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Convergence semigroup categories

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Abstract. Properties of the category consisting of all objects of the form (X, S, λ) are investigated, where X is a convergence space, S is a commutative semigroup, and $\lambda : X \times S \to X$ is a continuous action. A "generalized quotient" of each object is defined without making the usual assumption that for each fixed $g \in S$, $\lambda(., g) : X \to X$ is an injection.

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1. Introduction and Preliminaries

. The notion of a topological group acting continuously on a topological space has been the subject of numerous research articles. Park [13, 14] and Rath [16] studied these concepts in the larger category of convergence spaces. This is a more natural category to work in since the homeomorphism group on a space can be equipped with a coarsest convergence structure making the group operations continuous. Moreover, unlike in the topological context, quotient maps are productive in the category of all convergence spaces with continuous maps as morphisms. This property played a key role in the proof of several results contained in [3]; for example, see Theorem 4.5 [3].

Given a topological semigroup acting on a topological space, Burzyk et al. [5] introduced a "generalized quotient space." Elements of this space are equivalence classes determined by an abstraction of the method used to construct the rationals from the integers. Generalized quotient spaces are used in the study of generalized functions [10, 11, 12]. Moreover, generalized quotients in the category of convergence spaces are defined in [3] for the case whenever $\lambda(., g) : X \to X$ is injective, where λ is a continuous action of a convergence semigroup S on a convergence space X. Generalized quotients are defined and studied here without the requirement that $\lambda(., g)$ is injective. Furthermore, the category consisting of objects of the form (X, S, λ) is investigated. The terminology used here involving categories can be found in Adamek et al. [1].

Basic definitions and concepts needed in the area of convergence spaces are given in this section. Let X be a set, 2^X the power set of X, and let $\mathfrak{F}(X)$ denote the set of all filters on X. Recall that $\mathfrak{B} \subseteq 2^X$ is a base for a filter on X provided $\mathfrak{B} \neq \varnothing$, $\varnothing \notin \mathfrak{B}$, and $B_1, B_2 \in \mathfrak{B}$ implies that there exists $B_3 \in \mathfrak{B}$ such that $B_3 \subseteq B_1 \cap B_2$. Moreover, $[\mathfrak{B}]$ denotes the filter on X whose base is $\mathfrak{B}$; that is, $[\mathfrak{B}] = \{A \subseteq X : B \subseteq A \text{ for some } B \in \mathfrak{B}\}$. Fix $x \in X$, define $\dot{x}$ to be the filter whose base is $\mathfrak{B} = \{\{x\}\}$. If $f : X \to Y$ and $\mathcal{F} \in \mathfrak{F}(X)$, then $f^{\rightarrow}\mathcal{F}$ denotes the image filter on Y whose base is $\{f(F) : F \in \mathcal{F}\}$.

A **convergence structure** on X is a function $q : \mathfrak{F}(X) \to 2^X$ obeying:

(CS1) $x \in q(\dot{x})$ for each $x \in X$

(CS2) $x \in q(\mathcal{F})$ implies that $x \in q(\mathcal{G})$ whenever $\mathcal{F} \subseteq \mathcal{G}$.

The pair (X, q) is called a **convergence space**. The more intuitive notation $\mathcal{F} \xrightarrow{q} x$ is used for $x \in q(\mathcal{F})$. A map $f : (X, q) \to (Y, p)$ between two convergence spaces is called **continuous** whenever $\mathcal{F} \xrightarrow{q} x$ implies that $f^{\rightarrow}\mathcal{F} \xrightarrow{p} f(x)$. Let **CONV** denote the category whose objects consist of all the convergence spaces, and whose morphisms are all the continuous maps between objects. The collection of all objects in CONV is denoted by $|\mathrm{CONV}|$. If p and q are two convergence structures on X, then $\boldsymbol{p \leq q}$ means that $\mathcal{F} \xrightarrow{p} x$ whenever $\mathcal{F} \xrightarrow{q} x$. In this case, $p(q)$ is said to be **coarser**(**finer**) than $q(p)$, respectively. Also, for $\mathcal{F}, \mathcal{G} \in \mathfrak{F}(X)$, $\boldsymbol{\mathcal{F} \leq \mathcal{G}}$ means that $\mathcal{F} \subseteq \mathcal{G}$, and $\mathcal{F}(\mathcal{G})$ is called **coarser**(**finer**) than $\mathcal{G}(\mathcal{F})$, respectively.

It is well-known that CONV possesses initial and final convergence structures. In particular, if $(X_j, q_j) \in |\mathrm{CONV}|$ for each $j \in J$, then the **product convergence structure** r on $X = \underset{j \in J}{\times} X_j$ is given by $\mathcal{H} \xrightarrow{r} x = (x_j)$ iff $\pi_j^{\rightarrow}\mathcal{H} \xrightarrow{q_j} x_j$ for each $j \in J$, where π_j denotes the $\mathrm{j}^{\underline{\mathrm{th}}}$ projection map. Also, if $f : (X, q) \to Y$ is a surjection, then the **quotient convergence structure** σ on Y is defined by: $\mathcal{H} \xrightarrow{\sigma} y$ iff there exists $x \in f^{-1}(y)$ and $\mathcal{F} \xrightarrow{q} x$ such that $f^{\rightarrow}\mathcal{F} = \mathcal{H}$. In this case, σ is the finest convergence structure on Y making $f : (X, q) \to (Y, \sigma)$ continuous. Convergence quotient maps are defined and studied by Kent [7]. Moreover, Beattie and Butzmann [2] and Preuss [15] are good references for convergence space results.

Unlike the category of all topological spaces, CONV is cartesian closed and thus has suitable function spaces. In particular, let (X, q), $(Y, p) \in |\mathrm{CONV}|$ and let $C(X, Y)$ denote the set of all continuous functions from X to Y. Define $\omega : (X, q) \times C(X, Y) \to (Y, p)$ to be the evaluation map given by $\omega(x, f) = f(x)$. There exists a coarsest convergence structure **c** on $C(X, Y)$ such that w is jointly continuous. More precisely, c is defined by : $\Phi \xrightarrow{c} f$ iff $w^{\rightarrow}(\mathcal{F} \times \Phi) \xrightarrow{p} f(x)$ whenever $\mathcal{F} \xrightarrow{q} x$. This compatibility between (X, q) and $(C(X, Y), c)$ is an example of a continuous action in CONV.

Let **SG** denote the category whose objects consist of all the semigroups (with an identity element), and whose morphisms are all the homomorphisms between objects. Further, $(S,.,p)$ is said to be a **convergence semigroup** provided : $(S,.) \in |\mathrm{SG}|$, $(S,p) \in |\mathrm{CONV}|$, and $\gamma : (S,p) \times (S,p) \to (S,p)$ is continuous, where $\gamma(x,y) = x.y$. Let **CSG** be the category whose objects consist of all the convergence semigroups, and whose morphisms are all the continuous homomorphisms between objects.

An action of a semigroup on a topological space is used to define "generalized quotients" in [5]. Below is Rath's [16] definition of an action in the convergence space context. Let $(X,q) \in |\mathrm{CONV}|$, $(S,.,p) \in |\mathrm{CSG}|$, $\lambda : X \times S \to X$, and consider the following conditions:

(A1) $\lambda(x,e) = x$ for each $x \in X$ (e is the identity element)
(A2) $\lambda(\lambda(x,g),h) = \lambda(x,g.h)$ for each $x \in X$, $\quad g,h \in S$
(A3) $\lambda : (X,q) \times (S,.,p) \to (X,q)$ is continuous.

Then $(S,.)((S,.,p))$ is said to **act**(**act continuously**) on (X,q) whenever A1-A2 (A1-A3) are satisfied and, in this case, λ is called the **action** (**continuous action**), respectively. For sake of brevity, $(X,S) \in \mathbf{A}(\mathbf{AC})$ denotes the fact that $(S,.)$ is commutative and $(S,.,p) \in |\mathrm{CSG}|$ acts (acts continuously) on $(X,q) \in |\mathrm{CONV}|$, respectively. Moreover, $\boldsymbol{(X,S,\lambda)} \in \mathrm{A}$ indicates that the action is λ.

Remark 1.1. Fix a set X; then the set of all convergence structures on X with the ordering $p \leq q$ defined above is a complete lattice. Indeed, if $(X,q_j) \in |\mathrm{CONV}|$, $j \in J$, then $\sup_{j \in J} q_j = q^1$ is given by $\mathcal{F} \xrightarrow{q_1} x$ iff $\mathcal{F} \xrightarrow{q_j} x$, for each $j \in J$. Dually, $\inf_{j \in J} q_j = q^0$ is defined by $\mathcal{F} \xrightarrow{q0} x$ iff $\mathcal{F} \xrightarrow{q_j} x$, for some $j \in J$. It is easily verified that if $((X,q_j),(S,.,p),\lambda) \in \mathrm{AC}$ for each $j \in J$, then both $((X,q^1),(S,.,p),\lambda)$ and $((X,q^0),(S,.,p),\lambda)$ belong to AC.

The notion of "generalized quotients" determined by a commutative semigroup acting on a topological space is investigated in [5]. Elements of the semigroup in [5] are assumed to be injections on the given topological space.

Lemma 1.2 ([5])**.** *Suppose that $(S,X,\lambda) \in A$, and $\lambda(.,g) : X \to X$ is an injection, for each $g \in S$. Define $(x,g) \approx (y,h)$ on $X \times S$ iff $\lambda(x,h) = \lambda(y,g)$. Then $\approx$ is an equivalence relation on $X \times S$.*

In the context of Lemma 1.2, let $\langle (x,g) \rangle$ be the equivalence class containing (x,g), $\boldsymbol{B(X,S)}$ denote the quotient set $(X \times S)/\approx$, and define $\varphi : (X \times S, r) \to B(X,S)$ to be the canonical map, where $r = q \times p$ is the product convergence structure. Equip $B(X,S)$ with the convergence quotient structure $\boldsymbol{\sigma}$. Then $\mathcal{K} \xrightarrow{\sigma} \langle (y,h) \rangle$ iff there exist $(x,g) \approx (y,h)$ and $\mathcal{H} \xrightarrow{r} (x,g)$ such that $\varphi^{\to}\mathcal{H} = \mathcal{K}$. Properties of $(B(X,S),\sigma)$ are investigated in [3].

Whenever the hypothesis that $\lambda(.,g)$ is an injection for each $g \in S$ fails, one can still define a generalized quotient by extending $\approx$ to an equivalent relation as defined in (R2) below.

Assume that $((X,q),(S,.,p),\lambda) \in$ A, and consider the following relations:

(R1) $(x,f) \approx (y,g)$ in $X \times S$ iff $\lambda(x,g) = \lambda(y,f)$

(R2) $(x,f) \sim (y,g)$ in $X \times S$ iff there exists $(z_i,h_i) \in X \times S$, satisfying $(x,f) \approx (z_1,h_1) \approx (z_2,h_2) \approx \cdots \approx (z_n,h_n) \approx (y,g)$, for some $n \geq 1$

(R3) $x \simeq y$ in X iff there exists $g \in S$ such that $\lambda(x,g) = \lambda(y,g)$.

According to Lemma 1.1 [5], (R1) is an equivalence relation provided that $\lambda(.,g)$ is an injection for each $g \in S$. However, (R2) is an equivalence relation without assuming that $\lambda(.,g)$ is an injection for each $g \in S$. Moreover, it easily follows that (R3) is an equivalence relation.

Given $((X,q),(S,.,p),\lambda) \in$ A; denote $X_* = X/\simeq$, $\xi : X \to X_*$ the canonical map $\xi(x) = [x]$, and let q_* be the quotient structure on X_* in CONV determined by $\xi : (X,q) \to X_*$. Define $\varphi : X \times S \to B(X,S) = (X \times S)/\sim$ to be the canonical map $\varphi(x,g) = \langle (x,g) \rangle$, and let σ denote the quotient structure on $B(X,S)$ in CONV determined by $\varphi : (X \times S, r) \to B(X,S)$, where $r = q \times p$. Likewise, define $\varphi_* : X_* \times S \to B(X_*,S) = (X_* \times S)/\approx$ by $\varphi_*([x],g) = \langle\langle ([x],g) \rangle\rangle$. Let r_* denote the product structure on $X_* \times S$, and let σ_* be the quotient structure on $B(X_*,S)$ determined by $\varphi_* : X_* \times S \to B(X_*,S)$. Define $\lambda_* : X_* \times S \to X_*$ by $\lambda_*([x],g) = [\lambda(x,g)]$, and denote $\eta : B(X,S) \to B(X_*,S)$ by $\eta(\langle (x,g) \rangle) = \langle\langle ([x],g) \rangle\rangle$. It is shown below that these definitions are well-defined, and their properties are investigated.

