HYPERCYCLIC ALGEBRAS FOR CONVOLUTION AND COMPOSITION OPERATORS

J. BÈS, J. A. CONEJERO, AND D. PAPATHANASIOU

Abstract. We provide an alternative proof to those by Shkarin and by Bayart and Matheron that the operator D of complex differentiation supports a hypercyclic algebra on the space of entire functions. In particular we obtain hypercyclic algebras for many convolution operators not induced by polynomials, such as $\cos(D)$, De^D, or $e^D - aI$, where $0 < a \leq 1$. In contrast, weighted composition operators on function algebras of analytic functions on a plane domain fail to support supercyclic algebras.

1. Introduction

A special task in linear dynamics is to understand the algebraic and topological properties of the set

$$HC(T) = \{f \in X : \quad \{f, Tf, T^2 f, \dots\} \text{ is dense in } X\}$$

of hypercyclic vectors for a given operator T on a topological vector space X. It is well known that in general $HC(T)$ is always connected and is either empty or contains a dense infinite-dimensional linear subspace (but the origin), see [24]. Moreover, when $HC(T)$ is non-empty it sometimes contains (but zero) a closed and infinite dimensional linear subspace, but not always [7, 17]; see also [6, Ch. 8] and [19, Ch. 10].

When X is a topological algebra it is natural to ask whether $HC(T)$ can contain, or must always contain, a subalgebra (but zero) whenever it is non-empty; any such subalgebra is said to be a *hypercyclic algebra* for the operator T. Both questions have been answered by considering convolution operators on the space $X = H(\mathbb{C})$ of entire functions on the complex plane $\mathbb{C}$, endowed with the compact-open topology; that convolution operators (other than scalar multiples of the identity) are hypercyclic was established by Godefroy and Shapiro [16], see also [12, 20, 2], together with the fact that convolution operators on $H(\mathbb{C})$ are precisely those of the form

$$f \overset{\Phi(D)}{\mapsto} \sum_{n=0}^{\infty} a_n D^n f \quad (f \in H(\mathbb{C}))$$

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where $\Phi(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{C})$ is of (growth order one and finite) exponential type (i.e., $|a_n| \leq M \frac{R^n}{n!} (n = 0, 1, \dots)$, for some $M, R > 0$) and where D is the operator of complex differentiation. Aron et al [3, 4] showed that no translation operator τ_a on $H(\mathbb{C})$

$$\tau_a(f)(z) = f(z + a) \quad f \in H(\mathbb{C}), z \in \mathbb{C}$$

can support a hypercyclic algebra, in a strong way:

Theorem 1. (Aron, Conejero, Peris, Seoane) *For each integer $p > 1$ and each $f \in HC(\tau_a)$, the non-constant elements of the orbit of f^p under τ_a are those entire functions for which the multiplicities of their zeros are integer multiples of p.*

In sharp contrast with the translations operators, they also showed that the collection of entire functions f for which every power f^n $(n = 1, 2, \dots)$ is hypercyclic for D is residual in $H(\mathbb{C})$. Later Shkarin [23, Thm. 4.1] showed that $HC(D)$ contained both a hypercyclic subspace and a hypercyclic algebra, and with a different approach Bayart and Matheron [6, Thm. 8.26] also showed that the set of $f \in H(\mathbb{C})$ that generate an algebra consisting entirely (but the origin) of hypercyclic vectors for D is residual in $H(\mathbb{C})$, and by using the latter approach we now know the following:

Theorem 2. (Shkarin [23], Bayart and Matheron [6], Bès, Conejero, Papathanasiou [10]) *Let P be a non-constant polynomial with $P(0) = 0$. Then the set of functions $f \in H(\mathbb{C})$ that generate a hypercyclic algebra for $P(D)$ is residual in $H(\mathbb{C})$.*

Motivated by the above results we consider the following question.

Question 1. *Let $\Phi \in H(\mathbb{C})$ be of exponential type so that the convolution operator $\Phi(D)$ supports a hypercyclic algebra. Must Φ be a polynomial? Must $\Phi(0) = 0$?*

In Section 2 we answer both parts of Question 1 in the negative, by establishing for example that $\Phi(D)$ supports a hypercyclic algebra when $\Phi(z) = \cos(z)$ and when $\Phi(z) = ze^z$ (Example 10 and Example 11), as well as when $\Phi(z) = (a_0 + a_1 z^n)^k$ with $|a_0| \leq 1$ and $0 \neq a_1$ and when $\Phi(z) = e^z - a$ with $0 < a \leq 1$ (Corollary 9 and Example 12). All such examples are derived from our main result:

Theorem 3. *Let $\Phi \in H(\mathbb{C})$ be of finite exponential type so that the level set $\{z \in \mathbb{C} : \ |\Phi(z)| = 1\}$ contains a non-trivial, strictly convex compact arc Γ_1 satisfying*

$$\text{conv}(\Gamma_1 \cup \{0\}) \setminus (\Gamma_1 \cup \{0\}) \subseteq \Phi^{-1}(\mathbb{D}). \tag{1.1}$$

Then the set of entire functions that generate a hypercyclic algebra for the convolution operator $\Phi(D)$ is residual in $H(\mathbb{C})$.

Here for any $A \subset \mathbb{C}$ the symbol $\operatorname{conv}(A)$ denotes its convex hull, and $\mathbb{D}$ denotes the open unit disc. Also, an arc $\mathcal{C}$ is said to be strictly convex provided for each z_1, z_2 in $\mathcal{C}$ the segment $\operatorname{conv}(\{z_1, z_2\})$ intersects $\mathcal{C}$ at at most two points.

In Section 3 we consider the following question, motivated by Theorem 1:

Question 2. *Can a multiplicative operator on a F-algebra support a hypercyclic algebra? In particular, can a composition operator support a hypercyclic algebra on some space $H(\Omega)$ of holomorphic functions on a planar domain Ω?*

The study of hypercyclic composition operators on spaces of holomorphic functions may be traced back to the classical examples by Birkhoff [12] and by Seidel and Walsh [22], and is described in a recent survey article by Colonna and Martínez-Avendaño [14]. Grosse-Erdmann and Mortini showed that the space $H(\Omega)$ of holomorphic functions on a planar domain Ω supports a hypercyclic composition operator if and only if Ω is either simply connected or infinitely connected [18].

We show in Section 3 that a given multiplicative operator T on an F-algebra X supports a hypercyclic algebra if and only if T is hypercyclic and for each non-constant polynomial P vanishing at zero the map $X \to X$, $f \mapsto P(f)$ has dense range (Theorem 16). We use this to derive that for each $0 \neq a \in \mathbb{R}$ the translation operator τ_a supports a hypercyclic algebra on $C^\infty(\mathbb{R}, \mathbb{C})$ (Corollary 21) but fails to support a hypercyclic algebra on $C^\infty(\mathbb{R}, \mathbb{R})$ (Corollary 17). Here by $C^\infty(\mathbb{R}, \mathbb{K})$ we denote the Fréchet space of $\mathbb{K}$-valued infinitely differentiable functions on $\mathbb{R}$ whose topology is given by the seminorms

$$P_k(f) = \max_{0\leq j\leq k} \max_{t\in[-k,k]} |f^{(j)}(t)| \quad (f \in C^\infty(\mathbb{R}, \mathbb{K}),\ k \in \mathbb{N}).$$

Finally, we show that no weighted composition operator $C_{\omega,\varphi} : H(\Omega) \to H(\Omega)$, $f \mapsto \omega(f \circ \varphi)$, supports a supercyclic algebra (Theorem 22). Recall that a vector f in an F-algebra X is said to be *supercyclic* for a given operator $T : X \to X$ provided

$$\mathbb{C} \cdot \operatorname{Orb}(f, T) = \{\lambda T^n f : \ \lambda \in \mathbb{C}, n = 0, 1, \dots\}$$

is dense in X. Accordingly, any subalgebra of X consisting entirely (but zero) of supercyclic vectors for T is said to be a *supercyclic algebra*.

