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Dynamics of multivalued linear operators

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Abstract: We introduce several notions of linear dynamics for multivalued linear operators (MLO's) between separable Fréchet spaces, such as hypercyclicity, topological transitivity, topologically mixing property, and Devaney chaos. We also consider the case of disjointness, in which any of these properties are simultaneously satisfied by several operators. We revisit some sufficient well-known computable criteria for determining those properties. The analysis of the dynamics of extensions of linear operators to MLO's is also considered.

Keywords: Hypercyclicity, Topological transitivity, Topologically mixing property, Devaney chaos, Multivalued linear operators

MSC: 47A16

1 Introduction and preliminaries

Let X, Y be separable Fréchet spaces and $\{T_i\}_{i \in I}$ a family of mappings from X to Y . An element $x \in X$ is said to be *universal* if every element of Y can be approximated by certain $T_i x$. In other words, if the set $\{T_i x : i \in I\}$ is dense in Y . Once universality holds, the set of universal vectors is generic, more precisely, it is a dense G_δ set by an application of Baire Category Theorem.

During the 90's a particular type of universality attracted the interest of many researchers. This was the case when the family of operators was given by the powers of a single operator T , that is $T_i := T^i$, $i \in \mathbb{N}_0$. This particular case of universality is known as *hypercyclicity*, and such a vector x is known as a *hypercyclic vector*. In [1], examples and results in universality were collected, with special attention to the first remarkable results in hypercyclicity. This compendium helped to attract the interest of many researchers to this new area. The recent monographs [2, 3] collected the advances in the study of the dynamics of linear continuous operators in Banach and Fréchet spaces during last years.

One of the core results of linear dynamics is the Hypercyclicity Criterion (HC) [4, 5]. It states that a linear operator T is hypercyclic if it has a right inverse S and a dense subset X_0 , such that the orbits by T and by S of the elements of X_0 tend to 0. Some generalizations have been later introduced. In the situation where T is invertible, the operators T and T^{-1} are hypercyclic at the same time. The hypothesis of the (HC) can be significantly relaxed, see [3]. Additionally, there are equivalent formulations, such as the blow-up/collapse criterion, that can be formulated without explicit reference to the aforementioned right inverses, see [1, 6].

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Multivalued functions often arise as inverses of functions that are not injective. The study of the dynamics of functions that take sets as image of certain elements has been considered by several authors. Chaotic properties for set valued maps between spaces of compact sets endowed with the Hausdorff distance were initially considered by Román-Flores [7], Peris [8] and Banks [9]. Chaos on hyperspaces was considered in [10]. However the study of linear dynamics for multivalued linear operators has not been considered yet.

In this work, we consider hypercyclicity, topological transitivity, topologically mixing property, and chaos (in the sense of Devaney) for MLO's, as well as their disjoint versions. The paper is organized as follows: In Section 2, we recall the most important definitions from the theory of multivalued linear operators. Section 3 is completely devoted to survey the formulation of those linear dynamical properties for multivalued linear operators. Some well-known sufficient criteria are adapted to these settings. The scope of the results is determined by means of some illustrative examples. Finally, the dynamics of the extensions of linear operators to MLO's is considered in Section 4.

2 Multivalued linear operators

We present a brief overview of the necessary definitions and properties of multivalued linear operators. For more details, we refer the reader to the monographs of Cross [11], Favini and Yagi [12], Álvarez, Cross and Wilcox [13, 14], and by the third named author [15].

We use the standard notation throughout the paper. Let X be a separable Fréchet space, i.e. a complete, metrizable, locally convex space, over the field \mathbb{K} . The abbreviation $\text{cs}(X)$ stands for the fundamental system of continuous seminorms defining the topology of X . We also define the ball of center $x \in X$ and radius $\varepsilon > 0$ with respect to $p \in \text{cs}(X)$ as $B_p(x, \varepsilon) := \{y \in X : p(x - y) < \varepsilon\}$. If X is a Banach space, then the norm of an element $x \in X$ is denoted by $\|x\|$.

Assuming that Y is another separable Fréchet space, then we denote by $L(X, Y)$ the space consisting of all continuous linear operators from X to Y , and if $X = Y$ we simply denote it as $L(X)$. Let \mathcal{B} be the family of bounded subsets of X . Given $p \in \text{cs}(Y)$, $B \in \mathcal{B}$, we define $p_B(T) := \sup_{x \in B} p(Tx)$ for every $T \in L(X, Y)$. Then the set $\{p_B : B \in \mathcal{B}, p \in \text{cs}(Y)\}$ defines a fundamental system of seminorms on $L(X, Y)$ that induces the Hausdorff locally convex topology of the convergence over the bounded sets of X .

We now recall the definition of a multivalued linear operator. We denote by $P(Y)$ the family of subsets of Y .

Definition 2.1. Let X and Y be Fréchet spaces. A multivalued map (multimap) $\mathcal{A} : X \rightarrow P(Y)$ such that for every open set $V \in P(Y)$ satisfies that $\mathcal{A}^{-1}(V)$ is said to be a multivalued linear operator (MLO) if the following holds:

- (i) The domain of \mathcal{A} , $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$, is a linear subspace of X ;
- (ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x + y)$, for every $x, y \in D(\mathcal{A})$ and $\lambda \mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, for every $\lambda \in \mathbb{K}$, $x \in D(\mathcal{A})$.

We denote the set of these operators as $ML(X, Y)$. If $X = Y$, then we say that \mathcal{A} is an MLO in X , and we denote this set by $ML(X)$.

An almost immediate consequence of the definition is that $\mathcal{A}x + \mathcal{A}y = \mathcal{A}(x + y)$ and $\lambda \mathcal{A}x = \mathcal{A}(\lambda x)$ for all $x, y \in D(\mathcal{A})$, $\lambda \neq 0$. Furthermore, $\lambda \mathcal{A}x + \eta \mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$, for every $x, y \in D(\mathcal{A})$ and $\lambda, \eta \in \mathbb{K}$ with $|\lambda| + |\eta| \neq 0$.

If \mathcal{A} is an MLO, then the image of 0 by \mathcal{A} , $\mathcal{A}0$, is a linear manifold in Y and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. We denote the range of \mathcal{A} as $R(\mathcal{A}) := \bigcup \{\mathcal{A}x : x \in D(\mathcal{A})\}$.

Definition 2.2. Given $\mathcal{A}, \mathcal{B} \in ML(X, Y)$, we define

- the kernel of \mathcal{A} as $N(\mathcal{A}) := \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$,
- the sum $\mathcal{A} + \mathcal{B}$ as $(\mathcal{A} + \mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A} + \mathcal{B})$ with $D(\mathcal{A} + \mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$,
- the scalar multiplication $z\mathcal{A}$, with $z \in \mathbb{K}$, as $(z\mathcal{A})(x) := z\mathcal{A}x$, $x \in D(\mathcal{A})$ with $D(z\mathcal{A}) := D(\mathcal{A})$.