Lemma 1.3. *Assume that $((X,q),(S,.,p),\lambda) \in A$. Then*

(a) *$(x,f) \sim (y,g)$ iff there exists $h \in S$ such that $(x,hf) \approx (y,hg)$*

(b) *$x \simeq y$ iff for each $g \in S$, $\lambda(x,g) \simeq \lambda(y,g)$*

(c) *$\lambda_*(.,g) : X_* \to X_*$ is an injection, for each fixed $g \in S$*

(d) *$\eta : B(X,S,\lambda) \to B(X_*,S,\lambda_*)$ is a bijection.*

Proof. (a): Suppose that $(x,f) \sim (y,g)$. Then there exists $(z_i,h_i) \in X \times S$, $1 \leq i \leq n$, such that $(x,f) \approx (z_1,h_1)$, $(z_1,h_1) \approx (z_2,h_2)$,...,$(z_n,h_n) \approx (y,g)$. Verification is illustrated whenever $n = 2$; that is, $(x,f) \approx (z_1,h_1)$, $(z_1,h_1) \approx (z_2,h_2)$, and $(z_2,h_2) \approx (y,g)$. Then $\lambda(x,h_1) = \lambda(z_1,f)$, $\lambda(z_1,h_2) = \lambda(z_2,h_1)$, and $\lambda(z_2,g) = \lambda(y,h_2)$. It is shown that $(x,h_2h_1f) \approx (y,h_2h_1g)$. Indeed, since $(S,.)$ is commutative, $\lambda(x,h_2h_1g) = \lambda(\lambda(x,h_1),h_2g) = \lambda(\lambda(z_1,f),h_2g) = \lambda(\lambda(z_1,h_2),fg) = \lambda(\lambda(z_2,h_1),fg) = \lambda(\lambda(z_2,g),h_1f) = \lambda(\lambda(y,h_2),h_1f) = \lambda(y,h_2h_1f)$. Hence $(x,h_2h_1f) \approx (y,h_2h_1g)$ and, in general $h = h_1h_2\cdots h_n$.

Conversely, assume that $(x,hf) \approx (y,hg)$ for some $h \in S$. It is shown that $(x,f) \approx (\lambda(y,h),hg)$ and $(\lambda(y,h),hg) \approx (y,g)$. Indeed, by hypothesis, $\lambda(x,hg) = \lambda(y,hf) = \lambda(\lambda(y,h),f)$ and thus $(x,f) \approx (\lambda(y,h),hg)$. Also, $\lambda(\lambda(y,h),g) = \lambda(y,hg)$ and hence $(\lambda(y,h),hg) \approx (y,g)$. It follows that $(x,f) \sim (y,g)$.

(b): Suppose that $x \simeq y$; then there exists $h \in S$ such that $\lambda(x,h) = \lambda(y,h)$. Assume that $g \in S$. Then $(\lambda(x,g),h) \approx (\lambda(y,g),h))$. Indeed, since $(S,.)$ is commutative, $\lambda(\lambda(x,g),h) = \lambda(\lambda(x,h),g) = \lambda(\lambda(y,h),g) = \lambda(\lambda(y,g),h)$, and thus $\lambda(x,g) \simeq \lambda(y,g)$. Conversely, $x = \lambda(x,e) \simeq \lambda(y,e) = y$.

(c): Recall that $X_* = X/\simeq = \{[x] : x \in X\}$ and $\lambda_*([x], g) = [\lambda(x, g)]$. First, note that λ_* is well-defined. Indeed, if $x \simeq y$ and $g \in S$, then by (b), $\lambda(x, g) \simeq \lambda(y, g)$ implies that $\lambda_*([x], g) = \lambda_*([y], g)$, and thus λ_* is well-defined. Next, it is shown that for $g \in S$ fixed, $\lambda_*(., g)$ is an injection. Suppose that $\lambda_*([x], g) = \lambda_*([y], g)$; then $\lambda(x, g) \simeq \lambda(y, g)$ and it follows by (R3) that there exists $h \in S$ such that $\lambda(\lambda(x, g), h) = \lambda(\lambda(y, g), h)$. Equivalently, $\lambda(x, gh) = \lambda(y, gh)$, and again by (R3), $x \simeq y$. Hence $\lambda_*(., g)$ is an injection.

(d): Recall that $\eta(\langle(x, g)\rangle) = \langle([x], g)\rangle$, $x \in X$, $g \in S$. First, observe that η is well-defined. Indeed, assume that $(x, g) \sim (y, h)$; then by (a), there exists $k \in S$ such that $(x, kg) \approx (y, kh)$. Hence $\lambda(x, kh) = \lambda(y, kg)$, and thus $\lambda_*([x], kh) = [\lambda(x, kh)] = [\lambda(y, kg)] = \lambda_*([y], kg)$. Therefore $([x], kg) \approx ([y], kh)$ on $X_* \times S$, and thus by (a) and (c), $([x], g) \approx ([y], h)$. It follows that $\langle([x], g)\rangle = \langle([y], h)\rangle$, and hence η is well-defined.

Next, it is shown that η is an injection. Suppose that $\langle([x], g)\rangle = \eta(\langle(x, g)\rangle) = \eta(\langle(y, h)\rangle) = \langle([y], h)\rangle$; then $([x], g) \approx ([y], h)$ in $X_* \times S$. Hence $[\lambda(x, h)] = \lambda_*([x], h) = \lambda_*([y], g) = [\lambda(y, g)]$, and thus $\lambda(x, h) \simeq \lambda(y, g)$. According to (R3), there exists $k \in S$ such that $\lambda(x, hk) = \lambda(\lambda(x, h), k) = \lambda(\lambda(y, g), k) = \lambda(y, gk)$. Hence $(x, gk) \approx (y, hk)$, and by part (a), $(x, g) \sim (y, h)$. Therefore $\langle(x, g)\rangle = \langle(y, h)\rangle$, and thus η is an injection. Clearly η is a surjection, and consequently η is a bijection. $\square$

2. Action Categories

. Consider the triple $(X, (S, .), \lambda)$, where $X \in |\mathrm{SET}|$, $(S, .) \in |\mathrm{SG}|$, $(S, .)$ is commutative, and $\lambda : X \times S \to X$ is an action. Define $\boldsymbol{\mathcal{X}}$ to be the category whose objects consist of all triples $(X, (S, .), \lambda)$, and whose morphisms are all pairs $(f, k) : (X, (S, .), \lambda) \to (Y, (T, .), \mu)$ obeying:

(B1) $f : X \to Y$ is a map, $k : (S, .) \to (T, .)$ is a homomorphism
(B2) $f \circ \lambda = \mu \circ (f \times k)$.

Furthermore, let $\boldsymbol{\mathcal{C}}$ denote the category consisting of all objects $((X, q), (S, ., p), \lambda) \in \mathrm{AC}$, and whose morphisms $(f, k) : ((X, q), (S, .), \lambda) \to ((Y, q^Y), (T, ., p^T), \mu)$ satisfy:

(C1) $f : (X, q) \to (Y, q^Y)$ is continuous, $k : (S, ., p) \to (T, ., p^T)$ is a continuous homomorphism
(C2) $f \circ \lambda = \mu \circ (f \times k)$.

Clearly $\mathrm{id}_X \times \mathrm{id}_S : ((X, q), (S, ., p), \lambda) \to ((X, q), (S, ., p), \lambda)$ is the identity morphism in $\mathcal{C}$. Also, observe that the composition of two $\mathcal{C}$-morphisms is again a $\mathcal{C}$-morphism. Indeed, suppose that $(f, k) : ((X, q), (S, ., p), \lambda) \to ((Y, q^Y), (T, ., p^T), \mu)$ and $(h, l) : ((Y, q^Y), (T, ., p^T), \mu) \to ((Z, q^Z), (R, ., p^R), \delta)$ are two $\mathcal{C}$-morphisms. Clearly (C1) is satisfied. It remains to verify (C2). Since (f, k) and (h, l) each obeys (C2), $f \circ \lambda = \mu \circ (f \times k)$ and $h \circ \mu = \delta \circ (h \times l)$. If $(x, g) \in X \times S$, then $\delta \circ (h \times l) \circ (f \times k)(x, g) = \delta \circ (h \times l)(f(x), k(g)) = (h \circ \mu)(f(x), k(g)) = h(\mu \circ (f \times k))(x, g) = h(f \circ \lambda)(x, g) = ((h \circ f) \circ \lambda)(x, g)$, and it follows that $(h \circ f) \circ \lambda = \delta \circ (h \times l) \circ (f \times k)$. Hence (C2) is valid and $\mathcal{C}$ is a category; likewise, $\mathcal{X}$, is also a category.

If $\mathfrak{U} : \mathcal{C} \to \mathcal{X}$ denotes the faithful functor defined by $\mathfrak{U}((X,q),(S,.,p),\lambda) = (X,(S,.),\lambda)$, then $\boldsymbol{(\mathcal{C},\mathfrak{U})}$ is a concrete category over the base category $\mathcal{X}$. Let $\boldsymbol{\mathcal{D}}$ denote the full subcategory of $\mathcal{C}$ consisting of all objects $((X,q),(S,.,p),\lambda) \in |\mathcal{C}|$ such that $\lambda(.,g) : X \to X$ is an injection, for each fixed $g \in S$. Then $(\mathcal{D}, \mathfrak{U} \circ \mathfrak{E})$ is also a concrete category over $\mathcal{X}$, where $\mathfrak{E} : \mathcal{D} \to \mathcal{C}$ denotes the inclusion functor.

Theorem 2.1. *The category $\mathcal{D}$ is reflective in $\mathcal{C}$.*

Proof. Given $((X,q),(S,.p),\lambda) \in |\mathcal{C}|$, consider $((X_*,q_*),(S,.,p),\lambda_*)$, where $(X_*,q_*) \in |\mathrm{CONV}|$ and λ_* are defined in section 1. Then $\lambda_* : X_* \times S \to X_*$ is an action, and by Lemma 1.2(c), $\lambda_*(.,g) : X_* \to X_*$ is an injection. Recall that q_* is the quotient structure in CONV determined by the canonical map $\xi : (X,q) \to (X_*,q_*)$, $\xi(x) = [x]$.

Since quotient maps are productive in CONV, $\xi \times \mathrm{id}_S$ is a quotient map. Moreover, observe that $\lambda_* \circ (\xi \times \mathrm{id}_S) = \xi \circ \lambda$, $\xi \circ \lambda$ is continuous, and hence $\lambda_* : (X_*,q_*) \times (S,.,p) \to (X_*,q_*)$ is a continuous map. If follows that $((X_*,q_*),(S,.p),\lambda_*) \in |\mathcal{D}|$. Moreover, $(\xi,\mathrm{id}_S) : ((X,q),(S,.,p),\lambda) \to ((X_*,q_*),(S,.,p),\lambda_*)$ is a $\mathcal{C}$ morphism. Clearly (C1) is satisfied, and as mentioned above, $\xi \circ \lambda = \lambda_* \circ (\xi \times \mathrm{id}_S)$; hence (ξ,id_S) is a $\mathcal{C}$-morphism.