2. Proof of Theorem 3 and its consequences

The proofs of Theorem 2 and of its earlier versions exploit the shift-like behaviour of the operator D on $H(\mathbb{C})$ [23, 6, 10]. Our approach for Theorem 3 exploits instead the rich source of eigenfunctions that convolution operators on $H(\mathbb{C})$ have (i.e.,

$$\Phi(D)(e^{\lambda z}) = \Phi(\lambda)e^{\lambda z}$$

for each $\lambda \in \mathbb{C}$ and each $\Phi \in H(\mathbb{C})$ of exponential type) as well as the following key result by Bayart and Matheron:

Proposition 4. (Bayart-Matheron [6, Remark 8.26]) *Let T be an operator on a separable F-algebra X so that for each triple (U, V, W) of non-empty open subsets of X with $0 \in W$ and for each $m \in \mathbb{N}$ there exists $P \in U$ and $q \in \mathbb{N}$ so that*

$$\begin{cases} T^q(P^j) \in W & \text{for } 0 \leq j < m, \\ T^q(P^m) \in V. \end{cases} \tag{2.1}$$

Then the set of elements of X that generate a hypercyclic algebra for T is residual in X.

We start by noting the following invariant for composition operators with homothety symbol.

Lemma 5. *Let $\Phi \in H(\mathbb{C})$ be of exponential type, and let $\varphi : \mathbb{C} \to \mathbb{C}$, $\varphi(z) = az$ be a homothety on the plane, where $0 \neq a \in \mathbb{C}$. Then $\Phi_a := C_\varphi(\Phi)$ is of exponential type and*

$$C_\varphi(HC(\Phi_a(D))) = HC(\Phi(D)).$$

In particular, the algebra isomorphism $C_\varphi : H(\mathbb{C}) \to H(\mathbb{C})$ maps hypercyclic algebras of $\Phi_a(D)$ onto hypercyclic algebras of $\Phi(D)$.

Proof. For each $f \in H(\mathbb{C})$ we have $C_\varphi(f)(z) = f(az)$ $(z \in \mathbb{C})$, and thus

$$D^k C_\varphi(f) = a^k C_\varphi D^k(f) \quad (k = 0, 1, 2, \dots).$$

Hence given $\Phi(z) = \sum_{k=0}^\infty c_k z^k$ of exponential type $\Phi_a := C_\varphi(\Phi)$ is clearly of exponential type and

$$\begin{aligned} \Phi(D)C_\varphi(f) &= \sum_{k=0}^\infty c_k D^k C_\varphi(f) = \sum_{k=0}^\infty c_k a^k C_\varphi D^k(f) \\ &= C_\varphi \left(\sum_{k=0}^\infty c_k a^k D^k \right)(f) \\ &= C_\varphi \Phi_a(D)(f) \qquad (f \in H(\mathbb{C})). \end{aligned}$$

So $\Phi_a(D)$ is conjugate to $\Phi(D)$ via the algebra isomorphism C_φ. □

Remark 6.

(1) Lemma 5 is a particular case of the following Comparison Principle for Hypercyclic Algebras. *Any operator $T : X \to X$ on a Fréchet algebra X that is quasi-conjugate via a multiplicative operator $Q : Y \to X$ to an operator $S : Y \to Y$ supporting a hypercyclic algebra must also support a hypercyclic algebra. Indeed, if A is a hypercyclic algebra for S, then $Q(A) = \{Qy : y \in A\}$ is a hypercyclic algebra for T.*

(2) If $\Phi \in H(\mathbb{C})$ satisfies the assumptions of Theorem 3, then so will $\Phi_a := C_\varphi(\Phi)$ for any non-trivial homothety $\varphi(z) = az$. Indeed, notice that for any $r > 0$ we have

$$a\Phi_a^{-1}(r\partial\mathbb{D}) = \Phi^{-1}(r\partial\mathbb{D}).$$

Hence if $\Gamma \subset \Phi^{-1}(r\partial\mathbb{D})$ is a smooth arc satisfying

$$\mathrm{conv}(\Gamma \cup \{0\}) \setminus (\Gamma \cup \{0\}) \subset \Phi^{-1}(r\mathbb{D}),$$

then $\Gamma_a := \frac{1}{a}\Gamma \subset \Phi_a^{-1}(r\partial\mathbb{D})$ is a smooth arc satisfying

$$\mathrm{conv}(\Gamma_a \cup \{0\}) \setminus (\Gamma_a \cup \{0\}) \subset \Phi_a^{-1}(r\mathbb{D}).$$

Moreover, if Γ is a strictly convex, compact, simple and non-closed arc whose convex hull does not contain the origin, say, then Γ_a will share each corresponding property as these are invariant under homothecies. In particular, the angle difference between the endpoints of Γ is the same as the corresponding quantity in Γ_a.

The next result ellaborates on the geometric assumption of Theorem 3. Here for any $0 \neq z \in \mathbb{C}$ we denote by $\arg(z)$ the argument of z that belongs to $[0, 2\pi)$.

Proposition 7. *Let $\Phi \in H(\mathbb{C})$ and let $\Gamma \subset \Phi^{-1}(r\partial\mathbb{D})$ be a simple, strictly convex arc with endpoints z_1, z_2 satisfying $0 < \arg(z_1) < \arg(z_2) < \pi$ and $\mathrm{Re}(z_1) \neq \mathrm{Re}(z_2)$, where $r > 0$. Suppose that $0 \notin \mathrm{conv}(\Gamma)$ and that*

$$\Omega := \mathrm{conv}(\Gamma \cup \{0\}) \setminus (\Gamma \cup \{0\}) \subset \Phi^{-1}(r\mathbb{D}). \tag{2.2}$$

Then $S(0, z_1, z_2) \setminus \Gamma$ consists of two connected components of which Ω is the bounded one, where

$$S(0, z_1, z_2) = \{0 \neq w \in \mathbb{C} : \ \arg(z_1) \leq \arg(w) \leq \arg(z_2)\}.$$

Moreover,

$$\Omega = \{tz : \ (t, z) \in (0, 1) \times \Gamma\} = \{tz : \ (t, z) \in (0, 1) \times \mathrm{conv}(\Gamma)\},$$

and $\partial\Omega = [0, z_1) \cup (0, z_2) \cup \Gamma$. In addition,

$$\Gamma \cap (I \times (0, \infty)) = \mathrm{Graph}(f) \cup \{z_1, z_2\}$$

for some smooth function $f : I \to \mathbb{R}$, where I is the closed interval with endpoints $\mathrm{Re}(z_1)$ and $\mathrm{Re}(z_2)$ and where f is concave up if $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and concave down if $\mathrm{Re}(z_2) < \mathrm{Re}(z_1)$.

In Figure 1 we illustrate one case of the statement of this Proposition 7.

