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Balleans, hyperballeans and ideals

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Abstract

A ballean $\mathcal{B}$ (or a coarse structure) on a set X is a family of subsets of X called balls (or entourages of the diagonal in $X \times X$) defined in such a way that $\mathcal{B}$ can be considered as the asymptotic counterpart of a uniform topological space. The aim of this paper is to study two concrete balleans defined by the ideals in the Boolean algebra of all subsets of X and their hyperballeans, with particular emphasis on their connectedness structure, more specifically the number of their connected components.

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1. Introduction

1.1. **Basic definitions.** A *ballean* is a triple $\mathcal{B} = (X, P, B)$ where X and P are sets, $P \neq \varnothing$, and $B : X \times P \to \mathcal{P}(X)$ is a map, with the following properties:

(i) $x \in B(x, \alpha)$ for every $x \in X$ and every $\alpha \in P$;
(ii) *symmetry*, i.e., for any $\alpha \in P$ and every pair of points $x, y \in X$, $x \in B(y, \alpha)$ if and only if $y \in B(x, \alpha)$;

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(iii) *upper multiplicativity*, i.e., for any $\alpha, \beta \in P$, there exists a $\gamma \in P$ such that, for every $x \in X$, $B(B(x,\alpha),\beta) \subseteq B(x,\gamma)$, where $B(A,\delta) = \bigcup\{B(y,\delta) \mid y \in A\}$, for every $A \subseteq X$ and $\delta \in P$.

The set X is called *support of the ballean*, P – *set of radii*, and $B(x,\alpha)$ – *ball of centre x and radius α*.

This definition of ballean does not coincide with, but it is equivalent to the usual one (see [11] for details).

A ballean $\mathcal{B}$ is called *connected* if, for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x,\alpha)$. Every ballean (X,P,B) can be partitioned in its *connected components*: the *connected component* of a point $x \in X$ is

$$\mathcal{Q}_X(x) = \bigcup_{\alpha \in P} B(x,\alpha).$$

Moreover, we call a subset A of a ballean (X,P,B) *bounded* if there exists $\alpha \in P$ such that, for every $y \in A$, $A \subseteq B(y,\alpha)$. The empty set is always bounded. A ballean is *bounded* if its support is bounded. In particular, a bounded ballean is connected. Denote by $\flat(X)$ the family of all bounded subsets of a ballean X.

If $\mathcal{B} = (X,P,B)$ is a ballean and Y a subset of X, one can define the *subballean $\mathcal{B}\restriction_Y = (Y,P,B_Y)$ on Y induced by $\mathcal{B}$*, where $B_Y(y,\alpha) = B(y,\alpha) \cap Y$, for every $y \in Y$ and $\alpha \in P$.

A subset A of a ballean (X,P,B) is *thin* (or *pseudodiscrete*) if, for every $\alpha \in P$, there exists a bounded subset V of X such that $B_A(x,\alpha) = B(x,\alpha) \cap A = \{x\}$ for each $x \in A \setminus V$. A ballean is *thin* if its support is thin. Bounded balleans are obviously thin.

We note that to each ballean on a set X can be associated a *coarse structure* [12]: a particular family $\mathcal{E}$ of subsets of $X \times X$, called *entourages of the diagonal* Δ_X. The pair $(X,\mathcal{E})$ is called a *coarse space*. This construction highlights the fact that balleans can be considered as asymptotic counterparts of uniform topological spaces. For a categorical look at the balleans and coarse spaces as "two faces of the same coin" see [4].

Definition 1.1 ([11, 5])**.** Let $\mathcal{B} = (X,P,B)$ be a ballean. A subset A of X is called:

(i) *large in X* if there exists $\alpha \in P$ such that $B(A,\alpha) = X$;
(ii) *thick in X* if, for every $\alpha \in P$, there exists $x \in A$ such that $B(x,\alpha) \subseteq A$;
(iii) *small in X* if, for every $\alpha \in P$, $X \setminus B(A,\alpha)$ is large in X.

Let $\mathcal{B}_X = (X,P_X,B_X)$ and $\mathcal{B}_Y = (Y,P_Y,B_Y)$ be two balleans. Then a map $f\colon X \to Y$ is called

(i) *coarse* if for every radius $\alpha \in P_X$ there exists another radius $\beta \in P_Y$ such that $f(B_X(x,\alpha)) \subseteq B_Y(f(x),\beta)$ for every point $x \in X$;
(ii) *effectively proper* if for every $\alpha \in P_Y$ there exists a radius $\beta \in P_X$ such that

$$f^{-1}(B_Y(f(x),\alpha)) \subseteq B_X(x,\beta) \quad \text{for every} \quad x \in X;$$

(iii) a *coarse embedding* if it is both coarse and effectively proper;
(iv) an *asymorphism* if it is bijective and both f and f^{-1} are coarse or, equivalently, f is bijective and both coarse and effectively proper;
(v) an *asymorphic embedding* if it is an asymorphism onto its image or, equivalently, if it is an injective coarse embedding;
(vi) a *coarse equivalence* if it is a coarse embedding such that $f(X)$ is large in $\mathcal{B}_Y$.

We recall that a family $\mathcal{I}$ of subsets of a set X is an *ideal* if $A, B \in \mathcal{I}$, $C \subseteq A$ imply $A \cup B \in \mathcal{I}$, $C \in \mathcal{I}$. In this paper, we always impose that $X \notin \mathcal{I}$ (so that $\mathcal{I}$ is proper) and $\mathcal{I}$ contains the ideal $\mathfrak{F}_X$ of all finite subsets of X. Because of this setting, a set X that admits an ideal $\mathcal{I}$ is infinite, as otherwise $X \in \mathcal{I}$.

We consider the following two balleans with support X determined by $\mathcal{I}$.

Definition 1.2. (i) The *$\mathcal{I}$-ary ballean* $X_{\mathcal{I}\text{-}ary} = (X, \mathcal{I}, B_{\mathcal{I}\text{-}ary})$, with radii set $\mathcal{I}$ and balls defined by

$$B_{\mathcal{I}\text{-}ary}(x, A) = \{x\} \cup A, \text{ for } x \in X \text{ and } A \in \mathcal{I};$$

(ii) The *point ideal ballean* $X_{\mathcal{I}} = (X, \mathcal{I}, B_{\mathcal{I}})$, where

$$B_{\mathcal{I}}(x, A) = \begin{cases} \{x\} & \text{if } x \notin A, \\ \{x\} \cup A = A & \text{otherwise.} \end{cases}$$

The balleans $X_{\mathcal{I}\text{-}ary}$ and $X_{\mathcal{I}}$ are connected and unbounded. While $X_{\mathcal{I}}$ is thin, $X_{\mathcal{I}\text{-}ary}$ is never thin (this follows from Proposition 1.3 and results from [11] reported in Theorem 2.2).

For every connected unbounded ballean $\mathcal{B}$ with support X one can define the *satellite ballean* $X_{\mathcal{I}}$, where $\mathcal{I} = \flat(X)$ is the ideal of all bounded subsets of X.

Proposition 1.3. *For every ideal $\mathcal{I}$ on a set X, the map $id_X \colon X_{\mathcal{I}} \to X_{\mathcal{I}\text{-}ary}$ is coarse, but it is not effectively proper.*

Proof. Pick an arbitrary non-empty element $F \in \mathcal{I}$. Since $\mathcal{I}$ is a proper ideal, for every $K \in \mathcal{I}$, there exists $x_K \in X \setminus (F \cup K)$. Hence, in particular,

$$B_{\mathcal{I}\text{-}ary}(x_K, F) = \{x_K\} \cup F \not\subseteq \{x_K\} = B_{\mathcal{I}}(x_K, K).$$

□

Let $\mathcal{B} = (X, P, B)$ be a ballean. Then the radii set P can be endowed with a preorder $\leq_{\mathcal{B}}$ as follows: for every $\alpha, \beta \in P$, $\alpha \leq_{\mathcal{B}} \beta$ if and only if $B(x, \alpha) \subseteq B(x, \beta)$, for every $x \in X$. A subset $P' \subseteq P$ is *cofinal* if it is cofinal in this preorder (i.e., for every $\alpha \in P$, there exists $\alpha' \in P'$, such that $\alpha \leq_{\mathcal{B}} \alpha'$). If P' is cofinal, then $\mathcal{B} = (X, P', B')$, where $B' = B \restriction_{X \times P'}$. If $\mathcal{I}$ is an ideal on a set X, then both the preorders $\leq_{\mathcal{B}_{\mathcal{I}}}$ and $\leq_{\mathcal{B}_{\mathcal{I}\text{-}ary}}$ on $\mathcal{I}$ coincide with the natural preorder $\subseteq$ on $\mathcal{I}$, defined by inclusion.