Assume that $(f,k) : ((X,q),(S,.,p),\lambda) \to ((Y,q^Y),(T,.,p^T),\mu)$ is a $\mathcal{C}$-morphism, where $((Y,q^Y),(T,.,p^T),\mu) \in |\mathcal{D}|$. Then $f \circ \lambda = \mu \circ (f \times k)$. Define $f_* : X_* \to Y$ by $f_*([x]) = f(x)$. Observe that f_* is well-defined. Indeed, if $x_1 \in [x]$, then by (R3), $\lambda(x,g) = \lambda(x_1,g)$ for some $g \in S$. Hence $\mu(f(x),k(g)) = (\mu \circ (f \times k))(x,g) = (f \circ \lambda)(x,g) = (f \circ \lambda)(x_1,g) = (\mu \circ (f \times k))(x_1,g) = \mu(f(x_1),k(g))$, and thus by (R3), $f(x) \simeq f(x_1)$. Since $Y_* = Y$, $f_* : X_* \to Y$ is well-defined. Moreover, $f_* \circ \xi = f$ is continuous, $\xi : (X,q) \to (X_*,q_*)$ is a quotient map in CONV, and thus it follows that f_* is continuous. Finally, $(\mu \circ (f_* \times k))([x],g) = \mu(f(x),k(g)) = (\mu \circ (f \times k))(x,g) = (f \circ \lambda)(x,g) = (f_* \circ \lambda_*)([x],g)$, for each $([x],g) \in X_* \times S$, and thus $f_* \circ \lambda_* = \mu \circ (f_* \times k)$. Hence $(f_*,k) : ((X_*,q_*),(S,.,p),\lambda_*) \to ((Y,q^Y),(T,.,p^T),\mu)$ is a $\mathcal{D}$-morphism. Therefore $\mathcal{D}$ is a reflective subcategory of $\mathcal{C}$. $\square$

Theorem 2.2. *The concrete category $(\mathcal{C},\mathfrak{U})$ over $\mathcal{X}$ is topological.*

Proof. Assume that $(f_j,k_j) : ((X,(S,.),\lambda) \to \mathfrak{U}((X_j,q_j),(S_j,.,p_j),\lambda_j)$, $j \in J$, is a source in $\mathcal{X}$. Define $\mathcal{F} \xrightarrow{q} x$ ($\mathcal{G} \xrightarrow{p} g$) iff for each $j \in J$, $f_j^{\to}\mathcal{F} \xrightarrow{q_j} f_j(x)$($k_j^{\to}\mathcal{G} \xrightarrow{p_j} k_j(g)$), respectively. Then $q(p)$ is the initial structure in CONV (CSG) determined from $f_j : X \to (X_j,q_j)$ ($k_j : (S,.) \to (S_j,.,p_j)$), $j \in J$, respectively. Next, it is shown that $\lambda : (X,q) \times (S,p) \to (X,q)$ is continuous. Assume that $\mathcal{F} \xrightarrow{q} x$ and $\mathcal{G} \xrightarrow{p} g$; then $f_j^{\to}\mathcal{F} \xrightarrow{q_j} f_j(x)$ and $k_j^{\to}\mathcal{G} \xrightarrow{p_j} k_j(g)$, for each $j \in J$. Employing the hypothesis, $f_j \circ \lambda = \lambda_j \circ (f_j \times k_j)$, it follows that $f_j^{\to}(\lambda^{\to}(\mathcal{F} \times \mathcal{G})) = \lambda_j^{\to}(f_j \times k_j)^{\to}(\mathcal{F} \times \mathcal{G}) = \lambda_j^{\to}(f_j^{\to}\mathcal{F} \times k_j^{\to}\mathcal{G}) \xrightarrow{q_j} \lambda_j(f_j(x),k_j(g)) = (\lambda_j \circ (f_j \times k_j))(x,g) = f_j(\lambda(x,g))$, for each $j \in J$. Hence $\lambda^{\to}(\mathcal{F} \times \mathcal{G}) \xrightarrow{q} \lambda(x,g)$, and thus λ is a continuous action. Then $((X,q),(S,.,p),\lambda)$

$\in |\mathcal{C}|$, and thus $(f_j, k_j) : ((X, q), (S, ., p), \lambda) \to ((X_j, q_j), (S_j, ., p_j), \lambda_j)$ is a $\mathcal{C}$-morphism.

Finally, suppose that $(f, k) : \mathfrak{U}((Z, q^Z), (T, ., p^T), \mu) \to \mathfrak{U}((X, q), (S, ., p), \lambda)$ is a $\mathcal{X}$-morphism such that $(f_j \times k_j) \circ (f \times k) : ((Z, q^Z), (T, ., p^T), \mu) \to ((X_j, q_j), (S_j, ., p_j), \lambda_j)$ is a $\mathcal{C}$-morphism. Since $q(p)$ is the initial structure in CONV (CSG) determined by $f_j : X \to (X_j, q_j)$ $(k_j : (S, .) \to (S_j, ., p_j))$, $j \in J$, it follows that $f : (Z, q^Z) \to (X, q)$ and $k : (T, ., p^T) \to (S, ., p)$ are continuous, respectively. Moreover, p and q are the unique structures possessing these properties. Since (f, k) is an $\mathcal{X}$-morphism, $f \circ \mu = \lambda \circ (f \times k)$, and thus (f, k) is a $\mathcal{C}$-morphism. Therefore $(\mathcal{C}, \mathfrak{U})$ is topological. $\square$

Remark 2.3. Theorem 2.2 shows that products exist in $\mathcal{C}$. In particular, suppose that $((X_j, q_j), (S_j, ., p_j), \lambda_j) \in |\mathcal{C}|$, for each j belonging to set J. Denote the product set by $X = \underset{j \in J}{\times} X_j$, the product semigroup by $(S, .) = \underset{j \in J}{\times} (S_j, .)$, the j^{th} projection map by $\pi_{j1} \times \pi_{j2} : X \times S \to X_j \times S_j$, and define $\lambda : X \times S \to X$ by $\lambda((x_j)_{j \in J}, (g_j)_{j \in J}) = (\lambda_j(x_j, g_j))_{j \in J}$. It follows that λ is an action, and $(X, (S, .), \lambda) \in |\mathcal{X}|$. Observe that for each $j \in J$, $\pi_{j1} \circ \lambda = \lambda_j \circ (\pi_{j1} \times \pi_{j2})$, and thus $(\pi_{j1}, \pi_{j2}) : (X, (S, .), \lambda) \to \mathfrak{U}((X_j, q_j), (S_j, ., p_j), \lambda_j)$ is an $\mathcal{X}$-morphism. Then by Theorem 2.2, $((X, q), (S, ., p), \lambda)$ is the unique $\mathfrak{U}$-initial lift, where $q(p)$ are product structures in CONV (CSG), respectively.

It was shown in Theorem 2.2 that every $\mathfrak{U}$-structured source has a unique $\mathfrak{U}$-initial lift. This also implies that every $\mathfrak{U}$-structured sink has a unique $\mathfrak{U}$-final lift; for example, see Theorem 21.9 [1]. Quotient morphisms in $\mathcal{C}$ are considered in the next theorem. Recall that if $((X, q), (S, ., p), \lambda) \in |\mathcal{C}|$, then $B(X, S) = (X \times S)/\sim$ denotes the generalized quotient, where $\sim$ is the equivalence relation defined in (R2). Let $r = q \times p$; then $\varphi : (X \times S, r) \to (B(X, S), \sigma)$ is the quotient map $\varphi(x, g) = \langle (x, g) \rangle$, and σ is the corresponding quotient structure in CONV.

Theorem 2.4. *Assume that* $((X, q), (S, ., p), \lambda) \in |\mathcal{C}|$. *Then*

(a) *a surjective* $\mathcal{X}$*-morphism* $(f, k) : \mathfrak{U}((X, q), (S, .p), \lambda) \to (Y, (T, .), \mu)$ *has a unique* $\mathfrak{U}$*-final lift to a quotient map* $(f, k) : ((X, q), (S, ., p), \lambda) \to ((Y, q^Y), (T, ., p^T), \mu)$ *in* $\mathcal{C}$.

(b) $(\varphi, id_S) : ((X \times S, r), (S, ., p), \Lambda) \to ((B(X, S), \sigma), (S, ., p), \lambda_B)$ *is a quotient map in* $\mathcal{C}$, *where* $\Lambda((x, g), h) := (\lambda(x, h), g)$ *and* $\lambda_B(\langle (x, g) \rangle, h) = \langle (\lambda(x, h), g) \rangle$.

(c) $(\eta, id_S) : ((B(X, S, \lambda), \sigma), (S, ., p), \lambda_B) \to ((B(X_*, S, \lambda_*), (S, ., p), \lambda^*_B)$ *is a* $\mathcal{C}$*-isomorphism, where* $\lambda^*_B(\langle ([x], g) \rangle, h) = \langle (\lambda_*([x], h), g) \rangle = \langle ([\lambda(x, h)], g) \rangle$.

Proof. (a): Fix $(y, t) \in Y \times T$, and define $\mathcal{H} \xrightarrow{q^Y} y$ $(\mathcal{K} \xrightarrow{p^T} t)$ iff there exists $\mathcal{F} \xrightarrow{q} x \in f^{-1}(y)$ $(\mathcal{G} \xrightarrow{p} g \in k^{-1}(t))$ such that $f^{\rightarrow}\mathcal{F} = \mathcal{H}$ $(k^{\rightarrow}\mathcal{G} = \mathcal{K})$, respectively. Then $q^Y(p^T)$ is the quotient structure in CONV (CSG). It is shown that the action $\mu : (Y, q^Y) \times (T, ., p^T) \to (Y, q^Y)$ is continuous. Indeed, suppose that $\mathcal{H} \xrightarrow{q^Y} y$ and $\mathcal{K} \xrightarrow{p^T} t$; then there exist $\mathcal{F} \xrightarrow{q} x \in f^{-1}(y)$ and $\mathcal{G} \xrightarrow{p} g \in k^{-1}(t)$

such that $f^{\rightarrow}\mathcal{F} = \mathcal{H}$ and $k^{\rightarrow}\mathcal{G} = \mathcal{K}$. Since (f, k) is an $\mathcal{X}$-morphism, $\mu \circ (f \times k) = f \circ \lambda$, and thus $\mu^{\rightarrow}(\mathcal{H} \times \mathcal{K}) = \mu^{\rightarrow}(f^{\rightarrow}\mathcal{F} \times k^{\rightarrow}\mathcal{G}) = (\mu \circ (f \times k))^{\rightarrow}(\mathcal{F} \times \mathcal{G}) = (f \circ \lambda)^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \xrightarrow{q^Y} (f \circ \lambda)(x, g) = (\mu \circ (f \times k))(x, g) = \mu(y, t)$. Therefore μ is continuous, and (f, k) is a $\mathcal{C}$-morphism.

Next, suppose that $(F, K) : \mathfrak{U}((Y, q^Y), (T, \cdot, p^T), \mu) \to \mathfrak{U}((Z, q^Z), (R, \cdot, p^R), \delta)$ is an $\mathcal{X}$-morphism such that $(F, K) \circ (f, k) : ((X, q), (S, \cdot, p), \lambda) \to (Z, q^Z), (R, \cdot, p^R), \delta)$ is a $\mathcal{C}$-morphism. Since q^Y and p^T are quotient structures in CONV and CSG, F and K are continuous maps, and thus (F, K) is a $\mathcal{C}$-morphism. Moreover q^Y and p^T are unique and hence (f, k) is a quotient map in $\mathcal{C}$.

(b): Observe that Λ is an action. Indeed, $\Lambda((x, g), e) = (\lambda(x, e), g) = (x, g)$. Moreover, $\Lambda(\Lambda((x, g), h), k) = \Lambda((\lambda(x, h), g), k) = (\lambda(\lambda(x, h), k), g) = (\lambda(x, hk), g) = \Lambda((x, g), hk)$, and thus Λ is an action. Note that Λ is the composition of the continuous maps :$((x, g), h) \mapsto ((x, h), g) \mapsto (\lambda(x, h), g)$. Therefore Λ is a continuous action, and thus $((X \times S, r), (S, \cdot, p), \Lambda) \in |\mathcal{C}|$.

First, it is shown that λ_B is well-defined. It must be shown that if $(x, g) \sim (x_1, g_1)$, then $\lambda_B(\langle(x, g)\rangle, h) = \lambda_B(\langle(x_1, g_1)\rangle, h)$. According to Lemma 1.3(a), there exists $k \in S$ such that $(x, kg) \approx (x_1, kg_1)$, or $\lambda(x, kg_1) = \lambda(x_1, kg)$. Note that $\lambda(\lambda(x, h), kg_1) = \lambda(\lambda(x, kg_1), h) = \lambda(\lambda(x_1, kg), h) = \lambda(\lambda(x_1, h), kg)$, and thus $(\lambda(x, h), kg) \approx (\lambda(x_1, h), kg_1)$. Again, by Lemma 1.3(a), $(\lambda(x, h), g) \sim (\lambda(x_1, h), g_1)$, and thus λ_B is well-defined.

Next, λ_B is an action. Indeed, $\lambda_B(\langle(x, g)\rangle, e) = \langle(\lambda(x, e), g)\rangle = \langle(x, g)\rangle$, and $\lambda_B(\lambda_B(\langle(x, g)\rangle, h), k) = \lambda_B(\langle(\lambda(x, h), g)\rangle, k) = \langle(\lambda(\lambda(x, h), k), g)\rangle = \langle(\lambda(x, hk), g)\rangle = \lambda_B(\langle(x, g)\rangle, hk)$. Hence λ_B is an action.

