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On proximal fineness of topological groups in their right uniformity

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Abstract

A uniform space X is said to be proximally fine if every proximally continuous function defined on X into an arbitrary uniform pace Y is uniformly continuous. We supply a proof that every topological group which is functionally generated by its precompact subsets is proximally fine with respect to its right uniformity. On the other hand, we show that there are various permutation groups G on the integers $\mathbb{N}$ that are not proximally fine with respect to the topology generated by the sets $\{g \in G : g(A) \subset B\}$, $A, B \subset \mathbb{N}$.

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1. Introduction

A function $f : X \to Y$ between two uniform spaces is said to be *proximally continuous* if for every bounded uniformly continuous function $g : Y \to \mathbb{R}$, the composition function $g \circ f : X \to \mathbb{R}$ is uniformly continuous; the reals $\mathbb{R}$ being equipped with the usual metric. A uniform space X is said to be *proximally fine* if every proximally continuous function defined on X into an arbitrary uniform space Y is uniformly continuous. It is well-known that metric spaces and uniform products of metric spaces are proximally fine; however, there are many uniform spaces that are not proximally fine although they are

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topologically well-behaved (some may be locally compact and even discrete). We refer the reader to Hušek's recent paper [12] for more information about proximally fineness of general uniform spaces. We are interested here in the fine proximal condition of the topological groups when these spaces are endowed with their right uniformity. This subject seems to have no specific literature, although there are some questions which could have been naturally addressed in this setting, such as the Itzkowitz problem for metric and/or locally compact groups [13] (the link between Itzkowitz problem and proximity theory was made later in [6]).

In view of the close relationship between the right uniformity of topological groups and the system of neighborhoods of their identity, it is reasonable to expect that the proximal fineness of a given topological group G is satisfied provided that a not too much restrictive condition is imposed on the topology of G. This feeling is heightened by Corollary 2.6 in this note asserting that it is sufficient to assume that G is functionally generated (in Arkhangel'skiĭ's sense) by its precompact subsets. In fact, this is still true under much less restrictive conditions (Theorem 2.4 below). Let us mention that one part of Theorem 2.4 was asserted without proof in [5][1]. The main subject of this note can be examined in the context of G-sets (or G-spaces) without significantly altering its essence, so the main result is stated and established in a somewhat more general form (Theorem 2.5).

Some examples of non-proximally fine groups are given in Section 3. To do that, we consider the topology τ on $\mathbb{N}^{\mathbb{N}}$ of uniform convergence when (the target set) $\mathbb{N}$ is endowed with the Samuel uniformity. We show in Corollary 3.4 that for any permutation group H on $\mathbb{N}$ for which all but finitely many orbits are finite and uniformly bounded, the only group topology on H which is finer than τ and proximally fine is the discrete topology. The question of whether there is an abelian (or at least SIN) group which is non-proximally fine group is left open.

2. Main result

As usual, if $(X,\mathcal{U})$ is a uniform space, where $\mathcal{U}$ is the uniform neighborhoods of the diagonal of X, then for $A \subset X$ and $U \in \mathcal{U}$, $U[A]$ stands for the set of $y \in X$ such that $(x, y) \in U$ for some $x \in A$. For undefined terms we refer to the books [8] and [16]. One of the tools used here is Katetov's extension theorem of uniformly continuous bounded real-valued functions [14]. The following statement is also well-known (see [8]); its proof is outlined here for the sake of completeness and because it can be adapted to show a result in the same spirit that will be used in the proof Theorem 2.5.

Proposition 2.1. *Let $f : X \to Y$, where $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are two uniform spaces. Then the following are equivalent:*

(1) *$f : X \to Y$ is proximally continuous,*

[1]The author is indebted to Professor Michael D. Rice for his interest in this result and its proof, which motivated the present work.

(2) *for every* $A \subset X$ *and* $V \in \mathcal{V}$, *there exists* $U \in \mathcal{U}$ *such that* $f(U[A]) \subset V[f(A)]$.

Proof. To show that 1) implies 2), let d be a bounded uniformly continuous pseudometric on Y such that $(x, y) \in V$ whenever $d(x, y) < 1$ ([8]) and consider the uniformly continuous function ϕ on Y defined by $\phi(y) = d(y, f(A))$. Since $\phi \circ f$ is uniformly continuous, there is $U \in \mathcal{U}$ such that $(x, y) \in U$ implies $|\phi(f(x)) - \phi(f(y))| < 1$. Then $f(U[A]) \subset V[f(A)]$.