Proof. Since $|\Phi| \leq r$ on $\mathrm{conv}(\Gamma \cup \{0\})$ by (2.2), the maximum modulus principle ensures that

$$\Gamma \cap \mathrm{int}(\mathrm{conv}(\Gamma \cup \{0\})) = \emptyset. \tag{2.3}$$

We claim that

$$\Gamma \subset \{0 \neq w \in \mathbb{C} : \ \arg(w) \in [\arg(z_1), \arg(z_2)]\}. \tag{2.4}$$

To see this, notice that since $0 \notin \mathrm{conv}(\Gamma)$ the arc Γ cannot intersect the ray $\{te^{i(\arg(z_2)+\pi)} : t \geq 0\}$, and by (2.3) it cannot intersect the interior of the triangle $\mathrm{conv}\{0, z_1, z_2\}$, either. Also, notice that if H denotes the open half-plane not containing 0 and with boundary

$$\partial H = \{z_1 + t(z_2 - z_1) : \ t \in \mathbb{R}\},$$

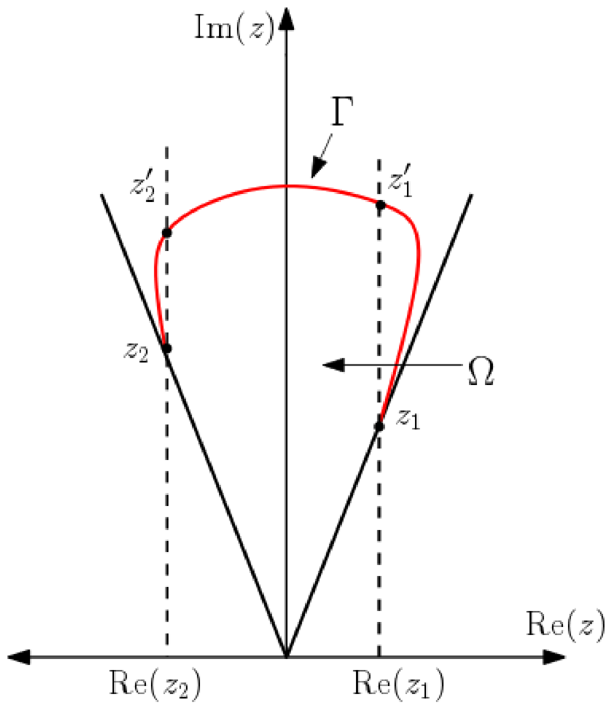

FIGURE 1. A representation of the sets appearing in Proposition 7.

then

$$\emptyset = \Gamma \cap H \cap \{0 \neq w \in \mathbb{C} : \ \arg(w) < \arg(z_1)\}, \tag{2.5}$$

as any $z \in \Gamma \cap H$ with $\arg(z) < \arg(z_1)$ would make $z_1 \in \operatorname{int}(\operatorname{conv}(\{z, z_2, 0\}))$, contradicting (2.3). Finally, since Γ is simple it now follows from (2.5) that

$$\emptyset = \Gamma \cap \{0 \neq w \in \mathbb{C} : \ \arg(w) \in [\pi + \arg(z_2), 2\pi) \cup [0, \arg(z_1))\},$$

and thus any $w \in \Gamma$ satisfies $\arg(z_1) \leq \arg(w)$. By a similar argument, each $w \in \Gamma$ satisfies $\arg(w) \leq \arg(z_2)$, and (2.4) holds. Next, using (2.3) and the continuity of the argument on $S(0, z_1, z_2)$ it is simple now to see that for each $\theta \in [\arg(z_1), \arg(z_2)]$ the ray

$$\{te^{i\theta} : \ t \geq 0\}$$

intersects Γ at exactly one point, giving the desired description for Ω. For the final statement, assume $\operatorname{Re}(z_2) < \operatorname{Re}(z_1)$ (the case $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ follows with a similar argument).

Notice that for each $x = t\operatorname{Re}(z_2) + (1-t)\operatorname{Re}(z_1)$ with $0 < t < 1$ there exists a unique $y \in \mathbb{R}$ so that

$$(x, y) \in \Gamma \quad \text{with} \quad y \in [t\operatorname{Im}(z_2) + (1-t)\operatorname{Im}(z_1), \infty). \tag{2.6}$$

Indeed, the continuous path Γ from z_1 to z_2 lies in $S(0, z_1, z_2)$ and only meets the closed triangle $\operatorname{conv}(\{0, z_1, z_2\})$ at z_1 and z_2, so the existence of a y verifying (2.6) follows (it also follows for the cases $t = 0, 1$, in which case there may exist up to two values per endpoint, by (2.4)). To see the uniqueness, if $y_2 > y_1 > t\,\mathrm{Im}(z_2) + (1-t)\,\mathrm{Im}(z_1)$ with $(x, y_1), (x, y_2) \in \Gamma$, then

$$(x, y_1) \in \mathrm{int}(\mathrm{conv}(\{z_1, z_2, x + iy_2\}) \cap \Gamma \subset \Omega \cap \Gamma = \emptyset,$$

a contradiction. Hence (2.6) defines a smooth function $f : [\mathrm{Re}(z_1), \mathrm{Re}(z_2)] \to (0, \infty)$ whose graph Γ_0 is a subarc of Γ, provided that if at either endpoint $x \in \{\mathrm{Re}(z_1), \mathrm{Re}(z_2)\}$ there are two values y satisfying $x + iy \in \Gamma$ we let $f(x)$ be the largest of such two values. □

Finally, Lemma 8 below will enable us to apply Proposition 4. Recall that for a planar smooth curve $\mathcal{C}$ with parametrization $\gamma : [0, 1] \to \mathbb{C}$, $\gamma(t) = x(t) + iy(t)$, its signed curvature at a point $P = \gamma(t_0) \in \mathcal{C}$ is given by

$$\kappa(P) := \frac{x'(t_0)y''(t_0) - y'(t_0)x''(t_0)}{|\gamma'(t_0)|^3}.$$

and its unsigned curvature at P is given by $|\kappa(P)|$. It is well-known that $|\kappa(P)|$ does not depend on the parametrization selected, and that the signed curvature $\kappa(P)$ depends only on the choice of orientation selected for $\mathcal{C}$. It is simple to see that any straight line segment has zero curvature. We say that $\mathcal{C}$ is *strictly convex* provided each segment with endpoints in the arc only intersects the arc at these points. Notice also that for the particular case when $\mathcal{C}$ is given by the graph of a function $y = f(x)$, $a \le x \le b$, (and oriented from left to right), its signed curvature at a point $P = (x_0, f(x_0))$ is given by

$$\kappa(P) = \frac{f''(x_0)}{(1 + (f'(x_0))^2)^{\frac{3}{2}}}.$$

In particular, $\kappa < 0$ on $\mathcal{C}$ if and only if $y = f(x)$ is concave down (i.e., $(1-s)f(a_1) + sf(b_1) < f((1-s)a_1 + sb_1)$ for any $s \in (0, 1)$ and any subinterval $[a_1, b_1]$ of $[a, b]$).

Lemma 8. *Let $\Phi \in H(\mathbb{C})$ be of exponential type supporting a non-trivial, strictly convex compact arc Γ_1 contained in $\Phi^{-1}(\partial\mathbb{D})$ so that*

$$\mathrm{conv}(\Gamma_1 \cup \{0\}) \setminus (\Gamma_1 \cup \{0\}) \subseteq \Phi^{-1}(\mathbb{D}).$$

Then for each $m \in \mathbb{N}$ there exist $r > 1$, a non-trivial, strictly convex smooth arc $\Gamma \subset \Phi^{-1}(r\partial\mathbb{D}) \cap \{tz : (t, z) \in (0, \infty) \times \Gamma_1\}$ and $\epsilon > 0$ so that

$$\mathrm{conv}(\Gamma \cup \{0\}) \setminus \Gamma \subseteq \Phi^{-1}(r\mathbb{D}). \tag{2.7}$$

and

$$\Lambda + \sum_{k=1}^{j} \frac{1}{m}\Gamma \subset \Omega \quad \textit{and} \quad \sum_{k=1}^{j} \frac{1}{m}\Gamma \subset \Omega \quad \textit{for each } 1 \le j < m, \tag{2.8}$$

where

$$\Omega := \mathrm{conv}(\Gamma_1 \cup \{0\}) \setminus (\Gamma_1 \cup \{0\})$$
$$\Lambda := \Omega \cap D(0, \epsilon) \cap \mathrm{conv}(\Gamma \cup \{0\}).$$

In Figure 2 we illustrate the different sets appearing in the statement of Lemma 8.