Since φ (id_S) are quotient maps in CONV (CSG), respectively, it remains to show that $\varphi \circ \Lambda = \lambda_B \circ (\varphi \times \mathrm{id}_S)$. Let $((x, g), h) \in (X \times S) \times S$; then $(\lambda_B \circ (\varphi \times \mathrm{id}_S))((x, g), h) = \lambda_B(\langle(x, g)\rangle, h) = \langle(\lambda(x, h), g)\rangle = \langle\Lambda((x, g), h)\rangle = (\varphi \circ \Lambda)((x, g), h)$. Therefore $\varphi \circ \Lambda = \lambda_B \circ (\varphi \times \mathrm{id}_S)$, and thus by part (a), λ_B is continuous, and $(\varphi, \mathrm{id}_S) : ((X \times S, r), (S, \cdot, p), \Lambda) \to ((B(X, S), \sigma), (S, \cdot, p), \lambda_B)$ is a quotient map in $\mathcal{C}$.

(c): Recall from Lemma 1.3(d) that η is a bijection, where $\eta : (B(X, S, \lambda), \sigma) \to ((B(X_*, S, \lambda_*), \sigma_*)$ is defined by $\eta(\langle(x, g)\rangle = \langle([x], g)\rangle$. It is shown that η is a homeomorphism in CONV. Observe that the diagram below commutes:

$$\begin{tikzcd}
(X \times S, r) \arrow[r, "\varphi"] \arrow[d, "\xi \times \mathrm{id}_S"] & (B(X, S), \sigma) \arrow[d, "\eta"] \\
(X_* \times S, r_*) \arrow[r, "\varphi_*"] & (B(X_*, S), \sigma_*)
\end{tikzcd}$$

Since φ is a quotient map in CONV, η is continuous iff $\eta \circ \varphi$ is continuous. However, $\eta \circ \varphi = \varphi_* \circ (\xi \times \mathrm{id}_S)$ is continuous by construction, and thus η is continuous. Also, φ_* is a quotient map in CONV, and hence η^{-1} is continuous iff $\eta^{-1} \circ \varphi_*$ is continuous. Since $\xi \times \mathrm{id}_S$ is a quotient map, $\eta^{-1} \circ \varphi_*$ is continuous iff $(\eta^{-1} \circ \varphi_*) \circ (\xi \times \mathrm{id}_S)$ is continuous. However, the latter map is simply

φ, and hence $\eta^{-1} \circ \varphi_*$ is continuous. Therefore η^{-1} is continuous, and thus $\eta : (B(X,S),\sigma) \to (B(X_*,S),\sigma_*)$ is a homeomorphism in CONV.

Since η is a homeomorphism, $((B(X,S),\sigma),(S,.,p),\lambda_B) \in |\mathcal{C}|$, $((B(X_*,S),\sigma_*),(S,.,p),\lambda_B^*) \in |\mathcal{C}|$, and it remains to show that $\eta \circ \lambda_B = \lambda_B^* \circ (\eta \times \mathrm{id}_S)$ and $\eta^{-1} \circ \lambda_B^* = \lambda_B \circ (\eta^{-1} \times \mathrm{id}_S)$. Let $(\langle(x,g)\rangle,h) \in B(X,S) \times S$; then $\lambda_B^* \circ (\eta \times \mathrm{id}_S)(\langle(x,g)\rangle,h) = \lambda_B^*(\langle([x],g)\rangle,h) = \langle(\lambda_*([x],h),g)\rangle = \langle([\lambda(x,h)],g)\rangle = \eta(\langle(\lambda(x,h),g)\rangle) = (\eta \circ \lambda_B)(\langle(x,g)\rangle,h)$. Hence $\eta \circ \lambda_B = \lambda_B^* \circ (\eta \times \mathrm{id}_S)$. Moreover, since η is a bijection, $\lambda_B = \eta^{-1} \circ \lambda_B^* \circ (\eta \times \mathrm{id}_S)$ and $\lambda_B \circ (\eta^{-1} \times \mathrm{id}_S) = \eta^{-1} \circ \lambda_B^*$. Therefore (η, id_S) is a $\mathcal{C}$-isomorphism. □

Remark 2.5. (i): Assume that $((X,q),(S,.,p),\lambda) \in |\mathcal{C}|$. An object in CONV is called **Hausdorff** whenever each filter converges to at most one element. It is shown in Theorem 4.1 [3] that $(B(X,S,\lambda),\sigma)$ is Hausdorff iff (X,q) is Hausdorff, provided that $\lambda(.,g)$ is an injection for each $g \in S$. Since $\lambda_*(.,g)$ is an injection for each $g \in S$, and $(B(X,S,\lambda),\sigma)$ and $(B(X_*,S,\lambda_*),\sigma_*)$ are homeomorphic, it follows that $(B(X,S,\lambda),\sigma)$ is Hausdorff iff (X_*,q_*) is Hausdorff.

(ii): Suppose that $(f,k) : ((X,q),(S,.,p),\lambda) \to ((Y,q^Y),(T,.,p^T),\mu)$ is an isomorphism in $\mathcal{C}$. In particular, f^{-1} (k^{-1}) is an isomorphism in CONV (CSG), respectively. Moreover, since $f \circ \lambda = \mu \circ (f \times k)$, then $\mu = f \circ \lambda \circ (f^{-1} \times k^{-1})$ and $\lambda = f^{-1} \circ \mu \circ (f \times k)$. Hence each action can be found from the other. It is not enough just to assume that f and k are isomorphisms in CONV and CSG, respectively.

3. Extensions

Consideration is given to embedding objects in $\mathcal{C}$ into objects which have nicer properties such as compactness. Recall that a convergence space (X,q) is said to be Hausdorff whenever each filter converges to at most one element. A (Hausdorff) convergence space (X,q) is **regular** (T3) provided $\mathrm{cl}_q\mathcal{F} \xrightarrow{q} x$ whenever $\mathcal{F} \xrightarrow{q} x$, respectively. Moreover, (X,q) is **compact** if each ultrafilter on X q-converges. Given $(X,q) \in |\mathrm{CONV}|$; $(X,\pi q)$ denotes the **pretopological modification** of (X,q), where $\mathcal{F} \xrightarrow{\pi q} x$ iff $\mathcal{F} \geq \cap\{\mathcal{G} : \mathcal{G} \xrightarrow{q} x\}$ (neighborhood filter at x). It is shown that in Theorem 1 [18] that $(X,q) \in |\mathrm{CONV}|$ has a T3 compactification iff πq is a Hausdorff completely regular topology, and q and πq agree on ultrafilter convergence. In this case, there exists a T3 compactification (X^*,q^*,δ) such that $\delta : (X,q) \to (X^*,q^*)$ is a dense embedding, and if $f : (X,q) \to (Y,\rho)$ is a continuous function, where (Y,ρ) is compact T3, then there exists a continuous map $f^* : (X^*,q^*) \to (Y,\rho)$ such that $f_* \circ \delta = f$

Lemma 3.1. *Suppose that (f_1,k_1) and (f,k) are $\mathcal{C}$-morphisms, $F(K)$ is a morphism in CONV (CSG), respectively, $f_1(X)$ and $k(S)$ are dense, and the diagram below commutes. Then (F,K) is also a $\mathcal{C}$-morphism.*

$$\begin{tikzcd}
((X,q),(S,.,p),\lambda) \arrow[r, "(f_1,k_1)"] \arrow[d, "(f,k)"'] & ((X_1,q_1),(S_1,.,p_1),\lambda_1) \arrow[dl, "(F,K)"'] \\
((Y,q^Y),(T,.,p^T),\mu) &
\end{tikzcd}$$

Proof. It remains to show that $F \circ \lambda_1 = \mu \circ (F \times K)$. Fix $(z,g) \in X_1 \times S_1$. Since $f_1(X)(k_1(S))$ is dense, there exists $\mathcal{F}(\mathcal{G}) \in \mathfrak{F}(X)(\mathfrak{F}(S))$ such that $f_1^{\to}\mathcal{F} \xrightarrow{q_1} z(k_1^{\to}\mathcal{G} \xrightarrow{p_1} g)$, respectively. Employing the assumptions, it follows that $(\mu \circ (F \times K))(z,g) = \mu(F(z),K(g)) = \mu(F(\lim f_1^{\to}\mathcal{F}), K(\lim k_1^{\to}\mathcal{G}) = \mu(\lim(F \circ f_1)^{\to}\mathcal{F}, \lim(K \circ k_1)^{\to}\mathcal{G}) = \lim \mu^{\to}(f^{\to}\mathcal{F} \times k^{\to}\mathcal{G}) = \lim(\mu \circ (f \times k))^{\to}(\mathcal{F} \times \mathcal{G}) = \lim(f \circ \lambda)^{\to}(\mathcal{F} \times \mathcal{G}) = \lim((F \circ f_1) \circ \lambda)^{\to}(\mathcal{F} \times \mathcal{G}) = \lim F^{\to}(f_1 \circ \lambda)^{\to}(\mathcal{F} \times \mathcal{G}) = \lim F^{\to}(\lambda_1 \circ (f_1 \times k_1))^{\to}(\mathcal{F} \times \mathcal{G}) = \lim(F \circ \lambda_1)^{\to}(f_1^{\to}\mathcal{F} \times k_1^{\to}\mathcal{G}) = (F \circ \lambda_1)(\lim(f_1^{\to}\mathcal{F} \times k_1^{\to}\mathcal{G})) = (F \circ \lambda_1)(z,g)$. Hence $F \circ \lambda_1 = \mu \circ (F \times K)$, and thus (F,K) is a $\mathcal{C}$-morphism. □

Theorem 3.2. *Assume that (X,q) has a T3-compactification, and $(f,k) : ((X,q),(S,.,p),\lambda) \to ((Y,q^Y),(T,.,p^T),\mu)$ is a $\mathcal{C}$-morphism, where (Y,q^Y) is compact T3, and p is the discrete structure. Then, using the notations above, (δ, id_S) is a dense $\mathcal{C}$-embedding and, moreover, (f^*,k) is a $\mathcal{C}$-morphism such that the diagram below commutes, for some λ^*:*

$$\begin{tikzcd}
((X,q),(S,.,p),\lambda) \arrow[r, "(\delta, id_S)"] \arrow[d, "(f,k)"'] & ((X^*,q^*),(S,.,p),\lambda^*) \arrow[dl, "(f^*,k)"'] \\
((Y,q^Y),(T,.,p^T),\mu) &
\end{tikzcd}$$

Proof. Fix $g \in S$. Then the subspace $X^* \times \{g\}$ of $X^* \times S$ is a T3 compactification of $X \times \{g\}$ which possesses the continuous extension property. Let $\lambda_g : X \times \{g\} \to X$ be the function $\lambda_g(x,g) = \lambda(x,g)$, $x \in X$. Since $\delta \circ \lambda_g$ is continuous, there exists a continuous extension λ_g^* such that the diagram below commutes:

$$\begin{tikzcd}
X \times \{g\} \arrow[r, "\delta \circ \lambda_g"] \arrow[d, "\delta \times \mathrm{id}_g"'] & X^* \\
X^* \times \{g\} \arrow[ur, "\lambda_g^*"] &
\end{tikzcd}$$

Define $\lambda^* : X^* \times S \to X^*$ by $\lambda^*(z,g) = \lambda^*_g(z,g)$, for each $z \in X^*$, $g \in S$. Since p is the discrete structure on S, it follows that λ^* is continuous and, moreover, since the diagram above commutes, $(\lambda^* \circ (\delta \times \mathrm{id}_S))(x,g) = (\lambda^*_g \circ (\delta \times \mathrm{id}_g))(x,g) = (\delta \circ \lambda_g)(x,g) = (\delta \circ \lambda)(x,g)$, for each $(x,g) \in X \times S$. Hence $\lambda^* \circ (\delta \times \mathrm{id}_S) = \delta \circ \lambda$, and thus $(\delta, \mathrm{id}_S) : ((X,q),(S,.,p),\lambda) \to ((X^*,q^*),(S,.,p),\lambda^*)$ is a $\mathcal{C}$-morphism.

Next, it is shown that $\lambda^* : X^* \times S \to X^*$ is a action. As shown above, $\lambda^* \circ (\delta \times \mathrm{id}_S) = \delta \circ \lambda$, and thus if $x \in X$, then $\lambda^*(\delta(x), e)) = (\lambda^* \circ (\delta \times \mathrm{id}_S))(x,e) = (\delta \circ \lambda)(x,e) = \delta(\lambda(x,e)) = \delta(x)$. Moreover, if $z \in X^* - \delta(X)$, then since $\delta(X)$ is dense in X, there exists a filter $\mathcal{F}$ on X such that $\delta^{\to}\mathcal{F} \xrightarrow{q^*} z$. However, λ^* is continuous whenever p is the discrete structure, and thus $\lambda^{*\to}(\delta^{\to}\mathcal{F} \times \dot{e}) \xrightarrow{q^*} \lambda^*(z,e)$. Moreover, $\lambda^{*\to}(\delta^{\to}\mathcal{F} \times \dot{e}) = (\lambda^{*\to} \circ (\delta \times \mathrm{id}_S))^{\to}(\mathcal{F} \times \dot{e}) = (\delta \circ \lambda)^{\to}(\mathcal{F} \times \dot{e}) = \delta^{\to}(\lambda^{\to}(\mathcal{F} \times \dot{e})) = \delta^{\to}\mathcal{F} \xrightarrow{q^*} z$. Since (X^*, q^*) is Hausdorff, $\lambda^*(z,e) = z$.