For the converse, let $\phi : Y \to \mathbb{R}$ be a bounded uniformly continuous function and let $\varepsilon > 0$. Define $V = \{(x, y) \in Y \times Y : |\phi(x) - \phi(y)| < \varepsilon/3\}$. Then $V \in \mathcal{V}$ and since ϕ is bounded, there is a finite set $F \subset Y$ so that $Y = V[F]$. Let $A_z = f^{-1}(V[z])$, $z \in F$, and choose $U \in \mathcal{U}$ such that $f(U[A_z]) \subset V[f(A_z)]$ for each $z \in F$. Then $|\phi \circ f(x) - \phi \circ f(y)| \leq \varepsilon$ for each $(x, y) \in U$. Indeed, there exists $z \in F$ such that $x \in A_z$ so $(f(x), z) \in V$. Since $y \in U[x]$ we have $f(y) \in f(U[A_z]) \subset V[f(A_z)]$ thus there exists $t \in A_z$ such that $(f(y), f(t)) \in V$. As $t \in A_z$ we also have $(f(t), z) \in V$. Therefore, $(f(x), f(y)) \in V \circ V \circ V$. □

Let us say that a topological group G is *proximally fine* if the uniform space $(G,\mathcal{U}_r)$ is proximally fine, where $\mathcal{U}_r$ is the right uniformity of G. It is equivalent to say that $(G,\mathcal{U}_l)$ is proximally fine, where $\mathcal{U}_l$ is the left uniformity of G. Recall that a basis of $\mathcal{U}_r$ (respectively, $\mathcal{U}_l$) is given by the sets of the form $\{(g, h) \in G \times G : gh^{-1} \in V\}$ (respectively, $\{(g, h) \in G \times G : g^{-1}h \in V\}$) as V runs over the set $\mathcal{V}(e)$ of neighborhoods of the unit e of G.

Proposition 2.2. *Let* G *be a topological group,* $(Y,\mathcal{U})$ *a uniform space and let* $f : G \to Y$ *be a function. For* $g \in G$, *let* $\psi_g : G \to Y$ *be the function defined by* $\psi_g(h) = f(gh)$. *Then, the following are equivalent:*

(1) f *is uniformly continuous,* G *being equipped with the right uniformity,*
(2) *the function* $\psi : g \in G \to \psi_g \in Y^G$ *is continuous when* Y^G *is endowed with the uniformity of uniform convergence on* G.

Proof. To show that (1) implies (2), suppose that f is right uniformly continuous and let $U \in \mathcal{U}$. There is a neighborhood V of the unit e in G such that $xy^{-1} \in V$ implies $(f(x), f(y)) \in U$. Let $g \in G$. Then Vg is a neighborhood of g in G and for each $h \in Vg$ and $x \in G$ we have $hx(gx)^{-1} \in V$. It follows that $(f(hx), f(gx)) \in U$, so $(\psi_h(x), \psi_g(x)) \in U$ for every $x \in G$.

To show that (2) implies (1), let $U \in \mathcal{U}$ and choose $V \in \mathcal{V}(e)$ such that $(\psi_g(x), \psi_e(x)) \in U$ for every $g \in V$ and $x \in G$. Then, for every $g, h \in G$ such that $h \in Vg$ we have $(\psi_{hg^{-1}}(x), \psi_e(x)) \in U$ for each $x \in G$, equivalently, $(f(hx), f(gx)) \in U$, for every $x \in G$. In particular, $(f(h), f(g)) \in U$. □

It is possible to expand substantially the framework of the starting topic of this note without major changes as follows: Let G be a topological group and let X be a G-set, that is, X is a nonempty set for which there is a map $* : G \times X \to X$ satisfying $(gh)*x = g*(h*x)$ for every $g, h \in G$ and $x \in X$. The function $*$ is called a left action of G on X. To simplify, write gx in place of $g*x$ and UA in place of $U * A$ if $U \subset G$ and $A \subset X$. No topology will be required on X. Let $(Y, \mathcal{V})$ be a uniform space and $f : X \to Y$ a function. It is consistent with the definitions given above to say that a f is *right uniformly continuous* if for each $V \in \mathcal{V}$, there is $U \in \mathcal{V}(e)$ such that $(f(gx), f(hx)) \in V$ for each $x \in X$, whenever $gh^{-1} \in U$. Similarly, the function f is said to be *right proximally continuous* if for each bounded uniformly continuous function $\phi : Y \to \mathbb{R}$, the function $\phi \circ f$ is right uniformly continuous. It is easy to check (see the proof of Proposition 2.2) that f is right uniformly continuous if and only if the function $\psi : g \in G \to \psi(g) \in Y^X$ (where $\psi(g)(x) = f(gx)$) is continuous when Y^X is endowed with the uniform convergence. Similarly, a simple adaptation of the proof of Proposition 2.1 shows that f is right proximally continuous if and only if for each $A \subset GX$ and $V \in \mathcal{V}$, there is $U \in \mathcal{V}(e)$ such that $f(UA) \subset V[f(A)]$.

The following properties are required for Theorem 2.5; we are formulating them separately to reduce the proof to its essential components. Let X be a G-set and $f : X \to Y$ a right proximally continuous, as defined above. Then, for each $A \subset G$:

- (c1) for every $x \in X$, the function $g \in G \to f(gx) \in Y$ is continuous,
- (c2) if $\psi_{|A}$ is right uniformly continuous, then $\psi_{|\overline{A}}$ is right uniformly continuous,
- (c3) if $a \in A$ is a point of continuity of $\psi_{|A}$, then a is a point of continuity of $\psi_{|\overline{A}}$.