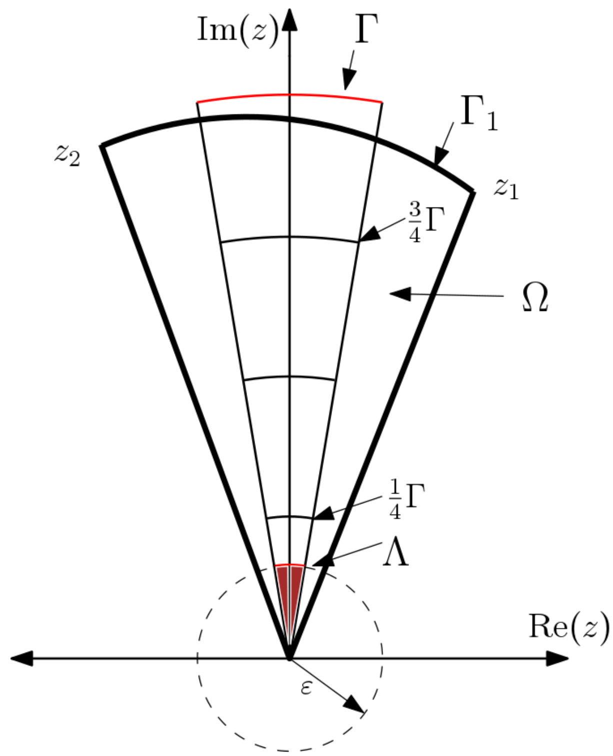

FIGURE 2. The sets appearing in Lemma 8, case $m = 4$.

Proof. Since Γ_1 is strictly convex, replacing it by a subarc if necessary we may further assume by Remark 6.(2) that Γ_1 is simple and with endpoints z_1, z_2 satisfying $0 < \arg(z_1) < \arg(z_2) < \pi$ and $\mathrm{Re}(z_2) < \mathrm{Re}(z_1)$ and so that $0 \notin \mathrm{conv}(\Gamma_1)$. By Proposition 7,

$$\Omega = \{tz : \ (t, z) \in (0, 1) \times \mathrm{conv}(\Gamma_1)\} \subset S(0, z_1, z_2), \tag{2.9}$$

with $\partial\Omega = [0, z_1) \cup \Gamma_1 \cup (0, z_2)$ and we may assume Γ_1 is the graph of a concave down function $f : [\mathrm{Re}(z_2), \mathrm{Re}(z_1)] \to (0, \infty)$ (i.e., replacing z_j by $z_j' = \mathrm{Re}(z_j) + if(\mathrm{Re}(z_j))$, $j = 1, 2$, if necessary). Now, pick $z_0 \in \Gamma_1 \setminus \{z_1, z_2\}$ with $\Phi'(z_0) \neq 0$, and let $w_0 := \Phi(z_0) = e^{i\theta_0}$, where $\theta_0 \in [0, 2\pi)$. Choose $\rho > 0$ small enough so that the only solution to

$$\Phi(z) = w_0$$

in $D(z_0, \rho)$ is at $z = z_0$, and so that $D(z_0, \rho) \cap ([0, z_1] \cup [0, z_2]) = \emptyset$. Next, pick

$$0 < s < \min\{|\Phi(z) - w_0| : \ |z - z_0| = \rho\}$$

and let $0 < \delta < \min\{1, s\}$ so that the polar rectangle

$$R_\delta := \{z = re^{i\theta} : \ (r, \theta) \in [1-\delta, 1+\delta] \times [\theta_0 - \delta, \theta_0 + \delta]\}$$

is contained in $D(w_0, s)$. Then

$$g : R_\delta \to D(z_0, \rho), \ g(w) = \frac{1}{2\pi i} \int\limits_{|z-z_0|=\rho} \frac{z\Phi'(z)}{\Phi(z) - w} dz$$

defines a univalent holomorphic function satisfying that

$$\Phi \circ g = \text{identity on } R_\delta, \tag{2.10}$$

see e. g. [15, p. 283]. So $W := g(R_\delta)$ is a connected compact neighborhood of z_0, and Φ maps W biholomorphically onto R_δ. Hence for each $1 - \delta \leq r \leq 1 + \delta$

$$\eta_r := g(R_\delta \cap r\partial\mathbb{D})$$

is a smooth arc contained in $W \cap \Phi^{-1}(r\partial\mathbb{D})$. In particular, $\eta_1 = W \cap \Gamma_1$ is a strictly convex subarc of Γ_1. Next, notice that since

$$W \cap \Omega \quad \text{and} \quad W \cap \text{Ext}(\Omega)$$

are the two connected components of $g(R_\delta \setminus \partial\mathbb{D}) = W \setminus \eta_1$ and $\Omega \subseteq \Phi^{-1}(\mathbb{D})$, by (2.10) the homeomorphism $g : R_\delta \setminus \partial\mathbb{D} \to W \setminus \eta_1$ must satisfy

$$g(R_\delta \cap \text{Ext}(\mathbb{D})) = W \cap \text{Ext}(\Omega)$$
$$g(R_\delta \cap \mathbb{D}) = W \cap \Omega.$$

Hence

$$W \cap \overline{\text{Ext}(\Omega)} = \bigcup_{1 \leq r \leq 1+\delta} \eta_r$$

and g induces a smooth homotopy among the curves $\{\eta_r\}_{1\leq r \leq 1+\delta}$. Namely, each η_r $(1 \leq r \leq 1 - \delta)$ has the Cartesian parametrization

$$\eta_r : \begin{cases} X(r,t) \\ Y(r,t) \end{cases} \qquad \theta_0 - \delta \leq t \leq \theta_0 + \delta,$$

where $X, Y : [1-\delta, 1+\delta] \times [\theta_0 - \delta, \theta_0 + \delta] \to \mathbb{R}$ are given by

$$X(r,t) := \text{Re}(g)(re^{it})$$
$$Y(r,t) := \text{Im}(g)(re^{it}).$$

Now, for any point $P = g(re^{i\theta})$ in W the (signed) curvature $\kappa^{\eta_r}(P)$ of η_r at P is given by

$$\kappa^{\eta_r}(P) = \frac{\frac{\partial X}{\partial t}(r,\theta)\frac{\partial^2 Y}{\partial^2 t}(r,\theta) - \frac{\partial Y}{\partial t}(r,\theta)\frac{\partial^2 X}{\partial^2 t}(r,\theta)}{\left((\frac{\partial X}{\partial t}(r,\theta))^2 + (\frac{\partial Y}{\partial t}(r,\theta))^2\right)^{\frac{3}{2}}}.$$

Hence the map $K : W \to \mathbb{R}$, $K(g(re^{it})) := \kappa^{\eta_r}(P)$, is continuous. Now, since η_1 is strictly convex there exists some $P = g(e^{i\theta_1})$ in η_1 for which each of

$\kappa^{\eta_1}(P)$ and $\frac{\partial X}{\partial t}(1,\theta_1)$ is non-zero. Hence by the continuity of K and of $\frac{\partial X}{\partial t}$ we may find some $0<\delta'<\delta$ so that the polar rectangle

$$R_{\delta'} := \{z = re^{i\theta} :\ (r,\theta) \in [1-\delta', 1+\delta'] \times [\theta_1 - \delta', \theta_1 + \delta']\}$$

is contained in the interior of R_δ and so that K and $\frac{\partial X}{\partial t}$ are bounded away from zero on $g(R_{\delta'})$ and on $R_{\delta'}$, respectively.

