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# Chaos for dynamics of birth-and-death models

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# Contents

# Sumario

En este trabajo vamos a estudiar el comportamiento caótico en espacios de Banach de sistemas lineales de dimensión infinita, concretamente estudiaremos los $C_0$-semigrupos de operadores solución de estos sistemas. Nos centraremos en unos modelos de la teoría cinética como es el caso del proceso de muerte, su opuesto, el proceso de nacimiento, y ambos a la vez, es en este último caso donde surgen más problemas. Probablemente los primeros en estudiar estos procesos fueron Azmy y Protopopescu en [**AP92**], más tarde se vieron algunas aplicaciones en los trabajos de Kimmel, Stivers, Świerniak y Polański, como [**KS94, KPS96, KPS98**] y más recientemente se generalizaron esos estudios en los trabajos de Banasiak, Lachowicz y Moszyński, como [**BL01, BLM07, BM11**].

Primeramente, en el Capítulo 1 daremos unas nociones básicas sobre espacios métricos y espacios de Banach, así como sobre teoría de operadores.

En el Capítulo 2, describiremos brevemente el concepto de sistema dinámico discreto y sus propiedades relacionadas con el concepto de caos en el sentido de Devaney, [**Dev89**]. En el Capítulo 3 introduciremos el concepto de $C_0$-semigrupo, de Problema de Cauchy Abstracto y diferentes nociones de estabilidad para $C_0$-semigrupos. Veremos también que la mayoría de las nociones para el caso discreto tienen su analogía en el caso continuo, extenderemos el concepto de caos para $C_0$-semigrupos y daremos algunos criterios para saber si un $C_0$-semigrupo es caótico.

En el Capítulo 4 mostraremos una recopilación de los resultados obtenidos por Banasiak y Lachowicz en [**BL01**] y una generalización por parte de Grosse-Erdmann y Peris en [**GP11**, Chapter 7] sobre el estudio del caos en el proceso de nacimiento y el proceso de muerte, por separado.

Finalmente en el Capítulo 5, mostraremos algunos resultados nuevos para el proceso de nacimiento-muerte, es decir, consideraremos los dos procesos anteriores como un proceso conjunto. Por una parte, consideraremos este proceso con coeficientes constantes, presentando una versión alternativa basada en el artículo [**BM11**] de Banasiak y Moszyński, donde se intentó generalizar este resultado sin éxito. Por otra parte, presentaremos el proceso anterior en su versión para coeficientes variables donde estudiaremos dos situaciones diferentes sobre los coeficientes, una basada en la versión anterior y otra de naturaleza totalmente diferente que tiene como caso particular los resultados mostrados por Banasiak, Lachowicz y Moszyński en [**BLM07**].

# Summary

In this work we will study the chaotic behaviour of infinite dimensional linear systems on Banach spaces, specially we will study the solution $C_0$-semigroups of operators of these systems. We will focus on the models of kinetic theory as is the case of the death model, the birth model and both together. It is in the last case when more problems appear. Azmy and Protopopescu studied these processes for the first time in [**AP92**]. Later in the works of Kimmel, Stivers, Świerniak and Polański, e.g. in [**KS94, KPS96, KPS98**]. In addition this subject has been recently studied for Banasiak, Lachowicz and Moszyński, e.g. in [**BL01, BLM07, BM11**].

Firstly, in Chapter 1 we will give some basic notion of metric and Banach spaces, as well as some concepts of operator theory.

In Chapter 2, we will briefly describe the concepts of discrete dynamical systems and the properties of these systems related to the concept of chaos in the sense of Devaney, [**Dev89**]. In Chapter 3 We also introduce $C_0$-semigroups, Abstract Cauchy Problems, and several formulations of stability for $C_0$-semigroups. We will see that most of the ideas for the discrete case have their analogy in the continuous case, and we will extend the definition of chaos in the sense of Devaney for $C_0$-semigroups. We also give some criteria to know if a $C_0$-semigroup is chaotic.

Chapter 4 is mainly of expository character. Here we will show the results on the study of chaos in the birth model and the death model, separately obtained by Banasiak and Lachowicz in [**BL01**] and a generalization by Grosse-Erdmann and Peris, [**GP11**, Chapter 7].

Finally, in Chapter 5, we will show some new results for a similar study to the joined birth-and-death model. On the one hand, we will consider the birth-and-death model with constant coefficients, presenting an alternative version, of the chaotic behaviour of this model, based in the work of Banasiak and Moszyński in [**BM11**], where it was attempted to generalize this result without success. On the other hand, we will also study this model for the case of nonconstant coefficients. These results generalize in part the owes obtained by Banasiak, Lachowicz and Moszyński in [**BLM07**].

CHAPTER 1

# Preliminaries

In this section we will introduce basic definitions, facts and some tools that it will be helpful in this work. The principal references are the books [**Rud74, MV97, HP57, EN00**]. The reader can be found the definitions, theorems, and their proofs in the above-mentioned books.

## 1. Metric and Banach spaces

We can start with a notion of metric and Banach spaces and this properties:

**Definition 1.1** (Metric space)**.** A real-valued function $d : X \times X \to \mathbb{R}$, defined for each pair of elements $x, y \in X$ is called a *metric* if it satisfies:

M1: $d(x, y) \geq 0$, $d(x, x) = 0$ and $d(x, y) > 0$ if $x \neq y$;
M2: $d(x, y) = d(y, x)$;
M3: $d(x, z) \leq d(x, y) + d(y, z)$, the triangle inequality.

A set $X$ provided with a metric is called *metric space* and $d(x, y)$ is called the distance between $x$ and $y$.

We will understand by *neighborhood* of a point $p \in X$ a set $\mathcal{U} \subset X$, which contains an open set $\mathcal{V}$ containing $p$.

A point $x$ in a metric space $X$ is called *isolated* if some neighbourhood of $x$ contains no other point from $X$.

A metric space is said to be *locally compact* if each points has a compact neighbourhood. Finally, we said that a metric space is *complete* if every Cauchy sequence in $X$ converges to an element of $X$.

**Theorem 1.2** (Baire category theorem)**.** *Let $(X, d)$ be a complete metric space and $\{G_n\}_n$ a sequence of nonempty dense open sets. Then $G := \bigcap_{n=1}^{\infty} G_n$, is a dense $G_\delta$-set in $X$.*

**Definition 1.3** (Normed space)**.** A complex vector space $X$ is said to be a *normed linear space* if to each $x \in X$ there is associated a nonnegative real number $\|x\|$, called the *norm of* $x$, such that:

N1: $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$,
N2: $\|\alpha x\| = |\alpha|\|x\|$ if $x \in X$ and $\alpha$ is a scalar,
N3: $\|x\| = 0$ implies $x = 0$.

Every normed linear space may be regarded as a metric space, being $\|x - y\|$ the distance between $x$ and $y$. A *Banach space* is a normed linear space which is complete with the metric defined by its norm.

**Proposition 1.4.** *Let $X$ and $Y$ be Banach spaces and let $T : X \to Y$ be a linear operator. The following four statements are equivalent:*

(i) *$T$ is continuous at $0$.*
(ii) *$T$ is continuous.*
(iii) *$T$ is uniformly continuous.*
(iv) *$T$ is bounded, i.e., there exists a constant $C > 0$ such that $\|Tx\|_Y \leq C\|x\|_X$ for all $x \in X$.*

**Definition 1.5.** Let $X$ and $Y$ be Banach spaces and $T : X \to Y$ be a continuous linear operator. We define

$$\|T\| := \inf\{C > 0 : \|Tx\|_Y \leq C\|x\|_X \text{ for all } x \in X\}$$

and refer to $\|T\|$ as the operator norm of $T$.

Equivalent formulations are

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\|_Y = \sup_{\|x\|=1} \|Tx\|_Y$$

**Definition 1.6.** An operator $T$ on a complex Banach space $X$ is called *compact* if for every $(x_n)_n$ in $X$ with $\|x_n\| \leq 1$, $n \geq 1$, the sequence $(Tx_n)_n$ has a convergent subsequence. This is equivalent to say that the image of the closed unit ball under $T$ is *relatively compact*, that is, its closure is compact.

**Theorem 1.7** (Banach-Steinhaus theorem)**.** *Let $X, Y$ be Banach spaces and $T_j : X \to Y$, $j \in I$, operators. If for every $x \in X$ we have $\sup_{j\in I} \|T_j x\| < \infty$, then $\sup_{j\in I} \|T_j\| < \infty$.*

**Definition 1.8.** Let $X$ and $Y$ be Banach spaces we denote by $\mathcal{L}(X, Y)$ the space of continuous linear operators $T : X \to Y$; under the operator norm. This space turns to be a Banach space whenever $Y$ is a Banach space. If $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$, the *dual* $X^* = \mathcal{L}(X, \mathbb{K})$ of a Banach space $X$ is the space of all continuous linear functionals on $X$. If $x^* \in X^*$ then we write,

$$x^*(x) = \langle x, x^*\rangle, \quad x \in X.$$

The *adjoint* $T^* : X^* \to X^*$ of an operator $T$ on $X$ is defined by $T^*x^* = x^* \circ T$, that is,

$$\langle x, T^*x^*\rangle = \langle Tx, x^*\rangle, \quad x \in X,\ x^* \in X^*.$$

**Definition 1.9.** Suppose that $f$ is a complex function defined on an open set $S$ of the plane. If $z_0 \in S$ and if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, we denote this limit by $f'(z_0)$ and call it the derivative of $f$ at $z_0$. If $f'(z_0)$ exists for every $z_0 \in S$, then we say that $f$ is *holomorphic* (or analytic) in $S$.

**Definition 1.10.** Let $\mathcal{U} \subset X$ be an open set. We say that a function $f : \mathcal{U} \to X$ is *weakly holomorphic* if the vector-valued function $\langle f,\ \phi\rangle :\ \mathcal{U} \to \mathbb{C}$, is holomorphic for every $\phi \in X^*$.

The space of all bounded linear operators on $X$ is denoted by $\mathcal{L}(X)$. Apart from the *uniform operator topology* on $\mathcal{L}(X)$, which is the one induced by the above operator norm, we frequently consider the *strong operator topology*, $\mathcal{L}_s(X)$, which is the topology of pointwise convergence on $(X, \|\cdot\|)$.

## 2. Classical Banach spaces

Now we introduce the following classical sequence and function spaces that we will use in this work. Here, $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$. The symbol $X$ will always stand for a Banach space.

If $p \in [1, +\infty)$, we denote by

$$\ell^\infty := \ell^\infty(X) := \{(x_n)_{n\in\mathbb{N}} \subset X : \sup_{n\in\mathbb{N}} \|x_n\|_X < \infty\}$$

with the norm $\|(x_n)_{n\in\mathbb{N}}\|_\infty := \sup_{n\in\mathbb{N}} \|x_n\|_X$,

$$\ell^p := \ell^p(X) := \{(x_n)_{n\in\mathbb{N}} \subset X : \sum_{n\in\mathbb{N}} \|x_n\|_X^p < \infty\}$$

with the norm $\|(x_n)_{n\in\mathbb{N}}\|_p := \left(\sum_{n\in\mathbb{N}} \|x_n\|_X^p\right)^{1/p}$ and

$$L^p(\Omega, \mu) := \{f : \Omega \to \mathbb{K} : f \text{ is p-integrable on a measurable space } \Omega \text{ respect to } \mu\}$$

with the norm $\|f\|_p := \left(\int_\Omega |f|^p(s)d\mu(s)\right)^{1/p}$.

The space

$$c_0(X) := \{(x_n)_{n\in\mathbb{N}} \subset X : \lim_{n\to\infty} x_n = 0\} \subset \ell^\infty$$

and if $\Omega$ is a locally compact space then we denote by

$$C_0(\Omega) := \{f \in C(\Omega) : f \text{ vanishes at infinity}\},$$

i.e., the space such of $f \in C(\Omega) := \{f : \Omega \to \mathbb{K} : f \text{ is continuous}\}$ such that for all $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset \Omega$ such that $|f(s)| < \varepsilon$ for all $s \in \Omega \backslash K_\varepsilon$, endowed with the sup-norm $\|f\|_\infty := \sup_{s\in\Omega} |f(s)|$.