Remark 1.4. Let (X, P_X, B_X) and (Y, P_Y, B_Y) be two balleans and $f\colon X \to Y$ be an injective map. We want to give some sufficient conditions that implies the effective properness of f.

(i) Suppose that there exist two cofinal subsets of radii P'_X and P'_Y of P_X and P_Y, respectively, and a bijection $\psi\colon P'_X \to P'_Y$ such that, for every $\alpha \in P'_X$ and every $x \in X$,

$$f(B_X(x,\alpha)) = B_Y(f(x), \psi(\alpha)) \cap f(X). \tag{1.1}$$

We claim that, under these hypothesis, f is a coarse embedding and then $f\colon X \to f(X)$ is an asymorphism.

First of all, let us check that f is coarse. Fix a radius $\alpha \in P_X$ and let $\alpha' \in P'_X$ such that $\alpha \leq \alpha'$. Hence

$$f(B_X(x,\alpha)) \subseteq f(B_X(x,\alpha')) \subseteq B_Y(f(x), \psi(\alpha')),$$

for every $x \in X$, where the last inclusion holds because of (1.1). As for the effective properness, since f is bijective, (1.1) is equivalent to

$$B_X(x,\alpha) = f^{-1}(B_Y(f(x), \psi(\alpha))),$$

for every $x \in X$, and this yelds to the thesis. In fact, for every $\beta \in P_Y$, there exists $\alpha' \in P'_X$ such that $\beta \leq \psi(\alpha')$ and thus, for every $x \in X$,

$$f^{-1}(B_Y(f(x),\beta)) \subseteq f^{-1}(B_Y(f(x),\psi(\alpha'))) = B_X(x,\alpha').$$

(ii) Note that $f\colon X \to Y$ is a coarse embedding if and only if $f\colon X \to f(X)$ is a coarse embedding, where $f(X)$ is endowed with the subballean structure inherited by Y. Suppose that $P'_X \subseteq P_X$ and $P''_{f(X)} \subseteq P_Y$ are cofinal subsets of radii in X and $f(X)$, respectively, and $\psi\colon P'_X \to P''_{f(X)}$ is a bijection such that (1.1) holds for every $x \in X$. Then f is a coarse embedding.

(iii) In notations of item (ii), in order to show that $P''_{f(X)}$ is cofinal in $f(X)$, it is enough to provide a cofinal subset of radii $P'_Y \subseteq P_Y$ in Y and a bijection $\varphi\colon P''_{f(X)} \to P'_Y$ such that, for every $y \in f(X)$ and every $\alpha \in P''_{f(X)}$, $B_Y(y,\alpha) \cap f(X) = B_Y(y, \varphi(\alpha)) \cap f(X)$.

1.2. **Hyperballeans.**

Definition 1.5. Let $\mathcal{B} = (X, P, B)$ be a ballean. Define its *hyperballean* to be $\exp(\mathcal{B}) = (\mathcal{P}(X), P, \exp B)$, where, for every $A \subseteq X$ and $\alpha \in P$,

$$\exp B(A,\alpha) = \{C \in \mathcal{P}(X) \mid A \subseteq B(C,\alpha),\, C \subseteq B(A,\alpha)\}. \tag{1.2}$$

It is not hard to check that this defines actually a ballean. Another easy observation is the following: for every ballean (X, P, B), $\mathcal{Q}_{\exp X}(\varnothing) = \{\varnothing\}$ and, in particular $\exp B(\{\varnothing\},\alpha) = \{\varnothing\}$ for every $\alpha \in P$, since $B(\varnothing,\alpha) = \varnothing$. Motivated by this, we shall consider also the subballean $\exp^*(X) = \exp(X) \setminus \{\varnothing\}$.

If $\mathcal{B} = (X, P, B)$ is a ballean, the subballean $X^\flat$ of $\exp\mathcal{B}$ having as support the family of all non-empty bounded subsets of $\mathcal{B}$ was already defined and studied in [10]. Note that, $\mathcal{B}$ is connected (resp., unbounded) if and only if $\mathcal{B}^\flat$ is connected (resp., unbounded).

In the sequel we focus our attention on four hyperballeans defined by an ideal $\mathcal{I}$ on a set X. In particular, we investigate $\exp X_{\mathcal{I}}$ and $\exp X_{\mathcal{I}\text{-}ary}$, as well as their subballeans $X^\flat_{\mathcal{I}\text{-}ary}$ and $X^\flat_{\mathcal{I}}$.

So $\exp X_{\mathcal{I}} = (\mathcal{P}(X), \mathcal{I}, \exp B_{\mathcal{I}})$, and, according to (1.2), for $A \subseteq X$ and $K \in \mathcal{I}$ one has

$$\exp B_{\mathcal{I}}(A,K) = \begin{cases} \{(A\setminus K)\cup Y \mid \varnothing \neq Y \subseteq K\} & \text{if } A\cap K \neq \varnothing, \\ \{A\} & \text{otherwise.}\end{cases} \tag{1.3}$$

In fact, fix $C \in \exp B_{\mathcal{I}}(A,K)$. If $A \cap K = \varnothing$, then $C = A$ (as $B_{\mathcal{I}}(A,K) = A$ and, for every $A' \subseteq A$, $B_{\mathcal{I}}(A',K) = A'$). Otherwise, $C \subseteq B_{\mathcal{I}}(A,K) = A \cup K$. Moreover, $A \subseteq B_{\mathcal{I}}(C,K)$ if and only if $C \cap K \neq \varnothing$ and $A \subseteq C \cup K$. In other words,

$$\exp B_{\mathcal{I}}(A,K) = \{Z \in \mathcal{P}(X) \mid A \setminus K \subsetneq Z \subseteq A \cup K\}, \quad \text{if } A \cap K \neq \varnothing.$$

Let us now compute the balls in $\exp X_{\mathcal{I}\text{-}ary} = (\mathcal{P}(X), \mathcal{I}, \exp B_{\mathcal{I}\text{-}ary})$. As mentioned above, $\exp B_{\mathcal{I}\text{-}ary}(\{\varnothing\},K) = \{\varnothing\}$ for every $K \in \mathcal{I}$. Fix now a non-empty subset A of X and a radius $K \in \mathcal{I}$. Then a non-empty subset $C \subseteq X$ belongs to $\exp B_{\mathcal{I}\text{-}ary}(A,K)$ if and only if

$$C \subseteq B_{\mathcal{I}\text{-}ay}(A,K) = A \cup K \quad \text{and} \quad A \subseteq B_{\mathcal{I}\text{-}ary}(C,K) = C \cup K,$$

since both A and C are non-empty. Hence

$$\begin{aligned} \exp B_{\mathcal{I}\text{-}ary}(A,K) &= \{(A\setminus K)\cup Y \mid Y \subseteq A,\ (A\setminus K)\cup Y \neq \varnothing\} = \\ &= \{Z \in \mathcal{P}(X) \mid A\setminus K \subseteq Z \subseteq A\cup K,\ Z \neq \varnothing\} \end{aligned}$$

for every $\varnothing \neq A \subseteq X$ and $K \in \mathcal{I}$.

By putting all together, one obtains that, for every $A \subseteq X$ and every $K \in \mathcal{I}$,

$$\exp B_{\mathcal{I}\text{-}ary}(A,K) = \begin{cases} \{Z \in \mathcal{P}(X) \mid A\setminus K \subseteq Z \subseteq A\cup K,\ Z\neq\varnothing\} & \text{if } A \neq \varnothing, \\ \{A\} & \text{otherwise } A = \varnothing.\end{cases} \tag{1.4}$$

Remark 1.6. Denote by $\mathbb{C}_X = \{0,1\}^X$ the Boolean ring of all function $X \to \{0,1\} = \mathbb{Z}_2$ and for $f \in \mathbb{C}_X$ let $\operatorname{supp} f = \{x \in X \mid f(x) = 1\}$. Then one has a ring isomorphism $\jmath = \jmath_X \colon \mathcal{P}(X) \to \mathbb{C}_X$, sending $A \in \mathcal{P}(X)$ to its characteristic function $\chi_A \in \mathbb{C}_X$, so $\jmath(\varnothing) = 0$, the zero function. Using $\jmath$, one can transfer the ball structure from $\exp B_{\mathcal{I}\text{-}ary}$ to $\mathbb{C}_X$: for $0 \neq f \in \{0,1\}^X$ and $A \in \mathcal{I}$ one has

$$\begin{aligned} \jmath(\exp B_{\mathcal{I}\text{-}ary}(\jmath^{-1}(f),A)) &= \{g \mid g(x) = f(x),\ x \in X\setminus A\} \\ &= \{g \mid g\restriction_{X\setminus A} = f\restriction_{X\setminus A}\}. \end{aligned} \tag{1.5}$$

While, according to (1.3) and (1.4), the empty set is "isolated" in both balleans $\exp X_{\mathcal{I}}$ and $X_{\mathcal{I}\text{-}ary}$, the set $\{g \mid g(x) = 0,\ x \in X \setminus A\} = \{g \mid g[X \setminus A] = \{0\}\}$ (i.e., the functions g with $\operatorname{supp} g \subseteq A$), still makes sense and seems a more natural candidate for a ball of radius A centered at the zero function.