In particular, either $\frac{\partial X}{\partial t} > 0$ or $\frac{\partial X}{\partial t} < 0$ on $R_{\delta'}$, and either $K>0$ or $K<0$ on $g(R_{\delta'})$. So each $\eta_r \cap g(R_{\delta'})$ $(1 \le r < 1+\delta')$ is the graph of a smooth function

$$f_r : (a_r, b_r) \to (0,\infty),$$

with

$$(a_r,b_r) = \begin{cases} (X(r,\theta_1-\delta'), X(r,\theta_1+\delta')) & \text{if } \frac{\partial X}{\partial t} > 0 \text{ on } R_{\delta'} \\ (X(r,\theta_1+\delta'), X(r,\theta_1-\delta')) & \text{if } \frac{\partial X}{\partial t} < 0 \text{ on } R_{\delta'}. \end{cases}$$

Since $g(re^{it}) \underset{r\to 1}{\to} g(e^{it})$ uniformly on $t \in [\theta_1-\delta, \theta_1+\delta]$, so

$$(a_r, b_r) \underset{r\to 1}{\to} (a_1, b_1)$$

and fixing a non-trivial compact subinterval $[a,b]$ of (a_1,b_1) there exists there exists $0<\delta''<\delta'$ so that

$$[a,b] \subset \cap_{1\le r\le 1+\delta''}(a_r,b_r).$$

So for each $1 < r \le 1+\delta''$

$$\eta_r' = \{(x, f_r(x));\ x\in[a,b]\}$$

is a subarc of η_r. In particular, $f_1 = f$ on $[a,b]$ must be a concave down function, and so must be each f_r with $1\le r\le 1+\delta''$. Thus choosing $r>1$ close enough to 1 the arc $\Gamma := \eta_r'$ satisfies

$$\operatorname{conv}(\Gamma\cup\{0\}) \setminus (\Gamma\cup\{0\}) \subset \Phi^{-1}(r\mathbb{D}) \cap \{tz :\ (t,z)\in(0,\infty)\times\Gamma_1\}$$

and

$$\sum_{k=1}^{j} \frac{1}{m}\Gamma \subset \Omega \ \text{ for } j=1,\dots,m-1.$$

By the compactness of Γ we may now get $\epsilon>0$ small enough so that the subsector

$$\Lambda := \Omega \cap D(0,\epsilon) \cap \operatorname{conv}(\Gamma\cup\{0\})$$

satisfies that

$$\Lambda + \sum_{k=1}^{j} \frac{1}{m}\Gamma \subset \Omega \ \text{ for } j=1,\dots,m-1,$$

and Lemma 8 holds. □

We are ready now to prove the main result.

Proof of Theorem 3. Let U, V and W be non-empty open subsets of $H(\mathbb{C})$, with $0 \in W$, and let $1 \leq m \in \mathbb{N}$ be fixed. By Proposition 4, it suffices to find some $f \in U$ and $q \in \mathbb{N}$ so that

$$\begin{aligned} \Phi(D)^q(f^j) &\in W \quad \text{for } j = 1, \dots, m-1, \\ \Phi(D)^q(f^m) &\in V. \end{aligned} \tag{2.11}$$

The case $m = 1$ is immediate as $\Phi(D)$ is topologically transitive, so we may assume $1 < m$. Now, let $r > 1$, let $\Gamma \subset \Phi^{-1}(r\partial\mathbb{D})$ and let Ω and the subsector Λ be given by Lemma 8. Since the arc Γ is non-trivial and Λ has non-empty interior, each of Γ and Λ has accumulation points in $\mathbb{C}$. Hence there exist $(a_k, b_k, \lambda_k, \gamma_k) \in \mathbb{C} \times \mathbb{C} \times \Lambda \times \Gamma$ $(k = 1, \dots, p)$ so that

$$(A, B) := \left(\sum_{k=1}^{p} a_k e^{\frac{\lambda_k z}{m}}, \sum_{k=1}^{p} b_k e^{\gamma_k z} \right) \in U \times V.$$

Next, set $R = R_q = \sum_{k=1}^{p} c_k e^{\frac{\gamma_k z}{m}}$, where for each $1 \leq k \leq p$ the scalar $c_k = c_k(q)$ is some solution of

$$z^m - \frac{b_k}{(\Phi(\gamma_k))^q} = 0.$$

Notice that for any $k = 1, \dots, p$ we have $|\Phi(\gamma_k)| = r > 1$ and thus $|c_k|^m = \frac{|b_k|}{|\Phi(\gamma_k)|^q} \underset{q \to \infty}{\to} 0$. So

$$R = R_q \underset{q \to \infty}{\to} 0. \tag{2.12}$$

For $1 \leq j \leq m$ we have

$$(A+R)^j = \sum_{\ell = (u,v) \in \mathcal{L}_j} \binom{j}{\ell} a^u \ c^v \ e^{(\frac{u \cdot \lambda + v \cdot \gamma}{m})z}$$

where $\mathcal{L}_j$ conists of those multiindexes $\ell = (u, v) \in \mathbb{N}_0^p \times \mathbb{N}_0^p$ satisfying $|\ell| := |u| + |v| = \sum_{k=1}^{p} u_k + \sum_{k=1}^{p} v_k = j$ and where for each $\ell = (u, v) \in \mathcal{L}_j$

$$\begin{aligned} a^u &:= a_1^{u_1} \ a_2^{u_2} \cdots a_p^{u_p}, \\ c^v &:= c_1^{v_1} \ c_2^{v_2} \cdots c_p^{v_p}, \text{ and} \\ \binom{j}{\ell} &= \frac{j!}{u_1! \cdots u_p! v_1! \cdots v_p!}. \end{aligned}$$

So for $1 \leq j \leq m$ we have

$$\Phi(D)^q((A+R)^j) = \sum_{\ell \in \mathcal{L}_j} U_{j,\ell},$$

where

$$\begin{aligned} U_{j,\ell} &= \binom{j}{\ell} a^u \ c^v \ \left(\Phi(\frac{u \cdot \lambda + v \cdot \gamma}{m}) \right)^q e^{(\frac{u \cdot \lambda + v \cdot \gamma}{m})z} \\ &= \binom{j}{\ell} a^u \ b^{\frac{v}{m}} \ \left(\frac{\Phi(\frac{u \cdot \lambda + v \cdot \gamma}{m})}{\prod_{k=1}^{p} \Phi(\gamma_k)^{\frac{v_k}{m}}} \right)^q e^{(\frac{u \cdot \lambda + v \cdot \gamma}{m})z}. \end{aligned}$$

Now, notice that if $\{e_1, \dots, e_p\}$ denotes the standard basis of $\mathbb{C}^p$, our selections of $(c_1, \dots, c_p)$ ensure that

$$\Phi^q(D)((A+R)^m) - B = \sum_{\ell \in \mathcal{L}_m^*} U_{m,\ell}, \tag{2.13}$$

where

$$\mathcal{L}_m^* = \{\ell = (u,v) \in \mathcal{L}_m : \ |u| \neq 0 \text{ or } \ v \notin \{me_1, \dots, me_p\}\}.$$

Also, for each $1 \le j \le m$ and $\ell = (u,v) \in \mathcal{L}_j$ with $|v| < m$ we have

$$U_{j,\ell} \underset{q\to\infty}{\to} 0,$$

as our selections of Λ and Γ give by (2.8) that $\frac{u\cdot\lambda + v\cdot\gamma}{m} \in \Omega$ and thus

$$\left|\Phi(\frac{u\cdot\lambda + v\cdot\gamma}{m})\right| < 1 < r = |\Phi(\gamma_1)| = \cdots = |\Phi(\gamma_p)|.$$

Hence since each $\mathcal{L}_j$ is finite we have

$$\Phi(D)^q((A+R_q)^j) \underset{q\to\infty}{\to} 0 \quad (1 \le j < m). \tag{2.14}$$

Finally, recall that by Lemma 8 we have

$$\operatorname{conv}(\Gamma_r) \setminus \Gamma_r \subseteq \Phi^{-1}(r\mathbb{D}).$$

Hence if $\ell = (u,v) \in \mathcal{L}_m^*$ with $|v| = m$ (so $\|v\|_\infty < m$ and $u = 0$) we also have that $U_{m,\ell} \underset{q\to\infty}{\to} 0$, as

$$\left|\Phi(\frac{u\cdot\lambda + v\cdot\gamma}{m})\right| = \left|\Phi(\frac{v\cdot\gamma}{m})\right| < r = |\Phi(\gamma_1)|^{\frac{v_1}{m}} \dots |\Phi(\gamma_p)|^{\frac{v_p}{m}}.$$

Thus

$$\Phi^q(D)((A+R_q)^m) \underset{q\to\infty}{\to} B,$$

and (2.11) follows by (2.12) and (2.14). □

2.1. **Some consequences of Theorem 3.** Theorem 3 complements [10, Thm. 1] and gives an alternative proof to those of Shkarin [23, Thm. 4.1] and Bayart and Matheron [6, Thm. 8.26] that D supports a hypercyclic algebra.

