

2019

## Parametrized solutions $X$ of the system $AXA = AY A$ and $A^k Y AX = XAY A^k$

David Ferreyra

*Universidad Nacional de Río Cuarto, Argentina, deferreyra@exa.unrc.edu.ar*

Marina Lattanzi

*Universidad Nacional de La Pampa, Argentina, mblatt@exactas.unlpam.edu.ar*

Fabián Levis

*Universidad Nacional de Río Cuarto, Argentina, flevis@exa.unrc.edu.ar*

Néstor Thome

*Universitat Politècnica de València, njthome@mat.upv.es*

Follow this and additional works at: <https://repository.uwyo.edu/ela>



Part of the [Algebra Commons](#)

---

### Recommended Citation

Ferreyra, David; Lattanzi, Marina; Levis, Fabián; and Thome, Néstor. (2019), "Parametrized solutions  $X$  of the system  $AXA = AY A$  and  $A^k Y AX = XAY A^k$ ", *Electronic Journal of Linear Algebra*, Volume 35, pp. 503-510.

DOI: <https://doi.org/10.13001/1081-3810.4051>

This Article is brought to you for free and open access by Wyoming Scholars Repository. It has been accepted for inclusion in *Electronic Journal of Linear Algebra* by an authorized editor of Wyoming Scholars Repository. For more information, please contact [scholcom@uwyo.edu](mailto:scholcom@uwyo.edu).



## PARAMETRIZED SOLUTIONS $X$ OF THE SYSTEM $AXA = AEA$ AND $A^k EAX = XAEA^k$ FOR A MATRIX $A$ HAVING INDEX $k^*$

D.E. FERREYRA<sup>†</sup>, M. LATTANZI<sup>‡</sup>, F.E. LEVIS<sup>†</sup>, AND N. THOME<sup>§</sup>

**Abstract.** Let  $A$  and  $E$  be  $n \times n$  given complex matrices. This paper provides a necessary and sufficient condition for the solvability to the matrix equation system given by  $AXA = AEA$  and  $A^k EAX = XAEA^k$ , for  $k$  being the index of  $A$ . In addition, its general solution is derived in terms of a G-Drazin inverse of  $A$ . As consequences, new representations are obtained for the set of all G-Drazin inverses; some interesting applications are also derived to show the importance of the obtained formulas.

**Key words.** Matrix equations, Index, G-Drazin inverses, Generalized inverses.

**AMS subject classifications.** 15A09, 15A24.

**1. Introduction.** Matrix equations arise in Matrix Theory as a valuable tool to solve an extensive and varied kind of problems. Sometimes, they appear from stating the model; other times, once the model is stated, after discretizing a continuous-time equation, etc. They have important applications in areas such as physics, mechanics, control theory, and many other fields [25]. For instance, the problem of finding all the solutions of the classical Yang-Baxter matrix equation  $AXA = XAX$  is a representative case [14, 17, 21]. On the other hand, some recent results finding algebraic solutions for operator equations such as  $AXB = B = BXA$  can be found in [22, 26].

Generalized inverses of matrices are very useful, among other topics, to tackle problems like the aforementioned ones and they play an important role in the study of matrix equations and partial orders [7, 8, 10, 11, 24, 27]. For example, the equation  $AX + YB = C$  or the more general version  $AXB + CYD = E$  (in the unknowns  $X$  and  $Y$ ) were solved by Baksalary and Kala in [1, 2] by means of projectors associated to generalized inverses.

On the other hand, among other authors, Dinčić investigated the Sylvester equation  $AX - XB = C$  under the condition  $\sigma(A) \cap \sigma(B) \neq \emptyset$  in [12] by using the Jordan canonical form.

In this paper, we completely solve a related matrix equation system and provide some applications.

Let  $\mathbb{C}^{m \times n}$  be the set of  $m \times n$  complex matrices. For  $A \in \mathbb{C}^{m \times n}$ , let  $A^*$ ,  $A^{-1}$ ,  $\text{rk}(A)$ , and  $\mathcal{R}(A)$  denote the conjugate transpose, the inverse ( $m = n$ ), the rank, and the range space of  $A$ , respectively. Moreover,

---

\*Received by the editors on May 22, 2019. Accepted for publication on September 4, 2019. Handling Editor: Oskar Maria Baksalary. Corresponding Author: Nestor Thome.