Taking into account this observation, we modify the ballean structure on $\mathbb{C}_X$, denoting by $\mathbb{C}(X, \mathcal{I})$ the new ballean, with balls defined by the unique formula suggested by (1.5):

(1.6) $B_{\mathbb{C}(X,\mathcal{I})}(f, A) = \{g \mid g(x) = f(x),\ x \in X \setminus A\} = \{g \mid g \restriction_{X\setminus A} = f \restriction_{X\setminus A}\}$,

where $A \in \mathcal{I}$; when no confusion is possible, we shall write shortly $B_{\mathbb{C}}(f, A)$. In this way

(1.7) $$\jmath \restriction_{\exp^*(X_{\mathcal{I}\text{-}ary})} : \exp^*(X_{\mathcal{I}\text{-}ary}) \to \mathbb{C}(X, \mathcal{I})$$

is an asymorphic embedding. The ballean $\mathbb{C}(X, \mathcal{I})$, as well as its subballean $\mathbb{M}(X, \mathcal{I})$, having as support the ideal $\{g \in \mathbb{C}_X \mid \operatorname{supp} g \in \mathcal{I}\}$ of the ring $\mathbb{C}_X$, will play a prominent role in the paper (note that $\mathbb{M}(X, \mathcal{I}) \setminus \{\varnothing\}$ coincides with $\jmath(X^\flat_{\mathcal{I}-ary})$). [1]

If $X = \mathbb{N}$ and $\mathcal{I} = \mathfrak{F}_{\mathbb{N}}$, then $\mathbb{M}(X, \mathcal{I})$ is the *Cantor macrocube* defined in [10]. Motivated by this, the ballean $\mathbb{M}(X, \mathcal{I})$, for an ideal $\mathcal{I}$ on a set X, will be called the $\mathcal{I}$-*macrocube* (o, shortly, a *macrocube*) in the sequel.

Remark 1.7. One of the main motivations for the above definitions comes from the study of topology of hyperspaces. For an infinite discrete space X, the set $\mathcal{P}(X)$ admits two standard non-discrete topologizatons via the Vietoris topology and via the Tikhonov topology.

In the case of the Vietoris topology, the local base at the point $Y \in \mathcal{P}(X)$ consists of all subsets of X of the form $\{Z \in \mathcal{P}(X) \mid K \subseteq Z \subseteq Y\}$, where K runs over the family of all finite subsets of Y. The Tikhonov topology arises after identification of $\mathcal{P}(X)$ with $\{0, 1\}^X$ via the characteristic functions of subsets of X. Given an ideal $\mathcal{I}$ on X, the point ideal ballean $X_{\mathcal{I}}$ can be considered as one of the possible asymptotic versions of the discrete space X, see Section 2. With these observations, one can look at $\exp X_{\mathcal{I}}$ as a counterpart of the Vietoris hyperspace of X, and the Tikhonov hyperspace of X has two counterparts $C(X, \mathcal{I})$ and $\exp X_{\mathcal{I}\text{-}ary}$. These parallels are especially evident in the case of the ideal $\mathfrak{F}_X$ of finite subsets of X.

1.3. **Main results.** In this paper we focus on hyperballeans of balleans defined by means of ideals, these are the point ideal balleans and the $\mathcal{I}$-ary balleans. It is known ([11]) that the point ideal balleans are precisely the thin balleans. Inspired by this fact, in §2, we give some further equivalent properties (Theorem 2.2).

[1] Sometimes we refer to $\mathbb{C}(X, \mathcal{I})$ as the $\mathcal{I}$-*Cartesian ballean*. Its ballean structure makes both ring operations on $\mathbb{C}(X, \mathcal{I})$ coarse maps, while $\exp(X_{\mathcal{I}\text{-}ary})$ fails to have this property.

As already anticipated the main objects of study will be the hyperballeans $\exp(X_{\mathcal{I}})$, $\exp(X_{\mathcal{I}\text{-}ary})$ and $\mathbb{C}(X,\mathcal{I})$, where $\mathcal{I}$ is an ideal of a set X. By restriction, we will gain also knowledge of their subballeans $X_{\mathcal{I}}^{\flat}$, $X_{\mathcal{I}\text{-}ary}^{\flat}$ and the $\mathcal{I}$-macrocube $\mathbb{M}(X,\mathcal{I})$. Since $X_{\mathcal{I}}$, $X_{\mathcal{I}\text{-}ary}$ and $\mathbb{C}(X,\mathcal{I})$ are pairwise different, it is natural to ask whether their hyperballeans are different or not. Section §3 is devoted to answering this question, comparing these three balleans from various points of view. In particular, we prove that $\exp(X_{\mathcal{I}})$ and $\exp(X_{\mathcal{I}\text{-}ary})$ are different (Corollary 3.2), although they have asymorphic subballeans (Theorem 3.3), the same holds for the pair $\exp X_{\mathcal{I}\text{-}ary}$ and $\mathbb{C}(X,\mathcal{I})$. Moreover, we show that $\mathbb{C}(X,\mathcal{I})$ (and in particular, $\exp^*(X_{\mathcal{I}\text{-}ary})$) is coarsely equivalent to a subballean of $\exp(X_{\mathcal{I}})$; so $\mathbb{M}(X,\mathcal{I})$ (and in particular, $X_{\mathcal{I}\text{-}ary}^{\flat}$) is coarsely equivalent to a subballean of $X_{\mathcal{I}}^{\flat}$

The final part of the section is dedicated to a special class of ideals defined as follows. For a cardinal κ and $\lambda \leq \kappa$ consider the ideal

$$\mathcal{K}_\lambda = \{F \subseteq \kappa \mid |F| < \lambda\}$$

of κ. A relevant property of this ideal is *homogeneity* (i.e., it is invariant under the natural action of the group $\mathrm{Sym}(\kappa)$ by permutations of κ [2]).

For the sake of brevity denote by $\mathcal{K}$ the ideal $\mathcal{K}_\kappa$ of κ. Theorem 3.5 provides a bijective coarse embedding of a subballean of $\exp(\kappa_{\mathcal{K}})$ into $\exp(\kappa_{\mathcal{K}\text{-}ary})$ and, under the hypothesis of regularity of κ, also $\exp(\kappa_{\mathcal{K}})$ itself asymorphically embeds into $\exp(\kappa_{\mathcal{K}\text{-}ary})$.

To measure the level of disconnectedness of a ballean $\mathcal{B}$, one can consider the number $\mathrm{dsc}(\mathcal{B})$ of connected components of $\mathcal{B}$. Although the two hyperballeans $\exp(X_{\mathcal{I}})$ and $\exp(X_{\mathcal{I}\text{-}ary})$ are different, they have the same connected components and in particular, $\mathrm{dsc}(\exp(X_{\mathcal{I}})) = \mathrm{dsc}(\exp(X_{\mathcal{I}\text{-}ary}))$. Moreover, this cardinal coincides with $\mathrm{dsc}(\mathbb{C}(X,\mathcal{I}))+1$ (Proposition 4.1). The main goal of Section §4 is to compute the cardinal number $\mathrm{dsc}(\mathbb{C}(X,\mathcal{I}))$. To this end we use a compact subspace $\mathcal{I}^\wedge$ of the Stone-Čech remained $\beta X \setminus X$ of the discrete space X. In this terms, $\mathrm{dsc}(\mathbb{C}(X,\mathcal{I})) = w(\mathcal{I}^\wedge)$.