Let $z \in X$ and $g,h \in S$; it is shown that $\lambda^*(\lambda^*(z,g),h) = \lambda^*(z,gh)$. First, suppose that $z = \delta(x)$. Then $\lambda^*(\lambda^*(\delta(x),g),h) = \lambda^*(\lambda^* \circ (\delta \times \mathrm{id}_S)(x,g),h) = \lambda^*((\delta \circ \lambda)(x,g),h) = \lambda^*(\delta(\lambda(x,g)),h) = (\lambda^* \circ (\delta \times \mathrm{id}_S))(\lambda(x,g),h) = (\delta \circ \lambda)(\lambda(x,g),h) = \delta(\lambda(\lambda(x,g),h)) = \delta(\lambda(x,gh)) = (\delta \circ \lambda)(x,gh) = (\lambda^* \circ (\delta \times \mathrm{id}_S))(x,gh) = \lambda^*(\delta(x),gh)$. Hence $\lambda^*(\lambda^*(\delta(x),g),h) = \lambda^*(\delta(x),gh)$. Further, assume that $z \in X^* - \delta(X)$; it is shown that $\lambda^*(\lambda^*(z,g),h) = \lambda^*(z,gh)$. There exists a filter $\mathcal{F}$ on X such that $\delta^{\to}\mathcal{F} \xrightarrow{q^*} z$. Since λ^* is continuous whenever p is the discrete structure, $\lambda^{*\to}(\delta^{\to}\mathcal{F} \times \dot{g}) \xrightarrow{q^*} \lambda^*(z,g)$. Employing $\lambda^* \circ (\delta \times \mathrm{id}_S) = \delta \circ \lambda$, $\lambda^{*\to}(\delta^{\to}\mathcal{F} \times \dot{g}) = [\lambda^* \circ (\delta \times \mathrm{id}_S)]^{\to}(\mathcal{F} \times \dot{g}) = (\delta \circ \lambda)^{\to}(\mathcal{F} \times \dot{g})$, and thus $\lambda^{*\to}[(\delta \circ \lambda)^{\to}(\mathcal{F} \times \dot{g}) \times \dot{h}] \xrightarrow{q^*} \lambda^*(\lambda^*(z,g),h)$. However, $\lambda^{*\to}[(\delta \circ \lambda)^{\to}(\mathcal{F} \times \dot{g}) \times \dot{h}] = [\lambda^* \circ (\delta \times \mathrm{id}_S)]^{\to}(\lambda^{\to}(\mathcal{F} \times \dot{g}) \times \dot{h}) = (\delta \circ \lambda)^{\to}(\lambda^{\to}(\mathcal{F} \times \dot{g}) \times \dot{h}) = \delta^{\to}[\lambda^{\to}(\lambda^{\to}(\mathcal{F} \times \dot{g}) \times \dot{h})] = \delta^{\to}(\lambda^{\to}(\mathcal{F} \times \dot{gh})) = [\lambda^* \circ (\delta \times \mathrm{id}_S)]^{\to}(\mathcal{F} \times \dot{gh}) = \lambda^{*\to}(\delta^{\to}\mathcal{F} \times \dot{gh}) \xrightarrow{q^*} \lambda^*(z,gh)$. Since (X^*,q^*) is Hausdorff, it follows that $\lambda^*(\lambda^*(z,g),h) = \lambda^*(z,gh)$, and thus λ^* is a continuous action.

It remains to show that $(\delta^{-1}_{\delta(X)}, \mathrm{id}_S) : ((\delta(X), q^*|_{\delta(X)}), (S,.,p), \lambda^*|_{\delta(X) \times S}) \to ((X,q),(S,.,p),\lambda)$ is a $\mathcal{C}$-morphism. Since δ is a embedding, only $(\delta^{-1} \circ \lambda^*)|_{\delta(X) \times S} = \lambda \circ (\delta^{-1}_{\delta(X)} \times \mathrm{id}_S)$ must be verified. Using the fact that $\lambda^* \circ (\delta \times \mathrm{id}_S) = \delta \circ \lambda$ and δ is an embedding, $\lambda^*|_{\delta(X) \times S} = \delta \circ \lambda \circ (\delta^{-1}_{\delta(X)} \times \mathrm{id}_S)$ and $\delta^{-1}_{\delta(X)} \circ \lambda^*|_{\delta(X) \times S} = \lambda \circ (\delta^{-1}_{\delta(X)} \times \mathrm{id}_S)$. Hence $(\delta^{-1}_{\delta(X)}, \mathrm{id}_S)$ is also a $\mathcal{C}$-morphism, and thus $(\delta, \mathrm{id}_S) : ((X,q),(S,.,p),\lambda) \to ((X^*,q^*),(S,.,p),\lambda^*)$ is an embedding in $\mathcal{C}$.

Finally, since (δ, id_S), (f,k) are $\mathcal{C}$-morphisms and $\delta(X)$ is dense in X^*, it follows from Lemma 3.1 that (f^*,k) is a $\mathcal{C}$-morphism. □

Suppose that (X,q) has the T3 compactification (X^*,q^*,δ) mentioned above, and assume that $((X,q),(S,.,p),\lambda) \in |\mathcal{C}|$. Define $\mathcal{G} \xrightarrow{p^*} g$ iff for each $\mathcal{H} \xrightarrow{q^*} z$, $\lambda^{*\to}(\mathcal{H} \times \mathcal{G}) \xrightarrow{q^*} \lambda^*(z,g)$.

Corollary 3.3. *Assume that* $((X,q),(S,.,p),\lambda) \in |\mathcal{C}|$, *and* (X^*,q^*,δ) *is the T3 compactification of* (X,q) *described above. Then* p^* *is the coarsest structure on* S *such that* $((X^*,q^*),(S,.,p^*),\lambda^*) \in |\mathcal{C}|$.

Proof. First, it is shown that $(S,p^*) \in |\mathrm{CONV}|$. Fix $g \in S$, Suppose that $\mathcal{H} \xrightarrow{q^*} z$; then according to Theorem 3.1, $\lambda^* : (X^*,q^*) \times (S,p) \to (X^*,q^*)$ is continuous whenever p has the discrete structure. Hence $\lambda^{*\to}(\mathcal{H} \times \dot{g}) \xrightarrow{q^*} \lambda^*(z,g)$, and thus $\dot{g} \xrightarrow{p^*} g$. Moreover, it easily follows that if $\mathcal{K} \xrightarrow{p^*} g$ and $\mathcal{L} \geq \mathcal{K}$, then $\mathcal{L} \xrightarrow{p^*} g$. Hence $(S,p^*) \in |\mathrm{CONV}|$. Next, assume that $\mathcal{G}_i \xrightarrow{p^*} g_i$, $i = 1,2$, and $\mathcal{H} \xrightarrow{q^*} z$; then $\lambda^{*\to}(\mathcal{H} \times \mathcal{G}_1\mathcal{G}_2) = \lambda^{*\to}(\lambda^{*\to}(\mathcal{H} \times \mathcal{G}_1) \times \mathcal{G}_2) \xrightarrow{q^*} \lambda^*(\lambda^*(z,g_1),g_2) = \lambda^*(z,g_1g_2)$. Hence $\mathcal{G}_1\mathcal{G}_2 \xrightarrow{p^*} g_1g_2$, and thus $(S,.,p^*) \in |\mathrm{CSG}|$. It follows from the definition that p^* is the coarsest structure such that $((X^*,q^*),(S,.,p^*),\lambda^*) \in |\mathcal{C}|$. $\square$

Let $(X,q) \in |\mathrm{CONV}|$. It is shown in [17] that if (X,q) is Hausdorff, then there exists a Hausdorff compactification $(\hat{X},\hat{q})$, where $j : (X,q) \to (\hat{X},\hat{q})$ denotes a dense embedding, and $\hat{X} = j(X) \cup \{\alpha : \alpha = \mathcal{G}$ is an ultrafilter on X which fails to q-converge$\}$. Let $\mathcal{F} \in \mathfrak{F}(X)$; then $\hat{\mathcal{F}}$ denotes the filter on $\hat{X}$ whose base is $\{\hat{F} : F \in \mathcal{F}\}$, and $\hat{F} = j(F) \cup \{\alpha \in \hat{X} : F \in \alpha\}$. Define $\mathcal{H} \xrightarrow{\hat{q}} j(x)$ iff $\mathcal{H} \geq \hat{\mathcal{F}}$ for some $\mathcal{F} \xrightarrow{q} x$, and $\mathcal{H} \xrightarrow{\hat{q}} \alpha$ iff $\mathcal{H} \geq \hat{\mathcal{G}}$ for some $\alpha = \mathcal{G}$. According to [17], if $f : (X,q) \to (Y,q^Y)$ is continuous, (Y,q^Y) is compact T3, then $\hat{f} : (\hat{X},\hat{q}) \to (Y,q^Y)$ is a continuous extension of f, where $\hat{f}(j(x) = f(x)$ and $\hat{f}(\alpha) = \lim f^{\to}\mathcal{F}$ in (Y,q^Y). Moreover, it is easily verified that the above results are valid whenever (X,q) fails to be Hausdorff.

Given any $(X,q) \in |\mathrm{CONV}|$, it is shown in Proposition 2.1 [8] that there exists a finest regular convergence rq which is coarser than q. Define $x \sim y$ iff $\dot{x} \xrightarrow{rq} y$; let sX be the set of all equivalence classes, and denote the corresponding quotient map by $\Phi : X \to sX$. Let sq be the quotient structure in CONV determined by: $\Phi : (X,rq) \to (sX,sq)$. According to Proposition 1.3 [9], (sX,sq) is T3. Moreover, it is shown in [8] and [9] that if f is continuous, then the maps below are continuous and the diagram commutes:

$$\begin{tikzcd}
(X,q) \arrow[r, "\mathrm{id}_X"] \arrow[d, "f"] & (X,rq) \arrow[r, "\Phi_X"] \arrow[d, "f"] & (sX,sq) \arrow[d, "f_s"'] \\
(Y,p) \arrow[r, "\mathrm{id}_Y"] & (Y,rp) \arrow[r, "\Phi_Y"] & (sY,sp)
\end{tikzcd} \quad \text{, where } f_s([x]) = f(x).$$

In particular, if $f : (X,q) \to (Y,q^Y)$ is continuous and (Y,q^Y) is compact T3, then the maps below are continuous and the diagram commutes:

$$
\begin{tikzcd}
(X,q) \arrow[r, "j"] \arrow[d, "f"] & (\hat{X},\hat{q}) \arrow[r, "\mathrm{id}_{\hat{X}}"] \arrow[d, "\hat{f}"] & (\hat{X},r\hat{q}) \arrow[r, "\Phi"] \arrow[d, "\hat{f}"] & (s\hat{X},s\hat{q}) \arrow[d, "\hat{f}_s"] \\
(Y,q^Y) \arrow[r, "\mathrm{id}_Y"] & (Y,q^Y) \arrow[r, "\mathrm{id}_Y"] & (Y,q^Y) \arrow[r, "\mathrm{id}_Y"] & (Y,q^Y)
\end{tikzcd}
$$

Theorem 3.4. *Assume that* $(f,k) : ((X,q),(S,.,p),\lambda) \to ((Y,q^Y),(T,.,p^T),\mu)$ *is a* $\mathcal{C}$*-morphism, where* (X,q) *is Hausdorff,* p *is the discrete structure on* S*, and* (Y,q^Y) *is compact T3. Using the notations defined above,* $(\hat{f},k)$ *and* $(\hat{f}_s,k)$ *are* $\mathcal{C}$*-morphisms such that the diagram below commutes:*

$$
\begin{tikzcd}
((X,q),(S,.,p),\lambda) \arrow[r, "(j,id_S)"] \arrow[dr, "(f,k)"'] & ((\hat{X},r\hat{q}),(S,.,p),\hat{\lambda}) \arrow[r, "(\Phi,id_S)"] \arrow[d, "(\hat{f},k))"'] & ((s\hat{X},s\hat{q}),(S,.,p),\hat{\lambda}_s) \arrow[dl, "(\hat{f}_s,k)"] \\
 & ((Y,q^Y),(T,.,p^T),\mu) &
\end{tikzcd}
$$