<sup>†</sup>Universidad Nacional de Río Cuarto, CONICET, FCEFQyN, RN 36 KM 601, 5800 Río Cuarto, Córdoba, Argentina (deferreyra@exa.unrc.edu.ar, flevis@exa.unrc.edu.ar). Partially supported by Universidad Nacional de Río Cuarto (grant PPI 18/C472) and CONICET (grant PIP 112-201501-00433CO).

<sup>‡</sup>Universidad Nacional de La Pampa, FCEyN, Uruguay 151, 6300 Santa Rosa, La Pampa, Argentina (mblatt@exactas.unlpam.edu.ar). Partially supported by Universidad Nacional de La Pampa, Facultad de Ingeniería (grant Resol. Nro. 155/14).

<sup>§</sup>Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, 46022 Valencia, Spain (njthome@mat.upv.es). Partially supported by Ministerio de Economía, Industria y Competitividad of Spain (grant Red de Excelencia MTM2017-90682-REDT) and by Universidad Nacional del Sur of Argentina (grant 24/L108).

$I_m$  stands for the  $m \times m$  identity matrix and  $0_{m \times n}$  denotes the  $m \times n$  zero matrix. If the size is clear from the context, it will be directly denoted by  $I$  or  $0$ . For  $A \in \mathbb{C}^{n \times n}$ , the *index* of  $A$  is the smallest nonnegative integer  $k$  such that  $\mathcal{R}(A^k) = \mathcal{R}(A^{k+1})$ , and is denoted by  $k = \text{ind}(A)$ . Let  $A \in \mathbb{C}^{n \times n}$  with  $k = \text{ind}(A)$ . We recall that the *Drazin inverse* of  $A$  is the unique matrix  $X \in \mathbb{C}^{n \times n}$  such that (2)  $XAX = X$ , (5)  $AX = XA$ , and (6)  $A^{k+1}X = A^k$  hold, and is denoted by  $A^D$ . If  $A$  satisfies  $\text{ind}(A) \leq 1$ , then the Drazin inverse of  $A$  is called the *group inverse* of  $A$  and is denoted by  $A^\#$ . A matrix  $X \in \mathbb{C}^{n \times m}$  that satisfies the equation  $AXA = A$  is called a  $\{1\}$ -inverse of  $A \in \mathbb{C}^{m \times n}$ , and it is denoted by  $A^-$ . The symbol  $A\{1\}$  denotes the set of all  $\{1\}$ -inverses of  $A$ . We also recall that the *Moore-Penrose inverse* of  $A \in \mathbb{C}^{m \times n}$  is the unique matrix  $X \in \mathbb{C}^{n \times m}$  such that (1)  $AXA = A$ , (2)  $XAX = X$ , (3)  $(AX)^* = AX$ , and (4)  $(XA)^* = XA$  hold, and is denoted by  $A^\dagger$ . A detailed analysis of all these generalized inverses can be found, for example, in [3].

The Drazin inverse was defined in [13] and it has proved helpful in analyzing Markov chains, difference and differential equations, iterative procedures, etc., as we can see in [4, 5]. Among other applications, this generalized inverse can be used to construct a generalized transfer function for singular descriptor systems [19]. Taking into account its importance, many computational techniques have been developed to calculate it. Using a determinantal technique, another matrix equation problems related to Drazin inverse were investigated by Krychei in [16] by solving (for  $X$ ) the weighted matrix equations  $WAWX = D$ ,  $XWAW = D$ , and  $W_1AW_1XW_2BW_2 = D$  on quaternions.

Campbell and Meyer introduced in [6] some modifications to the classic Drazin inverse by introducing weak Drazin inverses. A particular case of weak Drazin inverses was defined by H. Wang and X. Liu in [23]. More precisely, for a given  $A \in \mathbb{C}^{n \times n}$  of index  $k$ , a matrix  $X \in \mathbb{C}^{n \times n}$  satisfying

$$(1.1) \quad AXA = A, \quad XA^{k+1} = A^k, \quad \text{and} \quad A^{k+1}X = A^k,$$

is called a *G-Drazin inverse* of  $A$ , which is, in general, not unique. The symbol  $A\{GD\}$  stands for the set of all G-Drazin inverses of  $A$ ; an element of this set is denoted by  $A^{GD}$ . Recently, it was proved [9] that the set of the equations (1.1) is equivalent to the more simplified one given by