2. Characterisation of thin connected balleans

Let $\mathcal{B} = (X,P,B)$ be a bounded ballean. Then $\mathcal{B}$ is thin. Moreover, $\flat(X) = \mathcal{P}(X)$, while every proper subset of X is non-thick and the only small subset is the empty set. Hence, we now focus on unbounded balleans. It is known ([11]) that a connected unbounded ballean $\mathcal{B}$ is thin if and only if the identity mapping of X defines an asymorphism between $\mathcal{B}$ and its *satellite ballean*. It was also shown that these properties are equivalent to having all functions $f\colon X \to \{0,1\}$ being slowly oscillating (such a function is called *slowly oscillating* if, for every $\alpha \in P$ there exists a bounded subset V such that $|f(B(x,\alpha))| = 1$ for

[2]Consequently, all these permutations become automatically asymorphisms, once we endow κ with the point ideal ballean or the $\mathcal{K}_\lambda$-ary ideal structure. One can easily see that these are the only homogeneous ideals of κ.

each $x \in X \setminus V$; this is a specialisation (for $\{0,1\}$-valued functions) of the usual more general notion, [11]). Theorem 2.2 provides further equivalent properties.

For a ballean $\mathcal{B} = (X, P, B)$, we define a mapping $C\colon X \to \mathcal{P}(X)$ by $C(x) = X \setminus \{x\}$.

Lemma 2.1. *Let $\mathcal{B} = (X, P, B)$ be a connected unbounded ballean. If Y is a subset of X, then $C(Y)$ is bounded in $\exp(\mathcal{B})$ if and only if there exists $\alpha \in P$ such that $|B(y, \alpha)| > 1$, for every $y \in Y$.*

Proof. ($\to$) Since $C(Y)$ is bounded in $\exp(\mathcal{B})$, there exists $\alpha \in P$ such that, for every $x, y \in Y$ with $x \neq y$, $C(y) \in \exp B(C(x), \alpha)$. Hence $y \in X \setminus \{x\} \subseteq B(X \setminus \{y\}, \alpha)$ and $x \in X \setminus \{y\} \subseteq B(X \setminus \{x\}, \alpha)$, in particular, $y \in B(Y \setminus \{y\}, \alpha)$ and $x \in B(Y \setminus \{x\}, \alpha)$, from which the conclusion descends.

($\leftarrow$) Since, for every $y \in Y$, there exists $z \in Y \setminus \{y\}$ such that $y \in B(z, \alpha)$, $C(y) \in \exp B(X, \alpha)$. Hence $C(Y) \subseteq \exp B(X, \alpha)$, and the latter is bounded. $\square$

If $\mathcal{B}$ is a ballean, denote by $\mathcal{B}^{\mathcal{M}}$ the subballean of $\exp \mathcal{B}$ whose support is the family of all non-thick non-empty subsets of X. If $\mathcal{B}$ is unbounded, then so it is $\mathcal{B}^{\mathcal{M}}$. Moreover, $\mathcal{B}^{\flat}$ is a subballean of $\mathcal{B}^{\mathcal{M}}$. This motivation for the choice of $\mathcal{M}$ comes from the fact that non-thick subsets[3] were called *meshy* in [5] (this term will not be adopted here).

Theorem 2.2. *Let $\mathcal{B} = (X, P, B)$ be an unbounded connected ballean. Then the following properties are equivalent:*

(i) $\mathcal{B}$ is thin;
(ii) $\mathcal{B} = \mathcal{B}_{\mathcal{I}}$, where $\mathcal{I} = \flat(X)$, i.e., $\mathcal{B}$ coincides with its satellite ballean;
(iii) if $A \subseteq X$ is not thick, then A is bounded;
(iv) $\mathcal{B}^{\mathcal{M}}$ is connected;
(v) the map $C\colon X \to \mathcal{P}(X)$ is an asymorphism between X and $C(X)$;
(vi) every function $f\colon X \to \{0,1\}$ is slowly oscillating.

Proof. The implication (iii)$\to$(iv) is trivial, since item (iii) implies that $\mathcal{B}^{\mathcal{M}} = \mathcal{B}^{\flat}$ and the latter is connected. Furthermore, (i)$\leftrightarrow$(ii) and (i)$\leftrightarrow$(vi) have already been proved in [11].

(iv)$\to$(iii) Assume that $A \subseteq X$ is not thick. Fix arbitrarily a point $x \in X$. The singleton $\{x\}$ is bounded, hence non-thick. By our assumption, $\mathcal{B}^{\mathcal{M}}$ is connected and both A and $\{x\}$ are non-thick, so there must be a ball centred at x and containing A. Therefore, A is bounded.

(v)$\to$(i) If $\mathcal{B}$ is not thin then there is an unbounded subset Y of X satisfying Lemma 2.1. Since $C(Y)$ is bounded in $\exp \mathcal{B}$, we see that C is not an asymorphism.

(ii)$\to$(v) On the other hand, suppose that $\mathcal{B} = \mathcal{B}_{\mathcal{I}}$. Fix a radius $V \in \mathcal{I}$. Without loss of generality, suppose that V has at least two elements. Now,

[3]or, equivalently, those subsets whose complement is large

pick an arbitrary point $x \in X$. If $x \in V$, then

$$\begin{aligned} C(B_{\mathcal{I}}(x,V)) &= \{X \setminus \{y\} \mid y \in V\} = \{A \in C(X) \mid X \setminus (V \cup \{x\}) \subsetneq A \subseteq X\} \\ &= \exp B_{\mathcal{I}}(C(x),V) \cap C(X). \end{aligned}$$

If, otherwise, $x \notin V$, then

$$\begin{aligned} C(B_{\mathcal{I}}(x,V)) &= \{X \setminus \{x\}\} = \{A \in C(X) \mid (X \setminus \{x\}) \setminus V \subsetneq A \subseteq X \setminus \{x\}\} \\ &= \exp B_{\mathcal{I}}(C(x),V) \cap C(X). \end{aligned}$$

(i)→(iii) Suppose that $\mathcal{B}$ is thin and A is an unbounded subset of X. We claim that A is thick. Fix a radius $\alpha \in P$ and let $V \subseteq X$ be a bounded subset of X such that $B(x,\alpha) = \{x\}$, for every $x \notin V$. Since A is unbounded, there exists a point $x_\alpha \in A \setminus V$. Hence $B(x_\alpha,\alpha) = \{x_\alpha\} \subseteq A$, which shows that A is thick.

(iii)→(vi) Assume that X does not satisfy (vi), i.e, X has a non-slowly-oscillating function $f\colon X \to \{0,1\}$. Take a radius α such that, for every bounded V, there exists $x \in X \setminus V$ such that $|f(B(x,\alpha))| = 2$. Hence $A = \{x \in X \mid |f(B(x,\alpha))| = 2\}$ is unbounded. Decompose A as the disjoint union of

$$A_0 = \{x \in A \mid f(x) = 0\} \quad \text{and} \quad A_1 = \{x \in A \mid f(x) = 1\}.$$

Since $A = A_0 \cup A_1$, either A_0 or A_1 is unbounded. Moreover, for every $x \in A$, both $A_0 \cap B(x,\alpha) \neq \varnothing$ and $A_1 \cap B(x,\alpha) \neq \varnothing$ and thus A_0 and A_1 are not thick. □

Remark 2.3. (i) Let us see that one cannot weaken item (iii) in the above theorem by replacing "non-thick" by the stronger property "small". In other words, a ballean need not be thin provided that all its small subsets are bounded. To this end consider the ω-*universal* ballean (see [11, Example 1.4.6]): an infinite countable set X, endowed with the radii set

$$P = \{f\colon X \to [X]^{<\infty} \mid x \in f(x),\ \{y \in X \mid x \in f(y)\} \in [X]^{<\infty},\ \forall x \in X\},$$

and $B(x,f) = f(x)$, for every $x \in X$ and $f \in P$. Since it is *maximal* (i.e., it is connected, unbounded and every properly finer ballean structure is bounded) by [11, Example 10.1.1], then every small subset is finite (by application of [11, Theorem 10.2.1]), although it is not thin.

(ii) Let $\mathcal{B}$ be an unbounded connected ballean and X be its support. Consider the map $CB\colon \mathcal{B}^\flat \to \exp\mathcal{B}$ such that $CB(A) = X \setminus A$, for every bounded A. It is trivial that $C = CB\restriction_X$, where X is identified with the family of all its singletons. Hence, if CB is an asymorphic embedding, then C is an asymorphic embedding too, and thus $\mathcal{B}$ is thin, according to Theorem 2.2. However, we claim that CB is not an asymorphic embedding if $\mathcal{B}$ is thin and then item (v) in Theorem 2.2 cannot be replaced with this stronger property.