We check the validity of these properties for the benefit of the reader. Let $V \in \mathcal{V}$. For every $x \in X$ and $g \in G$, there is $U \in \mathcal{V}(e)$ such that $f(Ugx) \subset V[f(gx)]$. Since Ug is a neighborhood of g in G, (c1) holds. Property (c2) and (c3) follows from (c1). For, if $x \in X$, $U \in \mathcal{V}(e)$ are such that $(f(ax), f(bx)) \in V$ for every $a, b \in A$ with $a \in UUb$, then (c1) implies that $f(gx), f(hx)) \in V^2$ for each $g, h \in \overline{A}$ such that $g \in Uh$. Taking x arbitrary in X gives (c2). For (c3), let U be an open neighborhood of the unit in G such that $(f(gx), f(ax)) \in V$ for ever $g \in Ua \cap A$ and $x \in X$. Since $Ua \cap \overline{A} \subset \overline{Ua \cap A}$, it follows from (c1) that for every $g \in Ua \cap \overline{A}$ and $x \in X$, $(f(gx), f(ax)) \in V^2$.

To establish Theorem 2.5 we also need the next key lemma; this is a well-known tool in the theory of proximity spaces (most often with W^4 instead of W^3).

Lemma 2.3. *Let X be a set and W be a symmetric binary relation on X. Then, for every infinite cardinal η and for every sequence $(x_n, y_n)_{n<\eta} \subset X \times X$ such that $(x_n, y_n) \notin W^3$ for each $n < \eta$, there is a cofinal set $A \subset \eta$ such that $(x_n, y_m) \notin W$ for every $n, m \in A$.*

Proof. Replacing η by its cofinality $\mathrm{cf}(\eta)$, we may suppose that η is regular. Let $M \subset \eta$ be a maximal set satisfying $(x_n, y_m) \notin W$ for every $n, m \in M$. If M is cofinal in η, the proof is finished, so suppose that M is not cofinal in η. For each $j \in M$, let $A_j = \{n < \eta : (x_n, y_j) \in W\}$, $B_j = \{n < \eta : (x_j, y_n) \in W\}$, $C_j = A_j \cup B_j$ and $C = \cup_{j \in M} C_j$. The maximality of M implies that $\eta \subset M \cup C$. Since η is regular, there is $j \in M$ such that C_j is cofinal in η, therefore A_j or B_j is cofinal in η. We suppose that it is A_j, the other case is similar. Let

$n, m \in A_j$. Then $(x_n, y_j) \in W$ and $(x_m, y_j) \in W$, hence $(x_n, y_m) \notin W$ since $(x_m, y_m) \notin W^3$ (recall that W is symmetric). Similarly, $(x_m, y_n) \notin W$. □

We will now specify a few topological concepts that will be used in what follows. The first is a variant of Herrlich's notion of radial spaces [11]. Radial spaces were characterized by A.V. Arhangel'skiĭ [3] as follows: A space X is radial if and only if for each $x \in X$ and $A \subset X$ such that $x \in \overline{A}$, there is $B \subset A$ of regular cardinality $|B|$ such that $x \in \overline{C}$ for every $C \subset B$ having the same cardinality as B. Let us say that a subset A of X is *relatively o-radial* in X if for every collection $(O_i)_{i\in I}$ of open sets in X and $x \in A$ such that

$$x \in \overline{\cup_{i\in I} O_i \cap A} \setminus \cup_{i\in I} \overline{O_j},$$

there is a set $J \subset I$ of regular cardinality such that $x \in \overline{\cup_{j\in L} O_j}$ whenever $L \subset J$ and $|L| = |J|$. If the set J can always be chosen countable, then A is said to be *relatively o-Malykhin* in X. All closures are taken in X.

Every almost metrizable (in particular, Čech-Complete) group is o-Malykhin (in itself). More generally, every inframetrizable group [16] is o-Malykhin, see [6].

Let X be a topological space and $\mathcal{M}$ a collection of subspaces of X. Following Arhangel'skiĭ [2], the space X is said to be *functionally generated* by the collection $\mathcal{M}$ if for every discontinuous function $f : X \to \mathbb{R}$, there exists $A \in \mathcal{M}$ such that the restriction $f_{|A} : A \to \mathbb{R}$ of f to the subspace A of X has no continuous extension to X. The space X is said to be *strongly functionally generated* by $\mathcal{M}$ if for every discontinuous function $f : X \to \mathbb{R}$, there exists $A \in \mathcal{M}$ such that the restriction $f_{|A} : A \to \mathbb{R}$ of f to the subspace A of X is discontinuous.

The following is the main result of this note. The statement corresponding to the case (2) was asserted (without proof) in [5].

Theorem 2.4. *Let G be a topological group satisfying at least one of the following:*

(1) *G is functionally generated by the sets $\overline{A} \subset G$ such that AA^{-1} is relatively o-radial in G,*
(2) *G is strongly functionally generated by the sets $A \subset G$ such that A is relatively o-radial in G.*

Then G is proximally fine.

According to Proposition 2.2 and keeping the above notations, Theorem 2.4 is obtained from the following general result by considering the left action of G on itself.