$$(1.2) \quad AXA = A \quad \text{and} \quad A^kX = XA^k.$$

The purpose of this paper is to study the matrix equation system given by

$$(1.3) \quad AXA = AEA \quad \text{and} \quad A^kEAX = XAEA^k,$$

for a matrix  $A \in \mathbb{C}^{n \times n}$  of index  $k$ , and a fixed matrix  $E \in \mathbb{C}^{n \times n}$ . Clearly, the matrix equation system (1.3) is more general than (1.2) because, for example, by setting  $E = A^\dagger$  in (1.3) we get (1.2). We obtain general expressions of solutions of the system (1.3) and derive new formulas for all solutions of the system (1.2). We highlight the importance that the fact of making available a parametrized solution as that provided in this paper gives us the opportunity of using it in further applications, such as occurred with the one quoted in Lemma 2.1 below.

The paper is organized as follows. In Section 2, we provide a necessary and sufficient condition for the solvability to the matrix equation system (1.3) and we find all its solutions by giving a parametrized expression. In Section 3, we derive new representations for the set of all G-Drazin inverses of  $A$ . In particular, we give (Subsection 3.1) a more simplified representation for all G-Drazin inverses of a matrix  $A$  of index one. Moreover, we obtain (Subsection 3.2) some necessary conditions for any *G-Drazin inverse* of a matrix  $B$  to be a *G-Drazin inverse* of  $A$ .

**2. A parametrized general solution.** In this section, we extend the system given by (1.2) to a more general case, by introducing a sort of G-Drazin inverse of  $A$  with respect to a fixed matrix  $E$ . We solve this new system by finding a parametrized solution.

By mimicking the definition of affine linear variety of a linear system  $Ax = b$  (for given  $A \in \mathbb{C}^{m \times n}$  and  $b \in \mathbb{C}^m$ ), which is given by the set of all its solutions  $\{x \in \mathbb{C}^n : Ax = b\} = x_0 + S_H$  (where  $Ax_0 = b$  and  $S_H = \{x \in \mathbb{C}^n : Ax = 0\}$  is a vector subspace), we introduce the affine linear variety  $S$  of all solutions of (1.3):

$$S = \{X \in \mathbb{C}^{n \times n} : X \text{ satisfies equations in (1.3)}\},$$

the corresponding vector subspace

$$S_H = \{X \in \mathbb{C}^{n \times n} : AXA = 0 \text{ and } A^kEAX = XAEA^k\}$$

associated to the homogeneous case, and a fixed (but arbitrary) element  $X_0 \in S$ . It is also true that  $S = X_0 + S_H$  holds.

The following result will be useful in what follows.

LEMMA 2.1. ([3, p. 52]) *Let  $A, B, C \in \mathbb{C}^{n \times n}$ . The matrix equation  $AXB = C$  is consistent if and only if  $AA^-CB^-B = C$  for some  $A^- \in A\{1\}$  and  $B^- \in B\{1\}$ , in which case, the general solution of the equation is given by  $X = A^-CB^- + Z - A^-AZBB^-$ , for arbitrary  $Z \in \mathbb{C}^{n \times n}$ .*

It is well known that matrices  $A^-$  and  $B^-$  in Lemma 2.1 can be arbitrarily chosen as  $\{1\}$ -inverses of  $A$  and  $B$ , respectively. Next result will use the notations  $P_A := AA^-$  and  $Q_A := A^-A$ , where  $A^-$  is any fixed  $\{1\}$ -inverse of  $A$ .

THEOREM 2.2. *Let  $A \in \mathbb{C}^{n \times n}$  be a matrix of index  $k$  and  $E \in \mathbb{C}^{n \times n}$  be a fixed matrix. Then the matrix equation system*

$$(2.4) \quad AXA = AEA \quad \text{and} \quad A^kEAX = XAEA^k,$$

*is consistent if and only if*

$$(2.5) \quad A^{k+1}(EA)^2 = (AE)^2A^{k+1}$$

*holds. In this case, if  $A^{GD}$  is a fixed G-Drazin inverse of  $A$ , the general solution of (2.4) is given by*

$$(2.6) \quad X = A^{GD}AEAA^{GD} + T - Q_{A^kEA}T(I - P_A) - (I - Q_A)TP_{AEA^k} - Q_ATP_A,$$

*for arbitrary  $T \in \mathbb{C}^{n \times n}$ .*

*Proof.* Assume that  $X$  is any solution of (2.4). Then, pre- and post-multiplying by  $A$  the equality  $A^kEAX = XAEA^k$ , we arrive at  $A^{k+1}(EA)^2 = (AE)^2A^{k+1}$ , since  $AXA = AEA$ . Hence, (2.5) holds.