Since $\mathcal{B}$ is thin, we can assume that $\mathcal{B}$ coincides with its satellite $\mathcal{B}_{\mathcal{I}}$ (Theorem 2.2). Fix a radius $V \in \mathcal{I}$ of $\exp X_{\mathcal{I}}$ and suppose, without loss of generality, that V has at least two elements. For every radius $W \in \mathcal{I}$ of $X_{\mathcal{I}}^\flat$, pick an element

$A_W \in \mathcal{I}$ such that $A_W \subseteq X \setminus (W \cup V)$. Hence, $CB^{-1}(\exp B_{\mathcal{I}}(CB(A_W), V)) \not\subseteq B^\flat_{\mathcal{I}}(A_W, W) = \{A_W\}$, which implies that CB is not effectively proper. In fact, since $A_W \cup V \in \mathcal{I}$,

$$\exp B_{\mathcal{I}}(CB(A_W), V) = \{Z \subseteq X \mid X \setminus (A_W \cup V) \subsetneq Z \subseteq X \setminus A_W\} \subseteq CB(X^\flat_{\mathcal{I}}),$$

and thus $|\exp B_{\mathcal{I}}(CB(A_W), V) \cap CB(X^\flat_{\mathcal{I}})| > 1$.

A characterization of thin (and coarsely thin) balleans in terms of asymptotically isolated balls can be found in [8, Theorems 1, 2].

3. Further properties of $\exp(X_{\mathcal{I}})$, $\exp(X_{\mathcal{I}\text{-}ary})$ and $\mathbb{C}(X, \mathcal{I})$

Let $f\colon X \to Y$ be a map between sets. Then there is a natural definition for a map $\exp f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$, i.e., $\exp f(A) = f(A)$, for every $A \subseteq X$. If $f\colon X \to Y$ is a map between two balleans such that $f(A) \in \flat(Y)$, for every $A \in \flat(Y)$ (e.g., a coarse map), then the restriction $f^\flat = \exp f \restriction_{X^\flat}\colon X^\flat \to Y^\flat$ is well-defined.

The following proposition can be easily proved.

Proposition 3.1. *Let $\mathcal{B}_X = (X, P_X, B_X)$ and $\mathcal{B}_Y = (Y, P_Y, B_Y)$ be two balleans and let $f\colon X \to Y$ be a map between them. Then:*

- *(i) $f\colon \mathcal{B}_X \to \mathcal{B}_Y$ is coarse if and only if $\exp f\colon \exp\mathcal{B}_X \to \exp\mathcal{B}_Y$ is coarse if and only if $f^\flat\colon \mathcal{B}^\flat_X \to \mathcal{B}^\flat_Y$ is well-defined and coarse;*
- *(ii) $f\colon \mathcal{B}_X \to \mathcal{B}_Y$ is a coarse embedding if and only if $\exp f\colon \exp\mathcal{B}_X \to \exp\mathcal{B}_Y$ is a coarse embedding if and only if $f^\flat\colon \mathcal{B}^\flat_X \to \mathcal{B}^\flat_Y$ is well-defined and a coarse embedding.*

For the sake of simplicity, throughout this section, for every ideal $\mathcal{I}$ of a set X, the ballean $\mathbb{C}(X, \mathcal{I})$ will be identified with $\jmath^{-1}(\mathbb{C}(X, \mathcal{I}))$ (where $\jmath$ is defined in Remark 1.6), whose support is $\mathcal{P}(X)$. Hence, by this identification, if $A \subseteq X$ and $K \in \mathcal{I}$,

$$B_{\mathbb{C}}(A, K) = \{Y \mid A \setminus K \subseteq Y \subseteq A \cup K\}.$$

Corollary 3.2. *For every ideal $\mathcal{I}$ on X, the following statements hold:*

- *(i) $j = \exp id_X\colon \exp X_{\mathcal{I}} \to \exp X_{\mathcal{I}\text{-}ary}$ is coarse, but it is not an asymorphism;*
- *(ii) $j\colon \exp X_{\mathcal{I}\text{-}ary} \to \mathbb{C}(X, \mathcal{I})$ is coarse, but it is not an asymorphism;*
- *(iii) the same holds for the restriction $i = id^\flat_X\colon X^\flat_{\mathcal{I}} \to X^\flat_{\mathcal{I}\text{-}ary}$.*

Proof. Since $id_X\colon X_{\mathcal{I}} \to X_{\mathcal{I}\text{-}ary}$ is coarse, but it is not effectively proper (Proposition 1.3), items (i) and (iii) follow from Propositions 3.1. Item (ii) descends from the fact that $\exp(X_{\mathcal{I}\text{-}ary}) \restriction_{\mathcal{P}(X)\setminus\{\varnothing\}} = \mathbb{C}(X, \mathcal{I}) \restriction_{\mathcal{P}(X)\setminus\{\varnothing\}}$, and $\mathcal{Q}_{\exp X_{\mathcal{I}\text{-}ary}}(\varnothing) = \{\varnothing\}$, while $\mathcal{Q}_{\mathbb{C}(X,\mathcal{I})}(\varnothing) = \mathcal{I}$. $\square$

In spite of Corollary 3.2, we show now that a cofinal part of $\exp(X_{\mathcal{I}})$ asymorphically embeds in $\exp(X_{\mathcal{I}\text{-}ary})$.

For every ideal $\mathcal{I}$ on X and $x \in X$ consider the families $\mathcal{U}_x = \{U \subseteq X \mid x \in U\}$, the principal ultrafilter of $\mathcal{P}(X)$ generated by $\{x\}$, and $\mathcal{I}_x = \mathcal{U}_x \cap \mathcal{I} = \{F \in \mathcal{I} \mid x \in F\}$.

Theorem 3.3. *For every ideal $\mathcal{I}$ on X and $x \in X$, the following statements hold:*

- *(i) if $j\colon \exp(X_{\mathcal{I}}) \to \exp(X_{\mathcal{I}\text{-}ary})$ is the map defined in Corollary 3.2(i), its restriction $j \restriction_{\mathcal{U}_x}$ is an asymorphism between the corresponding subballeans;*
- *(ii) $\mathbb{C}(X,\mathcal{I})$ and, in particular, $\exp^*(X_{\mathcal{I}\text{-}ary})$ are coarsely equivalent to the subballean of $\exp(X_{\mathcal{I}})$ with support $\mathcal{U}_x$, witnessed by the map*

$$j \restriction_{\mathcal{U}_x}\colon \exp(X_{\mathcal{I}}) \restriction_{\mathcal{U}_x} \to \exp^*(X_{\mathcal{I}\text{-}ary}) \subseteq \mathbb{C}(X,\mathcal{I}).$$

Proof. (i) For every $C \in \mathcal{U}_x$ and $A \in \mathcal{I}_x$, we have

$$\exp B_{\mathcal{I}}(C,A) \cap \mathcal{U}_x = \{(C \setminus A) \cup Y \mid x \in Y \subseteq A\} = \exp B_{\mathcal{I}\text{-}ary}(C,A) \cap \mathcal{U}_x,$$

since $C \cap A \neq \varnothing$. Hence the conclusion follows by Remark 1.4(i), since $\mathcal{I}_x$ is a cofinal subset of radii of $\mathcal{I}$.

(ii) In view of item (i), it remains to see that $j(\mathcal{U}_x)$ is large in $\mathbb{C}(X,\mathcal{I})$. Indeed, for every $A \in \mathcal{U}_x$, $B_{\mathbb{C}}(A,\{x\}) = \{A, A \setminus \{x\}\}$, and so $B_{\mathbb{C}}(j(\mathcal{U}_x),\{x\}) = \mathbb{C}(X,\mathcal{I})$, where $\{x\} \in \mathcal{I}$. ☐

Since $X^{\flat}_{\mathcal{I}}$, $X^{\flat}_{\mathcal{I}\text{-}ary}$, and $\mathbb{M}(X,\mathcal{I})$ are subballeans of $\exp(X_{\mathcal{I}})$, $\exp(X_{\mathcal{I}\text{-}ary})$, and $\mathbb{C}(X,\mathcal{I})$ respectively, by taking by restrictions we obtain the following immediate corollary.

Corollary 3.4. *For every ideal $\mathcal{I}$ on X and $x \in X$, the following statements hold:*

- *(i) $j \restriction_{\mathcal{I}_x}$ is an asymorphism between the corresponding subballeans of $X^{\flat}_{\mathcal{I}}$ and $X^{\flat}_{\mathcal{I}\text{-}ary}$;*
- *(ii) $\mathbb{M}(X,\mathcal{I})$ and, in particular, $X^{\flat}_{\mathcal{I}\text{-}ary}$ are coarsely equivalent to the subballean of $X^{\flat}_{\mathcal{I}}$ with support $\mathcal{I}_x$, witnessed by the map $j \restriction_{\mathcal{I}_x}\colon X^{\flat}_{\mathcal{I}} \restriction_{\mathcal{I}_x} \to X^{\flat}_{\mathcal{I}\text{-}ary} \subseteq \mathbb{M}(X,\mathcal{I})$.*

3.1. **$\mathbb{C}(\kappa,\mathcal{K})$ and the hyperballeans $\exp(\kappa_{\mathcal{K}})$ and $\exp(\kappa_{\mathcal{K}\text{-}ary})$.** Now we focus our study on some more specific ideals. For an infinite cardinal κ and for its ideal

$$\mathcal{K} = [\kappa]^{<\kappa} = \{Z \subset \kappa \mid |Z| < \kappa\},$$

consider the two balleans $\kappa_{\mathcal{K}}$ and $\kappa_{\mathcal{K}\text{-}ary}$. Here we investigate some relationships between hyperballeans of those two balleans and the ballean $\mathbb{C}(\kappa,\mathcal{K})$.