Theorem 2.5. *Let G be a topological group and suppose that for each discontinuous bounded function $\alpha : G \to \mathbb{R}$, there is a set $A \subset G$ having at least one of the following conditions:*

(1) *$\alpha_{|\overline{A}}$ has no continuous extension to G and AA^{-1} is relatively o-radial in G,*

(2) $\alpha_{|\overline{A}}$ *is discontinuous at some point of* A *and* A *is relatively o-radial in* G.

Let X *be a* G*-set,* $(Y, \mathcal{V})$ *a uniform space and let* $f : X \to Y$ *be a right proximally continuous. Then* $f : X \to Y$ *is right uniformly continuous.*

Proof. We have to show that the function $\psi : g \in G \to \psi(g) \in Y^X$ (where $\psi(g)(x) = f(gx)$) is continuous. We proceed by contradiction by supposing that ψ is not continuous. Then, there is a bounded uniformly continuous function $\theta : Y^X \to \mathbb{R}$ such that $\theta \circ \psi$ is not continuous (see [8]). Let $A \subset G$ satisfying at least one of the conditions (1) and (2) with respect to the function $\theta \circ \psi$. In case (1), there is no compatible uniformity on G making uniformly continuous the function $\theta \circ \psi_{|\overline{A}}$; for, otherwise, Katetov's theorem would give us a continuous extension of $\theta \circ \psi_{|\overline{A}}$. In particular, $\psi_{|\overline{A}}$ is not right uniformly continuous. As remarked above, it follows from (c2) that $\psi_{|A}$ is not right uniformly continuous. There is then an open and symmetric $W \in \mathcal{U}$ such that for every $V \in \mathcal{V}(e)$, there exist $a_V \in V$, $g_V \in A$ and $x_V \in X$ satisfying $a_V g_V \in A$ and

$$(f(a_V g_V x_V), f(g_V x_V)) \notin W^6. \tag{2.1}$$

For each $V \in \mathcal{V}(e)$, let $h_V = g_V x_V$ and define

$$O_V = \{g \in G : (f(gh_V), f(a_V h_V)) \in W\}.$$

Since the functions $g \in G \to f(gh_V) \in Y$, $V \in \mathcal{V}(e)$, are continuous by the property (c1), each O_V is open in G. Since $a_V \in O_V \cap Ag_V^{-1} \subset AA^{-1}$ and $a_V \in V$ for each $V \in \mathcal{V}(e)$, it follows that

$$e \in \overline{\bigcup_{V \in \mathcal{V}(e)} O_V \cap (AA^{-1})}. \tag{2.2}$$

We also have $e \notin \overline{O}_V$, for each $V \in \mathcal{V}(e)$. Indeed, otherwise, there exist $V \in \mathcal{V}(e)$ and $g \in O_V$ such that $(f(gh_V), f(h_V)) \in W$. Therefore $(f(a_V h_V), f(h_V)) \in W^2$ which contradicts (2.1).

Since AA^{-1} is relatively o-radial in G, in view of (2.2), there is a set $\Gamma \subset \mathcal{V}(e)$ of regular cardinal such that for each set $I \subset \Gamma$ of the same cardinal as Γ, we have

$$e \in \overline{\bigcup_{V \in I} \{g \in G : (f(gh_V), f(a_V h_V)) \in W\}}. \tag{2.3}$$

By Lemma 2.3 and (2.1), there is $I \subset \Gamma$ such that $|I| = |\Gamma|$ (since $|\Gamma|$ is regular) and $(f(a_U h_U), f(h_V)) \notin W^2$ for every $U, V \in I$. Since f is right proximally continuous, there exists $V \in \mathcal{V}(e)$ such that

$$f(V\{h_U : U \in I\}) \subset W[f(\{h_U : U \in I\})]. \tag{2.4}$$

By (2.3) applied to I, there is $U_1 \in I$ such that $V \cap O_{U_1} \neq \varnothing$. Let $g \in V$ be such that $(f(gh_{U_1}), f(a_{U_1} h_{U_1})) \in W$. By (2.4), there is $U_2 \in I$ so that $(f(gh_{U_1}), f(h_{U_2})) \in W$. It follows that $(f(a_{U_1} h_{U_1}), f(h_{U_2})) \in W^2$, which is a contradiction. Therefore, ψ is continuous in case (1).

In case (2), A is o-radial in G and $\theta \circ \psi_{|\overline{A}}$ is discontinuous at some point $a \in A$. Since θ is continuous, $\psi_{|\overline{A}}$ is necessarily discontinuous at a. It follows from the property (c3) that $\psi_{|A}$ is discontinuous at a. Let $W \in \mathcal{U}$ be symmetric and open such that for every $V \in \mathcal{V}(e)$, there exist $a_V \in V$ and $x_V \in G$, such that $a_V a \in A$ and $(f(a_V a x_V), f(a x_V)) \notin W^6$. Taking $h_V = a x_V$ for each $V \in \mathcal{V}(e)$, we have

$$e \in \overline{\bigcup_{V \in \mathcal{V}(e)} \{g \in G : (f(g h_V), f(h_V)) \in W\} \cap (A a^{-1})}.$$

It is easy to see that Aa^{-1} is relatively o-radial in G, therefore the proof can be continued and concluded in the same way as in the first case. It should be noted that Katetov's theorem was not used in this case. □

Let $A \subset G$, where G is a topological group. It is proved in [6] that AA^{-1} is relatively o-Malykhin in G, provided that A is left and right precompact. Thus Theorem 2.4 yields:

Corollary 2.6. *Every topological group G which is functionally generated by the collection of its precompact subsets is proximally fine.*

In view of the role played by locally compact groups in many areas of mathematics, it is worth mentioning the following particular case of Corollary 2.6.