For the converse, let  $A^{GD}$  be a fixed G-Drazin inverse of  $A$ . If (2.5) holds, then  $X_0 := A^{GD}AEAA^{GD}$  is a solution of (2.4). In fact, we first note that  $AA^{GD}A = A$  holds. Now,

$$AX_0A = AA^{GD}AEAA^{GD}A = AEA.$$

Moreover,

$$\begin{aligned} A^k EAX_0 &= A^k EAA^{GD} AEAA^{GD} = A^k (EA)^2 A^{GD} = A^{GD} A^{k+1} (EA)^2 A^{GD} \\ &= A^{GD} (AE)^2 A^{k+1} A^{GD} = A^{GD} (AE)^2 A^k = A^{GD} AEAA^{GD} AEA^k \\ &= X_0 AEA^k. \end{aligned}$$

Hence, both equations in (2.4) hold. It then follows that  $S = A^{GD} AEAA^{GD} + S_H$ .

In order to obtain the general solution, it is enough to determine  $S_H$ . Applying Lemma 2.1 to the equation  $AXA = 0$ , its general solution is given by

$$(2.7) \quad X = Z - Q_A Z P_A,$$

for arbitrary  $Z \in \mathbb{C}^{n \times n}$ . Now, by substituting (2.7) in  $A^k EAX = X AEA^k$  and by virtue of the equalities  $A^k EAQ_A = A^k EA$  and  $P_A AEA^k = AEA^k$ , we obtain

$$(2.8) \quad A^k EAZ(I - P_A) = (I - Q_A)ZAEA^k.$$

Post-multiplying the equation (2.8) by the projector  $AA^{GD}$ , we have

$$(2.9) \quad (I - Q_A)ZAEA^k = 0.$$

Since  $I - Q_A$  is a projector,  $I - Q_A \in (I - Q_A)\{1\}$  holds. Then, by Lemma 2.1, the general solution (in  $Z$ ) of (2.9) is given by

$$(2.10) \quad Z = W - (I - Q_A)^-(I - Q_A)WAEA^k(AEA^k)^- = W - (I - Q_A)WAEA^k(AEA^k)^-,$$

for any  $(AEA^k)^- \in (AEA^k)\{1\}$  and arbitrary  $W \in \mathbb{C}^{n \times n}$ . Thus, by using (2.10) and the identity  $A(I - Q_A) = 0$ , we obtain

$$(2.11) \quad A^k EAZ(I - P_A) = A^k EAW(I - P_A).$$

However,  $W$  will be constrained in some sense because (2.8) must be satisfied. In fact, from (2.8), (2.9), and (2.11) we have

$$(2.12) \quad A^k EAW(I - P_A) = 0.$$

Again, by applying Lemma 2.1 to the equation (2.12) (now in the unknown  $W$ ) and using that  $I - P_A \in (I - P_A)\{1\}$ , we get

$$(2.13) \quad W = T - (A^k EA)^- A^k EAT(I - P_A),$$

for any  $(A^k EA)^- \in (A^k EA)\{1\}$  and arbitrary  $T \in \mathbb{C}^{n \times n}$ . Then, from (2.10) and (2.13), we obtain  $Z = T - (A^k EA)^- A^k EAT(I - P_A) - (I - Q_A)TAEA^k(AEA^k)^-$ , for arbitrary  $T \in \mathbb{C}^{n \times n}$ , because  $(I - P_A)A = 0$ . Finally, by substituting this last expression for  $Z$  in (2.7) and by taking into account again the identities  $(I - P_A)P_A = 0$  and  $Q_A(I - Q_A) = 0$ , we arrive at

$$(2.14) \quad S_H = \{T - Q_A A^k EA T(I - P_A) - (I - Q_A)TPAEA^k - Q_A T P_A : T \text{ is arbitrary}\}.$$

Then, the proof is complete. □

REMARK 2.3. If the condition (2.5) is not fulfilled in Theorem 2.2, we have that  $S = \emptyset$ .