## 3. Spectral theory

In this work is essential to deal differential equations and a good tool for the study these equations is the spectral theory. In several occasions we find the eigenvalues and eigenvectors of these equations and we will use a criterion to conclude if the system is chaotic or not.

**Definition 1.11.** Let $X$ be a complex Banach space $X$ and let $T$ be an operator on $X$. The *spectrum* $\sigma(T)$ of $T$ is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} \; ; \; \lambda I - T \text{ is not invertible}\}.$$

Moreover, each $0 \neq x \in X$ satisfying $Tx = \lambda x$ is an eigenvector for $T$ corresponding to $\lambda$.

The *point spectrum* $\sigma_p(T)$ is the set of eigenvalues of $T$.

The number

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

is called the *spectral radius* of $T$.

For the spectral radius we have that

$$r(T) = \lim_{n \to \infty} \|T^n\|^{1/n}.$$

**Theorem 1.12** (Spectral mapping theorem)**.** *Let $f$ be a holomorphic function on a neighbourhood of $\sigma(T)$. Then*

$$\sigma(f(T)) = f(\sigma(T)).$$

Its version for $C_0$-semigroups can be found in [**EN00**]. There is also a version for the point spectrum.

**Theorem 1.13** (Point spectral mapping theorem)**.** *Let $f$ be a holomorphic function on a open neighbourhood $O$ of $\sigma(T)$ that is not constant on any component of $O$. Then*

$$\sigma_p(f(T)) = f(\sigma_p(T)).$$

## 4. Jury test for quadratic polynomials

Finally, the following lemma is a generalization of the so-called *Jury test* for quadratic polynomials, originally formulated only for $w \in \mathbb{R}$. A proof of this result for $w \in \mathbb{C}$ can be find in [**BM11**]. In this section we give an alternative proof of the above result:

**Lemma 1.14.** *Consider the family of quadratic equations for $z \in \mathbb{C}$*

$$z^2 + wz + r = 0, \tag{1.1}$$

*where $w \in \mathbb{C}$ and $r \in \mathbb{R}$ are parameters. For a fixed $r$ let $E_r$ denote the set of all complex $w$ such that the absolute value of each root of (1.1) is less than 1. If $|r| < 1$, then*

$$E_r = \left\{ w \in \mathbb{C} \ : \ \left( \frac{Re\ w}{1+r} \right)^2 + \left( \frac{Im\ w}{1-r} \right)^2 < 1 \right\} \tag{1.2}$$

Proof. With the assumptions $r \in \mathbb{R}$ such that $|r| < 1$ and the notation of $z_1$, $z_2$ for the two roots of quadratic equation with $|z_i| < 1,\ i = 1, 2$:

If $r = 0$, the equation is $z^2 + wz = 0$ and the roots are $z_1 = 0$ and $z_2 = w$, thus $|z_2| < 1 \longleftrightarrow |w|^2 = (Re\ w)^2 + (Im\ w)^2 < 1$ and $w \in E_r$.

On the other hand, if $r \neq 0$, we have $z_1 + z_2 = -w$ and $z_1 z_2 = r$. This implies that if $r = r_1 r_2$, with $r_i \in \mathbb{R},\ i = 1, 2$, there exists $\theta \in [0,\ 2\pi[$ such that we can rewrite the roots as:

$$z_1 = r_1 e^{i\theta} \text{ and } z_2 = r_2 e^{-i\theta}.$$

We consider the following cases:

Case 1: We consider that $r \in ]0,\ 1[$, without loss of generality we can assume that $r_1$ and $r_2$ are real positive numbers, otherwise we can choose another $\theta$. Note that $|z_i| < 1$ if, and only if, $r_i \in ]r,\ 1[,\ i = 1, 2$. If $r_i \in ]r,\ 1[,\ i = 1, 2$ since $-w = z_1 + z_2 = (r_1 + r_2)\cos(\theta) + i(r_1 - r_2)\sin(\theta)$, and $1 + r = 1 + r_1 r_2 = r_1 + [(1 - r_1) + r_1 r_2] > r_1 + r_2$ holds, we obtain that:

$$\left(\frac{Re\ w}{1+r}\right)^2 + \left(\frac{Im\ w}{1-r}\right)^2 = \left(\frac{r_1 + r_2}{1+r}\right)^2 \cos^2(\theta) + \left(\frac{r_1 - r_2}{1-r}\right)^2 \sin^2(\theta) < 1.$$

Conversely, if $\left(\frac{r_1 + r_2}{1+r}\right)^2 \cos^2(\theta) + \left(\frac{r_1 - r_2}{1-r}\right)^2 \sin^2(\theta) < 1$ holds, without loss of generality we can suppose $r_1 \leq r_2$. Aiming for a contradiction, suppose that $r_2 > 1$, then we have that $r_1 < r < 1$, because $r = r_1 r_2$. This implies that $r_2 - r_1 > 1 - r$, and also $1 + r = r_1 + [(1 - r_1) + r_1 r_2] < r_1 + r_2$, and we obtain a contradiction.

Case 2: If $r \in ]-1,\ 0[$, this situation can be reduced to the first case if we consider without loss of generality $r_1 = -k_1$ and $r_2 = k_2$ with $k_i > 0,\ i = 1, 2$.

□

CHAPTER 2

# Linear Discrete Dynamical systems

In this chapter we will introduce basic definitions and some basic results of the theory of linear dynamical systems. We suggest [**BM09**] and [**GP11**] for an introduction to this topic.

The theory of dynamical systems study the behaviour of evolving systems. Let $X$ be a set of elements that described the admissible different states of a system. If $x_n \in X$ is the state of the system at time $n \geq 0$, then its evolution will be given by a linear map $T : X \to X$ such that $x_{n+1} = T(x_n)$. In this sense we need that $X$ was a metric space and $T$ was a continuous map.

**Definition 2.1** (Discrete dynamical system)**.** Let $X$ be an infinite compact metric space and let $T$ be a continuous map $T : X \to X$. A *discrete dynamical system* is a pair $(X, T)$. We define the *orbit* of a point $x \in X$ as the set $\mathcal{O}(x, T) = \{T^n(x) : n \in \mathbb{N}\}$, $T^n$ denotes the $n$th iterate of a map $T$. Often we will simply call $T$ or $T : X \to X$ a dynamical system. Moreover we adopt the notation used in operator theory to write $Tx$ for $T(x)$.

**Definition 2.2.** We say that $x \in X$ is a *fixed point for the dynamical system* $T : X \to X$ if $Tx = x$ and we say that $x \in X$ is a *periodic point for the dynamical system* $T$ if $T^n x = x$ for some $n \in \mathbb{N}$. The set of all periodic points of this system is denoted by $Per(T)$. If $x \in Per(T)$ then the smallest positive integer n such that $T^n x = x$ is called a *primary period* of $x$.

**Definition 2.3.** A dynamical system $T : X \to X$ is:

(i) *topologically transitive* if for any pair of nonempty open sets $\mathcal{U}, \mathcal{V} \subset X$ there exists an $n \in \mathbb{N}$ such that $T^n\mathcal{U} \cap \mathcal{V} \neq \emptyset$;
(ii) *weakly mixing* if the map $T \times T$ is topologically transitive;
(iii) *mixing* if for any pair of nonempty open sets $\mathcal{U}, \mathcal{V} \subset X$ there exists some $n_0 \in \mathbb{N}$ such that $T^n\mathcal{U} \cap \mathcal{V} \neq \emptyset$ for every integer $n \geq n_0$;

In 1989 Robert L. Devaney was the first to propose a good definition of chaos, see [**Dev89**], this concept reflects the unpredictability of chaotic systems because the definition contain a *sensitive dependence on initial conditions*, i.e.:

**Definition 2.4.** Let $(X, d)$ be a metric space without isolated points. Then the dynamical system $T : X \to X$ is said to have *sensitive dependence on initial conditions* if there exists some $\delta > 0$ such that, for every $x \in X$ and $\varepsilon > 0$, there exists some $y \in X$ with $d(x, y) < \varepsilon$ such that, for some $n \geq 0$, $d(T^n x, T^n y) > \delta$. The number $\delta$ is called a *sensitivity constant* for $T$.

**Definition 2.5** (Devaney chaos)**.** A dynamical system $T : X \to X$ is called chaotic in the sense of *Devaney* if it satisfies the following three properties:

(i) $T$ is topologically transitive,
(ii) $Per(T)$ is dense in $X$,
(iii) $T$ has sensitive dependence on initial conditions.

However, it was proved in 1992 by Banks, Brooks, Cairns, Davis and Stacey in [**BBCDS92**] that if $X$ is an infinite set, the sensitivity is a consequence of transitivity and dense periodicity.

**Theorem 2.6** ([**BBCDS92**])**.** *Let $X$ be a non-finite metric space. If a dynamical system $T : X \to X$ is topologically transitive and has a dense set of periodic points then $T$ has sensitive dependence on initial conditions with respect to any metric defining the topology of $X$.*

A link between chaos theory and linear operator theory has been established in the Transitivity theorem by Birkhoff in 1920 when it was realized that the topological transitivity was equivalent to the notion of hypercyclicity established by Beauzamy in 1986:

**Definition 2.7** ([**Bea86**])**.** Let $X$ be a topological vector space.
An operator $T : X \to X$ is said to be *hypercyclic* if there is an $x \in X$ whose orbit $\mathcal{O}(x, T)$ is dense in $X$. In that case, $x$ is called a *hypercyclic vector* for $T$. The set of hypercyclic vectors is denoted by $HC(T)$.

**Theorem 2.8** (Transitivity theorem, [**Bir20**])**.** *Let $X$ be a separable complete metric space without isolated points and let $T : X \to X$ a continuous map. Then the following assertions are equivalent:*

(i) *$T$ is topologically transitive;*
(ii) *$T$ is hypercyclic operator.*

*If one of these conditions holds then, by Theorem 1.2, the set $HC(T)$ of hypercyclic vectors is a dense $G_\delta$-set, i.e., $HC(T)$ is a countable intersection of open sets.*

In 1991 Godefroy and Shapiro adopted Devaney's definition also for linear chaos.

**Definition 2.9** ([**GS91**])**.** Let $X$ be a complete metric vector space. An operator $T : X \to X$ is called chaotic in the sense of Devaney, if:

(i) $T$ is hypercyclic,
(ii) $Per(T)$ is dense in $X$.

The reader can find the proofs of following results e.g in [**BM09**]. In addition the original proofs of some of these results can be found in [**Kit82**]:

**Proposition 2.10.** *Let $T$ be a hypercyclic operator on a (real or complex) Banach space $X$. Then we have:*

(i) *$T^*$ has no eigenvalues, that is, $\sigma_p(T^*) = \emptyset$;*
(ii) *the orbit of every $x^* \neq 0$ in $X^*$ under $T^*$ is unbounded.*

**Theorem 2.11.** *Let $T$ be a hypercyclic operator on a complex Banach space $X$. Then every connected component of $\sigma(T)$ meets the unit circle $\mathbb{T}$, i.e., $\sigma(T) \cap \mathbb{T} \neq \emptyset$.*

**Proposition 2.12.** *Let $T$ be a linear map on a complex vector space $X$. Then the set $Per(T)$ of periodic points of $T$ is given by*

$$span\{x \in X \; ; \; Tx = \lambda x \text{ for some } \lambda \in \mathbb{C} \text{ with } \lambda^n = 1 \text{ for some } n \in \mathbb{N}\}.$$

**Proposition 2.13.** *Let $T$ be a chaotic operator on a complex Banach space $X$. Then its spectrum has no isolated points and it contains infinitely many roots of unity; in particular, $\sigma(T) \cap \mathbb{T}$ is infinite.*

**Theorem 2.14.** *No compact operator is hypercyclic.*

CHAPTER 3

# Linear Continuous Dynamical systems

Now, we will introduce the concepts of strongly continuous semigroups of bounded linear operators (at now, $C_0$-semigroups[1]) in Banach spaces. We suggest [**DSW97**], [**EN00**] and also [**GP11**] for an introduction to this topic. If we consider a single operator and his iterates in the discrete case of the dynamical systems and the time $t_n > 0$ with $\lim\limits_{n\to\infty} t_n = \infty$ we can be viewed the $C_0$-semigroups as the case continuous case when $t > 0$.