Proof. It follows from (3.1) that $\hat{f}$, $\hat{f}_s$ are continuous, and the diagram commutes. Using the notations given above, define $\hat{\lambda} : \hat{X} \times S \to \hat{X}$ by $\hat{\lambda}(j(x),g) = (j \circ \lambda)(x,g)$, and if $\alpha = \mathcal{F}$, $\hat{\lambda}(\alpha,g) = \lim(j \circ \lambda)^{\rightarrow}(\mathcal{F} \times \dot{g})$ in $(\hat{X},\hat{q})$. Observe that $\hat{\lambda}(\hat{A} \times B) \subseteq \mathrm{cl}_{r\hat{q}}(j \circ \lambda)(A \times B)$ whenever $A \subseteq X$ and $B \subseteq S$, and thus, since p is discrete, $\hat{\lambda} : (\hat{X},\hat{q}) \times (S,p) \to (\hat{X},r\hat{q})$ is continuous. According to Theorem 6.3 [8], $r(\hat{q} \times p) = r\hat{q} \times p$, and thus $\hat{\lambda} : (\hat{X},r\hat{q}) \times (S,p) \to (\hat{X},r\hat{q})$ is continuous. Moreover, define $\hat{\lambda_s} : s\hat{X} \times S \to s\hat{X}$ by $\hat{\lambda}_s([z],g) = [\hat{\lambda}(z,g)]$. Note that $\hat{\lambda}_s$ is well-defined. Indeed, if $z_1 \sim z_2$, then $\dot{z_1} \xrightarrow{r\hat{q}} z_2$, and since $\hat{\lambda} : (\hat{X},r\hat{q}) \times (S,p) \to (\hat{X},r\hat{q})$ is continuous, $\hat{\lambda}(\dot{z_1},g) = \hat{\lambda}^{\rightarrow}(\dot{z_1} \times \dot{g}) \xrightarrow{r\hat{q}} \hat{\lambda}(z_2,g)$. Hence $\hat{\lambda}(z_1,g) \sim \hat{\lambda}(z_2,g)$, and thus $\hat{\lambda_s}$ is well-defined. Observe that the diagram below commutes and $\Phi \times \mathrm{id}_S$ is a quotient map:

$$
\begin{tikzcd}
(\hat{X},r\hat{q}) \times (S,p) \arrow[r, "\hat{\lambda}"] \arrow[d, "\Phi \times \mathrm{id}_S"] & (\hat{X},r\hat{q}) \arrow[d, "\Phi"] \\
(s\hat{X},s\hat{q}) \times (S,p) \arrow[r, "\hat{\lambda}_s"] & (s\hat{X},s\hat{q})
\end{tikzcd}
$$

Since $\Phi \times \mathrm{id}_S$ is a quotient map and $\Phi \times \hat{\lambda}$ is continuous, it follows that $\hat{\lambda}_s$ is continuous. Then $((\hat{X},r\hat{q}),(S,.,p),\hat{\lambda}),((s\hat{X},s\hat{q}),(S,.,p),\hat{\lambda}_s) \in |\mathcal{C}|$, and it is straightforward to verify that (j, id_S) and (Φ, id_S) are $\mathcal{C}$-morphisms. Moreover, employing Lemma 3.1, $(\hat{f}, k)$ and $(\hat{f}_s, k)$ are also $\mathcal{C}$-morphisms. □

A commutative semigroup can be embedded in a group iff it is cancellative. One way of constructing the group is by means of equivalence classes of ordered pairs just as one forms the rationals from the integers. Let **CG** denote the category whose objects consist of all the commutative convergence groups, and having all the continuous group homomorphisms as its morphisms. Let $(S, .) \in |\mathrm{SG}|$ be commutative and cancellative; then a special case of Theorem 1.24 [6] shows that $(S, .)$ is embedded in the group $(\bar{S}, .)$, where elements in $\bar{S}$ can be expressed in the form gh^{-1}, for some $g, h \in S$. The natural injection is denoted by $j : (S, .) \to (\bar{S}, .)$, where $j(g) = g$ for all $g \in S$. This notation is used below.

Lemma 3.5. *Assume that $(S, ., p) \in |CSG|$ is commutative and cancellative. Then there is a finest structure $\bar{p}$ on $\bar{S}$ such that $(\bar{S}, ., \bar{p}) \in |CG|$ and $j : (S, p) \to (\bar{S}, \bar{p})$ is continuous.*

Proof. Let $\mathcal{H} \in \mathfrak{F}(\bar{S})$; then $\mathcal{H}^{-1}$ denotes the filter on $\bar{S}$ whose base is $\{H^{-1} : H \in \mathcal{H}\}$. Define $\bar{p}$ on $\bar{S}$ as follows: $\mathcal{K} \xrightarrow{\bar{p}} gh^{-1}$ iff there exist $\mathcal{G} \xrightarrow{p} g_1$ and $\mathcal{H} \xrightarrow{p} h_1$ such that $\mathcal{K} \geq (j^{\to}\mathcal{G})(j^{\to}\mathcal{H})^{-1}$ and $g_1 h_1^{-1} = gh^{-1}$. Clearly $\dot{gh^{-1}} \xrightarrow{\bar{p}} gh^{-1}$ since $\dot{g} \xrightarrow{p} g$ and $\dot{h} \xrightarrow{p} h$. Also, if $\mathcal{L} \geq \mathcal{K}$ and $\mathcal{K} \xrightarrow{\bar{p}} gh^{-1}$, then $\mathcal{L} \xrightarrow{\bar{p}} gh^{-1}$. Hence $(\bar{S}, \bar{p}) \in |\mathrm{CONV}|$.

Next, it is shown that multiplication in $(\bar{S}, ., \bar{p})$ is a continuous operation. Suppose that $\mathcal{K}_i \xrightarrow{\bar{p}} g_i h_i^{-1}$, $\mathcal{G}_i \xrightarrow{p} g_i$, $\mathcal{H}_i \xrightarrow{p} h_i$ and $\mathcal{K}_i \geq (j^{\to}\mathcal{G}_i)(j^{\to}\mathcal{H}_i)^{-1}$, $i = 1, 2$. Since $(S, .)$ is commutative and j is a homomorphism, $\mathcal{K}_1.\mathcal{K}_2 \geq (j^{\to}\mathcal{G}_1.j^{\to}\mathcal{G}_2)\ (j^{\to}\mathcal{H}_1)^{-1}(j^{\to}\mathcal{H}_2)^{-1} = j^{\to}(\mathcal{G}_1.\mathcal{G}_2)(j^{\to}(\mathcal{H}_1.\mathcal{H}_2))^{-1}$. However, $\mathcal{G}_1.\mathcal{G}_2 \xrightarrow{p} g_1 g_2$, $\mathcal{H}_1.\mathcal{H}_2 \xrightarrow{p} h_1 h_2$, and thus $\mathcal{K}_1.\mathcal{K}_2 \xrightarrow{\bar{p}} (g_1 g_2)(h_1 h_2)^{-1} = (g_1 h_1^{-1})(g_2 h_2^{-1})$. Hence multiplication in $(\bar{S}, ., \bar{p})$ is continuous.

Finally, inversion is a continuous operation. Indeed, suppose that $\mathcal{K} \xrightarrow{\bar{p}} gh^{-1}$, $\mathcal{G} \xrightarrow{p} g$, $\mathcal{H} \xrightarrow{p} h$, and $\mathcal{K} \geq (j^{\to}\mathcal{G})(j^{\to}\mathcal{H})^{-1}$. Then $\mathcal{K}^{-1} \geq (j^{\to}\mathcal{H})(j^{\to}\mathcal{G})^{-1} \xrightarrow{\bar{p}} hg^{-1} = (gh^{-1})^{-1}$, and thus $(\bar{S}, .\bar{p}) \in |\mathrm{CG}|$

Assume that $(\bar{S}, ., r) \in |\mathrm{CG}|$ such that $j : (S, p) \to (\bar{S}, r)$ is continuous and $\mathcal{K} \xrightarrow{\bar{p}} gh^{-1}$. Then there exist $\mathcal{G} \xrightarrow{p} g_1$, $\mathcal{H} \xrightarrow{p} h_1$ such that $\mathcal{K} \geq (j^{\to}\mathcal{G})(j^{\to}\mathcal{H})^{-1}$ and $g_1 h_1^{-1} = gh^{-1}$. It follows that $j^{\to}\mathcal{H} \xrightarrow{r} g_1$, $j^{\to}\mathcal{H} \xrightarrow{r} h_1$, and since $(\bar{S}, ., r) \in |\mathrm{CG}|$, $(j^{\to}\mathcal{G}).(j^{\to}\mathcal{H})^{-1} \xrightarrow{r} g_1 h_1^{-1}$. Therefore $\mathcal{K} \xrightarrow{r} gh^{-1}$, and thus $\bar{p} \geq r$. □

Suppose that $((X, q), (S, ., p), \lambda) \in |\mathcal{C}|$, and recall that $(B(X, S), \sigma)$ denotes the generalized quotient space determined by $\varphi : (X \times S, r) \to (X \times S)/\sim = B(X, S)$. It is shown in Theorem 2.4(b) that $((B(X, S), \sigma), (S, ., p), \lambda_B) \in |\mathcal{C}|$, where $\lambda_B(\langle (x, g) \rangle, h) = \langle (\lambda(x, h), g) \rangle$. The next result shows that $(\bar{S}, ., \bar{p})$ also acts continuously on $(B(X, S), \sigma)$.

Lemma 3.6. *Assume that $((X,q),(S,.,p),\lambda) \in |\mathcal{C}|$, where $(S,.)$ is cancellative, and $j : (S,.,p) \to (\bar{S},\bar{p})$ denote the natural injection. Then there exists a continuous action $\bar{\lambda}_B$ such that $((B(X,S),\sigma),(\bar{S},.,\bar{p}),\bar{\lambda}_B) \in |\mathcal{C}|$ and $(id_B, j) : ((B(X,S),\sigma),(S,.,p),\lambda_B) \to ((B(X,S),\sigma),(\bar{S},.,\bar{p}),\bar{\lambda}_B)$ is a $\mathcal{C}$-morphism.*

Proof. Define $\bar{\lambda}_B : B(X,S)\times\bar{S} \to B(X,S)$ by $\bar{\lambda}_B(\langle(x,g)\rangle, kl^{-1}) = \langle(\lambda(x,k),gl)\rangle)$, where $x \in X$ and $g,k,l \in S$. First, it is shown that $\bar{\lambda}_B$ is well-defined. Assume that $(x,g) \sim (x_1,g_1)$ and $kl^{-1} = k_1l_1^{-1}$. Employing Lemma 1.3(a), there exists $h \in S$ such that $(x,hg) \approx (x_1,hg_1)$, and thus $\lambda(x,hg_1) = \lambda(x_1,hg)$. Then $(\lambda(x,k),hgl) \approx (\lambda(x_1,k_1),hg_1l_1)$; indeed, $\lambda(\lambda(x,k),hg_1l_1) = \lambda(x,khg_1l_1) = \lambda(\lambda(x,hg_1),kl_1) = \lambda(\lambda(x_1,hg),kl_1) = \lambda(\lambda(x_1,hg),k_1l) = \lambda(x_1,hgk_1l) = \lambda(\lambda(x_1,k_1),hgl)$. Hence $(\lambda(x,k),hgl) \approx (\lambda(x_1,k_1),hg_1l_1)$, and again by Lemma 1.3(a), $(\lambda(x,k),gl) \sim (\lambda(x_1,k_1),g_1l_1)$. Therefore, $\bar{\lambda}_B$ is well-defined.