It can be simply verified that $\mathcal{A} + \mathcal{B}, z\mathcal{A} \in ML(X, Y)$, too. Moreover, $(\omega z)\mathcal{A} = \omega(z\mathcal{A}) = z(\omega\mathcal{A})$, for every $z, \omega \in \mathbb{K}$.

– the direct sum of k copies of \mathcal{A} with itself, $k \in \mathbb{N}$, on $X^k = \underbrace{X \oplus \dots \oplus X}_k$ as

$$\begin{aligned} \underbrace{\mathcal{A} \oplus \dots \oplus \mathcal{A}}_k(x_1, x_2, \dots, x_k) &:= \{(y_1, y_2, \dots, y_k) : y_i \in \mathcal{A}x_i \text{ for all } i = 1, 2, \dots, k\} \\ &= \mathcal{A}x_1 \times \dots \times \mathcal{A}x_k \in P(Y^k). \end{aligned}$$

with $D(\underbrace{\mathcal{A} \oplus \dots \oplus \mathcal{A}}_k) := \underbrace{D(\mathcal{A}) \oplus \dots \oplus D(\mathcal{A})}_k$.

– the adjoint $\mathcal{A}^* : Y^* \rightarrow P(X^*)$ of \mathcal{A} by its graph

$$\mathcal{A}^* := \{(y^*, x^*) \in Y^* \times X^* : \langle y^*, y \rangle = \langle x^*, x \rangle \text{ for all pairs } (x, y) \in \mathcal{A}\}.$$

It is simply verified that $\mathcal{A}^* \in ML(Y^*, X^*)$, and that $\langle y^*, y \rangle = 0$ whenever $y^* \in D(\mathcal{A}^*)$ and $y \in \mathcal{A}0$.

If $X = Y$, we also define the product $\mathcal{B}\mathcal{A}x := \bigcup\{\mathcal{B}y : y \in D(\mathcal{B}) \cap \mathcal{A}x\}$ with $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$. Then, the integer powers of an MLO $\mathcal{A} \in ML(X)$ are recursively defined as follows: $\mathcal{A}^0 := I$, and if \mathcal{A}^{n-1} is defined, we set

$$\begin{aligned} \mathcal{A}^n x &:= (\mathcal{A}\mathcal{A}^{n-1})x = \bigcup\{\mathcal{A}y : y \in D(\mathcal{A}) \cap \mathcal{A}^{n-1}x\}, \text{ with} \\ D(\mathcal{A}^n) &:= \{x \in D(\mathcal{A}^{n-1}) : D(\mathcal{A}) \cap \mathcal{A}^{n-1}x \neq \emptyset\}. \end{aligned}$$

We define the inverse \mathcal{A}^{-1} of $\mathcal{A} \in ML(X)$ as

$$\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$$

on $D(\mathcal{A}^{-1}) := R(\mathcal{A})$. We can prove inductively that $(\mathcal{A}^n)^{-1} = (\mathcal{A}^{n-1})^{-1}\mathcal{A}^{-1} = (\mathcal{A}^{-1})^n =: \mathcal{A}^{-n}$, $n \in \mathbb{N}$ and $D((\lambda - \mathcal{A})^n) = D(\mathcal{A}^n)$, for every $n \in \mathbb{N}_0, \lambda \in \mathbb{K}$. Moreover, if \mathcal{A} is single-valued, then the above definitions are consistent with the usual definition of powers of \mathcal{A} . According to this, we set $D_\infty(\mathcal{A}) := \bigcap_{n \in \mathbb{N}} D(\mathcal{A}^n)$.

For every $\mathcal{A} \in ML(X, Y)$ we define $\check{\mathcal{A}} := \{(x, y) : x \in D(\mathcal{A}), y \in \mathcal{A}x\}$. Then \mathcal{A} is an MLO if, and only if, $\check{\mathcal{A}}$ is a linear relation in $X \times Y$, i.e., $\check{\mathcal{A}}$ is a linear subspace of $X \times Y$.

Suppose that \check{X} is a linear subspace of X and $\mathcal{A} \in ML(X, Y)$. Then we define the restriction of $\mathcal{A} \in ML(X, Y)$ to the subspace \check{X} , denoted by $\mathcal{A}|_{\check{X}}$ for short, as $\mathcal{A}|_{\check{X}}x := \mathcal{A}x, x \in D(\mathcal{A}|_{\check{X}})$ with $D(\mathcal{A}|_{\check{X}}) := D(\mathcal{A}) \cap \check{X}$. Clearly, $\mathcal{A}|_{\check{X}} \in ML(\check{X}, Y)$, too. It is well known that $\mathcal{A} \in ML(X, Y)$ is injective (resp., single-valued) if $\mathcal{A}^{-1}\mathcal{A} = I_{|D(\mathcal{A})}^X$ (resp., $\mathcal{A}\mathcal{A}^{-1} = I_{|R(\mathcal{A})}^Y$).

If $\mathcal{A} \in ML(X, Y)$ and $\mathcal{B} \in ML(X, Y)$, then we write $\mathcal{A} \subseteq \mathcal{B}$ if $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$. If $S \in L(X, Y)$ has domain $D(S) = D(\mathcal{A})$ and $S \subseteq \mathcal{A}$, then S is called a *section* of \mathcal{A} , and then we have $\mathcal{A}x = Sx + \mathcal{A}0, x \in D(\mathcal{A})$ and $R(\mathcal{A}) = R(S) + \mathcal{A}0$.

An element $\lambda \in \mathbb{K}$ is an *eigenvalue* of \mathcal{A} if there exists some vector $x \in X \setminus \{0\}$ such that $\lambda x \in \mathcal{A}x$; we call x an *eigenvector* of \mathcal{A} corresponding to the eigenvalue λ . Observe that, in the purely multivalued case, a vector $x \in X \setminus \{0\}$ can be an eigenvector of operator \mathcal{A} corresponding to different values of scalars λ . The point spectrum of \mathcal{A} , $\sigma_p(\mathcal{A})$ for short, is defined as the set of all eigenvalues of \mathcal{A} .

In our work, the multivalued linear operator $B^{-1}A = \{(x, y) \in X \times X : Ax = By\}$ has an important role, where A and B are single-valued linear operators acting between the spaces X and Y , and B is not necessarily injective.

3 Dynamical properties of MLO's

We recall some of the most significant notions of linear dynamical properties, further details can be found in [2, 3].

