ and $g^*(\infty) = \infty$. Then g^* is continuous.*

Proof. Follows easily using Lemma 2.1 and is left to the reader. □

Lemma 2.3. *Let X, G and Λ be as in Lemma* 2.1. *The mapping $\Lambda^* : X^* \times G \to X^*$ defined by*

$$\Lambda^*(x, g) = g^*(x) = \begin{cases} \Lambda(x, g) & \text{if } x \in X \\ g^*(\infty) = \infty & \text{if } x = \infty \end{cases}$$

is continuous

Proof. Follows easily using the property of Λ and is left to the reader. □

3. Construction of a new generalized quotient space

In this section we shall define a new generalized quotient space $\mathcal{B}^*$ which will be shown to be the one point compactification of the generalized quotient space $\mathcal{B}$ constructed in Section 2.

Let $\mathcal{A}^* = \{(f, \alpha) \in \chi^* \times \Delta /\ \alpha(i)(f(j)) = \alpha(j)(f(i))\ \forall\ i, j \in I\}$, where $\chi^* =$ [closure of χ in ${X^*}^I$] $\cup \{f_\infty\}$ with $f_\infty : I \to X^*$ is defined by $f_\infty(i) = \infty$ $\forall\ i \in I$. Define a relation $\sim$ on $\mathcal{A}^*$ as follows: $(f, \alpha) \sim (g, \beta)$ if $\alpha(i)(g(j)) = \beta(j)(f(i))\ \forall\ i, j \in I$. It is clear that this relation $\sim$ is an equivalence relation in $\mathcal{A}^*$ (note that each element $\alpha \in \Delta$ gives raise to a map $\alpha^* : X^* \to {X^*}^I$ defined by $\alpha^*|_X = \alpha$ and $\alpha^*(\infty) = f_\infty$ which is easily seen to be injective.

This observation is indeed crucial to the proof of the fact that $\sim$ is transitive in the same way as in the proof of the transitivity of $\backsim$ in $\mathcal{A}$). We now observe the following properties of χ^*.

a: $\alpha x \in \chi^*$ for all $\alpha \in \Delta$ and all $x \in X^*$.
b: $\alpha\psi \in \chi^*$ for all $\alpha \in \Delta$ and all $\psi \in \chi^*$.

Indeed property **(a)** can be easily proved where as property **(b)** can be proved using the properties of net convergence in X^* and ${X^*}^I$.

Now in a canonical manner we can define the generalized function space $\mathcal{B}^*$ by $\mathcal{B}^* = \mathcal{A}^*|_\sim$. We shall also give the quotient topology to $\mathcal{B}^*$ given by the map $p^* : \mathcal{A}^* \to \mathcal{B}^*$ defined by $p^*((f,\alpha)) = \left[\frac{f(j)}{\alpha(j)}\right]$.

Lemma 3.1. $\mathcal{A}^* = \mathcal{A} \cup \{(f_\infty, \alpha)/\ \alpha \in \Delta\}$.

Proof. It is clear that $\mathcal{A} \cup \{(f_\infty, \alpha)/\ \alpha \in \Delta\} \subset \mathcal{A}^*$. Let $(f,\alpha) \in \mathcal{A}^*$. If $f = f_\infty$ then there is nothing to prove. Therefore assume $f(i_0) \neq \infty$ for some $i_0 \in I$. Then $\alpha(i)(f(j)) = \alpha(j)(f(i))\ \forall\ i,j \in I$ will imply that $f(i) \neq \infty\ \forall\ i \in I$. Hence $f \in X^I$. But $(f,\alpha) \in \chi^* \times \Delta$ will imply that $f \in \chi^*$ ie., f is in the closure of χ in ${X^*}^I$. This implies that f is in the closure of χ in X^I (indeed if $W = V_{\alpha_1} \times V_{\alpha_2} \times \cdots V_{\alpha_n} \times \prod X$ is any basic open set of f in X^I then $W^* = V_{\alpha_1} \times V_{\alpha_2} \times \cdots V_{\alpha_n} \times \prod X^*$ is a basic open set of f in ${X^*}^I$. Hence $W \cap \chi = W^* \cap \chi \neq \phi$). Now χ is closed in X^I implies that $(f,\alpha) \in \chi \times \Delta$ and as $\alpha(i)(f(j)) = \alpha(j)(f(i))\ \forall\ i,j \in I$, $(f,\alpha) \in \mathcal{A}$. Thus $\mathcal{A}^* \subset \mathcal{A} \cup \{(f_\infty, \alpha)/\ \alpha \in \Delta\}$. This completes the proof. $\square$

Lemma 3.2. *$\mathcal{A}$ is open in $\mathcal{A}^*$.*

Proof. Equivalently we shall show that the set $D = \{(f_\infty, \alpha)/\ \alpha \in \Delta\}$ is closed in $\mathcal{A}^*$. Let $(f,\alpha) \in \mathcal{A}^*$ be a limit point of D. Suppose $f(i_0) \neq \infty$ for some $i_0 \in I$. Then there are open sets U and V in X^* containing $f(i_0)$ and ∞ respectively such that $U \cap V = \phi$. Then it is clear that $W_1 = U \times \prod\limits_{i \neq i_0} X^*$ is a basic open neighbourhood of f in ${X^*}^I$ such that $f_\infty \notin W_1$. If W_2 is any open set in G^I containing α then $W_1 \times W_2$ is an open set containing (f,α) in ${X^*}^I \times G^I$ but not containing any element of the form (f_∞, β), $\beta \in \Delta$. This shows that (f,α) is not a limit point of D in $\mathcal{A}^*$ which is a contradiction. Hence $f(i) = \infty\ \forall\ i \in I$ and hence $(f,\alpha) = (f_\infty, \alpha) \in D$. Thus D is closed in $\mathcal{A}^*$. $\square$

Lemma 3.3. *$\mathcal{A}^*$ is a compact Hausdorff space.*

Proof. Since ${X^*}^I \times G^I$ is Hausdorff and a subspace of a Hausdorff space is Hausdorff, $\mathcal{A}^*$ is Hausdorff. Since $\mathcal{A}^* \subset \chi^* \times \Delta$ which is compact (note that χ^* is a closed subset of the compact space ${X^*}^I$ and Δ is compact), we merely show that $\mathcal{A}^*$ is closed in $\chi^* \times \Delta$.