Furthermore, with κ as above, if $x < \kappa$, put

$$\mathcal{U}_{\geq x} = \{A \subseteq \kappa \mid \min A = x\} \quad \text{and} \quad \mathcal{K}_{\geq x} = \mathcal{U}_{\geq x} \cap \mathcal{K} = \{A \in \mathcal{K} \mid \min A = x\}.$$

For every pair of ordinals $\alpha \leq \beta < \kappa$, let $[\alpha,\beta] = \{\gamma \in \kappa \mid \alpha \leq \gamma \leq \beta\}$. Clearly, the cardinal κ is regular if and only if the family $P_{int} = \{[0,\alpha] \mid \alpha < \kappa\}$ is cofinal in $\mathcal{K}$. For a ballean $\mathcal{B} = (X, P, B)$, two subsets A, B of X are called *close* if A and B are in the same connected component of $\exp \mathcal{B}$.

Theorem 3.5. *Let κ be an infinite cardinal and $x < \kappa$. Then:*

- *(i) the subballean $\mathcal{U}_{\geq x}$ of $\exp(\kappa_{\mathcal{K}})$ is asymorphic to $\mathbb{C}(\kappa, \mathcal{K})$, so $\exp^*(\kappa_{\mathcal{K}})$ is the disjoint union of κ pairwise close copies of $\mathbb{C}(\kappa, \mathcal{K})$;*
- *(ii) if κ is regular, then $\exp(\kappa_{\mathcal{K}})$ asymorphically embeds into $\exp(\kappa_{\mathcal{K}\text{-}ary})$.*

Proof. (i) Fix a bijection $g\colon \kappa \to \mathcal{A}$, where $\mathcal{A}$ is the family of all ordinals α such that $x < \alpha < \kappa$. Define a map $f\colon \mathbb{C}(\kappa, \mathcal{K}) \to \mathcal{U}_{\geq x} \subseteq \exp(\kappa_{\mathcal{K}})$ such that, for every $X \subseteq \kappa$, $f(X) = g(X) \cup \{x\}$. We claim that f is the desired asymorphism. In order to prove it, we want to apply Remark 1.4(ii). Fix a radius $K \in \mathcal{K}$ (i.e., $|K| < \kappa$). Then, for every $X \subseteq \kappa$,

$$\begin{aligned} f(B_{\mathbb{C}}(X, K)) &= f(\{Y \subseteq \kappa \mid X \setminus K \subseteq Y \subseteq X \cup K\}) = \\ &= \{f(g^{-1}(Z)) \mid X \setminus K \subseteq g^{-1}(Z) \subseteq X \cup K\} = \\ &= \{Z \cup \{x\} \mid g(X) \setminus g(K) \subseteq Z \subseteq g(X) \cup g(K)\} = \\ &= \{W \in \mathcal{U}_{\geq x} \mid (g(X) \cup \{x\}) \setminus (g(K) \cup \{x\}) \subsetneq W \subseteq \\ &\qquad \subseteq (g(X) \cup \{x\}) \cup (g(K) \cup \{x\})\} = \\ &= \exp B_{\mathcal{K}}(g(X) \cup \{x\}, g(K) \cup \{x\}) \cap \mathcal{U}_{\geq x} \\ &= \exp B_{\mathcal{K}}(f(X), g(K) \cup \{x\}) \cap \mathcal{U}_{\geq x}. \end{aligned}$$

If we show that $\{g(K) \cup \{x\} \mid K \in \mathcal{K}\}$ is cofinal in $f(\kappa) = \mathcal{U}_{\geq x}$, then the conclusion follows, since we can apply Remark 1.4(ii) by putting $\psi(K) = g(K) \cup \{x\}$, for every $K \in \mathcal{K}$. It is enough to check that, for every $X \subseteq \kappa$ and $K \in \mathcal{K}$,

$$\exp B_{\mathcal{K}}(X, K) \cap \mathcal{U}_{\geq x} = \exp B_{\mathcal{K}}(X, \psi(K)) \cap \mathcal{U}_{\geq x},$$

which proves the cofinality of $\psi(\mathcal{K})$ in $f(\kappa)$, in virtue of Remark 1.4(iii).

The last assertion of item (i) follows from the facts that the family $\{\mathcal{U}_{\geq x} \mid x < \kappa\}$ is a partition of $\exp(\kappa_{\mathcal{K}})$, and, for every $x < y < \kappa, \mathcal{U}_{\geq x} \in \exp B_{\mathcal{K}}(\mathcal{U}_{\geq y}, [x, y])$, where $[x, y] \in \mathcal{K}$.

(ii) Every ordinal $\alpha \in \kappa$ can be written uniquely as $\alpha = \beta + n$, where β is a limit ordinal and n is a natural number. We say that α is *even* (*odd*) if n is even (odd). We denote by E the set of all odd ordinals from κ and fix a monotonically increasing bijection $\varphi\colon \kappa \to E$. For each non-empty $F \subseteq \kappa$, let $y_F \in \kappa$ such that $y_F + 1 = \min \varphi(F)$ and define $f(F) = \{y_F\} \cup \varphi(F)$. Moreover, we set $f(\varnothing) = \varnothing$. Let $S = f(\exp(\kappa_{\mathcal{K}}))$. Hence the elements of S are the empty set and those subsets A of κ, consisting of odd ordinals and precisely one even ordinal $\alpha \in A$ such that $\alpha = \min A$.

We claim that $f\colon \exp(\kappa_{\mathcal{K}}) \to S$ is an asymorphism.

Since κ is regular, $P_{int} \subseteq \mathcal{K}$ is a cofinal subset of radii. Now fix $[0, \alpha] \in P_{int}$. Take an arbitrary subset A of κ. We can assume A to be non-empty, since in that case, there is nothing to be proved. The thesis follows, once we prove that

$$f(\exp B_{\mathcal{K}}(A,[0,\alpha])) = \exp B_{\mathcal{K}\text{-}ary}(f(A),[0,\varphi(\alpha)]) \cap S, \tag{3.1}$$

since we can apply Remark 1.4(i) if we define the bijection $\psi([0,\beta]) = [0,\varphi(\beta)]$, for every $\beta < \kappa$, between cofinal subsets of radii.

If $A \cap [0,\alpha] = \varnothing$, then also $f(A)$ and $[0,\varphi(\alpha)]$ are disjoint, which implies that

$$\exp B_{\mathcal{K}\text{-}ary}(f(A),[0,\varphi(\alpha)]) \cap S = \{f(A)\}.$$

Otherwise, suppose that A and $[0,\alpha]$ are not disjoint. In particular $y_A \in [0,\varphi(\alpha)]$. We divide the proof of (3.1) in this case in some steps.

First of all we claim that, for every $\varnothing \neq Z \subseteq \kappa$,

$$\text{if } A \setminus [0,\alpha] \subseteq Z \subseteq A \cup [0,\alpha], \text{ then: } Z \neq A \setminus [0,\alpha] \text{ if and only if } y_Z \leq \varphi(\alpha). \tag{3.2}$$

In fact, if $Z = A \setminus [0,\alpha]$, then $\min Z > \alpha$ and so $\min \varphi(Z) > \varphi(\alpha)$. Since $\varphi(\alpha) \in E$, $\varphi(Z) \subseteq E$, and $y_Z \notin E$, we have that $y_Z > \varphi(\alpha)$. Conversely, if $Z \neq A \setminus [0,\alpha]$, there exists $z \in Z \cap [0,\alpha]$, since $Z \subseteq A \cup [0,\alpha]$. Hence $\min Z \leq z \leq \alpha$ and thus $y_Z < \min \varphi(Z) \leq \varphi(\alpha)$.