Corollary 2.7. *Every locally compact topological group is proximally fine.*

Recall that the topological group G is said to SIN (or with small invariant neighborhoods of the identity) if the left uniformity $\mathcal{U}_l$ and the right uniformity $\mathcal{U}_l$ of G are equal. It is well-known and easy to see that the property of being SIN for G is equivalent to the inclusion $\mathcal{U}_r \subset \mathcal{U}_l$ or, equivalently, to the uniform continuity of the identity map from $(G, \mathcal{U}_l)$ to $(G, \mathcal{U}_r)$. Following Protasov [15], the group G is said to be FSIN (or functionally balanced) if every bounded right uniformly continuous function $f : G \to \mathbb{R}$ is left uniformly continuous. Clearly, the property of being FSIN for G is equivalent to the proximal continuity of the identity map from $(G, \mathcal{U}_l)$ to $(G, \mathcal{U}_r)$. Consequently, every proximally fine FSIN group is SIN. The question whether every FSIN group is SIN is called Itzkowitz problem and is still open. We refer the reader to [5] for more information; see also [17] for a very recent contribution to this topic.

Since every proximally fine FSIN group is SIN, it follows from Theorem 2.4 that every FSIN group is SIN provided that it is strongly functionally generated by its relatively o-radial subsets. This result has already been formulated (implicitly and without proof) in [5]. This is supplemented by the next corollary of Theorem 2.4 :

Corollary 2.8. *Every FSIN group G which is functionally generated by the sets $\overline{A} \subset G$ such that AA^{-1} is relatively o-radial in G is a SIN group.*

To conclude this section, we would like to take this opportunity to comment on the parenthesized question of [5, Question 6] whether every bounded topological group is FSIN. The answer is of course no, since FSIN is a hereditary property (by Katetov's theorem) and every topological group is isomorphic both algebraically and topologically to a subgroup of a bounded group [10] (see also [9]).

3. Examples

In this section, we give some examples of non-proximally fine Hausdorff topological groups and examine their behavior towards the FSIN property. The set of positive integers is denoted by $\mathbb{N}$ and $\mathcal{U}$ is the Samuel uniformity of the uniform discrete space $\mathbb{N}$. The uniformity $\mathcal{U}$ is sometimes called the precompact reflection of the uniform discrete space $\mathbb{N}$. A basis of $\mathcal{U}$ is given by the sets $\cup_{i\leq n} A_i \times A_i$, where $A_1, \dots A_n$ is a partition of the integers. Let $\mathbb{N}^{\mathbb{N}}$ be endowed with the uniformity $\mathcal{V}$ of uniform convergence when $\mathbb{N}$ (the target space) is equipped with the uniformity $\mathcal{U}$. Let G denote the permutation group of the set $\mathbb{N}$ of positive integers and let S be the normal subgroup of G given by finitary permutations $g \in G$; that is, $g \in S$ if and only if the set $\mathrm{supp}(g) = \{x \in \mathbb{N} : g(x) \neq x\}$ is finite. It is easy to see that G (hence S) is a topological group when equipped with the topology τ induced by the uniformity $\mathcal{V}$. More precisely, G is a non-Archimedean group, since a basis of neighborhoods of its unit is given by the subgroups of G of the form

$$H_\pi = \{g \in G : g(A_i) = A_i, i = 1, \dots, n\},$$

where $\pi = \{A_1, \dots, A_n\}$ is a finite partition of $\mathbb{N}$. Another approach (which we will not adopt here) is to consider G as the group of all auto-homeomorphisms of $\beta\mathbb{N}$ equipped with the compact-open topology, where $\beta\mathbb{N}$ is the Stone-Čech-compactification of the integers.

The so-called natural Polish topology τ_0 on G, given by pointwise convergence, is coarser than τ and for every partition π of $\mathbb{N}$ the set H_π is τ_0-closed. This is to say that τ_0 is a cotopology for τ in the sense of [1]; in particular, (G, τ) is submetrizable and a Baire space. In what follows, unless otherwise stated, the groups G and S will be systematically considered under the topology τ.

For later use, we check that the quotient group G/S (with the quotient topology) is Hausdorff, that is, S is closed in G. Let $g \in G \setminus S$. A simple induction allows to construct an infinite set $A \subset \mathbb{N}$ such that $g(A) \subset \mathbb{N} \setminus A$ (alternatively, A is obtained from Lemma 2.3 applied to the set $\{(x, g(x)) : x \in \mathrm{supp}(g)\}$). Let $\pi = \{A, \mathbb{N} \setminus A\}$. Then gH_π is a τ-neighborhood of g and for each $h \in H_\pi$, $gh(A) = g(A) \subset \mathbb{N} \setminus A$. Hence $gH_\pi \cap S = \varnothing$.