Definition 3.1. Let $\mathcal{A} \in ML(X)$ and let $x \in X$. Then we say that:

- i) x is a hypercyclic vector of \mathcal{A} if $x \in D_\infty(\mathcal{A})$ and for each $n \in \mathbb{N}_0$ and each $j \in \{1, \dots, N\}$ there exists an element $y_{j,n} \in \mathcal{A}^n x$ such that the set $\{y_n : n \in \mathbb{N}\}$ is dense in X . In this case, we say that \mathcal{A} is hypercyclic.
- ii) x is a periodic point of \mathcal{A} if $x \in D_\infty(\mathcal{A})$ and there exists $n \in \mathbb{N}$ such that $x \in \mathcal{A}^n x$;
- iii) \mathcal{A} is topologically transitive if for every pair of non-empty open sets $U, V \subset X$ there exists $n \in \mathbb{N}$ such that $U \cap \mathcal{A}^{-n}(V) \neq \emptyset$.
- iv) \mathcal{A} is topologically mixing if for every pair of non-empty open sets $U, V \subset X$ there exists $n_0 \in \mathbb{N}$ such that $U \cap \mathcal{A}^{-n}(V) \neq \emptyset$ holds for $n \geq n_0$.
- v) \mathcal{A} is chaotic (in the sense of Devaney) if \mathcal{A} is topologically transitive and the set consisting on all periodic points of \mathcal{A} is dense in X .

With a similar argument as in the case of linear operators, by Baire’s Category Theorem, the notion of hypercyclicity is equivalent to the one of transitivity, as long as the MLO’s were defined on the whole space X [16, Th. 3.1.6]. Let $\mathcal{A} \in ML(X)$ and let $(O_n)_{n \in \mathbb{N}}$ be an open base of the topology of X . Then the set consisting of hypercyclic vectors of \mathcal{A} is denoted shortly by $HC(\mathcal{A})$ and it can be computed by

$$HC(\mathcal{A}) = \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \mathcal{A}^{-k}(O_n). \tag{1}$$

Remark 3.2. *The periodic points of \mathcal{A} form a linear submanifold of X : Suppose that x is a periodic point of \mathcal{A} . Then $x \in D_\infty(\mathcal{A})$ and there exists an integer $n \in \mathbb{N}$ such that $x \in \mathcal{A}^n x$. This implies the existence of $y_j \in X$ ($1 \leq j \leq n - 1$) such that $(x, y_1) \in \mathcal{A}, (y_1, y_2) \in \mathcal{A}, \dots, (y_{n-1}, x) \in \mathcal{A}$. Repeating this sequence, we easily get that $x \in \mathcal{A}^{kn} x$ for all $k \in \mathbb{N}$, and the statement holds.*

Remark 3.3. *On the one hand, one can replace the sequence of powers of $\mathcal{A} \in ML(X)$ by a sequence of MLO’s $(\mathcal{A}_n)_{n \in \mathbb{N}}$, and refer to the notion of universality instead of hypercyclicity, c.f. [1, Def. 2]).*

On the other hand, in Definition 3.1, our assumptions on X , as well as on $(\mathcal{A}^n)_{n \in \mathbb{N}}, \mathcal{A}, (\mathcal{A}_j^n)_{n \in \mathbb{N}}$ and \mathcal{A}_j can be substantially relaxed ($1 \leq j \leq N$). It suffices to suppose that X and Y are topological spaces as well as that \mathcal{A}^n are binary relations from X to Y . This notion will be considered in our follow-up research study [17].

Disjointness in linear dynamics was independently introduced by Bernal-González [18] and by Bès and Peris [19], see also [20]. For the sake of completeness, we explicitly state them:

Definition 3.4. *Let $N \geq 2, 1 \leq j \leq N$, and $\mathcal{A}_1, \dots, \mathcal{A}_N \in ML(X)$. Let $x \in X$. Then we say that:*

- i) x is a disjoint hypercyclic vector of \mathcal{A} if $x \in D_\infty(\mathcal{A}_j)$ and for each $n \in \mathbb{N}_0$ and each $j \in \{1, \dots, N\}$ there exists an element $y_{j,n} \in \mathcal{A}_j^n x$ such that the set $\{(y_{1,n}, \dots, y_{N,n}) : n \in \mathbb{N}\}$ is dense in $X \oplus \dots \oplus X$. In this case, we say that $\mathcal{A}_1, \dots, \mathcal{A}_N$ are disjoint hypercyclic, or simply d-hypercyclic.
- ii) $\mathcal{A}_1, \dots, \mathcal{A}_N$ are disjoint topologically transitive, or simply d-topologically transitive if for any choice of non-empty open sets $U, V_1, \dots, V_n \subset X$ there exists $n \in \mathbb{N}$ such that $U \cap \mathcal{A}^{-n}(V) \cap \dots \cap \mathcal{A}_N^{-n}(V) \neq \emptyset$.
- iii) $\mathcal{A}_1, \dots, \mathcal{A}_N$ are disjoint topologically mixing, or simply d-topologically mixing if for any choice of non-empty open sets $U, V_1, \dots, V_n \subset X$ there exists $n_0 \in \mathbb{N}$ such that $U \cap \mathcal{A}^{-n}(V) \cap \dots \cap \mathcal{A}_N^{-n}(V) \neq \emptyset$ for every $n \geq n_0$.
- iv) $\mathcal{A}_1, \dots, \mathcal{A}_N$ are disjoint chaotic, or simply d-chaotic if $\mathcal{A}_1, \dots, \mathcal{A}_N$ are d-hypercyclic and the set of periodic points, denoted by $\mathcal{P}(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_N) := \{(x_1, x_2, \dots, x_N) \in X^N : \exists n \in \mathbb{N} \text{ with } x_j \in \mathcal{A}_j^n x_j, j \in \{1, \dots, N\}\}$, is dense in X^N .

It is clear that any multivalued linear extension of a hypercyclic (chaotic) single-valued linear operator is again hypercyclic (chaotic). It is also worth noting that Definition 3.1 prescribes some cases in which even zero can be a hypercyclic vector: Let $\mathcal{A} := \{0\} \times W$, where W is a dense linear submanifold of Y . Then \mathcal{A} is hypercyclic, zero is the unique hypercyclic vector of \mathcal{A} and there is no single-valued linear restriction of \mathcal{A} that is hypercyclic (in particular, a hypercyclic MLO need not be densely defined and the inverse of a hypercyclic MLO need not be hypercyclic, in contrast to the single-valued linear case, see also [19, Prob. 1]). Furthermore, with this example, we can get that an MLO \mathcal{A} and some arbitrary multiples of it can be d-hypercyclic, in contrast to what is indicated in

[19, p. 299]. Moreover, the non-triviality of the manifold \mathcal{A}_0 is not essentially connected with hypercyclicity of an MLO: Just take $\mathcal{A} := X \times W$, with W a non-dense linear submanifold of X so as to \mathcal{A} cannot be hypercyclic.