Observe that $\bar{\lambda}_B$ is an action. Indeed, $\bar{\lambda}_B(\langle(x,g)\rangle,e) = \langle(\lambda(x,e),g)\rangle = \langle(x,g)\rangle$, and $\bar{\lambda}_B(\bar{\lambda}_B(\langle(x,g)\rangle,k_1l_1^{-1}),k_2l_2^{-1}) = \bar{\lambda}_B(\langle(\lambda(x,k_1),gl_1)\rangle,k_2l_2^{-1}) = \langle(\lambda(\lambda(x,k_1),k_2),gl_1l_2)\rangle =$
$\langle(\lambda(x,k_1k_2),gl_1l_2)\rangle = \bar{\lambda}_B(\langle(x,g)\rangle,k_1l_1^{-1}k_2l_2^{-1})$. Hence $\bar{\lambda}_B$ is an action. Furthermore, $\bar{\lambda}_B : (B(X,S),\sigma) \times (\bar{S},\bar{p}) \to (B(X,S),\sigma)$ is continuous. Suppose that $\mathcal{H} \xrightarrow{\sigma} \langle(x,g)\rangle$ and $\mathcal{M} \xrightarrow{\bar{p}} kl^{-1}$; then there exist $(x_1,g_1) \sim (x,g)$, $k_1l_1^{-1} = kl^{-1}$, $\mathcal{F} \xrightarrow{q} x_1$, $\mathcal{G} \xrightarrow{p} g_1$, $\mathcal{K} \xrightarrow{p} k_1$, and $\mathcal{L} \xrightarrow{p} l_1$ such that $\mathcal{H} \geq \varphi^{\to}(\mathcal{F}\times\mathcal{G})$ and $\mathcal{M} \geq \mathcal{K}.\mathcal{L}^{-1}$. It follows that $\bar{\lambda}_B^{\to}(\mathcal{H}\times\mathcal{M}) \geq \bar{\lambda}_B^{\to}(\varphi^{\to}(\mathcal{F}\times\mathcal{G})\times \mathcal{K}.\mathcal{L}^{-1}) = \varphi^{\to}(\lambda^{\to}(\mathcal{F}\times\mathcal{K})\times\mathcal{G}.\mathcal{L}) \xrightarrow{\sigma} \varphi(\lambda(x_1,k_1),g_1l_1) = \langle(\lambda(x_1,k_1),g_1l_1)\rangle = \bar{\lambda}_B(\langle(x_1,g_1)\rangle,k_1l_1^{-1}) = \bar{\lambda}_B(\langle(x,g)\rangle,kl^{-1})$ since $\bar{\lambda}_B$ is well-defined. Therefore $\bar{\lambda}_B$ is continuous, and thus $((B(X,S),\sigma),(\bar{S},.,\bar{p}),\bar{\lambda}_B) \in |\mathcal{C}|$

Finally, $(\mathrm{id}_B, j) : ((B(X,S),\sigma),(S,.,p),\lambda_B) \to ((B(X,S),\sigma),(\bar{S},.,\bar{p}),\bar{\lambda}_B)$ is a $\mathcal{C}$-morphism since $\mathrm{id}_B \circ \lambda_B = \bar{\lambda}_B \circ (\mathrm{id}_B \times j)$. $\square$

Lemma 3.7. *Suppose that $(f,k) : ((X,q),(S,.,p),\lambda) \to ((Y,q^Y),(T,.,p^T),\mu)$ is $\mathcal{C}$-morphism, and assume that $(T,.,p^T) \in |CG|$. If $(x,g),(y,h) \in X\times S$ and $(x,g) \sim (y,h)$, then $\mu(f(x),(k(g))^{-1})) = \mu(f(y),(k(h))^{-1})$.*

Proof. Since $(x,g) \sim (y,h)$, it follows from Lemma 1.3(a) that there exists $l \in S$ such that $(x,lg) \approx (y,lh)$; hence $\lambda(x,lh) = \lambda(y,lg)$ and thus $f(\lambda(x,lh)) = f(\lambda(y,lg))$. Observe that $\mu(f(x),(k(g))^{-1}) = \mu(f(x),(k(g))^{-1}k(lh)(k(lh))^{-1}) = \mu(\mu(f(x),k(lh)),(k(g))^{-1}(k(lh))^{-1}) = \mu((\mu\circ(f\times k))(x,lh),(k(g))^{-1}(k(lh))^{-1})$. Since (f,k) is a $\mathcal{C}$-morphism, $f\circ\lambda = \mu\circ(f\times k)$, and thus $(\mu\circ(f\times k))(x,lh) = f(\lambda(x,lh)) = f(\lambda(y,lg)) = ((\mu\circ(f\times k))(y,lg)$. Therefore, $\mu(f(x),(k(g))^{-1}) = \mu((\mu\circ(f\times k))(y,lg),(k(g))^{-1}(k(lh))^{-1}) =$
$\mu(\mu(f(y),k(l)k(g)),(k(g))^{-1}(k(h))^{-1}(k(l))^{-1}) = \mu(f(y),(k(h))^{-1})$. $\square$

Define $\beta : (X,q) \to (B(X,S),\sigma)$ by $\beta(x) = \langle(x,e)\rangle$, and observe that β is continuous. Indeed, if $\mathcal{F} \xrightarrow{q} x$, then $\beta^{\to}\mathcal{F} = \varphi^{\to}(\mathcal{F}\times\dot{e}) \xrightarrow{\sigma} \varphi(x,e) = \langle(x,e)\rangle$. Hence β is continuous. Therefore $(\beta,j) : ((X,q),(S,.,p),\lambda) \to ((B(X,S),\sigma),(\bar{S},.,\bar{p}),\bar{\lambda}_B)$ is a $\mathcal{C}$-morphism since $(\bar{\lambda}_B\circ(\beta\times j))(x,g) = \bar{\lambda}_B(\langle(x,e)\rangle,g) = \langle(\lambda(x,g),e)\rangle = \beta(\lambda(x,g)) = (\beta\circ\lambda)(x,g)$.

The following result was proved in the topological context under the assumptions that $\lambda(.,g)$ is injective for each fixed $g \in S$, and S is equipped with the discrete topology [4]. Here $j : (S,.) \to (\bar{S},.)$ is an embedding, and consequently $(S,.)$ must be cancellative. A canonical map is used in [4] which is not necessarily an embedding but does not require that $(S,.)$ be cancellative.

Theorem 3.8. *Assume $((X,q),(S,.,p),\lambda) \in |\mathcal{C}|$, $(S,.)$ is cancellative, and let $j : (S,.,p) \to (\bar{S},.,\bar{p})$ denote the injection $j(g) = g$ given in Lemma 3.5. If (f,k) is a $\mathcal{C}$-morphism and $(T,.,p^T) \in |CG|$, then there exists a $\mathcal{C}$-morphism (F,K) such that the diagram below is commutative:*

$$\begin{tikzcd}
((X,q),(S,.,p),\lambda) \arrow[r, "{(\beta,j)}"] \arrow[d, "{(f,k)}"'] & ((B(X,S),\sigma),(\bar{S},.,\bar{p}),\bar{\lambda}_B) \arrow[dl, "{(F,K)}"'] \\
((Y,q^Y),(T,.,p^T),\mu) &
\end{tikzcd}$$

Proof. Define $F : B(X,S) \to Y$ by $F(\langle(x,g)\rangle) = \mu(f(x),(k(g))^{-1})$, and $K : \bar{S} \to T$ as $K(mn^{-1}) = k(m)(k(n))^{-1}$. If follows from Lemma 3.7 that F is well-defined, and it is easily shown that K is a well-defined group homomorphism.

Observe that F is continuous. Indeed, assume that $\mathcal{H} \xrightarrow{\sigma} \langle((x,g))\rangle$; then there exist $(x_1,g_1) \sim (x,g)$, $\mathcal{F} \xrightarrow{q} x_1$, $\mathcal{G} \xrightarrow{p} g_1$ such that $\mathcal{H} \geq \varphi^{\to}(\mathcal{F} \times \mathcal{G})$. Hence $F^{\to}\mathcal{H} \geq F^{\to}(\varphi^{\to}(\mathcal{F} \times \mathcal{G})) = \mu^{\to}(f^{\to}\mathcal{F} \times (k^{\to}\mathcal{G})^{-1}) \xrightarrow{q^Y} \mu(f(x_1),(k(g_1))^{-1}) = F(\langle(x_1,g_1)\rangle) = F(\langle(x,g)\rangle)$, and thus F is continuous. It easily follows that K is continuous and the diagram commutes. It remains to show that $F \circ \bar{\lambda}_B = \mu \circ (F \times K)$. Note that $(\mu \circ (F \times K))(\langle(x,g)\rangle, mn^{-1}) = \mu(\mu(f(x),(k(g))^{-1}),k(m)(k(n))^{-1}) = \mu(f(x),k(m)(k(gn))^{-1})$. Since (f,k) is a $\mathcal{C}$-morphism, $f \circ \lambda = \mu \circ (f \times k)$, and therefore $f(\lambda(x,m)) = \mu(f(x),k(m))$.
Hence $(F \circ \bar{\lambda}_B)(\langle(x,g)\rangle, mn^{-1}) = F(\langle(\lambda(x,m),gn)\rangle) = \mu(f(\lambda(x,m)),(k(gn))^{-1}) = \mu(\mu(f(x),k(m)),(k(gn))^{-1}) = \mu(f(x),k(m)(k(gn))^{-1})$, and thus $F \circ \bar{\lambda}_B = \mu \circ (F \times K)$. Therefore, (F,K) is a $\mathcal{C}$-morphism. ∎

Corollary 3.9. *Suppose that the hypotheses of Theorem 3.8 are satisfied except that $(T,.)$ is cancellative, commutative, and $(T,.,p^T) \in |CSG|$. Then there exists a $\mathcal{C}$-morphism (F,K) such that the diagram below is commutative:*

$$\begin{tikzcd}
((X,q),(S,.,p),\lambda) \arrow[r, "(\beta_X, j_S)"] \arrow[d, "(f,k)"] & ((B(X,S),\sigma_X),(\bar{S},.,\bar{p}),\bar{\lambda_B}) \arrow[d, "(F,K)"] \\
((Y,q^Y),(T,.,p^T),\mu) \arrow[r, "(\beta_Y, j_T)"] & ((B(Y,T),\sigma_Y),(\bar{T},.,\bar{p^T}),\bar{\mu_B})
\end{tikzcd}$$

4. Examples and Special Cases

First, some examples of generalized quotient spaces are presented, and then the work is concluded with some results pertaining to the case whenever the semigroup is generated by a single element. In the following examples, N denotes a natural number, and let $\mathbb{N}$ designate the set of all natural numbers.

Example 4.1. Let $X = \mathcal{C}^\infty(\mathbb{R}^N)$ be the space of all continuous complex-valued functions defined on $\mathbb{R}^N$, and equipped with the topology of uniform convergence on compact sets. Denote

$$S = \{f \in \mathcal{C}^\infty(\mathbb{R}^N) \,:\, f \text{ has compact support}, f \geq 0, \text{ and } \int f = 1\}.$$

Then S is a semigroup with respect to the convolution

$$(f * g)(u) = \int_{\mathbb{R}^N} f(v)g(u-v)dv.$$

Define the action λ of S on the space X by convolution, that is,

$$\lambda(x, f) = x * f.$$

Note that λ is not injective. Then, for $x, y \in X$,

$$x \simeq y \text{ iff } x * f = y * f \text{ for some } f \in S.$$

Under this equivalence relation some functions will be identified with 0 and, moreover, $B(X_*, S)$ is a space of generalized functions. Clearly, not all continuous functions can be identified with elements of $B(X_*, S)$, and thus $B(X_*, S)$ is a proper extension of X_*. Every element of $B(X_*, S)$ has derivatives of all orders defined by

$$D^\alpha \langle([x], g)\rangle = \langle([D^\alpha x], g)\rangle.$$

It is not difficult to check that this operation is well-defined and continuous.

Consider the situation when S is generated by a single map. Let X be a nonempty set, and let $g : X \to X$ be a non-injective map. Then $S = \{g^n : n = 0, 1, 2, ...\}$ is a commutative semigroup with respect to composition, and

$$x \simeq y \text{ iff there exists an } n \text{ such that } \lambda(x, g^n) = \lambda(y, g^n).$$

If g is a surjection, then $\lambda_*(., g^n) : X_* \to X_*$ is a surjection for every $n \in \mathbb{N}$. Consequently, $B(X_*, S) = X_*$ (identifiable); hence, in order to produce a proper extension of X_*, it is necessary to start with a g that is not surjective.

Example 4.2. Let X be a normed space, $T : X \to X$ a norm preserving linear operator, and assume that S is the semigroup generated by T. Since T is injective, $X_* = X$. For sake of brevity, denote $\langle (x, T^n) \rangle$ by $\frac{x}{T^n} \in B(X,S)$, and define

$$\left\| \frac{x}{T^n} \right\| = \|x\| .$$

If $\frac{x}{T^n} = \frac{y}{T^m}$, then $T^m x = T^n y$, and hence

$$\left\| \frac{x}{T^n} \right\| = \|x\| = \|T^m x\| = \|T^n y\| = \|y\| = \left\| \frac{y}{T^m} \right\| .$$

Therefore, $\|\cdot\|$ is well-defined in $B(X,S)$. One can also verify that $\|\cdot\|$ is a norm in $B(X,S)$; in particular,

$$\begin{aligned}\left\| \frac{x}{T^n} + \frac{y}{T^m} \right\| &= \left\| \frac{T^m x + T^n y}{T^{m+n}} \right\| = \|T^m x + T^n y\| \\ &\le \|T^m x\| + \|T^n y\| = \|x\| + \|y\| \\ &= \left\| \frac{x}{T^n} \right\| + \left\| \frac{y}{T^m} \right\| .\end{aligned}$$

Consequently, this extension of a normed space $(X, \|\cdot\|)$ produces a normed space $(B(X,S), \|\cdot\|)$. Moreover, the map defined by

$$T \frac{x}{T^n} = \frac{Tx}{T^n}$$

is a norm preserving bijection on $B(X,S)$.