Recall that for a topological group H, the lower uniformity $\mathcal{U}_l \wedge \mathcal{U}_r$ on H is called the Roelcke uniformity and has base consisting of the sets $\{(x, y) \in H \times H : x \in VyV\}$, $V \in \mathcal{V}(e)$ (see [16]). Every (right) proximally fine group is proximally fine with respect to the Roelcke uniformity; therefore, the following shows in a strong way that the groups G and S are not proximally fine:

Proposition 3.1. *Let $k \in \mathbb{N}$ and $\phi : G \to \mathbb{N}$ be the evaluation function $\phi(g) = g(k)$, $\mathbb{N}$ being equipped with the discrete uniformity.*

(1) *The function $\phi : G \to \mathbb{N}$ is left uniformly continuous and right proximally continuous. In particular, ϕ is Roelcke-proximally continuous.*
(2) *If $\mathbb{N}$ is endowed with the uniformity $\mathcal{U}$, then ϕ is right uniformly continuous. Conversely, if $\mathcal{V}$ is a uniformity on $\mathbb{N}$ such that the restriction of $\phi_{|S} : S \to (\mathbb{N}, \mathcal{V})$ is right uniformly continuous, then $\mathcal{V} \subset \mathcal{U}$.*

In particular, G and S are not Roelcke proximally fine.

Proof. 1) Clearly, ϕ is left uniformly continuous with respect to the natural topology τ_0; since $\tau_0 \subset \tau$, ϕ is left uniformly continuous. To show that ϕ is right proximally continuous, let $L \subset G$ and put $\pi_0 = \{A, \mathbb{N} \setminus A\}$, where $A = \{g(k) : g \in L\}$, and let us verify that $\phi(H_{\pi_0} L) \subset \phi(L)$. Proposition 2.1 will then conclude the proof. Let $h_0 \in H_{\pi_0}$ and $g_0 \in L$. Then $h_0(g_0(k)) \in \{g(k) : g \in L\}$, hence we can write $h_0(g_0(k)) = g(k)$ for some $g \in L$, thus $\phi(h_0 g_0) \in \phi(L)$.
2) $\phi : G \to (\mathbb{N},\mathcal{U})$ is right uniformly continuous, since for every partition $\pi = \{A_1, \ldots, A_n\}$ of $\mathbb{N}$, we have $(g(k), h(k)) \in \cup_{i \leq n} A_i \times A_i$ provided that $hg^{-1} \in H_\pi$. For the converse, suppose that $\mathcal{V}$ is a uniformity on $\mathbb{N}$ satisfying $\mathcal{V} \not\subset \mathcal{U}$ and let $V \in \mathcal{V} \backslash \mathcal{U}$. We will check that for any partition $\pi = \{A_1, \ldots, A_n\}$ of $\mathbb{N}$, there are $g, h \in S$ having $g \in H_\pi h$ and $(\phi(g), \phi(h)) \notin V$. We may suppose that $A_1 = \{k\}$ and that A_2 contains two elements a and b such that $(a, b) \notin V$. Define $g \in S$ by $g(k) = a$, $g(a) = k$ and $g(x) = x$ for $x \notin \{k, a\}$. Define also $h \in S$ by $h(k) = b$, $h(a) = k$, $h(b) = a$ and $h(x) = x$ otherwise. Then $gh^{-1} \in H_\pi$, but $(\phi(g), \phi(h)) \notin V$. □

The next statements 3.2 and 3.4 give some extremal properties showing that the above examples provided by Proposition 3.1 are somehow optimal. For a subgroup H of G and $L \subset \mathbb{N}$, let $H_{(L)}$ stand for the pointwise stabilizers of L in H (that is, the set of $h \in H$ such that $h(x) = x$ for all $x \in L$).

Let us recall that a topological group H is said to be *strongly FSIN* if every real-valued right uniformly continuous function on H is left uniformly continuous.

Proposition 3.2. *Let H be subgroup of G and $F \subset \mathbb{N}$ a finite set. Let τ_1 be a group topology on H and for each $k \in F$, let $\phi_k : g \in H \to g(k) \in \mathbb{N}$, where $\mathbb{N}$ is endowed with the discrete uniformity.*

(1) *If for each $k \in F$, $\phi_k : (H, \tau_1) \to \mathbb{N}$ is right proximally continuous, then τ is coarser than τ_1 on $H_{(\mathbb{N} \backslash HF)}$.*
(2) *If for each $k \in F$, $\phi_k : (H, \tau_1) \to \mathbb{N}$ is right uniformly continuous, then τ_1 is discrete on $H_{(\mathbb{N} \backslash HF)}$.*
(3) *If $\tau_{0|H} \subset \tau_1$ and (H, τ_1) is strongly FSIN, then τ_1 is discrete on $H_{(\mathbb{N} \backslash HF)}$.*