**3. Some applications.** We can derive some interesting applications, by particularizing the parameter  $E$  considered in the statement of the problem (1.3).

**3.1. General representations for G-Drazin inverses.** In this subsection, we obtain all parametrized solutions for the set of all G-Drazin inverses of a matrix  $A$  based on any particular G-Drazin inverse of  $A$ . In particular, we obtain more simplified representations for all G-Drazin inverses of a matrix  $A$  of index one.

We start with the following auxiliary lemma.

LEMMA 3.1. *Let  $A \in \mathbb{C}^{n \times n}$  be a matrix of index  $k$ . If  $X \in A\{GD\}$ , then the following assertions are true:*

- (i)  $A^kX^k = X^kA^k$ ,      (ii)  $X^k \in A^k\{GD\}$ .

*Proof.* From  $A^kX = XA^k$ , it is obvious that (i) holds. If  $k \leq 1$ , the assertion (ii) trivially holds. For  $k \geq 2$ ,  $X^k \in A^k\{1\}$  since  $A^kX^kA^k = X^kA^kA^k = X^{k-1}(XA^{k+1})A^{k-1} = X^{k-1}A^kA^{k-1} = X^{k-2}(XA^{k+1})A^{k-2} = \dots = XA^kA = A^k$ . As  $ind(A^k) = 1$ , from (i) we get (ii).  $\square$

Now, by setting  $E = A^\dagger$  in Theorem 2.2, (1.3) reduces to (1.2), and using Lemma 3.1 we can derive an interesting result for G-Drazin inverses; namely, all parametrized solutions of the (equivalent) equation system that define them.

THEOREM 3.2. *Let  $A \in \mathbb{C}^{n \times n}$  be a matrix of index  $k$  and let  $A^{GD}$  be a fixed G-Drazin inverse of  $A$ . Then the set of all G-Drazin inverses of  $A$  is given by*

$$(3.15) \quad A\{GD\} = \{A^{GD} + (I - P_{A^k})T(I - P_A) + (I - Q_A)T(I - P_{A^k}) - (I - Q_A)T(I - P_A) : T \text{ is arbitrary}\},$$

or equivalently,

$$(3.16) \quad A\{GD\} = \{A^{GD} + (I - P_{A^k})U(I - P_A) + (I - Q_A)V(I - P_{A^k}) : U, V \text{ are arbitrary}\},$$

where  $P_A = AA^{GD}$ ,  $Q_A = A^{GD}A$ , and  $P_{A^k} = A^k(A^{GD})^k$ .

*Proof.* Set  $E = A^\dagger$ . We recall that the solution set of the system (1.3) is given by  $S = X_0 + S_H$  for some (fixed but arbitrary)  $X_0$ . For a fixed  $A^{GD} \in A\{GD\}$ , notice that clearly  $X_0 := A^{GD}$  is a solution of (1.2). In particular,  $A^{GD} \in A\{1\}$ . Now, by Lemma 3.1, we have  $(A^{GD})^k \in A^k\{1\}$ . Setting  $A^- = A^{GD}$  and  $(AEA^k)^- = (A^k)^- = (A^{GD})^k$  in Theorem 2.2, projectors become  $P_A = AA^{GD}$ ,  $Q_A = A^{GD}A$ ,  $P_{A^k} = A^k(A^{GD})^k$ , and  $Q_{A^k} = (A^{GD})^kA^k$ . By Lemma 3.1, it follows that  $P_{A^k} = Q_{A^k}$ . Since  $-Q_{A^k} = -P_{A^k} = (I - P_{A^k}) - I$ , we have  $-Q_{A^k}T(I - P_A) = (I - P_{A^k})T(I - P_A) - T(I - P_A)$ . Similarly,  $(I - Q_A)T(-P_{A^k}) = (I - Q_A)T(I - P_{A^k}) - T + Q_AT$ . Now, some computations in (2.14) lead to (3.15). In order to see the equality between expressions in (3.15) and (3.16), we first choose  $T = U(I - P_A) + (I - Q_A)V$ , with arbitrary  $U$  and  $V$  to be substituted in (3.15) and by using the facts that  $ATA = 0$  and  $P_AP_{A^k} = P_{A^k}Q_A = P_{A^k}$ , some calculations yield (3.16). For the other inclusion, by setting  $U = T$  and  $V = (I - Q_A)T - T(I - P_A)$ , with arbitrary  $T$ , (3.15) can be easily deduced from (3.16).  $\square$

REMARK 3.3. Observe that while expression (3.16) for  $A\{GD\}$  is expressed in terms of two parameters (namely,  $U$  and  $V$ ), expression (3.15) requires only one (namely,  $T$ ).