It is an elementary fact that the point spectrum of the adjoint of a hypercyclic continuous single-valued linear operator has to be empty [4], see also [2, 3]. The same holds for hypercyclic MLO's:

Theorem 3.5. *If $\mathcal{A} \in ML(X)$ is hypercyclic, then $\sigma_p(\mathcal{A}^*) = \emptyset$.*

Proof. Let x be a hypercyclic vector for \mathcal{A} . For every $n \in \mathbb{N}$, there exists an element $y_n \in \mathcal{A}^n x$ such that $\{y_n : n \in \mathbb{N}\}$ is dense in X . Suppose that there exist $\lambda \in \mathbb{K}$ and $x^* \in X^* \setminus \{0\}$ such that $\lambda x^* \in \mathcal{A}^* x^*$, i.e., that $\langle x^*, y \rangle = \lambda \langle x^*, x \rangle$, whenever $y \in \mathcal{A}x$. It is clear that $\langle x^*, y_n \rangle = \lambda^n \langle x^*, x \rangle$ for all $n \in \mathbb{N}$, so that the assumption $\langle x^*, x \rangle = 0$ implies $\langle x^*, y_n \rangle = 0$ for all $n \in \mathbb{N}$, and therefore, $x^* = 0$. So, $\langle x^*, x \rangle \neq 0$. If $|\lambda| \leq 1$, then $|\langle x^*, y_n \rangle| = |\lambda^n \langle x^*, x \rangle| \leq |\langle x^*, x \rangle|$ for all $n \in \mathbb{N}$. This would imply $|\langle x^*, u \rangle| \leq |\langle x^*, x \rangle|$ for all $u \in X$, which is a contradiction since $|\langle x^*, nu \rangle|$ diverges to ∞ for any $u \in X$ such that $\langle x^*, u \rangle \neq 0$. If $|\lambda| > 1$, then $|\langle x^*, y_n \rangle| = |\lambda^n \langle x^*, x \rangle| \geq |\langle x^*, x \rangle|$ for all $n \in \mathbb{N}$; this would imply $|\langle x^*, u \rangle| \geq |\langle x^*, x \rangle|$ for all $u \in X$, which is a contradiction since $|\langle x^*, u/n \rangle|$ converges to 0 for all $u \in X$, and $\langle x^*, x \rangle \neq 0$. \square

The Hypercyclicity Criterion, initially stated by Kitai [4] and by Gethner and Shapiro [5], gives some sufficient conditions in order to determine that an operator is hypercyclic. Those conditions were proved to be equivalent to be weakly mixing [21, 22]. Since then, several equivalent formulations have been given. We will state it in terms of the collapse / blow-up conditions [21, Def. 1.2] and [1, Th. 3.4] initially stated by Godefroy and Shapiro [6].

It is worth noting that this criterion can be formulated, in a certain way, for MLO's and that we do not need any type of continuity or closedness of operators under consideration for its validity; a straightforward proof of the next result is very similar to that of [23, Th. 9] (continuous version) and therefore it is omitted.

Criterion 3.6 (Blow-up/collapse for MLOs). *Let $\mathcal{A} \in ML(X)$. Suppose that $(m_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of positive integers.*

Let X_0 be the set of those elements $y \in X$ for which there exists a sequence $(y_n)_{n \in \mathbb{N}}$ in X such that $y_n \in \mathcal{A}^{m_n} y$, $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} y_n = 0$.

Let X_∞ consist of those elements $z \in X$ for which there exist a null sequence $(\omega_n)_{n \in \mathbb{N}}$ in X and a sequence $(u_n)_{n \in \mathbb{N}}$ in X such that $u_n \in \mathcal{A}^{m_n} \omega_n$, $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} u_n = z$.

If X_0, X_∞ are dense in X , then \mathcal{A} is hypercyclic.

We will also state a version of this criterion for disjoint hypercyclicity of MLOs; see [19, Prop. 2.6 & Th. 2.7] for single-valued case of linear operators. Its proof follows the same lines.

Proposition 3.7 (d-Blow-up/collapse Criterion for MLO's). *Let $N \in \mathbb{N}$ and let $\mathcal{A}_j \in ML(X)$, $j \in \mathbb{N} \cap [1, N]$ and $(m_n)_{n \in \mathbb{N}}$ a strictly increasing sequence of positive integers.*

Let X_0 be the set of those elements $y \in X$ satisfying that for every $1 \leq j \leq N$ there exists a sequence $(y_{n,j})_{n \in \mathbb{N}}$ in X such that $y_{n,j} \in \mathcal{A}_j^{m_n} y$, $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} y_{n,j} = 0$.

For each $j \in \mathbb{N} \cap [1, N]$, let us consider the set $X_{\infty,j}$, consisting of those elements $z \in X$ for which there exist elements $\omega_{n,i}(z)$ and $u_{n,i,j}(z)$ in X ($n \in \mathbb{N}$, $1 \leq i \leq N$) such that $(\omega_{n,j}(z))_{n \in \mathbb{N}}$ is a null sequence in X , $u_{n,i,j}(z) \in \mathcal{A}_j^{m_n} \omega_{n,i}(z)$, $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} u_{n,i,j} = \delta_{i,j} z$ ($1 \leq i \leq N$) where $\delta_{i,j}$ denotes the Kronecker delta.

If $X_0, X_{\infty,1}, \dots, X_{\infty,N}$, then the operators $\mathcal{A}_1, \dots, \mathcal{A}_N$ are d-hypercyclic.

Observe that the assertions of Proposition 3.6 and Proposition 3.7 can be formulated for sequences of multivalued linear operators (cf. [19, Rem. 2.8]) and for finite direct sums of each operator with $k - 1$ copies of itself, $k, n \in \mathbb{N}$, namely $\left(\underbrace{\mathcal{A} \oplus \dots \oplus \mathcal{A}}_k\right)^n = \underbrace{\mathcal{A}^n \oplus \dots \oplus \mathcal{A}^n}_k$, $k, n \in \mathbb{N}$. In this case, we just have to pass to the subsets $X_0^k, X_{\infty,j}^k$, $1 \leq j \leq N$, as well to the tuples $(z, \dots, z), (y_{n,j}, \dots, y_{n,j}), (w_{n,i}, \dots, w_{n,i}), (u_{n,i,j}, \dots, u_{n,i,j})$, each of which having exactly k components, and the proof follows as indicated.

The following theorem extends the criterion in [24, Th. 2.1], that is a kind of reformulation of Godefroy-Shapiro [6] and Desch-Schappacher-Webb Criterion (continuous version) [25], see also [3, Th. 7.30], for MLO's.