Proof. 1) We first show that for a given $A \subset \mathbb{N}$, there is a τ_1-neighborhood V_A of the unit (in H) such that $f(A \cap (HF)) \subset A$ for every $f \in V_A$. To do that, for each $k \in F$, define $L_k = \{g \in H : g(k) \in A\}$. According to Proposition 2.1, there is a τ_1-neighborhood V_A of the unit such that for every $k \in F$, $\phi_k(V_A L_k) \subset \phi_k(L_k)$. Let $n \in A \cap (HF)$ and $f \in V_A$. Choose $g \in H$

and $k \in F$ such that $g(k) = n$. Then $g \in L_k$, hence $\phi_k(fg) \in \phi_k(L_k)$ and thus $f(n) \in A$. This shows that $f(A \cap (HF)) \subset A$ for every $f \in V_A$. It follows that for any partition $\pi = \{A_1, \dots, A_1\}$ of $\mathbb{N}$, there is a τ_1-neighborhood of the unit in $H_{(\mathbb{N}\setminus HF)}$, namely $V = H_{(\mathbb{N}\setminus HF)} \cap V_{A_1} \cap \ldots \cap V_{A_n}$, such that $V \subset H_\pi$. Since τ_1 is a group topology, it follows that τ is coarser than τ_1 on $H_{(\mathbb{N}\setminus HF)}$.

2) Suppose that τ_1 is not discrete on $H_{(\mathbb{N}\setminus HF)}$ and let V be τ_1-neighborhood of the unit in H. We will show that there are $g, h \in H$ and $k \in F$ such that $gh^{-1} \in V$ and $g(k) \neq h(k)$, contradicting the right uniform continuity of ϕ_l for at least one l in the finite set F. Since for each $k \in F$, ϕ_k is τ_1-continuous, there is $f \in V \cap H_{(\mathbb{N}\setminus HF)}$ such that $f(k) = k$ for all $k \in F$ and $f(a) \neq a$ for some $a \in \mathbb{N}$. Then $a \in HF$, hence there are $h \in H$ and $k \in F$ such that $h(k) = a$. Taking $g = fh$, we get $gh^{-1} \in V$ and $\phi_k(g) \neq \phi_k(h)$.

3) If (H, τ_1) is strongly FSIN, then the functions ϕ_k, $k \in F$, are τ_1-right uniformly continuous on H because they are left uniformly continuous on (G, τ_0) and $\tau_{0|H} \subset \tau_1$. It follows form (2) that τ_1 is discrete on $H_{(\mathbb{N}\setminus HF)}$. □

Lemma 3.3. *Let H be a subgroup of G, $m \in \mathbb{N}$ and $L \subset \mathbb{N}$ such that $|L \cap K| \leq m$ for each orbit K of the action of H on $\mathbb{N}$. Then $H_{(L)} \in \tau$.*

Proof. There is a finite partition $A_0, \dots, A_m$ of $\mathbb{N}$ such that $A_0 = \mathbb{N} \setminus L$ and $|A_i \cap K| \leq 1$ for each $1 \leq i \leq m$ and every orbit K. Then $H_\pi \subset H_{(L)}$. Indeed, if $f \in H_\pi$ and $x \in K \cap A_i$ where $1 \leq i \leq m$, then $f(x) \in K$ and $f(x) \in A_i$, thus $f(x) = x$ since $|A_i \cap K| \leq 1$. □

Corollary 3.4. *Let H be a subgroup of G for which all but finitely many orbits are finite and uniformly bounded. Then the discrete topology is the only group topology on H that is both proximally fine and finer than $\tau_{|H}$.*

Proof. If τ_1 is a proximally fine group topology on H finer than τ, then the evualuation mappings $\phi_k : H \to \mathbb{N}$, $k \in \mathbb{N}$, are right uniformly continuous (with respect to τ_1). It follows from Proposition 3.2(2) that $H_{(\mathbb{N}\setminus HF)}$ is τ_1-discrete, where $F \subset \mathbb{N}$ is a finite set such that the cardinals of all orbits Hn, $n \notin F$, are finite and uniformly bounded. By Lemma 3.3, $H_{(\mathbb{N}\setminus HF)}$ is τ-open hence τ_1-open, consequently, τ_1 is discrete. □

Similarly, the next result follows from Proposition 3.2(3) and Lemma 3.3.

Corollary 3.5. *Let H be subgroup of G for which all but finitely many orbits are finite and uniformly bounded. If (H, τ) is strongly FSIN, then (H, τ) is discrete. Moreover, if H has finitely many orbits and (H, τ_0) is strongly FSIN, then H is a (closed) discrete subgroup of (G, τ_0).*

It is plain that strongly FSIN groups are FSIN, but it still isn't known whether the converse is true or not (see Question 3 in [5]). It follows from Proposition 3.1 that the groups G and S are not strongly FSIN, but this does not allow us to conclude that they are not FSIN, because none of the functions ϕ_k, $k \in \mathbb{N}$, is bounded. The next result shows that the groups G and S as well as their quotient G/S are not FSIN.

For a topological group H, let $R(H)$, respectively $U(H)$, stand for the real Banach spaces of bounded right uniformly continuous and of bounded right and left uniformly continuous functions on H. As usual, $\mathfrak{c}$ denotes the cardinal of $\mathbb{R}$.