Corollary 9. *Let $P(z) = (a_0 + a_1 z^k)^n$ with $|a_0| \le 1$, $a_1 \neq 0$, and $k, n \in \mathbb{N}$. Then $P(D)$ supports a hypercyclic algebra on $H(\mathbb{C})$.*

Proof. Notice first that $Q_1(z) = a_0 + z^k$ satisfies the assumptions of Theorem 3, and hence so does $Q_2(z) = a_0 + a_1 z^k$, by Remark 6. The conclusion now follows by a result due to Ansari [1] that the set of hypercyclic vectors for an operator T coincides with the corresponding set of hypercyclic vectors for any given iterate T^n $(n \in \mathbb{N})$. □

We may also apply Theorem 3 to convolution operators that are not induced by polynomials.

Example 10. The operators $\cos(aD)$ and $\sin(aD)$ support a hypercyclic algebra on $H(\mathbb{C})$ if $a \neq 0$. To see this, notice first that by Lemma 5 we may assume that $a = 1$. For the first example, notice that $\Phi(z) = \cos(z)$ is of exponential type and

$$|\Phi(z)|^2 = |\cos(z)|^2 = \cos^2(x) + \sinh^2(y) \quad (z = x + iy, x, y \in \mathbb{R}).$$

So $\Gamma = \{(x, f(x)) : \ 0 \leq x \leq \pi\} \subset \Phi^{-1}(\partial\mathbb{D})$ for the smooth function $f : [0, \pi] \to [0, \infty)$, $f(x) = \sinh^{-1}(\sin(x))$, which is concave down since its second derivative $f''(x) = \frac{-2\sin(x)}{(1+\sin^2(x))^{\frac{3}{2}}}$ is negative on $(0, \pi)$. Now

$$\mathrm{conv}(\Gamma \cup \{0\}) \setminus (\Gamma \cup \{0\})$$

is the region bounded by the graph of f and the x-axis, on which $|\Phi| < 1$, and $\cos(D)$ supports a hypercyclic algebra by Theorem 3. The proof for $\sin(D)$ follows similarly by considering instead the subarc

$$\Gamma_0 := \left\{ \left(x - \frac{\pi}{2}, \sinh^{-1}(\sin(x)) \right) : \ 0 \leq x \leq \pi \right\}$$

of $\{z \in \mathbb{C} : \ |\sin(z)| = 1\}$.

The next two examples should be contrasted with [3, Corollary 2.4].

Example 11. The operator $T = D\tau_1 = De^D$ on $H(\mathbb{C})$, where τ_1 is the translation operator $g(z) \mapsto g(z+1), g \in H(\mathbb{C})$ supports a hypercyclic algebra.

Let $\Phi(z) = ze^z$. Clearly Φ is of exponential type, so we may check the conditions of Theorem 3. Writing $z = x + iy$ we get

$$|f(z)| = 1 \Leftrightarrow y^2 = e^{-2x} - x^2 \tag{2.15}$$

The above equation has solutions provided the function $\phi(x) = e^{-2x} - x^2$ satisfies that $\phi(x) \geq 0$. By doing some elementary calculus, one shows that ϕ is strictly decreasing on $\mathbb{R}$ and has a unique solution say $r \in (0, 1)$. Thus the graph of the function

$$h(x) = \sqrt{e^{-2x} - x^2}, \quad x \in (-\infty, r]$$

lies in $f^{-1}(\partial\mathbb{D})$. Taking derivatives, we get that $h' < 0$ and $h'' < 0$ on $(0, r)$, thus h is strictly decreasing and concave down on $[0, r]$. Furthermore, it is evident that the sector

$$S = \{z = x + iy \in \mathbb{C} : 0 \leq x < r, 0 \leq y < h(x)\}$$

lies in $f^{-1}(\mathbb{D})$. Thus, the strictly convex arc

$$\Gamma_1 = \{z = x + iy \in \mathbb{C} : 0 \leq x \leq r, y = h(x)\}$$

satisfies the conditions of Theorem 3, which guarantees the existence of a hypercyclic algebra for the operator $f(D)$.

Example 12. For each $0 < a \leq 1$, the operator $T = \tau_1 - aI = e^D - aI$ supports a hypercyclic algebra. To see this, we will show that the exponential type function $\Phi(z) = e^z - a$ satisfies the assumptions of Theorem 3. If $z = x + iy$ then an easy calculation shows that

$$|\Phi(z)| \leq 1 \Leftrightarrow e^{2x} - 2a\cos(y)e^x + a^2 - 1 \leq 0. \tag{2.16}$$

If we restrict $y \in [0, \frac{\pi}{4}]$ the inequality (2.16) has solution $x \leq \log(a\cos(y) + \sqrt{1 - a^2\sin^2(y)})$. Hence, setting

$$\Gamma_1 = \left\{ z = x + iy \in \mathbb{C} : 0 \leq y \leq \frac{\pi}{4}, x = \log(a\cos(y) + \sqrt{1 - a^2\sin^2(y)}) \right\}$$

we get that $\Gamma_1 \subset \Phi^{-1}(\partial\mathbb{D})$, and that

$$\{z = x + iy \in \mathbb{C} : 0 \leq y \leq \frac{\pi}{4}, x < \log(a\cos y + \sqrt{1 - a^2\sin^2(y)})\} \subset \Phi^{-1}(\mathbb{D}).$$

Moreover, since $0 < a \leq 1$ and $0 \leq y \leq \frac{\pi}{4}$, it follows that

$$x = \log\left(a\cos(y) + \sqrt{1 - a^2\sin^2(y)}\right) > 0$$

and that

$$\frac{dx}{dy} = -\frac{a\sin(y)}{\sqrt{1 - a^2\sin^2(y)}} < 0,$$
$$\frac{d^2x}{dy^2} = -\frac{a\cos(y)}{(1 - a^2\sin^2(y))^{3/2}} < 0.$$

Hence, the function $x = \log\left(a\cos(y) + \sqrt{1 - a^2\sin^2(y)}\right), y \in [0, \frac{\pi}{4}]$ is positive, decreasing and concave down. It follows that $\mathrm{conv}(\Gamma_1) \setminus \Gamma_1 \subset \Phi^{-1}(\mathbb{D})$, and hence by Theorem 3 that $\Phi(D)$ has a hypercyclic algebra as claimed.

The following observation by Godefroy and Shapiro allows to conclude the existence of hypercyclic algebras for differentiation operators on $C^\infty(\mathbb{R}, \mathbb{C})$.