**Definition 3.1** ($C_0$-semigroups of operators)**.** Let $X$ be an infinite-dimensional separable Banach space. A one-parameter family $\mathcal{T} = \{T_t : X \to X \ ; \ t \geq 0\}$ is a *strongly continuous semigroup of operators*, from now on *$C_0$-semigroup*, if the following three conditions are satisfied:

(i) $T_0 = I$.
(ii) $T_t T_s = T_{t+s}$, for all $t, s \geq 0$.
(iii) $\lim\limits_{t\to s} T_t x = T_s x$, for each $x \in X$ and $s \geq 0$.

**Remark 3.2.** The third condition is associated to the continuity of the operators in the $C_0$-semigroup respect to the strong operator topology. If we replace this third condition by $\lim\limits_{t\to s} T_t = T_s$ respect to the norm of operators associated to the one of $X$ then we say that the $C_0$-semigroup is uniformly continuous. The Theorem 1.7 yields that the family $\{T_t\}_{t\geq 0}$ is locally equicontinuous, that is,

$$\forall M > 0 \ \ \exists C > 0 \text{ such that } \|T_t x\| \leq C\|x\|, \quad \forall t \in [0, M], \quad \forall x \in X.$$

This fact easily implies that the map $f : [0, +\infty[ \times X \to X$ given by $f(t, x) = T_t x$, $t \geq 0$, $x \in X$, is continuous. Moreover, one can establish an exponential bound for the operator norm of the $C_0$-semigroup.

**Proposition 3.3.** *If $\{T_t\}_{t\geq 0}$ is a $C_0$-semigroup, then there exists $\omega \in \mathbb{R}$ and $M \geq 1$ such that $\|T_t\| \leq M e^{\omega t}$, for each $t \geq 0$.*

**Definition 3.4** (Continuous dynamical system)**.** Let $X$ be an infinite compact metric space and let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a $C_0$-semigroup. A *continuous dynamical system* is a pair $(X, \mathcal{T})$. We define the *orbit* of a point $x \in X$ as the set $\mathcal{O}(x, \mathcal{T}) = \{T_t x : t \geq 0\}$.

The $C_0$-semigroups have a infinitesimal generator, usually denoted by $A$, which is an operator that can be a bounded (i.e., continuous) or an unbounded (densely defined and with closed graph). Is in the second case when the chaos in linear $C_0$-semigroups is applicable to linear partial differential equations and as in the case that we will study in this work, also in infinite linear systems of ordinary differential equations.

[1] "$C_0$" has its origin in the abbreviation of "Cesàro summable of order 0"

$C_0$-semigroups are, in a natural way, associated to Abstract Cauchy Problems (ACP):

$$\begin{cases} \frac{d}{dt}u(t) = Au(t) & \text{for } t \geq 0, \\ \\ u(0) = x, \end{cases} \tag{3.1}$$

where $x \in X$ is the initial value, and $A$ is a linear map defined on a dense subspace $D(A) \subset X$. If the above problem has a unique solution $u(\cdot, x)$, then $T_t x := u(t, x)$, $t \geq 0$, $x \in X$, defines a solution $C_0$-semigroup $\{T_t\}_{t\geq 0}$. The pair $(A, D(A))$ is called the *generator* of the solution $C_0$-semigroup.

Conversely, given a $C_0$-semigroup $\{T_t\}_{t\geq 0}$, one can consider the derivative at 0

$$Ax := \lim_{t\to 0} \frac{1}{t}(T_t x - x),$$

for those $x \in X$ such that the above limit exists. It turns out that it exists for each $x$ in a dense subspace $D(A)$ of $X$ (the domain of $A$), and that

$$A : D(A) \subset X \to X$$

is a linear map with closed graph, i.e., if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = y$, then $Ax = y$. We have that $\{T_t\}_{t\geq 0}$ is the solution $C_0$-semigroup to the ACP problem (3.1) associated to $A$. The generator determines the $C_0$-semigroup uniquely.

A basic fact that will be used through this section concerns the eigenvectors of the generator $(A, D(A))$. More precisely, by the Theorem 1.12,

$$Ax = \lambda x \text{ for some } \lambda \in \mathbb{K} \Longrightarrow T_t x = e^{it\lambda}x, \quad \text{for every } t \geq 0.$$

From the definition of the generator and the $C_0$-semigroup properties we also deduce $T_t(D(A)) \subset D(A)$ and $AT_t x = T_t Ax$ for every $t \geq 0$ and for each $x \in D(A)$.

In the special case when $D(A) = X$ and $A$ is an operator, we obtain a explicit formula for the operators of the $C_0$-semigroup:

$$T_t = e^{tA} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}, \quad t \geq 0. \tag{3.2}$$

In this case the $C_0$-semigroup is called *uniformly continuous* and we have that $\lim_{t\to s} \|T_t - T_s\| = 0$ for each $s \geq 0$.

We recall that if $A$ is bounded, then for each $t \geq 0$ we have

$$e^{tA}\mathbf{x} = \sum_{k=0}^{\infty} \frac{t^k A^k \mathbf{x}}{k!}, \qquad \mathbf{x} \in X \tag{3.3}$$

and the $C_0$-semigroup is continuous in the uniform operator topology.

The version of *Point spectral mapping theorem* for $C_0$-semigroups can be found in the book of Engel and Nagel, [**EN00**]:

**Theorem 3.5** (Point spectral mapping theorem for $C_0$-semigroups)**.** *Let $(A, D(A))$ be the generator of a $C_0$-semigroup $\{T_t\}_{t\geq 0}$ defined on a complex Banach space $X$. Then we have the following identities:*

(i) $\sigma_p(T_t)\backslash\{0\} = e^{t\sigma_p(A)}$, *for $t \geq 0$,*

(ii) $\ker(A-\lambda I)=\bigcap_{t\geq 0}\ker(T_t - e^{t\lambda}I)$, *for* $\lambda\in\mathbb{C}$,

(iii) $\ker(T_s - e^{s\lambda}I)=\overline{span}\bigcup_{n\in\mathbb{Z}}\ker\left(A-\left(\lambda+\frac{2\pi ni}{s}\right)I\right)$, *for* $s\geq 0$.

We begin with a brief discussion about the notion of stability in a general setting. For more details see [**EN00**]:

**Definition 3.6.** A $C_0$-semigroup $\{T_t\}_{t\geq 0}=(e^{tA})_{t\geq 0}$ on a Banach space $X$ is called:

(i) *(uniformly) exponentially stable* if there exists $\varepsilon>0$ such that

(3.4) $$\lim_{t\to+\infty} e^{\varepsilon t}|||e^{tA}|||=0, \qquad \text{where } |||\cdot||| \text{ denotes the operator norm.}$$

(ii) *(uniformly) exponentially stable on a subspace* $Y\subset X$ if there exists $\varepsilon>0$ such that for any $\mathbf{y}\in Y$

(3.5) $$\lim_{t\to+\infty} e^{\varepsilon t}\|e^{tA}\mathbf{y}\|_X=0.$$

(iii) *uniformly stable* if

(3.6) $$\lim_{t\to+\infty}|||e^{tA}|||=0,$$

(iv) *strongly stable* if

(3.7) $$\lim_{t\to+\infty}\|e^{tA}\mathbf{x}\|_X=0 \qquad \text{for all } x\in X,$$

(v) *weakly stable* if

(3.8) $$\lim_{t\to+\infty}\langle e^{tA}\mathbf{x},\mathbf{z}\rangle=0 \qquad \text{for all } x\in X, \text{ and } \mathbf{z}\in X^*.$$

**Proposition 3.7.** *For a $C_0$-semigroup $\{T_t\}_{t\geq 0}=(e^{tA})_{t\geq 0}$ on a Banach space $X$, the following assertions are equivalent:*

(i) $(e^{tA})_{t\geq 0}$ *is uniformly exponentially stable.*
(ii) $(e^{tA})_{t\geq 0}$ *is uniformly stable.*
(iii) *There exists $\varepsilon>0$ such that for any $x\in X$*

(3.9) $$\lim_{t\to+\infty} e^{\varepsilon t}\|e^{tA}\boldsymbol{x}\|_X=0.$$

The concepts of hypercyclicity, transitivity, mixing, weakly mixing and chaos have a version for $C_0$-semigroups. Now, we establish this relations for $C_0$-semigroups and for their discretizations. In this way, we will establish a feedback between continuous and discrete dynamical systems.

**Definition 3.8.** A *discretization* of $\{T_t\}_{t\geq 0}$ is a sequence of operators $(T_{t_n})_n$ in the $C_0$-semigroup, where $\lim_{n\to\infty} t_n=\infty$. If there is $t_0\neq 0$ such that $t_n=nt_0$ for each $n\in\mathbb{N}$, then $(T_{t_n})_n=(T_{t_0}^n)_n$ is called an *autonomous discretization* of $\{T_t\}_{t\geq 0}$.

**Definition 3.9.** A $C_0$-semigroup $\mathcal{T}=\{T_t : X\to X\ ;\ t\geq 0\}=\{T_t\}_{t\geq 0}$ on a Banach space $X$ is called *hypercyclic* if there is an $x\in X$ whose orbit $\mathcal{O}(x,\mathcal{T})=\{T_t x\ :\ t\geq 0\}$ under $\mathcal{T}$ is dense in $X$. In that a case, $x$ is called a *hypercyclic vector* for $\mathcal{T}$. We denote by $HC(\mathcal{T})$ the set of hypercyclic vectors of the $C_0$-semigroup.

An easy observation yields that a $C_0$-semigroup is hypercyclic if, and only if, it admits a hypercyclic discretization $(T_{t_n})_n$.

**Definition 3.10.** A $C_0$-semigroup $\mathcal{T}$ is called *topologically transitive* if, for any pair of nonempty open sets $\mathcal{U}, \mathcal{V} \subset X$, there is $t \geq 0$ such that $T_t(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$.

**Definition 3.11.** A $C_0$-semigroup $\mathcal{T}$ is *mixing* if, for any pair of nonempty open sets $\mathcal{U}, \mathcal{V} \subset X$, there exists some $t_0 \geq 0$ such that $T_t(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$, for all $t \geq t_0$.
The $C_0$-semigroup $\mathcal{T}$ is *weakly mixing* if, $\{T_t \oplus T_t : X \oplus X \to X \oplus X \ : \ t \geq 0\}$ is transitive.

**Definition 3.12.** The set of *periodic points* of a $C_0$-semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ is $Per(\mathcal{T}) := \{x \in X \ : \ T_t x = x \textit{ for some } t > 0\}$.
The $C_0$-semigroup $\mathcal{T}$ is said to be *chaotic* if it satisfies the following properties:

(i) $\mathcal{T}$ is hypercyclic,
(ii) $Per(\mathcal{T})$ is dense in $X$.

The following results provide characterizations of mixing and weakly mixing $C_0$-semigroups in terms of their discretizations, see details in [**CP09**].