Proposition 3.6. *The groups G, S and G/S are not FSIN. Moreover, the density character of the quotient Banach space $R(G/S)/U(G/S)$ is at least $2^{\mathfrak{c}}$.*

Proof. We shall exhibit, in (1) below, a real-valued bounded function which is left uniformly continuous on G but not right uniformly continuous when restricted to S. It will follow that G and S are not FSIN. As for G/S, our strategy is as follows: For each nonprincipal ultrafilter p on $\mathbb{N}$, we shall give in (2) a bounded right uniformly continuous ϕ_p defined on G which is not left uniformly continuous. This function is in addition constant on every coset of S, so it factorizes to G/S. Then, we show that for each bounded left uniformly continuous function ψ on G, we have $||\psi_p+\psi_q+\psi|| \geq 1$ for any distinct nonprincipal ultrafilters p, q. Since the quotient map $G \to G/S$ is both left and right uniformly continuous, this will imply that the Banach space $R(G/S)/U(G/S)$ contains a uniformly discrete set of cardinal $2^{\mathfrak{c}}$ (as $\beta\mathbb{N} \setminus \mathbb{N}$, see [8]).
(1) Let $\chi : G \to \{0,1\}$ be the function defined by $\chi(f) = 1$ if $f(1) \leq f(2)$ and $\chi(f) = 0$ otherwise. Then, clearly, χ is left uniformly continuous. Let us show that it is not right uniformly continuous on S. Let $A_1, \dots, A_n$ be a partition of $\mathbb{N}$. We may suppose that $A_1 = \{a, b\}$ with $a \neq b$. Define f and g in S by $f(1) = g(2) = a$, $f(2) = g(1) = b$, and $f(x) = g(x)$ for $x \notin \{1, 2\}$. Then $f^{-1}(A_i) = g^{-1}(A_i)$ for each $i = 1, \dots, n$, but $\chi(f) \neq \chi(g)$.

(2) Let p be a nontrivial ultrafilter on $\mathbb{N}$ and fix an infinite $A \subset \mathbb{N}$ such that $\mathbb{N}\setminus A$ is infinite. Let $\psi_p : G \to \{0,2\}$ be the function given by $\psi_p(f) = 2$ if $f^{-1}(A) \in p$. Clearly, the function ψ_p is bounded and right uniformly continuous. To show that ψ_p is constant on every coset of S, let $g \in G$ and $f \in S$. Then, for every $B \subset \mathbb{N}$, $g^{-1}(B) \setminus \mathrm{supp}(f) \subset (gf)^{-1}(B)$. Thus, taking $B = A$ if $g^{-1}(A) \in p$ or $B = \mathbb{N} \setminus A$ if not, we get that $\psi_p(gf) = \psi_p(g)$.

Let p and q be two distinct nonprincipal ultrafilters on $\mathbb{N}$ and let us verify that $||\psi_p + \psi_q + \psi|| \geq 1$ for each bounded left uniformly continuous $\psi : G \to \mathbb{R}$. It will follow that the quotient $R(G/S)/U(G/S)$ contains a norm 1 discrete set of cardinal $|\beta\mathbb{N} \setminus \mathbb{N}|$. Let $\varepsilon > 0$ and $\pi = \{B_1, \dots, B_n\}$ be a partition of $\mathbb{N}$ such that $|\psi(f) - \psi(g)| < \varepsilon$ for every $f, g \in G$ such that $g \in fH_\pi$. We may suppose that $B_1 = C \cup D$ with $C \in p$, $D \in q$ and $C \cap D = \varnothing$. Write again $D = D_1 \cup D_2$, where D_1 and D_2 are infinite, disjoint and $D_1 \in q$. Finally, let $\{E, F, K\}$ be a partition of of $\mathbb{N} \setminus A$, with E and F infinite and $|K| = |\mathbb{N} \setminus B_1|$. There are certainly $f, g \in G$ such that $f(C) = A$, $f(D) = E \cup F$, $g(C) = F$, $g(D_1) = E$, $g(D_2) = A$ and $f = g$ on $\mathbb{N} \setminus B_1$. We have $f^{-1}g \in H_\pi$ (i.e, $f(B_i) = g(B_i)$ for each $i \leq n$), $\psi_p(f) + \psi_q(f) = 2$ and $\psi_p(g) + \psi_q(g) = 0$. Since $|\psi(f) - \psi(g)| < \varepsilon$, it follows that $|\psi_p(f) + \psi_q(f) + \psi(f)| \geq 1 - \varepsilon$ or $|\psi_p(g) + \psi_q(g) + \psi(g)| \geq 1 - \varepsilon$. Since ε is arbitrary, we have $||\psi_p + \psi_q + \psi|| \geq 1$. □

Let us mention that the corollary of 3.6 that the Hausdorff group G/S is not discrete (being not FSIN) was established and used by T. Banakh *et al.* in [4] to answer a question by D. Dikranjan and A. Giordano Bruno in [7].

Knowing that the symmetric group G and its finitary subgroup S are highly nonabelian (their centers are trivial) and taking Corollary 3.5 into account, we are naturally led to conclude by asking the following:

Question 3.7. *Is there a Hausdorff topological group that is abelian* (*or at least SIN*) *and non-proximally fine?*

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