Theorem 3.8. *Let $\Omega \subseteq \mathbb{C}$ be an open connected set intersecting the unit circle $\Omega \cap S_1 \neq \emptyset$. Let $f : \Omega \rightarrow X \setminus \{0\}$ be an analytic mapping such that $\lambda f(\lambda) \in \mathcal{A}f(\lambda)$ for all $\lambda \in \Omega$. Set $\tilde{X} := \overline{\text{span}\{f(\lambda) : \lambda \in \Omega\}}$. Then the operator $\mathcal{A}|_{\tilde{X}}$ is topologically mixing in the space \tilde{X} and the set of periodic points of $\mathcal{A}|_{\tilde{X}}$ is dense in \tilde{X} , so that it is also chaotic.*

Proof. The proof is very similar to that of [23, Th. 5], which is based on an application of the Hahn-Banach Theorem. We will only outline the most relevant details.

Without loss of generality, we may assume that $\tilde{X} = X$. If $\Omega_0 \subseteq \Omega$ admits a cluster point in Ω , then the (weak) analyticity of mapping $\lambda \mapsto f(\lambda)$, $\lambda \in \Omega$ shows that $\Psi(\Omega_0) := \text{span}\{f(\lambda) : \lambda \in \Omega_0\}$ is dense in X .

Further on, it is clear that there exist $\lambda_0 \in \Omega \cap \{z \in \mathbb{C} : |z| = 1\}$ and $\delta > 0$ such that any of the sets $\Omega_{0,+} := \{\lambda \in \Omega : |\lambda - \lambda_0| < \delta, |\lambda| > 1\}$ and $\Omega_{0,-} := \{\lambda \in \Omega : |\lambda - \lambda_0| < \delta, |\lambda| < 1\}$ admits a cluster point in Ω .

Take two open sets $\emptyset \neq U, V \subseteq X$. Then there exists $y, z \in X, \epsilon > 0, p, q \in \text{cs}(X)$ such that $B_p(y, \epsilon) \subseteq U$ and $B_q(z, \epsilon) \subseteq V$. We may assume that $y = \sum_{i=1}^n \beta_i f(\lambda_i) \in \Psi(\Omega_{0,-})$, $z = \sum_{j=1}^m \gamma_j f(\tilde{\lambda}_j) \in \Psi(\Omega_{0,+})$, with $\alpha_i, \beta_j \in \mathbb{C} \setminus \{0\}$, $\lambda_i \in \Omega_{0,-}$ and $\tilde{\lambda}_j \in \Omega_{0,+}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

Set $z_t := \sum_{j=1}^m \frac{\gamma_j}{\lambda_j^t} f(\tilde{\lambda}_j)$ and $x_t := y + z_t, t \geq 0$. Then $\{x_t, y, z_t\} \subseteq D_\infty(\mathcal{A}), t \geq 0$ and it can be easily seen that $z \in \mathcal{A}^n z_n, n \in \mathbb{N}$. Consider $\omega_n := z + \sum_{i=1}^n \beta_i \lambda_i^n f(\lambda_i) \in \mathcal{A}^n x_n$, for every $n \in \mathbb{N}$. There exists $n_0(\epsilon) \in \mathbb{N}$ such that, for every $n \geq n_0(\epsilon), x_n \in B_p(y, \epsilon)$ and $w_n \in B_q(z, \epsilon)$. Therefore, \mathcal{A} is topologically mixing.

Since the set $\Omega \cap \exp(2\pi i \mathbb{Q})$ has a cluster point in Ω , the proof that the set of periodic points of \mathcal{A} is dense in X can be given as in that of [24, Th. 2.1].

Finally, the validity of implication:

$$\langle x^*, f(\lambda) \rangle = 0, \lambda \in \Omega \text{ for some } x^* \in X^* \Rightarrow x^* = 0.$$

yields that $\tilde{X} = X$. □

Remark 3.9. *Suppose that Ω is an open connected subset of $\mathbb{K} = \mathbb{C}$ satisfying $\Omega \cap S_1 \neq \emptyset$, as well as that $N \geq 2$ and $\mathcal{A}_1, \dots, \mathcal{A}_N \in ML(X)$. Let $f : \Omega \rightarrow X \setminus \{0\}$ be an analytic mapping such that $\lambda f(\lambda) \in \mathcal{A}_j f(\lambda)$ for all $\lambda \in \Omega$ and $j \in \mathbb{N}_N$. Set $\tilde{X} := \overline{\text{span}\{f(\lambda) : \lambda \in \Omega\}}$. Then it is very simple to show that, for every $l \in \mathbb{N}$, the set $\mathcal{P}((\mathcal{A}_1|_{\tilde{X}})^l, \dots, (\mathcal{A}_N|_{\tilde{X}})^l)$ is dense in \tilde{X}^N ; cf. also the proof of Theorem 3.8.*

In the following example, we will illustrate Theorem 3.8 with a concrete example that provides the existence of a substantially large class of topologically mixing multivalued linear operators with dense set of periodic points.

Example 3.10. *Suppose that $A \in L(X)$ with closed range satisfying that there exist an open connected subset $\emptyset \neq \Lambda$ of \mathbb{C} and an analytic mapping $g : \Lambda \rightarrow X \setminus \{0\}$ such that $Ag(v) = vg(v), v \in \Lambda$. Let $P(z)$ and $Q(z)$ be non-zero complex polynomials, let $R := \{z \in \mathbb{C} : P(z) = 0\}, \Lambda' := \Lambda \setminus R$, and let $\tilde{X} := \overline{\text{span}\{g(\lambda) : \lambda \in \Lambda\}}$. Suppose that*

$$\frac{Q}{P}(\Lambda') \cap S_1 \neq \emptyset.$$

Then Theorem 3.8 implies that the parts of MLO's $Q(\mathcal{A})P(\mathcal{A})^{-1}$ and $P(\mathcal{A})^{-1}Q(\mathcal{A})$ in \tilde{X} are topologically mixing in the space \tilde{X} . It can be easily checked that the sets of periodic points of these operators are dense in \tilde{X} .

Theorem 3.8 has a disjoint analogue in [18, Th. 4.3] for single-valued operators. The next result is the corresponding reformulation for MLO's, that can be obtained with a similar proof.

Theorem 3.11. *Suppose that $N \in \mathbb{N}$ and $\mathcal{A}_1, \dots, \mathcal{A}_N \in ML(X)$. For each natural number $0 \leq p \leq N$ there exists a total set D_p (that is, the linear span of D_p is dense in X) such that the following conditions hold for every $1 \leq j \leq N$:*

- (i) For every $e \in D_p$ there exists an eigenvalue $\lambda_{j,p}(e)$ of A_j for which $\lambda_{j,p}(e)e \in A_j e$.
 - (ii) For every $e \in D_0$ we have $\lambda_{j,0}(e) \in \text{int}(S_1)$
 - (ii) For every $e \in D_j$ we have $\lambda_{j,j}(e) \in \text{ext}(S_1)$.
 - (iii) For every $1 \leq i \neq j \leq N$ and every $e \in D_i$ we have $|\lambda_{j,i}(e)| < |\lambda_{i,i}(e)|$.
- Then the operators A_1, \dots, A_N are d -topologically mixing.