Let $(f,\alpha) \in \chi^* \times \Delta$ be a limit point of $\mathcal{A}^*$. Then there is a net $(f_\gamma, \alpha_\gamma)$ of points from $\mathcal{A}^*$ such that $(f_\gamma, \alpha_\gamma) \to (f,\alpha)$ in $\chi^* \times \Delta$. From this we have

$f_\gamma(i) \to f(i)$ in X^* and $\alpha_\gamma(i) \to \alpha(i)$ in G for each $i \in I$. Since $(f_\gamma, \alpha_\gamma) \in \mathcal{A}^*$ we have , $\alpha_\gamma(i)(f_\gamma(j)) = \alpha_\gamma(j)(f_\gamma(i))\ \forall\, i, j \in I$. Using Lemma 2.3 we now have $\alpha(i)(f(j)) = \alpha(j)(f(i))$ for each $i, j \in I$. Hence $(f, \alpha) \in \mathcal{A}^*$. This completes the proof. □

Lemma 3.4. *The set kerp*$^* = \{((f,\alpha),(g,\beta)) \in \mathcal{A}^* \times \mathcal{A}^* /\ (f,\alpha) \sim (g,\beta)\}$ *is closed in* $\mathcal{A}^* \times \mathcal{A}^*$.

Proof. Follows using net convergences and arguments similar to the one given in Lemma 3.3 and is left to the reader. □

Let us now recall the following theorem (see [7] , pp 183).

Theorem 3.5. *Let X be a non-compact topological space. Then X is locally compact and Hausdorff if and only if there exists a topological space Y satisfying the following conditions*

(1) *X is a subspace of Y.*
(2) *The set $Y \setminus X$ consists on a single point.*
(3) *Y is a compact Hausdorff space.*

We now make the following observations.

(1) $(f,\alpha) \sim (g,\beta)$ in $\mathcal{A}^*$ if and only if $(f,\alpha),(g,\beta) \in \mathcal{A}$ and $(f,\alpha) \sim (g,\beta)$ or $(f,\alpha) = (f_\infty,\alpha),(g,\beta) = (f_\infty,\beta)$. In particular $\mathcal{B}^* = \mathcal{B} \cup \left\{ \left[\frac{f_\infty(j)}{\alpha(j)} \right] \right\}$.
(2) Since X is a subspace of X^* (ie., the original topology of X is the same as the subspace topology of X in X^*), the product space $X^I \times G^I$ is a subspace of the product space ${X^*}^I \times G^I$. In particular $\mathcal{A}$ is a subspace of $\mathcal{A}^*$ (in the above sense) and hence $\mathcal{B}$ is a subspace of $\mathcal{B}^*$.
(3) $\mathcal{B}^* = p^*(\mathcal{A}^*) \Rightarrow \mathcal{B}^*$ is compact (note that $\mathcal{A}^*$ is compact and p^* is continuous).
(4) Since *kerp** is closed (Lemma 3.4) and $\mathcal{A}^*$ is a compact Hausdorff space it follows that $\mathcal{B}^*$ is Hausdorff.
(5) Since $\mathcal{B}^* \setminus \mathcal{B}$ is a singleton and $\mathcal{B}$ is non-compact we have $\overline{\mathcal{B}} = \mathcal{B}^*$.

Using all the above observations together with Theorem 3.5 we now get the following main theorem.

Theorem 3.6. *$\mathcal{B}$ is locally compact, Hausdorff and its one point compactification is $\mathcal{B}^*$.*

References

[1] T. K. Boehme, *The support of Mikusinski operators*, Trans. Amer. Math. Soc. **176** (1973), 319–334.

[2] V. Karunakaran and C. Ganesan, *Topology and the notion of convergence on generalized quotient spaces*, Int. J. Pure Appl. Math. **44**, no. 5 (2008), 797–808.

[3] J. Mikusinski and P. Mikusinski, *Quotients de suites et leurs applications dans l'analyse fonctionnelle*, Comptes Rendus **293**, serie I (1981), 463–464.

[4] J. Mikusinski and P. Mikusinski, *Quotients of sequences*, Proc. of the II conference on Convergence Szezyrk (1981), 39–45.
[5] P. Mikusinski, *Convergence of Boehmians*, Japan J. Math **9** (1983), 159–179.
[6] P. Mikusinski, *Generalized quotients with applications in analysis*, Methods Appl. Anal. **10** (2004), 377–386.
[7] J. R. Munkres, *Topology, second edition*, Prentice-Hall of India, private limited, New Delhi (2003).
[8] A. H. Zemanian, *Generalized integral transformation*, John Wiley and Sons, Inc., New York, (1968).

Received June 2009

Accepted March 2010

V. Karunakaran (vkarun_mku@yahoo.co.in)
Senior Professor, School of Mathematics, Madurai Kamaraj University, Tamil Nadu, Madurai - 625 021, India.

C. Ganesan (mkuganc@yahoo.com)
Research Scholar, School of Mathematics, Madurai Kamaraj University, Tamil Nadu, Madurai - 625 021, India.