**Proposition 3.13** ([**CP09**]). *Let $\{T_t\}_{t\geq 0}$ be a $C_0$-semigroup on a separable Banach space $X$. The following assertions are equivalent:*

(i) *$\{T_t\}_{t\geq 0}$ is weakly mixing.*
(ii) *$\{T_t\}_{t\geq 0}$ admits a mixing discretization.*
(iii) *$\{T_t\}_{t\geq 0}$ admits a weakly mixing discretization.*

**Proposition 3.14** ([**CP09**]). *Let $\{T_t\}_{t\geq 0}$ be a $C_0$-semigroup on a separable Banach space $X$. The following assertions are equivalent:*

(i) *$\{T_t\}_{t\geq 0}$ is mixing.*
(ii) *Every discretization of $\{T_t\}_{t\geq 0}$ is mixing.*
(iii) *Every discretization of $\{T_t\}_{t\geq 0}$ is weakly mixing.*
(iv) *Every discretization of $\{T_t\}_{t\geq 0}$ is transitive.*
(v) *There exists a mixing autonomous discretization of $\{T_t\}_{t\geq 0}$.*

**Theorem 3.15** ([**OU41**]). *If $\{T_t\}_{t\geq 0}$ is a hypercyclic $C_0$-semigroup on a separable Banach space $X$, and $x \in X$ is a hypercyclic vector, then there is a dense $G_\delta$-subset $J \subset ]0, \infty[$ such that the vector $x$ is hypercyclic for $T_t$, for each $t \in J$.*

**Theorem 3.16** ([**CP09**]). *Let $\{T_t\}_{t\geq 0}$ be a $C_0$-semigroup in the space of all operators on $X$. The following assertions are equivalent:*

(i) *$\{T_t\}_{t\geq 0}$ is weakly mixing.*
(ii) *All autonomous discretizations are weakly mixing.*

The problem of hypercyclic discretizations of $C_0$-semigroups asks if, given a hypercyclic $C_0$-semigroup $\{T_t\}_{t\geq 0}$ on a Banach space, also every single operator $T_t$, $t \geq 0$, is hypercyclic. This problem was solved in [**CMP07**] by Conejero, Müller and Peris in the following result:

**Theorem 3.17.** *Let $\{T_t\}_{t\geq 0}$ be a hypercyclic $C_0$-semigroup on a Banach space $X$. If $x \in X$ is hypercyclic vector for $\{T_t\}_{t\geq 0}$, then it is hypercyclic for each operator $T_t$, $t \geq 0$.*

The following criterions will be the main tools for prove the chaos in this work, for the details of proof see [**DSW97**], [**Mou06**] or [**GP11**].

**Proposition 3.18.** *Let $X$ be a complex separable infinite-dimensional Banach space and let $(A, D(A))$ be the generator of a $C_0$-semigroup $\{T_t\}_{t\geq 0}$ on $X$. Assume that there exists an open connected subset $\mathcal{U}$ and a weakly holomorphic function $f : \mathcal{U} \to X$ such that:*

(i) $\mathcal{U} \cap i\mathbb{R} \neq \emptyset$,

(ii) $f(\lambda) \in \ker(\lambda\mathbb{I} - A)$ *for every* $\lambda \in \mathcal{U}$,

(iii) *if for some $\phi \in X^*$ the function $h(\lambda) = \langle f(\lambda),\ \phi\rangle$ is identically zero on $\mathcal{U}$, then $\phi = 0$.*

*Then the $C_0$-semigroup $\{T_t\}_{t\geq 0}$ is chaotic and mixing.*

**Proposition 3.19.** *Let $X$ be a complex separable infinite-dimensional Banach space and let $(A, D(A))$ be the generator of a $C_0$-semigroup $\{T_t\}_{t\geq 0}$ on $X$. Assume that there exists a compact interval $I \subset \mathbb{R}$ and a continuous function $f : I \to X$ such that:*

(i) $f(\lambda) \in \ker(i\lambda\mathbb{I} - A)$ *for every* $\lambda \in I$,

(ii) *$span\{f(\lambda); \lambda \in I\}$ is dense in $X$.*

*Then the $C_0$-semigroup $\{T_t\}_{t\geq 0}$ is chaotic and mixing.*

**Theorem 3.20.** *Let $\{T_t\}_{t\geq 0}$ be a hypercyclic $C_0$-semigroup generated by $A$ in a Banach space $X$. Then the adjoint $A^*$ of $A$ and the dual $C_0$-semigroup $\{T_t^*\}_{t\geq 0}$ have the following properties:*

(i) *if $\phi \in X^*$, $\phi \neq 0$, then the orbit $\{T_t^*\}_{t\geq 0}$ is unbounded;*

(ii) *the point spectrum of $A^*$ is empty.*

**Proposition 3.21.** *If $(A, D(A))$ is the generator of a chaotic $C_0$-semigroup on a complex Banach space $X$, then $\sigma_p(A) \cap i\mathbb{R}$ is infinite and, moreover,*

$$X = \overline{span} \bigcup_{\lambda \in i\mathbb{R}} \ker(A - \lambda\mathbb{I}).$$

CHAPTER 4

# Applications of $C_0$-semigroups to differential equations

The main purpose of this chapter is to present some applications of $C_0$-semigroups to the asymptotic behavior of solutions to infinite linear systems of ordinary differential equations associated with the evolution of a cell population discussed in [**BL01**] by Jacek Banasiak and Miroslav Lachowicz, and generalized in [**GP11**] by Karl-Goswin Grosse-Erdmann and Alfredo Peris.

In [**AP92**] the authors undertook a detailed study of the death part of the birth-and-death process. Following the same line that [**AP92**] we consider an immobile medium that host particles indexed by non-negative integers $n \in \mathbb{N}_0$, and related to internal levels of energetic excitation. These particles collide and interact with the host medium as follows. After collision, particles of internal energy level $n \geq 1$ are absorbed by the medium at a rate $\alpha > 0$ and re-emitted as particles of internal energy level $n-1$ at a rate $\beta \geq \alpha$. Particles with internal energy $n = 0$ are absorbed and cease to exist.

We denote by $f_n(t)$ the distribution function corresponding to the particles of internal energy $n$, that satisfies:

$$\frac{df_n}{dt} = -\alpha f_n + \beta f_{n+1}, \quad n \in \mathbb{N}_0 \tag{4.1}$$

From now on, $X := \ell^1\left(L^1([0, +\infty[)\right)$ denote the space such that $0 \leq \mathbf{f} = (f_n)_{n\geq 0} \in X$, with the norm $\|\mathbf{f}\| = \|\mathbf{f}\|_X = \sum_{n=0}^{\infty} \|f_n\|_{L^1([0,+\infty[)} = \sum_{n=0}^{\infty} \|f_n\|_1$ that represents the total number of particles and

$$X^p := \ell^p\left(L^1([0, +\infty[)\right),\ 1 \leq p < \infty.$$

Note that, if $\frac{1}{p} + \frac{1}{q} = 1$ then

$$X^* = \ell^\infty\left(L^\infty([0,\ +\infty[)\right) \text{ and } (X^p)^* := \ell^q\left(L^\infty([0, +\infty[)\right).$$

We denote by $\mathcal{T} = \{T_t\}_{t\geq 0}$ the $C_0$-semigroup solution of (4.1) associated with the bounded operator defined by the right-hand-side of the above equation. It follows that the point spectrum of the operator is $\{-\alpha + \beta\mu : |\mu| < 1\}$, because $\mu = \frac{\alpha+\beta}{\lambda}$, with eigenvectors given by $h_\mu = (\mu,\ \mu^2,\ \mu^3,\ \ldots)$. Note that the assumptions of Proposition 3.18 are satisfied and $\mathcal{T}$ is chaotic in $X$.

In the next sections we consider a generalization of the model discussed earlier, the same problem (the death model) but now with variable coefficients and birth model with variable coefficients.

## 1. Death model with variable coefficients: First approach

In 2001, Jacek Banasiak and Miroslav Lachowicz generalized the result in [**AP92**].

The generalization consists in allowing variable coefficients $\alpha_n,\ \beta_n$, but keeping the assumption $0 < \alpha_n < \beta_n$ for any $n \in \mathbb{N}_0$. Moreover we assume that:

$$(4.2)\qquad \begin{aligned} &(A1) \quad \alpha_n = \alpha + a'_n, \text{ for some } \alpha \geq 0 \text{ and with } \lim_{n\to\infty} a'_n = 0;\\ &(A2) \quad \beta_n = \beta b_n, \text{ for some } \beta \geq \alpha \text{ and with } \lim_{n\to\infty} b_n = 1. \end{aligned}$$

In $X$ we consider the following system of equations:

$$(4.3)\qquad \frac{df_n}{dt} = (Lf)_n = -\alpha_n f_n + \beta_n f_{n+1}, \quad n \in \mathbb{N}_0$$

Consider the problem for eigenvectors:

$$(4.4)\qquad \lambda h_n = -\alpha_n h_n + \beta_n h_{n+1}, \quad n \in \mathbb{N}_0.$$

We denote by $h(\lambda) = \{h_n(\lambda)\}_{n\in\mathbb{N}_0}$ the eigenfunctions:

$$(4.5)\qquad \begin{aligned} h_0(\lambda) &= 1,\\ h_n(\lambda) &= \prod_{i=0}^{n-1} \frac{\lambda + \alpha_i}{\beta_i}, \text{ for } n \geq 1. \end{aligned}$$

**Lemma 4.1** ([**BL01**]). *Under the assumptions (4.2) the circle $\{\lambda \in \mathbb{C} : |\lambda+\alpha| < \beta\}$ belongs to the point spectrum $\sigma_p(L)$ of the operator $L$, and $\lambda = 0$ belongs to the interior of $\sigma_p(L)$.*

We will use the Proposition 3.18 with $\mathcal{U} = \{\lambda \in \mathbb{C} : |\lambda + \alpha| < \beta\}$ for study the chaos of the solution.

If $(\phi_n)_{n\in\mathbb{N}} \in X^*$ is an arbitrary sequence and $H_\phi(\lambda) = \sum_{k=0}^{\infty} \phi_k h_k(\lambda)$ is uniformly convergent function (at least in the circle $\mathcal{U}$).

We shall $H_\varphi(\mu) = H_\phi(\lambda) = \sum_{k=0}^{\infty} \varphi_k \prod_{i=0}^{k-1} (\mu + a_i)$ where:

$$\varphi_0 = \phi_0, \quad \varphi_k = \frac{\phi_k}{\prod_{i=0}^{k-1} b_i}$$

$$\mu = \frac{\lambda+\alpha}{\beta}, \quad a_n = \frac{a'_n}{\beta}$$

The function $H_\varphi(\mu)$ is analytic for $|\mu| < 1$. By the assumption (4.2) (A2), if $(\phi_n)_{n\in\mathbb{N}} \in X^*$, then also $(\varphi_n)_{n\in\mathbb{N}} \in X^*$. Thus is sufficient showing that $H_\varphi(\mu) \equiv 0$ yields $\varphi_n = 0$ for $n \in \mathbb{N}_0$ for any bounded $(\varphi_n)_{n\in\mathbb{N}}$.

As the series is absolutely convergent, we shall write $H_\varphi(\mu)$ as a power series in $\mu$.

If

$$A_{jn} = \begin{cases} 0, & \textit{for } j > n \\ 1, & \textit{for } n = j \\ \displaystyle\sum_{0 \le i_1 < \cdots < i_{n-j} \le n-1} a_{i_1} \cdots a_{i_{n-j}}, & \textit{otherwise} \end{cases}$$

We have

$$H_\varphi(\mu) = \sum_{j=0}^{\infty} \mu^j \sum_{n=j}^{\infty} \varphi_n A_{jn}$$

To be able to use the Proposition 3.18 we have to show that the only bounded solution of the infinite upper-triangular linear system is the zero solution:

$$\begin{aligned} \varphi_0 + A_{01}\varphi_1 + A_{02}\varphi_2 + \cdots &= 0 \\ \varphi_1 + A_{12}\varphi_2 + \cdots &= 0 \\ \varphi_2 + \cdots &= 0 \\ \cdots &= 0 \end{aligned}$$

Denote by $\mathcal{A}$ the operator generated by the matrix $\{A_{ij}\}_{0 \le i,j \le \infty}$ in $X^*$.

**Lemma 4.2** ([**BL01**]). *Assume that there exists: $q < 1$, and $k_0$ such that $|a_k| \le q^{k+1}$ for $k \ge k_0$. Then $\mathcal{A}$ is a bounded operator in $X^*$.*

**Lemma 4.3** ([**BL01**]). *There is $q < 1$ such that if $|a_k| \le q^{k+1}$ for $k \in \mathbb{N}_0$, then $\mathcal{A}$ is an isomorphism in $X^*$.*

Finally combining the above results:

**Theorem 4.4** ([**BL01**]). *Suppose that the sequences $\{\alpha_n\}_{n \in \mathbb{N}_0}$ and $\{\beta_n\}_{n \in \mathbb{N}_0}$ satisfy the assumptions (4.2), and the assumptions of Lemma 4.3. Then the $C_0$-semigroup generated by system (4.3) is chaotic in any $X^p$, $1 \le p < \infty$, and $c_0$.*

## 2. Death model with variable coefficients: general approach

In this section we will see a generalization of the assumptions (4.2) in the same problem, which was observed by K.-G. Grosse-Erdmann and A. Peris in [**GP11**, Chapter 7].