We illustrate Theorem 3.11 with two interesting examples pointing out that the continuity of operators can be neglected from the formulation of [18, Theorem 4.3], as well as that there exists a great number of Banach function spaces where this extended version of the aforementioned theorem can be applied (cf. also [18, Final questions, 2.]). Other examples involving single-valued or multivalued linear operators can be similarly given, by using the analysis from Example 3.10; cf. also [26], [27, Ex. 3.8 & Ex. 3.10] and [28] for some other unbounded differential operators that we can employ here.

Example 3.12. Let $p > 2$ and let X be a symmetric space of non-compact type of rank one, let P_p be the parabolic domain defined in the proof of [29, Th. 3.1], and let $c_p > 0$ be the apex of

$$P_p = \left\{ \|\rho\|^2 + z^2 : z \in \mathbb{C}, |\Im(z)| \leq \|\rho\| \cdot \left| \frac{2}{p} - 1 \right| \right\} \subseteq \mathbb{C}. \tag{2}$$

It is known that $\text{int}(P_p) \subseteq \sigma_p(\Delta_{X,p}^{\natural})$ [29, 30], where $\Delta_{X,p}^{\natural}$ denotes the corresponding Laplace-Beltrami operator acting on $L_{\natural}^p(X)$, the space of K -invariant functions in $L^p(X)$, see [29, Sec. 2.3], where K is a maximal compact subgroup of the non-compact semi-simple Lie group $\text{Isom}^0(X)$.

Furthermore, there exists an analytic function $g : \text{int}(P_p) \rightarrow L_{\natural}^p(X)$ such that $\Delta_{X,p}^{\natural} g(\lambda) = \lambda g(\lambda)$, $\lambda \in \text{int}(P_p)$ and that the set $\Psi(\Omega) := \{g(\lambda) : \lambda \in \Omega\}$ is total in X for any non-empty open subset $\Omega \subseteq \text{int}(P_p)$.

Take $N \in \mathbb{N}$ with $N \geq 2$ and numbers a_1, \dots, a_N satisfying $-1 - c_p < a_1 < a_2 < \dots < a_N < 1 - c_p$. For every $1 \leq i \leq N$ there exists a point $\lambda_i \in (P_p)$ such that $|\lambda_i + a_i| > \max(1, |\lambda_i + a_j|)$ for all $j \neq i$ with $1 \leq j \leq N$. Now taking Ω_0 as a small ball around c_p , and $D_i = \Psi(\Omega_i)$ with Ω_i a small ball around λ_i , $1 \leq i \leq N$, we can apply Theorem 3.11, and then we conclude that the operators $\Delta_{X,p}^{\natural} + a_1, \Delta_{X,p}^{\natural} + a_2, \dots, \Delta_{X,p}^{\natural} + a_N$ are d -topologically mixing.

Furthermore, by Remark 3.8, the set $\mathcal{P}(\Delta_{X,p}^{\natural} + a_1, \dots, \Delta_{X,p}^{\natural} + a_N)$ is dense in $(L_{\natural}^p(X))^N$. Hence, the operators $\Delta_{X,p}^{\natural} + a_1, \Delta_{X,p}^{\natural} + a_2, \dots, \Delta_{X,p}^{\natural} + a_N$ are d -chaotic.

The next example is inspired by some results from [31–33].

Example 3.13. Let $b > c/2 > 0$, $\Omega := \{\lambda \in \mathbb{C} : \Re \lambda < c - b/2\}$ and let us consider the bounded perturbation of the one-dimensional Ornstein-Uhlenbeck operator $\mathcal{A}_c u := u'' + 2bxu' + cu$ is acting on $L^2(\mathbb{R})$, with domain $D(\mathcal{A}_c) := \{u \in L^2(\mathbb{R}) \cap W_{loc}^{2,2}(\mathbb{R}) : \mathcal{A}_c u \in L^2(\mathbb{R})\}$. Then \mathcal{A}_c generates a strongly continuous semigroup and $\Omega \subseteq \sigma_p(\mathcal{A}_c)$.

For any non-empty open connected subset $\Omega' \subseteq \Omega$, which admits a cluster point in Ω , we have $X = \overline{\text{span}\{g_i(\lambda) : \lambda \in \Omega', i = 1, 2\}}$, where $g_1 : \Omega \rightarrow X$ and $g_2 : \Omega \rightarrow X$ are defined by $g_1(\lambda) := \mathcal{F}^{-1}(e^{-\frac{\xi^2}{2b}} \xi |\xi|^{-(2+\frac{\lambda-c}{b})})(\cdot)$, and $g_2(\lambda) := \mathcal{F}^{-1}(e^{-\frac{\xi^2}{2b}} |\xi|^{-(1+\frac{\lambda-c}{b})})(\cdot)$, $\lambda \in \Omega$ denoting by \mathcal{F}^{-1} the inverse Fourier transform on the real line.

Furthermore, $\mathcal{A}_c g_i(\lambda) = \lambda g_i(\lambda)$, $\lambda \in \Omega$, $i = 1, 2$. Suppose that $P_i(z)$ and $Q_i(z)$ are non-zero complex polynomials ($1 \leq i \leq N$), and Ω' denotes the set obtained by removing all zeroes of polynomials $P_i(z)$ from Ω . Let $\Lambda \subseteq \mathbb{C}$ be a non-empty open connected subset intersecting $S_1 = \{z \in \mathbb{C} : |z| = 1\}$ such that

$$\Lambda \subseteq \bigcap_{1 \leq j \leq N} \frac{Q_j}{P_j}(\Omega'). \tag{3}$$

In addition, let us suppose that there exist non-empty open connected subsets $\Omega_p \subseteq \Omega'$, $0 \leq p \leq N$, such that

$$\frac{Q_j}{P_j}(\Omega_0) \subseteq \text{int}(S_1), \text{ and } \frac{Q_j}{P_j}(\Omega_j) \subseteq \text{ext}(S_1), \quad 1 \leq j \leq N. \tag{4}$$

Finally, applying Theorem 3.11 and Remark 3.9, as in Example 3.10, we get that the multivalued linear operators $Q_1(\mathcal{A}_c)P_1(\mathcal{A}_c)^{-1}, \dots, Q_N(\mathcal{A}_c)P_N(\mathcal{A}_c)^{-1}$ are d -topologically mixing and that the set of periodic points of these operators are dense in X^N .

Furthermore, the same conclusion holds if we replace some of the operators $Q_i(\mathcal{A}_c)^{-1}P_i(\mathcal{A}_c)$ with $P_i(\mathcal{A}_c)Q_i(\mathcal{A}_c)^{-1}$, $1 \leq i \leq N$.