Fix now a subset $Z \subseteq \kappa$. If $f(Z) \in \exp B_{\mathcal{K}\text{-}ary}(f(A),[0,\varphi(\alpha)])$, then, by applying the definitions,

$$\varphi(A)\setminus[0,\varphi(\alpha)] = f(A)\setminus[0,\varphi(\alpha)] \subseteq \varphi(Z)\cup\{y_Z\} \subseteq f(A)\cup[0,\varphi(\alpha)] = \varphi(A)\cup[0,\varphi(A)].$$

Note that $\varphi(Z) \subseteq E$ and $y_Z \notin E$. Hence

$$\varphi(A \setminus [0,\alpha]) = \varphi(A) \setminus \varphi([0,\alpha]) \subseteq \varphi(Z) \subseteq \varphi(A) \cup \varphi([0,\alpha]) = \varphi(A \cup [0,\alpha])$$

and

$$y_Z \leq \varphi(\alpha).$$

Since φ is a bijection, we can apply (3.2) and obtain that $A \setminus [0,\alpha] \subsetneq Z \subseteq A \cup [0,\alpha]$, which means that $Z \in \exp B_{\mathcal{K}}(A,[0,\alpha])$. Hence we have proved the inclusion $(\supseteq)$ of (3.1). Since all the previous implications can be reverted, then (3.1) finally follows. $\square$

Corollary 3.6. *Let κ be an infinite cardinal and $x < \kappa$. Then:*

- *(i) the subballean $\mathcal{K}_{\geq x}$ of $\kappa^\flat_{\mathcal{K}}$ is asymorphic to $\mathbb{M}(\kappa,\mathcal{K})$, so $\kappa^\flat_{\mathcal{K}}$ is the disjoint union of κ pairwise close $\mathcal{K}$-macrocubes $\mathbb{M}(\kappa,\mathcal{K})$;*
- *(ii) if κ is regular then $\kappa^\flat_{\mathcal{K}}$ asymorphically embeds into $\kappa^\flat_{\mathcal{K}\text{-}ary}$.*

The proof of item (ii), specified for $\kappa = \omega$, can be found in [10].

4. The number of connected components of $\exp(X_{\mathcal{I}})$

Recall that $\operatorname{dsc}(\mathcal{B})$ denotes the number of connected components of a ballean $\mathcal{B}$. Clearly,

$$\operatorname{dsc}(\exp(\mathcal{B})) = \operatorname{dsc}(\exp^*(\mathcal{B})) + 1 \geq 2 \tag{4.1}$$

for every non-empty ballean $\mathcal{B}$. We begin with the following crucial observation.

Proposition 4.1. *For an ideal $\mathcal{I}$ on a set X, one has*

(i) the non-empty subsets Y, Z of X are close in $\exp(X_{\mathcal{I}-ary})$ if and only if $Y \triangle Z \in \mathcal{I}$;

(ii) two functions $f, g \in \mathbb{C}_X$ are close in $\mathbb{C}(X, \mathcal{I})$ if and only if $\operatorname{supp} f \triangle \operatorname{supp} g \in \mathcal{I}$;

(iii) for every $A \subseteq X$, $\mathcal{Q}_{\exp(X_{\mathcal{I}})}(A) = \mathcal{Q}_{\exp(X_{\mathcal{I}-ary})}(A)$, and in particular,

$$\begin{gathered}\operatorname{dsc}(\exp(X_{\mathcal{I}})) = \operatorname{dsc}(\exp(X_{\mathcal{I}\text{-}ary})) \text{ and} \\ \operatorname{dsc}(\exp^*(X_{\mathcal{I}})) = \operatorname{dsc}(\exp^*(X_{\mathcal{I}\text{-}ary})) = \operatorname{dsc}(\mathbb{C}(X, \mathcal{I}))\end{gathered} \tag{4.2}$$

(iv) $\operatorname{dsc}(\exp(X_{\mathcal{I}})) = \operatorname{dsc}(\mathbb{C}(X, \mathcal{I})) + 1$.

Proof. (i) Two non-empty subsets Y and Z of $\exp(X_{\mathcal{I}\text{-}ary})$ are close if and only if there exists $K \in \mathcal{I}$ such that $Y \in \exp B_{\mathcal{I}\text{-}ary}(Z, K)$, i.e.,

$$Y \subseteq Z \cup K \quad \text{and} \quad Z \subseteq Y \cup K. \tag{4.3}$$

If (4.3) holds, then

$$Y \triangle Z = (Y \setminus Z) \cup (Z \setminus Y) \subseteq ((Z \cup K) \setminus Z) \cup ((Y \cup K) \setminus Y) = K \in \mathcal{I}.$$

Conversely, if $Y \triangle Z \in \mathcal{I}$, then $K = Y \triangle Z$ trivially satisfies (4.3).

(ii) Let $Y = \operatorname{supp} f$ and $Z = \operatorname{supp} g$. If both f, g are non-zero, then Y, Z are non-empty and the assertion follows from (i) and the asymorphism between $\exp^*(X_{\mathcal{I}\text{-}ary})$ and $\mathbb{C}(X, \mathcal{I}) \setminus \{0\}$. If $g = 0$, then f is close to g if and only if $f \in \jmath(\mathcal{I})$, i.e., $Y = \jmath^{-1}(f) \in \mathcal{I}$. As $Z = \varnothing$, this proves the assertion in this case as well.

(iii) Fix a subset A of X. The inclusion $\mathcal{Q}_{\exp(X_{\mathcal{I}})}(A) \subseteq \mathcal{Q}_{\exp(X_{\mathcal{I}\text{-}ary})}(A)$ follows from Corollary 3.2(i).

Let us check the inclusion $\mathcal{Q}_{\exp(X_{\mathcal{I}})}(A) \supseteq \mathcal{Q}_{\exp(X_{\mathcal{I}-ary})}(A)$. If $A = \varnothing$, the claim is trivial, since $\mathcal{Q}_{\exp Y}(\varnothing) = \{\varnothing\}$, for every ballean Y. Otherwise, fix an element $x \in A$. Let $C \in \exp B_{\mathcal{I}\text{-}ary}(A, K)$, for some $K \in \mathcal{I}$. Then $C \neq \varnothing$, so we can fix also a point $y \in C$ and let $K' = K \cup \{x, y\} \in \mathcal{I}$. Then

$$C \subseteq B_{\mathcal{I}\text{-}ary}(A, K) = A \cup K = B_{\mathcal{I}}(A, K')$$

and

$$A \subseteq B_{\mathcal{I}\text{-}ary}(C, K) = C \cup K = B_{\mathcal{I}}(C, K'),$$

which shows that $C \in \exp B_{\mathcal{I}}(A, K')$. Hence, $C \in \mathcal{Q}_{\exp(X_{\mathcal{I}})}(A)$.

This proves the equality $\mathcal{Q}_{\exp(X_{\mathcal{I}})}(A) = \mathcal{Q}_{\exp(X_{\mathcal{I}-ary})}(A)$. It implies the first as well as the second equality in (4.2). To prove the last equality in (4.2), it suffices to note that $\mathcal{Q}_{\mathbb{C}(X,\mathcal{I})}(0) = \mathcal{I}$, by virtue of (ii). Hence, $\operatorname{dsc}(\mathbb{C}(X, \mathcal{I})) = \operatorname{dsc}(\mathbb{C}(X, \mathcal{I}) \setminus \{0\})$. To conclude, use the fact that $\exp^*(X_{\mathcal{I}\text{-}ary})$ is asymorphic to $\mathbb{C}(X, \mathcal{I}) \setminus \{0\}$.

Item (iv) follows from (iii) and (4.1) applied to $\mathcal{B} = \exp(X_{\mathcal{I}})$. $\square$

Proposition 4.1 allows us to reduce the computations of the number of connected components of all hyperballeans involved to the computation of the cardinal $\operatorname{dsc}(\mathbb{C}(X, \mathcal{I})$. In the sequel we simply identify $\mathbb{C}_X$ with the Boolean ring

$\mathcal{P}(X)$. So that functions $f \in \mathbb{C}_X$ are identifies with their support and the ideals $\mathcal{I}$ of X are simply the proper ideals of the Boolean ring $\mathbb{C}_X = \{0,1\}^X = \mathbb{Z}_2^X$, containing $\bigoplus_X \mathbb{Z}_2$.

According to Proposition 4.1, the connected components of $\mathbb{C}(X,\mathcal{I})$ are precisely the cosets $f+\mathcal{I}$ of the ideal $\mathcal{I}$, therefore, $\mathrm{dsc}(\mathbb{C}(X,\mathcal{I}))$ coincides with the cardinality of the quotient ring $\mathbb{C}_X/\mathcal{I}$:

$$\mathrm{dsc}(\mathbb{C}(X,\mathcal{I})) = |\mathbb{C}_X/\mathcal{I}| = |\mathcal{P}(X)/\mathcal{I}|. \tag{4.4}$$

In particular, for every infinite set X and an ideal $\mathcal{I}$ of X one has $\mathrm{dsc}(\mathbb{C}(X,\mathcal{I})) = 2$ if and only if $\mathcal{I}$ is a maximal ideal. This is an obvious consequence of (4.4) as $|\mathbb{C}_X/\mathcal{I}| = 2$ if and only if the ideal $\mathcal{I}$ is maximal.