We consider in $X$ the system of equations:

$$\frac{df_n}{dt} = -\alpha_n f_n + \beta_n f_{n+1}, \quad n \ge 1 \tag{4.6}$$

with $f_n \in L^1([0, +\infty[)$. If $\mathbf{f} = (f_n)_{n \in \mathbb{N}} \in X$, on $X$ we define:

$$A\mathbf{f} = (-\alpha_n f_n + \beta_n f_{n+1})_{n \in \mathbb{N}} \in X,$$

$\mathbf{f} = (f_n)_{n \in \mathbb{N}} \in D(A) \subset X$. We denote by $\{T_t\}_{t \ge 0}$ the $C_0$-semigroup solution of the Cauchy problem (4.6), where $A$ is a densely defined linear map on $X$ with closed graph.

**Proposition 4.5** ([**GP11**]). *Let $\alpha_n \in ]0, +\infty[$ and $\beta_n \in \mathbb{R}$, $n \in \mathbb{N}$, such that*

$$\alpha := \sup_k \alpha_k < \beta := \liminf_k \beta_k. \tag{4.7}$$

*Then the solution $C_0$-semigroup to the Cauchy problem (4.6) is chaotic and mixing.*

Proof. Let $\frac{\alpha}{2} < \mu < \frac{\beta}{2}$. We fix $\mathcal{U} \subset \mathbb{C}$ as the open disk centered at $\frac{-\alpha}{2}$ of radius $\mu$, which intersects the imaginary axis $i\mathbb{R}$. We can calculate the eigenvectors of A for each $\lambda \in \mathcal{U}$:

$$A\mathbf{f} = \lambda\mathbf{f} \Longrightarrow \lambda f_n = -\alpha_n f_n + \beta_n f_{n+1},\ n \geq 1.$$

Therefore,

$$f_n(\lambda) = \gamma_n f_1,\ \text{with } \gamma_1 := 1,\ \text{and } \gamma_n := \prod_{k=1}^{n-1} \frac{\lambda + \alpha_k}{\beta_k},\ n > 1.$$

Let $\lambda \in \mathcal{U}$ and fix $\delta \in ]2\mu,\ \beta[$. There is $n_0 \in \mathbb{N}$ such that $\beta_n > \delta$ for every $n \geq n_0$. Thus, since our assumptions imply that $-\alpha_n \in \mathcal{U}$ for all $n \in \mathbb{N}$, we have

$$\frac{|\lambda + \alpha_n|}{\beta_n} \leq \frac{2\mu}{\delta} < 1,$$

for all $n \geq n_0$. That is,

$$f(\lambda) = (f_n(\lambda))_n = (\gamma_n f_1)_n \in X,$$

and $Af(\lambda) = \lambda f(\lambda)$. The map $f : \mathcal{U} \to X$ is weakly holomorphic. Indeed, if

$$\phi \in X^* = \ell^\infty \left(L^\infty([0,\ +\infty[)\right),$$

then

$$h(\lambda) := \langle f(\lambda),\ \phi_n \rangle = \sum_{n\geq 1} \int_0^\infty f_n \phi_n = \eta_1 + \sum_{n\geq 2} \eta_n \prod_{k=1}^{n-1} (\lambda + \alpha_k),$$

where $\eta_1 = \int_0^\infty f_1\phi_1$ and $\eta_n = \left(\prod_{k=1}^{n-1} \frac{1}{\beta_k}\right) \int_0^\infty f_1 \phi_n,\ n \geq 2.$

By the selection of $\{\alpha_n\}_n$ and $\{\beta_n\}_n$ and $\mathcal{U}$, we obtain that $h$ is holomorphic on $\mathcal{U}$
(because the partial sums are a polynomial and the series is uniformly converge in the compacts of $\mathcal{U}$). Since we want to apply Proposition 3.18, it only remains to show that, if $\phi \in X^*$ vanishes on every $f(\lambda),\ \lambda \in \mathcal{U}$, and for each $f_1 \in L^1([0, +\infty[)$, then $\phi = 0$. To do so, let

$$0 = h(\lambda) = \eta_1 + \sum_{n\geq 2} \eta_n \prod_{k=1}^{n-1} (\lambda + \alpha_k).$$

We substitute $h(-\alpha_1) = 0$, which implies $\eta_1 = 0$, and we can decompose $h$ as a product

$$h(\lambda) = (\lambda + \alpha_1) \left( \eta_2 + \sum_{n\geq 3} \eta_n \prod_{k=2}^{n-1} (\lambda + \alpha_k) \right).$$

The right part of the product vanishes on $\mathcal{U}$, and a substitution of $\lambda = -\alpha_2$ into the right part gives $\eta_2 = 0$. We can decompose

$$h(\lambda) = (\lambda + \alpha_1)(\lambda + \alpha_2) g(\lambda),$$

and $g$ vanishes on $\mathcal{U}$. We inductively obtain $\eta_k = 0,\ k \in \mathbb{N}$, which yields $\int_0^\infty f_1\phi_k = 0$ for each $k \in \mathbb{N}$, and for every $f_1 \in L^1([0, +\infty[)$. This implies $\phi = 0$, and the $C_0$-semigroup is chaotic and mixing by Proposition 3.18. □

### 3. Birth model with variable coefficients

In the article of Jacek Banasiak and Miroslav Lachowicz in [**BL01**] the authors also discuss the opposite process, the birth model, to the one considered above.

The rate equations are:

$$\frac{df_0}{dt} = -\alpha_0 f_0,$$
$$\frac{df_n}{dt} = -\alpha_n f_n + \beta_{n-1} f_{n-1},\ n \in \mathbb{N}. \tag{4.8}$$

If we have the same assumptions (4.2), of the death model (first approach), on the sequences $\{\alpha_n\}_{n\in\mathbb{N}_0}$ and $\{\beta_n\}_{n\in\mathbb{N}_0}$. Then it is clear that the matrix:

$$K^T = \begin{pmatrix} -\alpha_0 & 0 & 0 & 0 & \dots \\ \beta_0 & -\alpha_1 & 0 & 0 & \dots \\ 0 & \beta_1 & -\alpha_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

defines a bounded operator $\mathcal{K}_0^T$ in $c_0$, and $\mathcal{K}_p^T$ in $X^p,\ 1 \le p \le \infty$, respectively.

We define the $C_0$-semigroup $\{S_p(t)\}_{t\ge0}$ generated by $\mathcal{K}_p^T$, $p \in \{0\} \cup [1,\ +\infty]$. If the $C_0$-semigroup is not hypercyclic then is not chaotic. Then we have:

**Theorem 4.6** ([**BL01**])**.** *The $C_0$-semigroup $\{S_p(t)\}_{t\ge0}$ is not hypercyclic for any $p \in \{0\} \cup [1,\ +\infty)$*

CHAPTER 5

# Study for dynamics of birth-and-death processes

In this chapter we study stability and chaos of the $C_0$-semigroup associated with the infinite birth-and-death systems with proliferation (with constant and variable coefficients).

In particular we discuss the stability and chaos results of [**BK99**] and extend in [**BM11**]. A recent model discussed in *op. cit.*, concern development of drug resistance in cancer cells. We consider as in the above chapter a medium with a population of cells divided in subpopulations and this subpopulations are characterised by the number of copies of a drug-resistant gene. If we consider a subpopulation $j$ in a population of cells and we assume that in a single event this subpopulation can generate cells with the drug-resistance gene only in the neighbouring subpopulations $j-1$ or $j+1$. Further, the rate of this events are $b_j$ and $d_j$. Moreover, in the same subpopulation $j$ the rate of this event are $a_j$. This is the called birth-and-death processes with proliferation.

In this chapter the space and the norm are the same that Chapter 4:

$$X := \ell^1\left(L^1([0,+\infty[)\right), \quad X^* = \ell^\infty\left(L^\infty([0,\ +\infty[)\right),$$

$$0 \le \mathbf{f} = (f_n)_{n\ge 1} \in X, \qquad \text{and } \|\mathbf{f}\| = \sum_{n=1}^{\infty} \|f_n\|_1.$$

## 1. Birth-and-death model with constant coefficients

We can write the model with constant coefficients as following:

$$\begin{aligned} \frac{df_1}{dt} &= af_1 + df_2, \\ \frac{df_n}{dt} &= bf_{n-1} + af_n + df_{n+1}, \quad n \ge 2. \end{aligned} \tag{5.1}$$

As discussed in [**BM11**] is essential that the system of equations have infinite dimension, because in e.g. [**KS94, KPS96, KPS98**] the authors obtain a result in the following case, when $0 < b < d$ and for a initial condition $f_1(0) = 1,\ f_i(0) = 0$ for $i \ge 2$, the drug-resistant population $\displaystyle\sum_{i=1}^{\infty} f_i$ decays exponentially to 0 if, and only if,

$$(\sqrt{d} - \sqrt{b})^2 \ge a + b + d, \tag{5.2}$$

and as we will seen later in this conditions we have chaos. The conclusion are clear, we can not use the results in the finite dimensional sampling in the behaviour of an infinite-dimensional system.

Following the work of Jacek Banasiak and Marcin Moszyński in [**BM11**], we have a few observations of this model.

$$\text{If } \ a,\ b,\ d \in \mathbb{R} \ \text{ and } \ b,\ d \neq 0, \tag{5.3}$$

let us denote by $\mathcal{L}$ the infinite matrix:

$$\begin{pmatrix} a & d & & & \\ b & a & d & & \\ & b & a & d & \\ & & b & a & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{5.4}$$

Note that if $X = X_1$, we also can use the spaces $X_s := \ell^1_s\left(L^1([0,+\infty[)\right)$ which are the spaces $X$ with the weight $s^n$, $s > 0$, and the norms are:

$$\|\mathbf{f}\|_s := \sum_{n=1}^{\infty} \|f_n\|_1 s^n.$$

Note also that $X_s \subseteq X$ are dense because,

$$\|\mathbf{f}\| \le s^{-1}\|\mathbf{f}\|_s, \quad \mathbf{f} \in X_s \text{ and for } s \ge 1, \tag{5.5}$$

The matrix $\mathcal{L}$ represents a bounded operator in $X_s$ for any $s > 0$ which we denote by $\widetilde{L}_s$ and $L := \widetilde{L}_1$.

Following [**BK99, BM11**], we observe that $U_s\mathbf{f} := \left(\frac{f_n}{s^n}\right)_{n\ge 1}$ defines an isometry from $X$ onto $X_s$ hence we can transfer $\tilde{L}_s$ to $X$ as $L_s := U_s^{-1}\widetilde{L}_s U_s$. Then we have

$$L_s = aI + C_s, \tag{5.6}$$

where $C_s$ is an operator in $X$ represented by

$$\begin{pmatrix} 0 & d/s & & & \\ sb & 0 & d/s & & \\ & sb & 0 & d/s & \\ & & sb & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{5.7}$$

To shorten notation, we put $C := C_1$, and note that also $L = L_1 = \widetilde{L}_1$.

**1.1. Stability.** In this subsection we focus on stability of the $C_0$-semigroups related to (5.1). You can see some general notions in Definitions 3.3 and 3.6, and details of this results in [**BM11**].