COROLLARY 3.4. *Let  $A \in \mathbb{C}^{n \times n}$  be a matrix of index one and let  $A^{GD}$  be a fixed G-Drazin inverse of  $A$ . Then the set of all G-Drazin inverses of  $A$  is given by*

$$A\{GD\} = \{A^{GD} + (I - A^{GD}A)T(I - AA^{GD}) : T \text{ is arbitrary}\}.$$

COROLLARY 3.5. Let  $A \in \mathbb{C}^{n \times n}$  be a matrix of index  $k$ . Then the general solution of the system

$$(3.17) \quad AXA = AA^D A \quad \text{and} \quad A^k X = X A^k$$

is given by

$$X = A^D + T - AA^D T + AA^D T P_A - T A A^D + Q_A T A A^D - Q_A T P_A,$$

for arbitrary  $T \in \mathbb{C}^{n \times n}$ .

*Proof.* By setting  $E = A^D$  in Theorem 2.2, (1.3) reduces to (3.17). Since  $AA^D$  is a projector and  $AA^D = A^D A$ , it is easy to see that  $A^{GD} AA^D = A^D AA^{GD} = A^D$  and  $(A^D)^k$  is a  $\{1\}$ -inverse of  $A^k$ . Thus, the particular solution has the form  $A^{GD} AA^D AA^{GD} = A^D$ . Now, applying Theorem 2.2 with  $(A^k)^- = (A^D)^k$ , we obtain the result.  $\square$

Next, as a consequence, we derive the following result about commuting  $\{1\}$ -inverses which was proved in [20, Lemma 4.5.2].

COROLLARY 3.6. Let  $A \in \mathbb{C}^{n \times n}$  be a matrix of index 1. Then the general solution of the system  $AXA = A$  and  $AX = XA$  is given by  $X = A^\# + (I - A^\# A)T(I - AA^\#)$ , for arbitrary  $T \in \mathbb{C}^{n \times n}$ .

**3.2. What about range spaces of  $A$  and  $B$  whenever  $B\{GD\} \subseteq A\{GD\}$  holds?** This subsection gives necessary conditions for any  $G$ -Drazin inverse of a matrix  $B$  to be a  $G$ -Drazin inverse of  $A$ . Next lemma plays an important role in the subsequent result.

LEMMA 3.7. Let  $A, B \in \mathbb{C}^{n \times n}$ . If  $AXB = 0$  for all  $X \in \mathbb{C}^{n \times n}$ , then  $A = 0$  or  $B = 0$ .

*Proof.* Let  $E_{i,j}$  be the matrix whose entries are all 0's except its  $(i, j)$ -entry which is 1. Then, we can consider  $A = [a_{ij}]$ , and note that  $0 = E_{kj} A E_{ik} B = a_{ij} E_{kk} B$  for all integer  $1 \leq k \leq n$ . Hence,  $a_{ij} B = 0$  for all integer  $1 \leq i, j \leq n$  from which we conclude that  $A = 0$  or  $B = 0$ .  $\square$

In [23], the condition  $B\{GD\} \subseteq A\{GD\}$  was studied related to the  $\mathcal{G}$ -based partial order that the  $G$ -Drazin determines. Next result goes more deeply in this sense.

THEOREM 3.8. Let  $A, B \in \mathbb{C}^{n \times n}$  be such that  $B\{GD\} \subseteq A\{GD\}$ . Let  $k = \text{ind}(A)$  and  $\ell = \text{ind}(B)$ . Then the following properties hold:

- (i)  $\mathcal{R}(A) \subseteq \mathcal{R}(B)$  or  $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$ ,      (ii)  $\mathcal{R}(A^k) \subseteq \mathcal{R}(B^\ell)$  and  $\mathcal{R}((A^k)^*) \subseteq \mathcal{R}((B^\ell)^*)$ .