Remark 4.2. The cardinality $|\mathbb{C}_X/\mathcal{I}|$ is easy to compute in some cases, or to get at least an easily obtained estimate for $|\mathbb{C}_X/\mathcal{I}|$ from above as we see now. To this end let

$$\iota(\mathcal{I}) = \min\{\lambda \mid \mathcal{I} \text{ is an intersection of } \lambda \text{ maximal ideals}\}.$$

Then

$$|\mathbb{C}_X/\mathcal{I}| \leq \min\{2^{\iota(\mathcal{I})}, 2^{|X|}\}. \tag{4.5}$$

Indeed, if $\mathcal{I} = \bigcap\{\mathfrak{m}_i \mid i < \iota(\mathcal{I})\}$, where $\mathfrak{m}_i$ are maximal ideals of B, then $B/\mathcal{I}$ embeds in the product $\prod_{i<\iota(\mathcal{I})} \mathbb{C}_X/\mathfrak{m}_i$ having size $\leq 2^{\iota(\mathcal{I})}$ as $\mathbb{C}_X/\mathfrak{m}_i \cong \mathbb{Z}_2$ for all i. To conclude the proof of (4.5) it remains to note that obviously $|\mathbb{C}_X/\mathcal{I}| \leq |\mathbb{C}_X| = 2^{|X|}$.

If $\iota(\mathcal{I}) = n$ is finite, then $\mathrm{dsc}(\mathbb{C}(X,\mathcal{I})) = 2^n$. Indeed, now $\mathcal{I} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ is a finite intersection of maximal ideals and the Chinese Remainder Theorem, applied to the Boolean ring $\mathbb{C}_X$ and the maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$, provides a ring isomorphism

$$\mathbb{C}_X/\mathcal{I} \cong \prod_{i=1}^n \mathbb{C}/\mathfrak{m}_i \cong \mathbb{Z}_2^n.$$

In particular, $|\mathbb{C}_X/\mathcal{I}| = |\mathbb{Z}_2^n| = 2^n$. By (4.4), we deduce

$$\mathrm{dsc}(\mathbb{C}(X,\mathcal{I})) = 2^n. \tag{4.6}$$

Let us conclude now with another example. For every infinite set X and the ideal $\mathcal{I} = \mathfrak{F}_X$ one has

$$\mathrm{dsc}(\exp(X_{\mathfrak{F}_X})) = \mathrm{dsc}(\mathbb{C}(X, \mathfrak{F}_X) = 2^{|X|}.$$

This follows from (4.4) and $|\mathfrak{F}_X| = |X| < 2^{|X|}$, which implies $|\mathbb{C}_X/\mathfrak{F}_X| = |\mathbb{C}_X| = 2^{|X|}$.

In order to obtain some estimate from *below* for $|\mathbb{C}_X/\mathcal{I}|$, we need a deeper insight on the spectrum $\mathrm{Spec}\,\mathbb{C}_X$ of $\mathbb{C}_X$. Since $\mathbb{C}_X$ is a Boolean ring, $\mathrm{Spec}\,\mathbb{C}_X$ coincides with the space of all maximal ideals of $\mathbb{C}_X$, which can be identified with the Stone–Čech compactification βX when we endow X with the discrete topology. As usual,

- we identify the Stone–Čech compactification βX with the set of all ultrafilters on X;
- the family $\{\overline{A} \mid A \subseteq X\}$, where $\overline{A} = \{p \in \beta X \mid A \in p\}$, forms the base for the topology of βX; and
- the set X is embedded in βX by sending $x \in X$ to the principal ultrafilter generated by x.

For a filter φ on X, define a closed subset $\overline{\varphi}$ of βX as follows:

$$\overline{\varphi} = \bigcap \{\overline{A} \mid A \in \varphi\}.$$

An ultrafilter $p \in \beta X$ belongs to $\overline{\varphi}$ if and only if p contains the filter φ. In other words,

$$\varphi = \bigcap \{u \mid u \in \overline{\varphi}\}. \tag{4.7}$$

For an ideal $\mathcal{I}$ on X, we consider the filter $\varphi_{\mathcal{I}} = \{X \backslash A \mid A \in \mathcal{I}\}$, and we simply write φ when there is no danger of confusion. Similarly, for a filter φ we define the ideal $\mathcal{I}_\varphi = \{X \backslash A \mid A \in \varphi\}$ and we simply write $\mathcal{I}$ when there is no danger of confusion.

Finally, let $\mathcal{I}^\wedge = \overline{\varphi_{\mathcal{I}}}$, and note that all ultrafilters in $\mathcal{I}^\wedge$ are non-fixed, i.e., $\mathcal{I}^\wedge \subseteq \beta X \setminus X$, as φ is contained in the Fréchet filter of all co-finite sets on X (since $\mathcal{I} \supseteq \mathfrak{F}_X$). Moreover, for a subset A of X one has

$$A \in \mathcal{I} \ \text{ if and only if } \ A \notin u \ \text{ for all } u \in \overline{\varphi_{\mathcal{I}}} = \mathcal{I}^\wedge. \tag{4.8}$$

As pointed out above, for any X the compact space βX coincides with the spectrum $\operatorname{Spec} \mathbb{C}_X$ of the ring $\mathbb{C}_X$. For an ideal $\mathcal{I}$ on X, $\overline{\varphi_{\mathcal{I}}}$ is the set of ultrafilters on X containing φ. For $u \in \overline{\varphi}_{\mathcal{I}}$ the ideal $\mathcal{I}_u$ is maximal and contains $\mathcal{I}$. More precisely, $\mathcal{I} = \bigcap_{u \in \mathcal{I}^\wedge} \mathcal{I}_u$. The maximal ideals $\mathcal{I}_u$, when u runs over $\overline{\varphi}$, bijectively correspond to the maximal ideals of the quotient $\mathbb{C}_X/\mathcal{I}$; in particular, $|\mathcal{I}^\wedge| = |\operatorname{Spec}(\mathbb{C}_X/\mathcal{I})|$. Along with Remark 4.2, this gives:

Proposition 4.3. *Let $\mathcal{I}$ be an ideal on set X. If $|\mathcal{I}^\wedge| = n$ is finite, then* $\operatorname{dsc}(\mathbb{C}(X,\mathcal{I})) = 2^n$. *Otherwise,* $\operatorname{dsc}(\mathbb{C}(X,\mathcal{I})) = w(\mathcal{I}^\wedge)$.

Here $w(\mathcal{I}^\wedge)$ denotes the weight of the space $\mathcal{I}^\wedge$. The second assertion follows from (4.4) and the equality $w(\mathcal{I}^\wedge) = |\mathcal{P}(X)/\mathcal{I}|$, its proof can be found in [3, §2].

Corollary 4.4. *Let $\mathcal{I}$ be an ideal on a countably infinite set set X such that $\mathcal{I}^\wedge$ is infinite. Then* $\operatorname{dsc}(\mathbb{C}(X,\mathcal{I})) = 2^\omega$.

Proof. Being an infinite compact subset of $\beta X \setminus X$, $\mathcal{I}^\wedge$ contains a copy of $\beta\mathbb{N}$. Therefore, $w(\mathcal{I}^\wedge) = 2^\omega$. Now Proposition 4.3 applies. ☐

In this section we have thoroughly investigated the number of connected components of $\exp(X_{\mathcal{I}})$, where $\mathcal{I}$ is an ideal of a set X. This leaves open the question to estimate $\operatorname{dsc}(\exp(X))$, where X is an arbitrary connected ballean.

Let Y be a subballean of X. In particular, $\mathrm{dsc}(\exp(X)) \geq \mathrm{dsc}(\exp(Y))$. If Y is thin, we can apply Theorem 2.2 so that $Y = Y_{\flat(Y)}$. The results from this section give a lower bound of $\mathrm{dsc}(\exp(Y))$, providing in this way also a lower bound for $\mathrm{dsc}(\exp(X))$, since

$$\mathrm{dsc}(\exp(X)) \geq \sup\{\mathrm{dsc}(\exp(Z)) \mid Z \text{ is a thin subballean of } X\}. \tag{4.9}$$

Unfortunately, (4.9) doesn't provide any useful information in the case when every thin subballean of X is bounded. In fact, if Z is a non-empty bounded subballean, then $\exp^*(Z)$ is connected and so $\mathrm{dsc}(\exp(Z)) = 2$.

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