**Observation 1** ([**BM11**]). *Let us make an observation which will be used later.*

1. *If for some $s \ge 1$ the $C_0$-semigroup $(e^{t\widetilde{L}_s})_{t\ge 0}$ is exponentially stable then, $(e^{tL})_{t\ge 0}$ is exponentially stable on the subspace $X_s \subset X$.*
2. *For any $s > 0$ the $C_0$-semigroup $(e^{t\widetilde{L}_s})_{t\ge 0}$ is exponentially stable if, and only if, $(e^{tL_s})_{t\ge 0}$ is exponentially stable.*

By (5.6) we have

$$e^{tL_s} = e^{at} e^{tC_s}, \tag{5.8}$$

where $e^{tC_s}$ is given by (3.3). It is easy to see that

$$|||e^{tC_s}||| \leq e^{t|||C_s|||} \leq e^{t(|b|s+|d|/s)}. \tag{5.9}$$

The following lemma from [BM11] is a correction of [BK99]:

**Lemma 5.1.** *We have*

$$(C_s^k \boldsymbol{f})_n = \sum_{i=0}^{k} \left[ \binom{k}{i} - \binom{k}{k-(n+i)} \right] (sb)^{k-i} (\frac{d}{s})^i f_{n-k+2i}, \tag{5.10}$$

*where* $\boldsymbol{f} = (f_1, f_2, \ldots)$, $f_i = 0$ *for* $i \leq 0$ *and the Newton symbol is also* $0$ *for the negative entries.*

With this correction, the following theorem is valid with the original proof of [BK99].

**Theorem 5.2.** *Let* $b, d > 0$. *The* $C_0$*-semigroup* $(e^{tL_s})_{t\geq 0}$ *is exponentially stable if, and only if,* $a < -2\sqrt{bd}$ *and* $s \in (\sigma_-,\ \sigma_+)$ *where*

$$\sigma_{\pm} = \frac{-a \pm \sqrt{a^2 - 4bd}}{2b}. \tag{5.11}$$

An extension of the above result are:

**Theorem 5.3** ([BM11])**.** *Let (5.3) holds,* $a < -2\sqrt{|b||d|}$ *and* $s \in (s_-,\ s_+)$ *where*

$$s_{\pm} = \frac{-a \pm \sqrt{a^2 - 4|b||d|}}{2|b|}, \tag{5.12}$$

*then* $s > 0$ *and the* $C_0$*-semigroup* $(e^{tL_s})_{t\geq 0}$ *is exponentially stable.*

We note that for any $s > 0$ we have $X_{fin} \subset X_s$, where $X_{fin}$ is the space of sequences with finitely many nonzero entries. And finally:

**Theorem 5.4** ([BM11])**.** *Suposse (5.3). If*

***(i)*** *:* $a < -(|b| + |d|)$

***or***

***(ii)*** *:* $-(|b| + |d|) \leq a < -2\sqrt{|b||d|}$ *and* $a < -2|b|$,

*then there is a dense subspace of* $X$*, containing* $X_{fin}$*, on which* $(e^{tL})_{t\geq 0}$ *is exponentially stable.*

*In particular the assertion holds if*

$$a < -2\sqrt{|b||d|} \quad \text{and} \quad 0 < |b| \leq |d|. \tag{5.13}$$

**1.2. Chaos.** In this subsection we will present an alternative version of the article of Banasiak and Moszyński [BM11]. We will determine a range of the parameters $a,\ b,\ d \in \mathbb{R}$ for witch $(e^{tL})_{t\geq 0}$ is chaotic in the sense of Devaney, but the final conditions are the same that in [BM11].

We consider $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ an eigenvalue such that $\lambda \mathbf{f} = \mathcal{L}\mathbf{f}$.

We will find the eigenvectors such that $\mathbf{f} = (f_n)_{n\geq 1} \in X$ satisfies:

$$bf_{n-1} + (a-\lambda)f_n + df_{n+1} = 0, \quad n \geq 2, \tag{5.14}$$

and the initial condition

$$f_2 = \frac{\lambda - a}{d} f_1. \tag{5.15}$$

Note that (5.14) is equivalent to:

$$f_{n+1} = \frac{\lambda - a}{d} f_n - \frac{b}{d} f_{n-1}. \tag{5.16}$$

We rewrite the above equation:

$$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{b}{d} & \frac{\lambda-a}{d} \end{pmatrix} \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix}. \tag{5.17}$$

We denote by $C = \begin{pmatrix} 0 & 1 \\ -\frac{b}{d} & \frac{\lambda-a}{d} \end{pmatrix}$.

We observe that the equation (5.17) is equivalent to:

$$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = C^{n-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{5.18}$$

From now on, we will find the eigenvalues, the diagonal matrix such the $C = PDP^{-1}$ and the conditions for the coefficients such that $C^k = PD^kP^{-1}$, $k \in \mathbb{N}$ converge, to have the elements of $D = \begin{pmatrix} z_+ & 0 \\ 0 & z_- \end{pmatrix}$ with absolute value strictly smaller than 1.

If $\mathbb{I}$ is the identity matrix we consider the determinant of the characteristic matrix

$$0 = \det(C - z\mathbb{I}) = \det \begin{pmatrix} -z & 1 \\ -\frac{b}{d} & \frac{\lambda-a}{d} - z \end{pmatrix}.$$

The characteristic equation is $dz^2 + (a-\lambda)z + b = 0$, then

$$z^2 + \frac{a-\lambda}{d} z + \frac{b}{d} = (z - z_+)(z - z_-) = 0 \tag{5.19}$$

From the characteristic equation, we have

$$z_+ z_- = \frac{b}{d}, \quad \text{and} \quad z_+ + z_- = \frac{\lambda - a}{d}. \tag{5.20}$$

**Lemma 5.5.** *Given $f_0 \in L^1([0, +\infty[)$, if the roots of the characteristic equation (5.19) are so that $|z_\pm| < 1$, then the vector $\boldsymbol{f} = (f_n)_{n\geq 1} \in X$ defined by*

$$f_n = ((z_+)^n - (z_-)^n) f_0, \quad n \in \mathbb{N}, \tag{5.21}$$

*is an eigenvector of $\mathcal{L}$ to the eigenvalue $\lambda$.*

Proof. Note that the roots $z_+$ and $z_-$ are solution of the characteristic equation (5.19), then $d(z_\pm)^2 + az_\pm + b = \lambda z_\pm$ and we know that $|z_\pm| < 1$, thus $z_\pm^n \to 0$, if $n \to \infty$. Moreover, $\mathcal{L}\mathbf{f} = (\mathcal{L}_n\mathbf{f})_{n\geq 1}$ and we observe that if $f_1 = (z_+ - z_-)f_0$ we can use the initial condition (5.15) and we obtain that

$$(z_+^2 - z_-^2)f_0 = (z_+ + z_-)(z_+ - z_-)f_0 = (z_+ + z_-)f_1 = f_2.$$

If $n = 1$, then $\mathcal{L}_1\mathbf{f} = af_1 + df_2$ and we get that $\mathcal{L}_1\mathbf{f} = af_1 + df_2 = af_1 + d\left(\frac{\lambda - a}{d}\right) f_1 = \lambda f_1$, in other case, for $n \geq 2$ we get:

$$\begin{aligned}
\mathcal{L}_n\mathbf{f} &= bf_{n-1} + af_n + df_{n+1} \\
&= f_1\left[b((z_+)^{n-1} - (z_-)^{n-1}) + a((z_+)^n - (z_-)^n) + d((z_+)^{n+1} - (z_-)^{n+1})\right] \\
&= f_1\left[(z_+)^{n-1}(d(z_+)^2 + az_+ + b) - (z_-)^{n-1}(d(z_-)^2 + az_- + b)\right] \\
&= f_1\left[(z_+)^{n-1}\lambda z_+ - (z_-)^{n-1}\lambda z_-\right] \\
&= \lambda((z_+)^n - (z_-)^n)f_1 \\
&= \lambda f_n
\end{aligned}$$

□

We will find now the conditions that ensure $|z_\pm| < 1$, and that the criterion for chaos is satisfied. Since

$$\left|\frac{b}{d}\right| = |z_+z_-| < 1 \longrightarrow |b| < |d|. \tag{5.22}$$

Also, we would like to find an open set $\mathcal{U} \subset \mathbb{C}$ such that $\mathcal{U}$ intersects the imaginary axis, an such that $|z_\pm(\lambda)| < 1$ if $\lambda \in \mathcal{U}$. We then use Lemma 1.14 for $w = \frac{a-\lambda}{d}$, $r = \frac{b}{d}$ and $\lambda = iy$ with $y \in \mathbb{R}$ to get

$$\frac{a^2}{(b+d)^2} + \frac{y^2}{(d-b)^2} < 1. \tag{5.23}$$

Thus, on the one hand $|a| < |b+d|$ is necessary, and the above condition is satisfied if $\lambda = iy$ is close enough to 0. We fix $y_1 > 0$ with

$$\frac{a^2}{(b+d)^2} + \frac{y_1^2}{(d-b)^2} < 1, \tag{5.24}$$

let $y_0 := \frac{y_1}{2}$. For every $\lambda \in i[y_0,\ y_1]$ we then have $|z_\pm(\lambda)| < 1$.

Let us consider the curve $\lambda(s) := siy_1 + (1-s)iy_0$, $s \in [0,\ 1]$. We observe that $\gamma(s) := (\lambda(s) - a)^2 - 4bd$, $s \in [0,\ 1]$, is an injective curve and $0 \notin \gamma^*$, being $\gamma^*$ the image of $\gamma$. In fact, if we consider $x_1,\ x_2 \in [0,\ 1]$ and $\gamma(x_1) = \gamma(x_2)$ then we have that

$$(\lambda(x_1) - \lambda(x_2))(\lambda(x_1) + \lambda(x_2)) = 2a(\lambda(x_1) - \lambda(x_2)).$$

If $\lambda(x_1) \neq \lambda(x_2)$ then we get a contradiction because $\lambda(x_1) + \lambda(x_2) = 2a \in \mathbb{R}$ and $\lambda(s) \in i\mathbb{R}$. In the case that $a = 0$ we also have a contradiction because as $y_1 > 0$, this implies that $\lambda(x_1) \neq -\lambda(x_2)$. If $\lambda(x_1) = \lambda(x_2)$, simplifying we get that $(x_1 - x_2)y_1 = (x_1 - x_2)y_0$, if $x_1 \neq x_2$ this is a contradiction with $y_0 := \frac{y_1}{2}$ and finally $x_1 = x_2$ and $\gamma(s)$ is an injective curve. Note that $\lambda(s) = 0 \longleftrightarrow s = -1$ but $-1$ is not in the domain and this implies that $0 \notin \gamma^*$. In other case, if we suppose that there exists $s \in [0,\ 1]\ :\ \gamma(s) = 0$ then $\lambda(s) = a \pm 2\sqrt{bd} \in \mathbb{C}\backslash i\mathbb{R}$ and we get a contradiction because $\lambda(s) \in i\mathbb{R}$.

This means that there is a branch of the logarithm on a neighbourhood $\mathcal{U}$ of $\gamma^*$, that we suppose without loss of generality simply connected. As a consequence, we find a holomorphic map $\varphi : \mathcal{U} \to \mathbb{C}$ such that $\varphi(\lambda)^2 = (\lambda - a)^2 - 4bd$ for all $\lambda \in \mathcal{U}$. We then define, for $\lambda \in \mathcal{U}$,

$$z_+(\lambda) = \frac{(\lambda - a) + \varphi(\lambda)}{2d} \quad \text{and} \quad z_-(\lambda) = \frac{(\lambda - a) - \varphi(\lambda)}{2d}, \tag{5.25}$$

which are holomorphic maps on $\mathcal{U}$. Moreover, $z_+(\lambda)$ and $z_-(\lambda)$ are precisely the roots of the characteristic equation (5.19).

Since we know that $|z_\pm(\lambda(s))| \in \left]\frac{|b|}{|d|},\ 1\right[$ for all $s \in [0,\ 1]$, without loss of generality we suppose that $\mathcal{U}$ is small enough so that $|z_\pm(\lambda)| \in \left]\frac{|b|}{|d|},\ 1\right[$ for all $\lambda \in \mathcal{U}$.

If we assume that for some $\phi = (\phi_n)_{n\geq 1} \in X^*$ the function $h(\lambda) = \langle \mathbf{f}(\lambda), \phi\rangle$ is identically zero on $\mathcal{U}$, we will prove then $\phi = 0$, and that the $C_0$-semigroup is chaotic and mixing by Proposition 3.18.