*Proof.* If  $\ell = 0$ , items (i) and (ii) are obvious. In order to show (i) and (ii) for  $\ell \geq 1$ , we consider a fixed  $G$ -Drazin inverse  $B^{GD}$  of  $B$ . By using the projectors  $P_B = BB^{GD}$ ,  $Q_B = B^{GD}B$ , and  $P_{B^\ell} = B^\ell(B^{GD})^\ell$ , from Theorem 3.2, it follows that the formula

$$(3.18) \quad X = B^{GD} + (I - P_{B^\ell})U(I - P_B) + (I - Q_B)V(I - P_{B^\ell}),$$

provides all  $G$ -Drazin inverses of  $B$  for arbitrary  $U \in \mathbb{C}^{n \times n}$  and  $V \in \mathbb{C}^{n \times n}$ .

(i) Since  $X, B^{GD} \in B\{GD\} \subseteq A\{GD\}$ , the equalities  $AXA = AB^{GD}A = A$  hold. Hence, by setting  $U = 0$  in (3.18), it follows that  $A(I - Q_B)V(I - P_{B^\ell})A = 0$  for arbitrary  $V$ . Therefore, from Lemma 3.7 we get  $(I - P_{B^\ell})A = 0$  or  $A(I - Q_B) = 0$ , which is equivalent to  $\mathcal{R}(A) \subseteq \mathcal{R}(B^\ell) \subseteq \mathcal{R}(B)$  or  $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$ , respectively.



(ii) Since  $X, B^{GD} \in B\{GD\} \subseteq A\{GD\}$ , we have  $XA^{k+1} = B^{GD}A^{k+1} = A^k$ . By setting  $U = 0$  in (3.18) we get  $(I - Q_B)V(I - P_{B^\ell})A^k = (I - Q_B)V(I - P_{B^\ell})A^{k+1}A^D = 0$ , for arbitrary  $V$ . Now, Lemma 3.7 implies  $(I - P_{B^\ell})A^k = 0$  because  $\ell \neq 0$ . Hence,  $\mathcal{R}(A^k) \subseteq \mathcal{R}(B^\ell)$ . Similarly, by setting  $V = 0$  in (3.18) we arrive at  $A^k(I - P_{B^\ell})U(I - P_B) = 0$ , for arbitrary  $U$ . Then  $A^k(I - P_{B^\ell}) = 0$ , i.e.,  $A^k = A^k B^\ell (B^{GD})^\ell = A^k (B^{GD})^\ell B^\ell$  by Lemma 3.1. Thus,  $\mathcal{R}((A^k)^*) \subseteq \mathcal{R}((B^\ell)^*)$ .  $\square$

REMARK 3.9. We point out that most of the results provided in this paper are also valid for Hilbert space operators, under an adequate interpretation. Let  $\mathcal{H}$  be a complex Hilbert space and denote by  $\mathcal{B}(\mathcal{H})$  the algebra of all bounded linear operators on  $\mathcal{H}$ . Now, the matrix equation system (1.3) should be interpreted as operator equations, where  $A \in \mathcal{B}(\mathcal{H})$  has closed range and  $k = \text{ind}(A)$  is the Drazin index of  $A$  (see [15]). The crucial fact for solving these operator equations is that Lemma 2.1 remains valid. In fact, if  $A, B \in \mathcal{B}(\mathcal{H})$  have closed ranges, then  $AXB = C$  has a solution  $X \in \mathcal{B}(\mathcal{H})$  if and only if  $AA^-CB^-B = C$ . In this case, the general solution is given by  $X = A^-CB^- + Z - A^-AZB^-B$ , for arbitrary  $Z \in \mathcal{B}(\mathcal{H})$  (see [10]). Finally, we would like to highlight that is not necessary to use Hilbert space operator techniques for solving the new system. It is enough to apply strictly algebraic techniques such as those used in the proof of Theorem 2.2. In consequence, a natural further question is to ask about considering our results in rings.

**Acknowledgements.** We would like to thank the referees for their valuable comments and suggestions which helped us to improve the presentation of the paper.