4 Dynamics of extensions of MLO's

Let $\mathcal{A} \in ML(X)$. In the sequel, we will often identify \mathcal{A} with its associated linear relation $\check{\mathcal{A}}$, which will also be denoted by the same symbol \mathcal{A} . It is clear that \mathcal{A} is contained in $X \times X$, which is hypercyclic, chaotic and topologically mixing (transitive, resp). Let us denote

$$S(\mathcal{A}) := \{Z : Z \text{ is a linear subspace of } X \times X \text{ and } \mathcal{A} \subseteq Z\}. \tag{5}$$

We say that $\mathcal{B} \in ML(X)$ is a *hypercyclic (chaotic, topologically mixing, topologically transitive) extension* of \mathcal{A} if $\mathcal{B} \in S(\mathcal{A})$ and \mathcal{B} is hypercyclic (chaotic, topologically mixing, topologically transitive).

Further on, let $N \in \mathbb{N}$, and let $\mathcal{A}_1, \dots, \mathcal{A}_N \in ML(X)$. Then the MLO's $X \times X, \dots, X \times X$, totally counted N times, are d -hypercyclic, d -chaotic and d -topologically mixing, d -topologically transitive. Denote

$$S(\mathcal{A}_1, \dots, \mathcal{A}_N) := \left\{ (\mathcal{B}_1, \dots, \mathcal{B}_N) : \begin{array}{l} \mathcal{B}_i \text{ is a linear subspace of } X \times X \\ \text{and } \mathcal{A}_i \subseteq \mathcal{B}_i \text{ for all } 1 \leq i \leq N \end{array} \right\}.$$

We say that the N -tuple $(\mathcal{B}_1, \dots, \mathcal{B}_N)$, with $\mathcal{B}_i \in ML(X)$, $1 \leq i \leq N$, is a *d -hypercyclic (d -chaotic, d -topologically mixing, d -topologically transitive) extension* of $(\mathcal{A}_1, \dots, \mathcal{A}_N)$ if $(\mathcal{B}_1, \dots, \mathcal{B}_N) \in S(\mathcal{A}_1, \dots, \mathcal{A}_N)$ and $\mathcal{B}_1, \dots, \mathcal{B}_N$ are d -hypercyclic (d -chaotic, d -topologically mixing, d -topologically transitive).

We proceed with some examples to illustrate these notions.

Example 4.1. (i) Suppose that $X := \mathbb{K}^n$ and $\mathcal{A} := A \in L(X)$. Then \mathcal{A} is not hypercyclic, and $\mathbb{K}^n \times \mathbb{K}^n$ is the only hypercyclic (chaotic, topologically mixing, topologically transitive) extension of \mathcal{A} ; a similar statement holds on any arbitrary finite-dimensional space X .

(ii) Suppose that X is infinite-dimensional, W is a non-dense linear subspace of X and $\mathcal{A} = X \times W$. Then any hypercyclic MLO extension of \mathcal{A} has the form $X \times W'$, where W' is a dense linear subspace of X containing W .

(iii) Any hypercyclic MLO extension \mathcal{A} of the identity operator I on X has the form $\mathcal{A}x = x + W$, $x \in X$, where $W = \mathcal{A}0$ is a dense linear subspace of X and x is a hypercyclic vector of \mathcal{A} .

The next examples depend on the following statement. Let us consider $A \in L(X)$. Then any MLO extension \mathcal{A} of A has the form $\mathcal{A}x = Ax + W$, $x \in X$, where $W = \mathcal{A}0$ is a linear subspace of X . Inductively,

$$\mathcal{A}^n x = A^n x + \sum_{j=0}^{n-1} A^j(W), \quad n \in \mathbb{N}, x \in X. \tag{6}$$

Example 4.2. Suppose that W is a dense linear subspace of X , and $\mathcal{A}x = Ax + W$, $x \in X$. Let U and V be two arbitrary open non-empty subsets of X . Then (6) implies that, for every $n \in \mathbb{N}$ and $u \in U$, there exists an element $\omega \in W$ such that $A^n u + \omega \in V \cap \mathcal{A}^n u$; in particular, \mathcal{A} is topologically mixing.

Suppose that X is infinite-dimensional, $A \in L(X)$ and $\{y_k : k \in \mathbb{N}\}$ is a dense subset of X . Denote by \mathfrak{T} the set consisting of all linear manifolds W' of X such that there exists $x \in X$ with the property that, for every $n \in \mathbb{N}$, there exist elements $z_{n,j} \in W'$ ($0 \leq j \leq n-1$) such that the set $\{A^n x + \sum_{j=0}^{n-1} A^j z_{n,j} : n \in \mathbb{N}\}$ is dense in X . Then \mathfrak{T} is non-empty because $X \in \mathfrak{T}$ (with $x = 0$, $z_{n,j} = 0$, $1 \leq j \leq n-1$, $z_{n,0} = y_n$), and $\check{\mathcal{A}} = A + \bigcap \mathfrak{T}$. Now we consider some examples:

Example 4.3. Let X be an infinite-dimensional complex Hilbert space with the complete orthonormal basis $\{e_n : n \in \mathbb{N}\}$. Let us consider A as the forward shift operator acting on an arbitrary element $x = \sum_{n=1}^{\infty} x_n e_n \in X$, as $A(\sum_{n=1}^{\infty} x_n e_n) := \sum_{n=1}^{\infty} x_n e_{n+1}$. The operator A satisfies $\|A\| = 1$ and therefore A is not hypercyclic.

On the one hand, any linear manifold W' belonging to \mathfrak{T} has to contain the linear span of $\{\omega\}$, for some element ω of X such that $\langle \omega, e_1 \rangle \neq 0$. On the other hand, let W be the linear span of $\{e_1\}$. In what follows, we will prove that $A + W$ is a hypercyclic extension of A , with zero being the corresponding hypercyclic vector: It is clear that there exists a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ of positive integers such that

$$\left\| y_k - \sum_{j=0}^{n_k-1} \alpha_{n_k, j} e_{j+1} \right\| < 2^{-k},$$

for some scalars $\alpha_{n_k, j}$ ($0 \leq j \leq n_k - 1$). In addition, the linear span of $\{e_1, \dots, e_l\}$ is contained in $\sum_{j=0}^{l-1} A^j(W)$ for any $l \in \mathbb{N}$. Plugging $z_{n, j} := 0$ ($0 \leq j \leq n - 1$) if $n \neq n_k$ for all $k \in \mathbb{N}$, and $z_{n, j} := \alpha_{n_k, j} e_1$ ($0 \leq j \leq n_k - 1$) if $n = n_k$ for some $k \in \mathbb{N}$, we can simply deduce that zero is a hypercyclic vector of $A + W$, as claimed.