Remark 13. (Godefroy and Shapiro) The restriction operator

$$\mathcal{R} : H(\mathbb{C}) \to C^\infty(\mathbb{R}, \mathbb{C})$$
$$f = f(z) \mapsto f(x)$$

is continuous, of dense range, and multiplicative, and for any complex polynomial $P = P(z)$ we have

$$\mathcal{R}P\left(\frac{d}{dz}\right) = P\left(\frac{d}{dx}\right)\mathcal{R}$$

By Theorem 3, Remark 6, Remark 13 and [10, Thm. 1] we have:

Corollary 14. *Let $P \in H(\mathbb{C})$ be either a non-constant polynomial vanishing at zero or so that the level set $\{z \in \mathbb{C} : |\Phi(z)| = 1\}$ contains a non-trivial, strictly convex compact arc Γ_1 satisfying*

$$\mathrm{conv}(\Gamma_1 \cup \{0\}) \setminus (\Gamma_1 \cup \{0\}) \subseteq P^{-1}(\mathbb{D}).$$

Then the operator $P(\frac{d}{dx})$ supports a hypercyclic algebra on $C^\infty(\mathbb{R},\mathbb{C})$. In particular, $T = aI + b\frac{d}{dx}$ supports a hypercyclic algebra on $C^\infty(\mathbb{R},\mathbb{C})$ whenever $|a| \le 1$ and $0 \neq b$.

Remark 15. The geometric assumption (1.1) in Theorem 3 does not seem to be a necessary one, as the following example by Félix Martínez suggests. The polynomial $P(z) := \frac{9^{9/8}}{8} z(z^8 - 1)$ vanishes at zero, so $P(D)$ supports a hypercyclic algebra by [10, Thm. 1]. On the other hand, numerical evidence suggests that the level set $\{z \in \mathbb{C} : \ |P(z)| = 1\}$ does not contain any non-trivial strictly convex compact arc Γ so that $\mathrm{conv}(\Gamma \cup \{0\}) \setminus (\Gamma \cup \{0\}) \subset P^{-1}(\mathbb{D})$, see Figure 3.

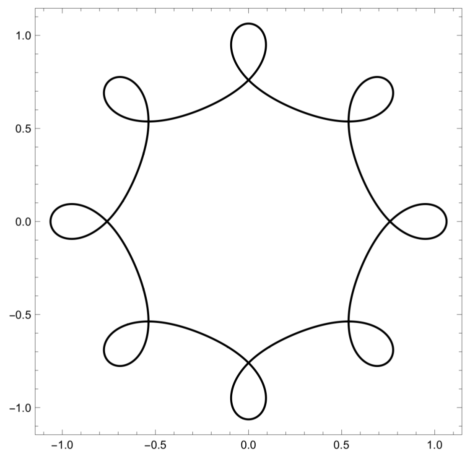

FIGURE 3. The level curve $\{z \in \mathbb{C} : \ |P(z)| = 1\}$ for $P(z) := \frac{9^{9/8}}{8} z(z^8 - 1)$.

We conclude the section by posing the following problem.

Problem 1. Let $\Phi(D) : H(\mathbb{C}) \to H(\mathbb{C})$ be a hypercyclic convolution operator *not* supporting a hypercyclic algebra.

(i) (Aron) Can Φ be a polynomial?
(ii) Must $\Phi \in H(\mathbb{C})$ be of the form $\Phi(z) = e^{az}$, for some $a \neq 0$?

3. HYPERCYCLIC ALGEBRAS AND COMPOSITION OPERATORS

3.1. **Translations on the algebra of smooth functions.** As observed in Theorem 1 by Aron et al [3] no translation operator $\tau_a(f)(\cdot) = f(\cdot + a)$ supports a hypercyclic algebra on $H(\mathbb{C})$. We show in Corollary 21 below that in contrast each non-trivial translation τ_a supports a hypercyclic algebra on $C^\infty(\mathbb{R},\mathbb{C})$. The proof is based on the following general fact.

Theorem 16. *Let T be a hypercyclic multiplicative operator on a separable F-algebra X over the real or complex scalar field $\mathbb{K}$. The following are equivalent:*

(a) *The operator T supports a hypercyclic algebra.*
(b) *For each non-constant polynomial $P \in \mathbb{K}[t]$ with $P(0) = 0$, the map $\widehat{P} : X \to X$, $f \mapsto P(f)$, has dense range.*
(c) *Each hypercyclic vector for T generates a hypercyclic algebra.*

Proof. The implication $(c) \Rightarrow (a)$ is immediate. To see $(a) \Rightarrow (b)$, let $g \in X$ generating a hypercyclic algebra $A(g)$ for T. In particular, for each polynomial P vanishing at the origin the multiplicativity of T gives that $\widehat{P}(\mathrm{Orb}(T,g)) = \mathrm{Orb}(T, P(g))$ is dense in X. Finally, to show the implication $(b) \Rightarrow (c)$, let $f \in X$ be hypercyclic for T and let $P \in \mathbb{K}[t]$ be a non-constant polynomial with $P(0) = 0$. Given $V \subset X$ open and non-empty, by our assumption the set $\widehat{P}^{-1}(V)$ is open and non-empty. So there exists $n \in \mathbb{N}$ for which $T^n(f) \in \widehat{P}^{-1}(V)$. The multiplicativity of the operator T now gives

$$T^n(P(f)) = \widehat{P}(T^n(f)) \in V.$$

So $P(f)$ is hypercyclic for T for each non-constant polynomial P that vanishes at zero. □

Corollary 17. *For each $0 \neq a \in \mathbb{R}$ the translation operator*

$$T_a : C^\infty(\mathbb{R},\mathbb{R}) \to C^\infty(\mathbb{R},\mathbb{R}),\ T_a(f)(x) = f(x+a),\ x \in \mathbb{R},$$

is weakly mixing but does not support a hypercyclic algebra.

Proof. Notice first that $J\tau_a = (T_a \oplus T_a)J$ for the $\mathbb{R}$-linear homeomorphism $J : C^\infty(\mathbb{R},\mathbb{C}) \to C^\infty(\mathbb{R},\mathbb{R}) \times C^\infty(\mathbb{R},\mathbb{R})$, $J(f) = (\mathrm{Re}(f), \mathrm{Im}(f))$. So T_a is weakly mixing. But the multiplicative operator T_a does not support a hypercyclic algebra, since $\{f^2 :\ f \in C^\infty(\mathbb{R},\mathbb{R})\}$ is not dense in $C^\infty(\mathbb{R},\mathbb{R})$. □

The next two lemmas are used to establish Proposition 20.

Lemma 18. *Let $B \in C^\infty(\mathbb{R},\mathbb{C})$ be of compact support. Then for each countable subset F of $\mathbb{C}$ and each $\epsilon > 0$ there exists $a \in D(0,\epsilon)$ such that*

$$\mathrm{Range}(B) \cap (F - a) = \emptyset.$$

Proof. Let $N \in \mathbb{N}$ such that B is supported in $[-N, N]$. Notice that the restriction of B to $[-N, N]$ is a closed rectifiable planar curve. Now, suppose that $\mathrm{Range}(B) \cap (F - a) \neq \emptyset$ for each $a \in D(0,\epsilon)$. Then

$$\begin{aligned} D(0,\epsilon) &= \bigcup_{y \in F} \{a \in D(0,\epsilon) : y - a \in \mathrm{Range}(B)\} \\ &= \bigcup_{y \in F} [D(0,\epsilon) \cap (y - \mathrm{Range}(B))], \end{aligned}$$

and since F is countable we must have for some $y \in F$ that

$$m(D(0,\epsilon) \cap (y - \mathrm{Range}(B))) > 0,$$

where m denotes the two dimensional Lebesgue measure. But since B is rectifiable $m(y - \mathrm{Range}(B)) = m(\mathrm{Range}(B)) = 0$, a contradiction. □

Lemma 19. *Let P be a polynomial of degree $m \in \mathbb{N}$ and let $B \in C^\infty(\mathbb{R}, \mathbb{C})$ be constant outside of a compact set such that*

$$\mathrm{Range}(B) \cap P(\{P' = 0\}) = \emptyset.$$

Then there exists $g \in C^\infty(\mathbb{R}, \mathbb{C})$ such that $P(g) = B$.