#### REFERENCES

- [1] J.K. Baksalary and R. Kala. The matrix equation  $AX + YB = C$ . *Linear Algebra Appl.*, 25:41–43, 1979.
- [2] J.K. Baksalary and R. Kala. The matrix equation  $AXB + CYD = E$ . *Linear Algebra Appl.*, 30:141–147, 1980.
- [3] A. Ben-Israel and T.N.E. Greville. *Generalized Inverses: Theory and Applications*. Springer-Verlag, New York, 2003.
- [4] S.L. Campbell. *Recent Applications of Generalized Inverses*. Pitman, London, 1982.
- [5] S.L. Campbell and C.D. Meyer. *Generalized Inverses of Linear transformations*. SIAM, Philadelphia, 2009.
- [6] S.L. Campbell and C.D. Meyer. Weak Drazin inverses. *Linear Algebra Appl.*, 20:167–178, 1978.
- [7] J.L. Chen, H.H. Zhu, P. Patrício, and Y.L. Zhang. Characterizations and representations of core and dual core inverses. *Canad. Math. Bull.*, 60:269–282, 2017.
- [8] D.S. Cvetković-Ilić, D. Mosić, and Y. Wei. Partial orders on  $B(H)$ . *Linear Algebra Appl.*, 481:115–130, 2015.
- [9] C. Coll, M. Lattanzi, and N. Thome. Weighted G-Drazin inverses and a new pre-order on rectangular matrices. *Appl. Math. Comput.*, 317:12–24, 2018.
- [10] A. Dajić and J.J. Koliha. Equations  $ax = c$  and  $xb = d$  in rings with involution with applications to Hilbert space operators. *Linear Algebra Appl.*, 429:1779–1809, 2008.
- [11] M.S. Djikić. Lattice properties of the core-partial order. *Banach J. Math. Anal.*, 11:398–415, 2017.
- [12] N.Č. Dinčić. Solving the Sylvester equation  $AX - XB = C$  when  $\sigma(A) \cap \sigma(B) \neq \emptyset$ . *Electron. J. Linear Algebra*, 35:1–23, 2019.
- [13] M.P. Drazin. Pseudo inverses in associative rings and semigroups. *Amer. Math. Monthly*, 65:506–514, 1958.
- [14] Q. Huang, M. Saeed Ibrahim Adam, J. Ding, and L. Zhu. All non-commuting solutions of the Yang-Baxter matrix equation for a class of diagonalizable matrices. *Oper. Matrices*, 13:187–195, 2019.
- [15] J.J. Koliha. A generalized Drazin inverse. *Glasg. Math. J.*, 38:367–381, 1996.
- [16] I. Kyrchei. Determinantal representations of the  $W$ -weighted Drazin inverse over the quaternion skew field. *Appl. Math. Comput.*, 264:453–465, 2015.
- [17] S. Mansour, J. Ding, and Q. Huang. Explicit solutions of the Yang-Baxter-like matrix equation for an idempotent. *Appl. Math. Lett.*, 63:1–76, 2017.
- [18] S.K. Mitra, P. Bhimasankaram, and S. Malik. *Matrix Partial Orders, Shorted Operators and Applications*. World Scientific, New Jersey, 2010.
- [19] P.N. Paraskevopoulos and M.A. Christodoulou. On the computation of the transfer function matrix of singular systems. *Franklin Inst.*, 317:403–411, 1984.



- [20] C.R. Rao and S.K. Mitra. *Generalized Inverse of Matrices and its Applications*. John Wiley & Sons, Inc., New York, 1971.
- [21] H. Tian. All solutions of the Yang-Baxter-like matrix equation for rank-one matrices. *Appl. Math. Lett.*, 51:55–59, 2016.
- [22] M. Vosough. Solutions of the system of operator equations  $BXA = B = AXB$  via the  $*$ -order. *Electron. J. Linear Algebra*, 32:172–183, 2017.
- [23] X. Wang and X. Liu. Partial orders based on core-nilpotent decomposition. *Linear Algebra Appl.*, 488:235–248, 2016.
- [24] Y. Wei, P. Stanimirović, and M. Petković. *Numerical and Symbolic Computations of Generalized Inverses*. World Scientific, Singapore, 2018.
- [25] C. Yan and M. Ge. *Braid Group, Knot Theory, and Statistical Mechanics*. World Scientific, Singapore, 1989.
- [26] X. Zhang and G. Ji. Solutions to the system of operator equations  $AXB = C = BXA$ . *Acta Math. Sci. Ser. B Engl. Ed.*, 38:1143–1150, 2018.
- [27] H.H. Zhu and P. Patrício. Several types of one-sided partial orders in rings. *RACSAM Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Mat.*, 113:3177–3184, 2019.