In particular, given a vector space X, let $\{e_1, e_2, ...\}$ be a Hamel basis, and define a norm by

$$\left\| \sum_{j=1}^{k} \alpha_j e_{p_j} \right\| = \sum_{j=1}^{k} |\alpha_j|.$$

Then

$$T(\sum_{j=1}^{k} \alpha_j e_{p_j}) = \sum_{j=1}^{k} \alpha_j e_{p_{j+1}}$$

and

$$U(\sum_{j=1}^{k} \alpha_j e_{p_j}) = \sum_{j=1}^{k} \alpha_j e_{2p_j}$$

are norm preserving operators on X. In both cases, $B(X,S)$ is a proper extension of X.

Example 4.3. **(i)** Consider an arbitrary nonempty set Z and denote $X = Z^{\mathbb{N}}$. Define $g : X \to X$ by $g((x_n)) = (x_{n+1})$, and let $S = \{g^n : n = 0, 1, 2, ...\}$. Then

$$(x_n) \simeq (y_n) \text{ iff there exists an } n_0 \text{ such that } x_n = y_n \text{ for all } n \geq n_0.$$

That is, two sequences are equivalent whenever they are eventually equal. Note that g is a surjection, and thus $B(X_*, S) = X_*$.

(ii) Let E be a vector space, $X = E^{\mathbb{N}}$, and denote

$$S = \{(\alpha_n) \in \{0, 1\}^{\mathbb{N}} : \text{Card}\{n \in \mathbb{N} : \alpha_n \neq 0\} < \mathcal{N}_0\}.$$

Then S is a commutative semigroup with respect to termwise multiplication, and define

$$\lambda((x_n), (\alpha_n)) = (\alpha_n x_n).$$

Observe that $\lambda(., (\alpha_n))$ is neither injective nor surjective, and $B(X_*, S) = X_*$.

Recall that a continuous surjection $f : (X, q) \to (Y, p)$ in CONV is called a **quotient map** if whenever $\mathcal{F} \xrightarrow{p} y$, then there exists $\mathcal{G} \xrightarrow{q} x \in f^{-1}(y)$ such that $f^{\to}\mathcal{G} = \mathcal{F}$. Moreover, a continuous surjection f is said to be a **proper map** provided that for each ultrafilter $\mathcal{F}$ on X, $f^{\to}\mathcal{F} \xrightarrow{p} y$ implies that $\mathcal{F} \xrightarrow{q} x$, for some $x \in f^{-1}(y)$. Given that $S = \{g^n : n = 0, 1, 2, ...\}$, two special cases are considered below by requiring that g be either a quotient or proper map.

Theorem 4.4. *Assume that* $(X, q) \in |CONV|$, $g : (X, q) \to (X, q)$ *is a quotient map in CONV,* $S = \{g^n : n = 0, 1, 2, ...\}$, *and define the action by* $\lambda(x, g^n) = g^n(x)$, $n = 0, 1, 2, ...$ *Then*

(a) $\lambda_*(., g^n) : (X_*, q_*) \to (X_*, q_*)$ *is a homeomorphism, for each fixed* $n = 0, 1, 2, ...$

(b) $\beta_* : (X_*, q_*) \to (B(X_*, S), \sigma_*)$ *is a homeomorphism whenever* S *is equipped with the discrete topology, where* $\beta_*([x]) = \langle([x], e)\rangle$.

Proof. (a): Denote $\lambda_*([x], g) = [\lambda(x, g)]$ by $g_*([x]) = [g(x)]$. Consider the commutative diagram below:

$$\begin{tikzcd} (X, q) \arrow[r, "g"] \arrow[d, "\xi"] & (X, q) \arrow[d, "\xi"] \\ (X_*, q_*) \arrow[r, "g_*"] & (X_*, q_*) \end{tikzcd}$$

Since ξ is a quotient map in CONV, g_* is continuous iff $g_* \circ \xi$ is continuous. However, $g_* \circ \xi = \xi \circ g$ is continuous, and thus g_* is continuous. Moreover, g_* is injective. Indeed, if $[g(x)] = g_*([x]) = g_*([z]) = [g(z)]$, then $g(x) \simeq g(z)$ and thus $g^{n+1}(x) = g^n(g(x)) = g^n(g(z)) = g^{n+1}(z)$ for some $n \geq 0$, and thus

$[x] = [z]$. Hence g_* is injective. Since g is onto, it follows that g_* is a continuous bijection. Next, it is shown that g_*^{-1} is continuous.

Again, since ξ is a quotient map, g_*^{-1} is continuous iff $g_*^{-1} \circ \xi$ is continuous. Since the diagram commutes, $g_* \circ \xi = \xi \circ g$, and thus $\xi = g_*^{-1} \circ \xi \circ g$. Assume that $\mathcal{F} \xrightarrow{q} x$; it must be shown that $(g_*^{-1} \circ \xi)^{\rightarrow}\mathcal{F} \xrightarrow{q_*} g_*^{-1}([x])$. Since g is a quotient map in CONV, there exists $\mathcal{H} \xrightarrow{q} z$ such that $g^{\rightarrow}\mathcal{H} = \mathcal{F}$ and $g(z) = x$. Then $\xi^{\rightarrow}\mathcal{H} \xrightarrow{q_*} [z]$, and thus $(g_*^{-1} \circ \xi)^{\rightarrow}\mathcal{F} = (g_*^{-1} \circ \xi \circ g)^{\rightarrow}\mathcal{H} = \xi^{\rightarrow}\mathcal{H} \xrightarrow{q_*} [z]$. Since $g(z) = x$, $g_*([z]) = [x]$, and thus $(g_*^{-1} \circ \xi)(x) = g_*^{-1}([x]) = [z]$. Hence $(g_* \circ \xi)^{\rightarrow}\mathcal{F} \rightarrow (g_*^{-1} \circ \xi)(x))$, and thus $g_*^{-1} \circ \xi$ is continous. Therefore g_*^{-1} is continuous, and hence g_* is a homeomorphism. Further, since the finite composition of quotient maps is again a quotient in CONV, g^n is also a quotient map, and it follows that $(g^n)_*$ is also a homeomorphism, $n = 0, 1, 2, ...$

(b): Note that if $\langle([x], e)\rangle = \beta_*([x]) = \beta_*([z]) = \langle([z], e)\rangle$, then $[x] = \lambda_*([x], e) = \lambda_*([z], e) = [z]$, and thus β_* is injective. Given $\langle([z], g^n)\rangle \in B(X_*, S)$, since g^n is onto, $g^n(x) = z$ for some $x \in X$. Then $\beta_*([x]) = \langle([x], e)\rangle = \langle([z], g^n)\rangle$, and thus β_* is a bijection. Observe that $\beta_* = \varphi_* \circ \sigma$, where $\sigma([x]) = ([x], e)$, and hence β_* is a continuous bijection.

It remains to show that β_*^{-1} is continuous. Assume that $\mathcal{F} \in \mathfrak{F}(X_*)$ such that $\beta_*^{\rightarrow}\mathcal{F} \xrightarrow{\sigma_*} \beta_*([x])$; it must be shown that $\mathcal{F} \xrightarrow{q_*} [x]$. Since $\varphi_* : (X_* \times S, r_*) \rightarrow (B(X_*, S), \sigma_*)$ is a quotient map in CONV and $\beta_*^{\rightarrow}\mathcal{F} \xrightarrow{\sigma_*} \beta([x]) = \langle([x], e)\rangle$, there exists $\mathcal{K} \xrightarrow{r_*} ([z], g^n)$ such that $\varphi_*^{\rightarrow}\mathcal{K} = \beta_*^{\rightarrow}\mathcal{F}$, where $r_* = q_* \times p$ and $\langle([z], g^n)\rangle = \langle([x], e)\rangle$. In particular, $\lambda_*([z], e) = \lambda_*([x], g^n)$, or $[z] = [g^n(x)]$. Since S has the discrete topology, there exists $\mathcal{H} \xrightarrow{q_*} [z]$ such that $\mathcal{H} \times \dot{g^n} \leq \mathcal{K}$, and thus $\varphi_*^{\rightarrow}(\mathcal{H} \times \dot{g^n}) \leq \varphi_*^{\rightarrow}\mathcal{K} = \beta_*^{\rightarrow}\mathcal{F}$. Fix $H \in \mathcal{H}$; then there exists $F \in \mathcal{F}$ such that $\beta_*(F) \subseteq \varphi_*(H \times \{g^n\})$. If $[s] \in F$, then $\beta_*([s]) = \langle([s], e)\rangle = \langle([t], g^n)\rangle$ for some $[t] \in H$. In particular, $[g^n(s)] = \lambda_*([s], g^n) = \lambda_*([t], e) = [t]$, and thus $(g^n)_*(F) \subseteq H$. Therefore $(g^n)_*^{\rightarrow}\mathcal{F} \geq \mathcal{H} \xrightarrow{q_*} [z]$, and since $(g^n)_*$ is a homeomorphism, $\mathcal{F} \xrightarrow{q_*} (g^n)_*^{-1}([z])$. However, $(g^n)_*([x]) = [g^n(x)] = [z]$, and thus $(g^n)_*^{-1}([z]) = [x]$. Hence $\mathcal{F} \xrightarrow{q_*} [x]$, and it follows that β_* is a homeomorphism. □

The final result is the analogue of Theorem 4.4 whenever g is a proper map. Object $(X, q) \in |\text{CONV}|$ is said to be a **Choquet space** provided that $\mathcal{F} \xrightarrow{q} x$ whenever each ultrafilter $\mathcal{G} \geq \mathcal{F}$, $\mathcal{G} \xrightarrow{q} x$; that is, $q-$convergence is determined by $q-$convergence of the ultrafilters on X. Given any $(X, q) \in |\text{CONV}|$, define $\hat{q}$ by : $\mathcal{F} \xrightarrow{\hat{q}} x$ iff for each ultrafilter $\mathcal{G} \geq \mathcal{F}$, $\mathcal{G} \xrightarrow{q} x$. Note that $\hat{q}$ and q agree on ultrafilter convergence. Moreover, it follows that $\hat{q}$ is the finest Choquet structure on X which is coarser than q. Suppose that $f : (X, q) \rightarrow (Y, p)$ is a map between two convergence spaces such that $f^{\rightarrow}\mathcal{F} \xrightarrow{p} f(x)$ whenever $\mathcal{F}$ is an ultrafilter on X for which $\mathcal{F} \xrightarrow{q} x$. It is easily shown that $f : (X, \hat{q}) \rightarrow (Y, \hat{p})$ is continuous. Verification of the next result makes use of the preceding fact along with an argument similar to that given in Theorem 4.4. The proof is omitted.

Theorem 4.5. *Suppose that $(X, q) \in |CONV|$, $g : (X, q) \to (X, q)$ is a proper map, $S = \{g^n : n = 0, 1, 2, ...\}$, and action $\lambda(x, g^n) = g^n(x)$, $n \geq 0$. Then*

(a) $\lambda_*(., g^n) : (X_*, \hat{q_*}) \to (X_*, \hat{q_*})$ *is a homeomorphism, for each fixed* $n \geq 0$

(b) $\beta_* : (X_*, \hat{q_*}) \to (B(X_*, S), \gamma_*)$ *is a homeomorphism whenever S has the discrete topology p, and γ_* denotes the quotient structure in CONV determined from* $\varphi_* : (X_*, \hat{q_*}) \times (S, p) \to (X_* \times S)/\approx$.

Employing Theorem 4.5, along with the fact that a continuous surjection $g : (X, q) \to (X, q)$ is a proper map whenever $(X, q) \in |\mathrm{CONV}|$ is compact and Hausdorff, gives the concluding result.

Corollary 4.6. *Assume that $(X, q) \in |CONV|$ is compact, Hausdorff , and $g : (X, q) \to (X, q)$ is a continous surjection. Suppose that $S = \{g^n : n = 0, 1, 2, ...\}$ has the discrete topology. Then $\beta_* : (X_*, \hat{q_*}) \to (B(X_*, S), \gamma_*)$ is a homeomorphism.*

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