Proof. Let $N \in \mathbb{N}$ such that B is constant outside $[-N, N]$. Consider the family $\mathcal{M}$ of tuples $((-\infty, b), h)$ with $b \in (-\infty, \infty]$ and $h \in C^\infty((-\infty, b), \mathbb{C})$ satisfying that $P \circ h = B$ on $(-\infty, b)$. Endow $\mathcal{M}$ with the partial order $\leq$ given by $((-\infty, b_1), h_1) \leq ((-\infty, b_2), h_2)$ if and only if both $b_1 \leq b_2$ and $h_1 = h_2$ on $(-\infty, b_1)$. Observe that $\mathcal{M} \neq \emptyset$ as by picking $c \in P^{-1}(B(-N))$ and letting $h : (-\infty, -N) \to \mathbb{C}$, $h(x) = c$, we have $((-\infty, -N), h) \in \mathcal{M}$. Also, any totally ordered subfamily $\{((-\infty, b_j), h_j)\}_{j \in J}$ of $\mathcal{M}$ has $((-\infty, b), h) \in \mathcal{M}$ as an upper bound, where $b = \sup_{j \in J} b_j$ and where $h : (-\infty, b) \to \mathbb{C}$ is defined by $h(x) = h_j(x)$ for $x \in (-\infty, b_j)$. It follows by Zorn's Lemma that $\mathcal{M}$ contains some maximal element $((-\infty, b), g)$. We claim that $b = \infty$. Now, if $b < \infty$ then since $\mathrm{Range}(B) \cap P(\{P' = 0\}) = \emptyset$ there exist m distinct points $z_1, \ldots, z_m$ such that $P(z_1) = \cdots = P(z_m) = B(b)$. Furthermore, we may find a neighbourhood W of $B(b)$ and pairwise disjoint open sets $U_1, \ldots, U_m$ with $(z_1, \ldots, z_m) \in U_1 \times \cdots \times U_m$ and biholomorphisms $g_j : W \to U_j$ such that $g_j \circ P(z) = z$ for $z \in U_j$ $(j = 1, \ldots, m)$. Now, pick an open interval $(a, c) \subset B^{-1}(W)$ with $b \in (a, c)$. Notice that $g((a, b)) \subset U_j$ for some unique j. Define h on $(-\infty, c)$ by $h(x) = g(x)$ if $x \in (-\infty, b)$ and $h(x) = g_j \circ B(x)$ if $x \in (a, c)$. Then h is well defined and in $C^\infty((-\infty, c), \mathbb{C})$ and $((-\infty, b), g) < ((-\infty, c), h)$ contradicting the maximality of $((-\infty, b), g)$. So $b = \infty$ and $P(g) = B$, concluding the proof. □

We note that Lemma 19 also follows from an (albeit longer) compactness argument that does not require Zorn's Lemma.

Proposition 20. *Let P be a non-constant polynomial with complex coeficients and which vanishes at zero. Then $\widehat{P} : C^\infty(\mathbb{R}, \mathbb{C}) \to C^\infty(\mathbb{R}, \mathbb{C})$, $f \mapsto P(f)$, has dense range.*

Proof. Given $U \subset C^\infty(\mathbb{R}, \mathbb{C})$ open and non-empty, pick $B \in U$ with compact support. By Lemma 18 there exists $a \in \mathbb{C}$ such that $B_1 := B + a \in U$ and $\mathrm{Range}(B_1) \cap P(\{P' = 0\}) = \emptyset$. By Lemma 19 there exists some $g \in C^\infty(\mathbb{R}, \mathbb{C})$ such that $P(g) = B_1$. □

Corollary 21. *For each $0 \neq a \in \mathbb{R}$ the translation operator τ_a supports a hypercyclic algebra on $C^\infty(\mathbb{R}, \mathbb{C})$.*

Proof. The hypercyclicity of τ_a on $C^\infty(\mathbb{R}, \mathbb{C})$ follows from Birkhoff's theorem and the so-called Comparison Principle; notice that $\tau_a \mathcal{R} = \mathcal{R} \tau_a$ where the translation on the right hand side is acting on $H(\mathbb{C})$. The conclusion now follows by Proposition 20 and Theorem 16. □

3.2. **Composition operators on $H(\Omega)$.** Recall that given a domain Ω in the complex plane, each $\omega \in H(\Omega)$ and $\varphi : \Omega \to \Omega$ holomorphic induce a weighted composition operator

$$C_{\omega,\varphi} : H(\Omega) \to H(\Omega), f \mapsto \omega(f \circ \varphi).$$

When $C_{\omega,\varphi}$ is supercyclic, the weight symbol ω must be zero-free and the compositional symbol φ must be univalent and without fixed points. Moreover, these conditions are sufficient for the hypercyclicity of $C_{\omega,\varphi}$ when Ω is simply connected [9, Proposition 2.1 and Theorem 3.1]. We conclude the paper by noting the following extension of Theorem 1.

Theorem 22. *Let $\Omega \subset \mathbb{C}$ be a domain. Then no weighted composition operator $C_{\omega,\varphi} : H(\Omega) \to H(\Omega)$ supports a supercyclic algebra.*

Proof. Given any $f \in H(\Omega)$ and $a \in \Omega$, the polynomial $g(z) = (z - a)^3$ is not in the closure of

$$\mathbb{C}\mathrm{Orb}(C_{\omega,\varphi}, f^2) = \mathrm{span}(\{f^2\}) \cup \left\{ \lambda \prod_{j=0}^{n-1} C_\varphi^j(\omega)\; C_\varphi^n(f^2) : \; n \in \mathbb{N}, \lambda \in \mathbb{C} \right\}.$$

Indeed, if (n_k) is a strictly increasing sequence of positive integers and (λ_k) is a scalar sequence satisfying

$$\lambda_k \left(\prod_{j=0}^{n_k-1} C_\varphi^j(\omega) \right) C_\varphi^{n_k}(f^2) = \lambda_k C_{\omega,\varphi}^{n_k}(f^2) \underset{k\to\infty}{\to} g$$

then by Hurwitz theorem [15, page 231] there exists a disc $D(a, \delta) \subset \Omega$ centered at a so that for each large k

$$\lambda_k \left(\prod_{j=0}^{n_k-1} C_\varphi^j(\omega) \right) C_\varphi^{n_k}(f^2) = \lambda_k \left(\prod_{j=0}^{n_k-1} C_\varphi^j(\omega) \right) (C_\varphi^{n_k}(f))^2$$

has exactly three zeroes (counted with multiplicity) on $D(a, \delta)$ which is impossible since ω is zero-free. □

When $\Omega \subset \mathbb{C}$ is simply connected and C_φ is a hypercyclic composition operator on $H(\Omega)$, then any operator in the algebra of operators generated by C_φ is also hypercyclic [8, Theorem 1]. Hence it is natural to ask:

Question 3. Let C_φ be a hypercyclic composition operator on $H(\Omega)$, where Ω is simply connected, and let P be a non-constant polynomial with $P(0) = 0$. Can $P(C_\varphi)$ support a hypercyclic algebra?

The answer must be affirmative if Problem 1(ii) has an affirmative answer. Finally, notice that in contrast with Theorem 22, by Corollary 21 it is possible for a composition operator to support a hypercyclic algebra on $C^\infty(\mathbb{R}, \mathbb{C})$. The hypercyclic weighted composition operators on $C^\infty(\Omega, \mathbb{C})$, where $\Omega \subset \mathbb{R}^d$ is open, have been characterized in [21], see also [13]. We conclude the paper with the following question.

Question 4. Let $\Omega \subset \mathbb{R}^d$ be open and nonempty. Which weighted composition operators on $C^\infty(\Omega, \mathbb{C})$ support a hypercyclic algebra?

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