If we have $0 = h(\lambda) = \langle \mathbf{f}(\lambda), \phi\rangle = \sum_{n\geq 1} \langle f_n(\lambda), \phi_n\rangle$, then

$$\begin{aligned} 0 &= \sum_{n\geq 1} \langle f_0, \phi_n\rangle \left((z_+(\lambda))^n - (z_-(\lambda))^n\right) \\ &= \sum_{n\geq 1} \langle f_0, \phi_n\rangle \left((z_+(\lambda))^n - \left(\frac{b/d}{z_+(\lambda)}\right)^n\right) \\ &= \sum_{n=-\infty}^{+\infty} g_n (z_+(\lambda))^n, \end{aligned}$$

where

$$g_n = \begin{cases} \langle f_0, \phi_n\rangle & \text{for } n \geq 1, \\ 0 & \text{for } n = 0, \\ -\langle f_0, \phi_{-n}\rangle \left(\frac{b}{d}\right)^{-n} & \text{for } n \leq -1. \end{cases}$$

We define $g : D \longrightarrow \mathbb{C}$ by

$$g(z) = \sum_{n=-\infty}^{+\infty} g_n z^n, \quad z \in D := \left\{\frac{|b|}{|d|} < |z| < 1\right\}.$$

We know that

$$z \in D' := \{z_+(\lambda)\ :\ \lambda \in \mathcal{U}\} \subset D, \text{ and}$$

$$0 = h(\lambda) = g(z_+(\lambda)), \quad \lambda \in \mathcal{U}.$$

Hence, by analyticity, $g$ is the zero on $D'$, therefore in $D$ and uniqueness of the Laurent expansion yields $g_n = 0, \quad \forall n \in \mathbb{Z}$. In particular, $\langle f_0, \phi_n\rangle = 0, \quad \forall n \in \mathbb{N}$,

$f_0 \in L^1[0,\ +\infty[$, which gives $\phi_n = 0$, $\forall n \in \mathbb{N}$. A direct consequence of these arguments is:

**Theorem 5.6.** *If $0 < |b| < |d|$ and $|a| < |b+d|$, then $(e^{tL})_{t\geq 0}$ is chaotic in $X$.*

Another results related with this problem are:

**Corollary 5.7** ([**BM11**]). *If $0 < |b| < |d|$ and $|a| < |b+d|$, then $(e^{tL})_{t\geq 0}$ is chaotic in $X^p$ and $c_0(L^1([0+\infty[))$.*

**Theorem 5.8** ([**BM11**]). *If $1 < p \leq \infty$ and $|d| < |b|$ (or if $p = 1$ and $|d| \leq |b|$) and $|a| < |b+d|$, then $(e^{tL})_{t\geq 0}$ is not hypercyclic in $X^p$, and thus it is not chaotic.*

### 2. Birth-and-death model with variable coefficients

In this section we will study the above problem but now with variable coefficients. We consider the same space $X$ and the same norm with the following problem:

$$\begin{aligned} \frac{df_1}{dt} &= a_1 f_1 + d_1 f_2, \\ \frac{df_n}{dt} &= b_n f_{n-1} + a_n f_n + d_n f_{n+1}, \quad n \geq 2, \end{aligned} \tag{5.26}$$

$\mathbf{f} = (f_n)_{n\geq 1} \in X$ and if $a_n,\ b_n,\ d_n \in \mathbb{R}$, with $f_0 = 0$, and let us denote by $\mathcal{L}$ the infinite matrix:

$$\begin{pmatrix} a_1 & d_1 & & & \\ b_2 & a_2 & d_2 & & \\ & b_3 & a_3 & d_3 & \\ & & b_4 & a_4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \tag{5.27}$$

We will find conditions in the coefficient for witch the $C_0$-semigroup solution for the above problem generate chaos or sub-chaos in $X$. For this problem we have a result since 2007 in [**BLM07**] by Banasiak, Lachowicz and Moszyński. They prove the following results:

**Theorem 5.9** ([**BLM07**]). *If $L_{max} = \mathcal{L}_{|D(L_{max})}$, with $D(L_{max}) = \{f \in X_p : \mathcal{L}f \in X_p\}$ Suppose that $1 \leq p < \infty$ and that there exists $N_0 \geq 1$ such that:*

$$\begin{aligned} &a_n = an + \alpha, \quad d_n = d(n-1) + \delta, \quad b_n = b(n+1) + \beta, \text{ for } n \geq N_0, \\ &\text{with } a = -(b+d), \quad b, d \geq 0, \quad \alpha,\ \beta,\ \delta \in \mathbb{R} \end{aligned} \tag{5.28}$$

*holds with $d > b$ and $\alpha + \beta + \delta - \frac{d-b}{p} > 0$.*
*Then the $C_0$-semigroup generated by $L_{max}$ in $X_p$ is sub-chaotic.*

**Theorem 5.10** ([**BLM07**]). *Suppose that assumptions (5.28) is satisfied, $p \in [1,\ +\infty)$, and either of two cases hold:*

- *$b > d$.*
- *$d_{m_0} = 0$ for some $m_0 \geq 1$*

*Then the $C_0$-semigroup generated by $L_{max}$ is not topologically chaotic.*

Note that the $C_0$-generated by $L_{max}$ is **sub-chaotic**, but **not chaotic** in the sense of Devaney, see the Definition 5.11.

**Definition 5.11** ([**BL05**])**.** Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a $C_0$-semigroup in a Banach space $X$ ($X \neq \{0\}$). We call that $\mathcal{T}$ is sub-chaotic if there exists a closed subspace $Y \neq \{0\}$ invariant under $\mathcal{T}$, such that $\mathcal{S} = \{T_t\mid_Y\}_{t\geq 0}$ is topologically chaotic (as a $C_0$-semigroup in $Y$).

**Remark 5.12** ([**BL05**])**.** Each topologically chaotic $C_0$-semigroup is also sub-chaotic. The dimension of any space of sub-chaotic is infinite.

The finality of this work is find a generalization of this problem. Is possible find chaos on $X := \ell^1\left(L^1([0,+\infty[)\right)$ in general? I don't have answer for this question, but now we will study another possibilities for find answers for this problem in the future.

We consider $\lambda \in \mathbb{C}\backslash\{0\}$ an eigenvalue such that $\lambda\mathbf{f} = \mathcal{L}\mathbf{f}$.
We will find the eigenvectors such that $\mathbf{f} = (f_n)_{n\geq 1} \in X$ satisfies:

(5.29)
$$b_n f_{n-1} + (a_n - \lambda) f_n + d_n f_{n+1} = 0, \quad n \geq 2,$$

with, $f_1 \in L^1([0,+\infty[)$, $d_1 \neq 0$ and the initial condition

(5.30)
$$f_2 = \frac{\lambda - a_1}{d_1} f_1.$$

Note that, providing that $d_n \neq 0$, (5.29) is equivalent to:

(5.31)
$$f_{n+1} = \frac{\lambda - a_n}{d_n} f_n - \frac{b_n}{d_n} f_{n-1}, \quad n \geq 2.$$

Without loss of generality we can assume that $d_n \neq 0, \ \ \forall n$.
We rewrite the above equation as follow:

(5.32)
$$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{b_n}{d_n} & \frac{\lambda - a_n}{d_n} \end{pmatrix} \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix}.$$

We denote by $A_n = \begin{pmatrix} 0 & 1 \\ -\frac{b_n}{d_n} & \frac{\lambda - a_n}{d_n} \end{pmatrix}$.
We note that the equation (5.32) is equivalent to:

(5.33)
$$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \left[\prod_{i=2}^{n} A_i\right] \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Following the notation of the previous section, we have that the characteristic equation is: $\det(A_n - z\mathbb{I}) = d_n z^2 + (a_n - \lambda) z + b_n = 0$, i.e.,

(5.34)
$$z_n^{\pm}(\lambda) = \frac{(\lambda - a_n) \pm \sqrt{(\lambda - a_n)^2 - 4 b_n d_n}}{2 d_n}.$$

It should be noted that the roots are simple if the discriminant is non zero, i.e., $\lambda \neq a_n \pm 2\sqrt{b_n d_n}$ and $|z_n^{\pm}(\lambda)| \neq 0$ if $b_n \neq 0$. In that case, we have different roots and we can decompose $A_n$ as $A_n = P_n D_n P_n^{-1}$ where,

(5.35)
$$P_n = \begin{pmatrix} 1 & 1 \\ z_n^{+}(\lambda) & z_n^{-}(\lambda) \end{pmatrix}, \quad P_n^{-1} = \frac{1}{z_n^{-}(\lambda) - z_n^{+}(\lambda)} \begin{pmatrix} z_n^{-}(\lambda) & -1 \\ -z_n^{+}(\lambda) & 1 \end{pmatrix},$$

and

$$D_n = \begin{pmatrix} z_n^+(\lambda) & 0 \\ 0 & z_n^-(\lambda) \end{pmatrix}. \tag{5.36}$$

A basic reasonable assumption is that $\lim_{n\to\infty} P_n = P$ (and $\lim_{n\to\infty} P_n^{-1} = P^{-1}$), in other words,

$$\lim_{n\to\infty} z_n^{\pm}(\lambda) = z^{\pm}(\lambda). \tag{5.37}$$

Another assumption is $0 < |z^{\pm}(\lambda)| < 1$.

Note that, the convergence of $P^{-1}$ implies that the roots $z_n^{\pm}(\lambda)$ are different and also $z^-(\lambda) \neq z^+(\lambda)$.

The above equation implies two natural cases to consider:
If $0 \neq d \in \mathbb{R}$ and $a,\ b,\ \alpha,\ \beta \in \mathbb{R}$,

$$\begin{cases} \text{Case 1:} \quad \lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty} b_n = b, \quad \lim_{n\to\infty} d_n = d, \\ \\ \text{Case 2:} \quad \lim_{n\to\infty} \dfrac{a_n}{d_n} = \alpha, \quad \lim_{n\to\infty} \dfrac{b_n}{d_n} = \beta, \quad \lim_{n\to\infty} d_n = \infty. \end{cases} \tag{5.38}$$

A fast observation is that:

- In the **Case 1**, $z^{\pm}(\lambda) = \frac{(\lambda-a)\pm\sqrt{(\lambda-a)^2-4bd}}{2d}$ with $\lambda \neq a \pm 2\sqrt{bd}$,
- and in the **Case 2**, $z^{\pm}(\lambda) = \frac{-\alpha\pm\sqrt{\alpha^2-4\beta}}{2}$ is a constant and also $\alpha^2 \neq 4\beta$.

This simple observation implies that in **Case 1** we have that $z^+(\lambda)$ and $z^-(\lambda)$ are either simple reals roots or complex roots with depending of $\lambda$ and in the **Case 2** we have the same case, but now these roots do not depend of $\lambda$. These observations are essential for the results because if we do not have simple roots, the matrix $P^{-1}$ does not exist.

The first problem to solve chaos of (5.26) is if $\mathbf{f} \in X$ whenever $\mathcal{L}\mathbf{f} = \lambda\mathbf{f}$.

**Lemma 5.13.** *Assume* $\lim_{n\to\infty} z_n^{\pm}(\lambda) = z^{\pm}(\lambda)$ *with* $|z^{\pm}(\lambda)| < 1$, *then*

$$\sum_{n\geq 2} \|A_n \cdots A_2\| < +\infty.$$

*Consequently, if* $\mathcal{L}\boldsymbol{f} = \lambda\boldsymbol{f}$ *then* $\boldsymbol{f} \in X$.

Proof. We recall that $z_n^{\pm}(\lambda) \to z^{\pm}(\lambda)$ then, we fix $\delta > 0$ such that $0 < |z_n^{\pm}(\lambda)| < \delta < 1$.

We have there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0,\ \|D_n\| \leq \max\{|z_n^+(\lambda)|,\ |z_n^-(\lambda)|\} < \delta$. As $P_n$ and $P_n^{-1}$ converge to $P$ and $P^{-1}$ respectively, then $\exists\ M > 0$ such that $\|P_m\| < M$ and $\|P_m^{-1}\| < M$, for all $m \in \mathbb{N}$, and if we fix $j > \dfrac{\delta^2}{1-\delta}$ we obtain: exists $n_1 \geq n_0$ such that for every $m \geq n_1$, $\|P_{m+1}^{-1}P_m - \mathbb{I}\| < \frac{\delta}{j} < 1$.