Consider the following MLO extension of $A \in ML(X)$

$$\tilde{A} := \bigcap \{Z \in S(A) : Z \text{ is hypercyclic}\}. \tag{7}$$

We call \tilde{A} the quasi-hypercyclic extension of A and similarly define the quasi-chaotic (quasi-topologically transitive, quasi-topologically mixing) extension of A .

Example 4.4. Let $X := l_2(\mathbb{Z})$, let $(a_n)_{n \in \mathbb{Z}}$ be a bounded subset of $(0, \infty)$, and let

$$Ax := \sum_{n=-\infty}^{\infty} x_n a_n e_{n+1} \text{ for any } x = \sum_{n=-\infty}^{\infty} x_n e_n \in X; \tag{8}$$

where $\{e_n : n \in \mathbb{Z}\}$ denotes the complete orthonormal basis of X . The hypercyclicity of bilateral weighted shift A has been characterized by Salas in [34, Th.2.1]: Given $\epsilon > 0$ and $q \in \mathbb{N}$, there exists a sufficiently large integer $n \in \mathbb{N}$ such that $\prod_{s=0}^{n-1} a_{s+j} < \epsilon$ and $\prod_{s=1}^n a_{j-s} > 1/\epsilon$ for all $|j| \leq q$. Suppose that this condition does not hold, then it can be simply verified from (6) that $\text{span}\{e_{n_k} : k \in \mathbb{N}\} \in \mathfrak{T}$ for any strictly decreasing sequence $(n_k)_{k \in \mathbb{N}}$ of negative integers. Hence, $\tilde{A} = A$.

Example 4.5. Suppose that W is a linear subspace of X , $Ax = Ax + W$, $x \in X$, and for each $\omega \in W$ there exist an integer $n \in \mathbb{N}$ and elements $\omega_j \in W$ such that $\sum_{j=1}^{n-1} A^j \omega_j = -A^n \omega$. This, in particular, holds provided that there exists an integer $n \in \mathbb{N}$ such that $A^n(W) \subseteq W$, as it is the case of the forward shift operator. Then (6) implies that $\omega \in A^n \omega$, $w \in W$, so that operator W consists of solely periodic points of A . Suppose now that $R(A)$ is dense in X , hence $R(A^j)$ is dense in X for all $j \in \mathbb{N}$. Using this fact and Example 4.2, we get that $Ax = Ax + R(A^j)$, $x \in X$ is a chaotic extension of A for all $j \in \mathbb{N}$; this is no longer true if $R(A)$ is not dense in X , even if dimension of $X \setminus \overline{R(A)}$ equals 1, cf. Example 4.3. In the case that the operator A is nilpotent and W is a dense submanifold of X , then it can be easily seen that $Ax = Ax + W$, $x \in X$ is a chaotic extension of A .

We can similarly introduce the notion of a d -quasi-hypercyclic extension $\widetilde{(A_1, \dots, A_N)}$ (d -quasi-topologically transitive extension, d -quasi-topologically mixing extension) of any N -tuple (A_1, \dots, A_N) of elements of $ML(X)$

$$\widetilde{(A_1, \dots, A_N)} := \bigcap \{(\mathcal{B}_1, \dots, \mathcal{B}_N) \in S(A_1, \dots, A_N) : \mathcal{B}_1, \dots, \mathcal{B}_N \text{ are } d\text{-hypercyclic}\}. \tag{9}$$

Suppose that X is infinite-dimensional and $A_1, \dots, A_N \in L(X)$. Then any MLO extension of the tuple (A_1, \dots, A_N) has the form $(A_1 + W_1, \dots, A_N + W_N)$, where W_i is a linear subspace of X , $1 \leq i \leq N$. Using (6), we have that $(A_1 + W_1, \dots, A_N + W_N)$ is d -hypercyclic if there exists some $x \in X$ with the property that, for every $n \in \mathbb{N}$ there exist elements $z_{n, j, l} \in W_l$ ($0 \leq j \leq n - 1$, $1 \leq l \leq N$) such that the set $\{(A_1^n x + \sum_{j=0}^{n-1} A_1^j z_{n, j, 1}, \dots, A_N^n x + \sum_{j=0}^{n-1} A_N^j z_{n, j, N}) : n \in \mathbb{N}\}$ is dense in X^N . Then, $(A_1 + X, \dots, A_N + X)$ is d -hypercyclic with $x = 0$ being the corresponding d -hypercyclic vector. Arguing as in Example 4.3, it can be easily seen that the following examples hold:

- Example 4.6.** (i) Let A be the operator considered in Example 4.3. Then $(A + W_1, A^2 + W_2, \dots, A^N + W_N)$ is a d -hypercyclic extension of the N -tuple (A, A^2, \dots, A^N) , where W_i is the linear span of $\{e_1, e_2, \dots, e_i\}$ for $1 \leq i \leq N$.
- (ii) Any d -hypercyclic extension of the tuple (I, \dots, I) has the form $(I + W_1, \dots, I + W_N)$, where W_i is a dense linear subspace of X , $1 \leq i \leq N$.
- (iii) Let $X := l_2(\mathbb{Z})$, let $(a_n)_{n \in \mathbb{Z}}$ be a bounded subset of $(0, \infty)$, and let A be defined through (8). Suppose that $(n_{k,i})_{k \in \mathbb{N}}$ is any strictly decreasing sequence of negative integers possessing the property that, for every $s \in \mathbb{N}$ and $j \in \mathbb{N}_{i-1}^0$, there exists $l \in \mathbb{Z}$ such that $l < -s$ and $n_{l,i} \equiv j \pmod{i}$, $1 \leq i \leq N$. Set $W_i := \text{span}(\{e_{n_{k,i}} : k \in \mathbb{N}\})$, $1 \leq i \leq N$. Then $(A + W_1, A^2 + W_2, \dots, A^N + W_N)$ is a d -hypercyclic extension of the tuple (A, A^2, \dots, A^N) .
- (iv) Let $A_1, \dots, A_N \in L(X)$, with $(A_1, \dots, A_N) = (A_1 + W_1, \dots, A_N + W_N)$, where W_i is a dense linear subspace of X , $1 \leq i \leq N$. Then (A_1, \dots, A_N) is a d -topologically mixing extension of (A_1, \dots, A_N) .
- (v) Suppose that the range of A_i is dense in X for every $1 \leq i \leq N$. Then $(A_1 + R(A_1^{j_1}), \dots, A_N + R(A_N^{j_N}))$ is a d -chaotic extension of (A_1, \dots, A_N) for all $j_1, \dots, j_N \in \mathbb{N}$. In the case that the operator A_i is nilpotent and W_i is a dense subspace of X , $1 \leq i \leq N$, then it can be easily seen that $(A_1 + W_1, \dots, A_N + W_N)$ is a d -chaotic extension of (A_1, \dots, A_N) .

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