Thus $\|P_{m+1}^{-1}P_m\| - \|\mathbb{I}\| \leq \|P_{m+1}^{-1}P_m - \mathbb{I}\| < \frac{\delta}{j}$ and this implies that:

$$\|P_{m+1}^{-1}P_m\| < 1 + \frac{\delta}{j}. \tag{5.39}$$

Finally, if we denote by $N := \|A_{n_1-1}\cdots A_1\|$ and $k := (n - n_1)$ then exists $\delta_1 := \left(\delta + \dfrac{\delta^2}{j}\right) < 1$ such that:

$$\begin{aligned}\|A_n \cdots A_1\| &\leq N\|P_n D_n (P_n^{-1}P_{n-1})D_{n-1}\cdots(P_{n_1+1}^{-1}P_{n_1})D_{n_1}P_{n_1}^{-1}\| \\ &\leq \|P_n\|\|P_{n_1}^{-1}\|\left(\prod_{m=n_1}^{n}\|D_m\|\right)\left(\prod_{m=n_1}^{n-1}\|P_{m+1}^{-1}P_m\|\right) \\ &< M^2 N\delta^k\left(1+\frac{\delta}{j}\right)^k \\ &= M^2 N\delta_1^k.\end{aligned}$$

□

Note that, exists $m_0 \in \mathbb{N}$ such that, for every $n \geq m_0$, $\quad 0 < |z_n^{\pm}(\lambda)| < 1$.

In this assumption, for every $n \geq m_0$, by above discussion in the previous section, we consider the following cases:

- In the **Case 1**, we define $K_1 := \lambda^*$, being $\lambda^*$ the image of $\lambda$, defined in the previous section. We observe that $K_1$ is a compact interval and $K_1 \subset i\mathbb{R}$.
- In the **Case 2**, the limit case implies that $z^{\pm}(\lambda)$ is a constant respect to $\lambda$ and, if $|z^{\pm}(\lambda)| < 1$, we choose $K_2 := B(0,1)$ the unit ball. Note that if we apply the Lemma 1.14 with $w = \alpha$ and $r = \beta$ then $|z^{\pm}(\lambda)| < 1$ if, and only if, $|\beta| < 1$ and $\alpha^2 < (1+\beta)$.

In order to obtain sub-chaos, we choose the subspace $Y$ in $X$ as follows,

$$Y := \overline{span}\{f(\lambda)\ : \mathcal{L}f(\lambda) = \lambda f(\lambda),\quad \lambda \in K_i\},\text{ where } i = 1,\ 2,\text{ depend of the case.}$$

Obviously, $\{0\} \neq Y$ and $Y$ is a closed subspace of $X$. Moreover, $Y$ is clearly a invariant subspace under $\mathcal{T} = \{T_t\}_{t\geq 0}$ (i.e., $T_tY \subset Y$, if $t \geq 0$), because if $f(\lambda) \in Y$ we can rewrite $T_tf(\lambda) = \lambda T_tf(\lambda)$.

**Theorem 5.14.** *Assume $\lim_{n\to\infty} z_n^{\pm}(\lambda) = z^{\pm}(\lambda)$ with $|z^{\pm}(\lambda)| < 1$, and the subspace*

$$Y := \overline{span}\{f(\lambda)\ : \mathcal{L}f(\lambda) = \lambda f(\lambda),\quad \lambda \in K_i\},\text{ where } i = 1,\ 2,\text{ depend of the case.}$$

*Then the $C_0$-semigroup $\mathcal{S} = \{T_t\ |_Y\}_{t\geq 0}$ is chaotic and mixing on $Y$, i.e., $\mathcal{T}$ is sub-chaotic in $X$.*

Proof.

- In the **Case 1**, if we consider the continuous function $\lambda \to f(\lambda) = (f_n(\lambda))_{n\geq 1}$ for $\lambda \in K_1$, begin $f_n$ is the same function that in the equation (5.32). We conclude that the $C_0$-semigroup $\mathcal{S} = \{T_t\ |_Y\}_{t\geq 0}$, is chaotic and mixing in $Y$ by the Proposition 3.19.
- In the **Case 2**, if we consider the same function for $\lambda \in K_2$. Note that we can write $f_n(\lambda) = P_n(\lambda)f_1$, where $P_n(\lambda)$ is a polynomial. This implies that $h(\lambda) = \langle f(\lambda), \phi\rangle$ is a holomorphic function on $K_2$ and as $f_1 \in L^1([0,+\infty[)$, if $h(\lambda) = 0$ then $\phi = (\phi_n)_{n\geq 1} = 0$. We conclude that the $C_0$-semigroup $\mathcal{S}$ is also chaotic and mixing in $Y$ by the Proposition 3.18.

□

# Bibliography


[Poin85] Poincaré, H. *Sur les Equations Lineaires aux Differentielles Ordinaires et aux Differences Finies*. Amer. J. Math. 7 (1885), no. 3, 203–258.

[Bir20] Birkhoff, George David. *Surface transformations and their dynamical applications*. Acta Math. 43 (1920), 1-119.

[Perr21] Perron, O. *Über Summengleichungen und Poincarésche Differenzengleichungen*. Math. Ann. 84 (1921), no. 1-2, 1–15.

[OU41] Oxtoby, J. C.; Ulam, S. M. *Measure-preserving homeomorphisms and metrical transitivity*. Ann. of Math. 42 (1941), no. 2, 874-920.

[Kell55] Kelley, John L. *General Topology*. D. Van Nostrand Company, Inc., Toronto-New York-London, 1955.

[HP57] Hille, Einar; Phillips, Ralph S. *Functional analysis and semi-groups*. American Mathematical Society Colloquium Publications, 31, 1957.

[RR64] Robertson, A. P.; Robertson, W. J. *Topological vector spaces*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 53 Cambridge University Press, New York 1964.

[Spa66] Spanier, Edwin H. *Algebraic topology*. McGraw-Hill, New York-Toronto, Ont.-London, 1966.

[Rud74] Rudin,W. *Real and complex analysis*. Second edition. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1974.

[Kit82] Kitai, Carol. *Invariant closed sets for linear operators*. Thesis (Ph.D.)-University of Toronto, Toronto, Canada, 1982.

[Wim84] Wimp, J. *Computation with recurrence relations*. Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[Bea86] Beauzamy, Bernard. *Un opérateur, sur l'espace de Hilbert, dont tous les polynômes sont hypercycliques*. C. R. Acad. Sci. Paris Sér. I Math. 303 (1986), no. 18, 923-925.

[Dev89] Devaney, Robert L. *An introduction to chaotic dynamical systems*. Second edition. Addison-Wesley Studies in Nonlinearity. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989.

[GS91] Godefroy, Gilles; Shapiro, Joel H. *Operators with dense, invariant, cyclic vector manifolds*. J. Funct. Anal. 98 (1991), no. 2, 229-269.

[AP92] Azmy Y.Y.; Protopopescu V. *Topological chaos for a class of linear models*. Math. Models Methods Appl. Sci. 2 (1992), no. 1, 79–90.

[BBCDS92] Banks, J.; Brooks, J.; Cairns, G.; Davis, G.; Stacey, P.. *On Devaney's definition of chaos*. Amer. Math. Monthly 99 (1992), no. 4, 332–334.

[KS94] Kimmel, Marek; Stivers, D.N. *Time-continuous branching walk models of unstable gene amplification*. Bull. Math. Biol., 56, no. 2 (1994), 337–357.

[KPS96] Kimmel, Marek; Świerniak, Andrzej; Polański, Andrzej. *Control problems arising in chemotherapy under evolving drug resistance*. Proceedings of the 13th World Congress of IFAC, Volume B (1996), 411-416.

[DSW97] Desch, Wolfgang; Schappacher, Wilhelm; Webb, Glenn F. *Hypercyclic and chaotic semigroups of linear operators*. Ergodic Theory Dynam. Systems 17 (1997), no. 4, 793–819.

[MV97] Meise, Reinhold; Vogt, Dietmar. *Introduction to functional analysis*. Oxford Graduate Texts in Mathematics, 2. The Clarendon Press, Oxford University Press, New York, 1997.

[KPS98] Kimmel, Marek; Świerniak, Andrzej; Polański, Andrzej. *Infinite-dimensional model of evolution of drug resistance of cancer cells*. J. Math. Systems Estim., 8 (1998), no. 1, 1–16.

[BK99] Bobrowski, Adam; Kimmel, Marek. *Asymptotic behaviour of an operator exponential related to branching random walk models of DNA repeats*. Journal of Biological Systems, 7 (1999), no. 1, 33-43.

[EN00] Engel, Klaus-Jochen; Nagel, Rainer. *One-parameter semigroups for linear evolution equations*. Graduate Texts in Mathematics, 194. Springer-Verlag, New York, 2000.

[BL01] Banasiak, Jacek; Lachowicz, Miroslaw. *Chaos for a class of linear kinetic models*. Compt. Rend. Acad. Sci. Paris, 329, Série IIb, 439–444, 2001.

[BMP01] Bonet, José and Martínez-Giménez, Félix and Peris, Alfredo. *A Banach space which admits no chaotic operator*. Bull. London Math. Soc., 33 (2001), no. 2, 196–198.

[VW01] Van Lint, J. H.; Wilson, R. M. *A course in combinatorics*. Second edition. Cambridge University Press, Cambridge, 2001.

[Ban05] Banasiak, Jacek. *Birth-and-death type systems with parameter and chaotic dynamics of some linear kinetic models*. Z. Anal. Anwendungen 24 (2005), no. 4, 675–690.

[BL05] Banasiak, Jacek; Moszyński, Marcin. *A generalization of Desch-Schappacher-Webb criteria for chaos*. Discrete Contin. Dyn. Syst. 12 (2005), no. 5, 959–972.

[Ela05] Elaydi, Saber. *An Introduction to difference equations*. Third edition. Undergraduate Texts in Mathematics. Springer, New York, 2005.

[BLM06] Banasiak, Jacek; Lachowicz, Miroslaw; Moszyński, Marcin. *Semigroups for generalized birth-and-death equations in $l^p$ spaces*. Semigroup Forum 73 (2006), no. 2, 175–193.

[EN06] Engel, Klaus-Jochen; Nagel, Rainer. *A short course on operator semigroups*. Universitext. Springer, New York, 2006.

[Mou06] El Mourchid, Samir. *The imaginary point spectrum and hypercyclicity*. Semigroup Forum 73 (2006), no. 2, 313–316.

[BLM07] Banasiak, Jacek; Lachowicz, Miroslaw; Moszyński, Marcin. *Chaotic behavior of semigroups related to the process of gene amplification–deamplification with cell proliferation*. Math. Biosci. 206 (2007), no. 2, 200–215.

[CMP07] Conejero, José A.; Müller, Vladimir; Peris, Alfredo. *Hypercyclic behaviour of operators in a hypercyclic $C_0$-semigroup*. J. Funct. Anal. 244 (2007), no. 1, 342–348.

[BM08] Banasiak, Jacek; Moszyński, Marcin. *Hypercyclicity and chaoticity spaces of $C_0$-semigroups*. Discrete Contin. Dyn. Syst. 20 (2008), no. 3, 577–587.

[Gros08] Karl-Goswin Grosse-Erdmann. *Chaotic infinite-dimensional dynamics*. Winter School, RTNS 2008, Cullera:
`http://www.dance-net.org/files/material/rtns2008_notes_grosse-erdmann.pdf`

[BM09] Bayart, Frédéric and Matheron, Étienne. *Dynamics of linear operators*. Cambridge Tracts in Mathematics, Cambridge University Press. 179 (2009).

[CP09] Conejero, José A.; Peris, Alfredo. *Hypercyclic translation $C_0$-semigroups on complex sectors*. Discrete Contin. Dyn. Syst. 25 (2009), no. 4, 1195-1208.

[BM11] Banasiak, Jacek; Moszyński, Marcin. *Dynamics of birth-and-death processes with proliferation - Stability and chaos*. To appear, Discrete Contin. Dyn. Syst. 29 (2011), no. 1, 67-79.

[GP11] Grosse-Erdmann, Karl-Goswin; Peris, Alfredo. *Linear chaos*. To appear, Universitext, Springer-Verlag.