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Dynamic properties of the dynamical system $(\mathcal{SF}^n_m(X), \mathcal{SF}^n_m(f))$

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Abstract

Let X be a continuum and let n be a positive integer. We consider the hyperspaces $\mathcal{F}_n(X)$ and $\mathcal{SF}_n(X)$. If m is an integer such that $n > m \geq 1$, we consider the quotient space $\mathcal{SF}^n_m(X)$. For a given map $f : X \to X$, we consider the induced maps $\mathcal{F}_n(f) : \mathcal{F}_n(X) \to \mathcal{F}_n(X)$, $\mathcal{SF}_n(f) : \mathcal{SF}_n(X) \to \mathcal{SF}_n(X)$ and $\mathcal{SF}^n_m(f) : \mathcal{SF}^n_m(X) \to \mathcal{SF}^n_m(X)$. In this paper, we introduce the dynamical system $(\mathcal{SF}^n_m(X), \mathcal{SF}^n_m(f))$ and we investigate some relationships between the dynamical systems (X, f), $(\mathcal{F}_n(X), \mathcal{F}_n(f))$, $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ and $(\mathcal{SF}^n_m(X), \mathcal{SF}^n_m(f))$ when these systems are: exact, mixing, weakly mixing, transitive, totally transitive, strongly transitive, chaotic, irreducible, feebly open and turbulent.

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1. Introduction

A continuum is a nonempty compact connected metric space. By a (discrete) dynamical system we mean a continuum with a continuous self-surjection. This class of dynamical systems belongs to the area of topological dynamics, which is a branch of dynamical systems and topology where the qualitative and asymptotic properties of dynamical systems are studied. In the last 30 years, dynamical systems had been greatly developed, this is because they are very useful to model problems of other sciences such as Physics, Biology and Economics. Currently, we can find several types of dynamical systems: exact, mixing, weakly mixing, transitive, totally transitive, strongly transitive, chaotic, minimal and sensitive, see [2, 3, 8, 10, 19, 21, 23, 24, 32].

Concerning hyperspaces theory, given a continuum X, the hyperspaces of X most studied are: the hyperspace 2^X which consists of all the nonempty compact subsets of X; given a natural number n, the hyperspace $\mathcal{C}_n(X)$ consisting of the elements of 2^X that have at most n components; and the hyperspace $\mathcal{F}_n(X)$ formed by the elements of 2^X which have at most n points. Each of them is topologized with the Hausdorff metric. These hyperspaces are extendly studied in continuum theory, see [20, 28, 30].

On the other hand, given a continuum X and a positive integer n, in 1979 [29], the study of quotient spaces of hyperspace was initiated with the introduction of the space $\mathcal{C}_1(X)/\mathcal{F}_1(X)$. Later the space $\mathcal{C}_n(X)/\mathcal{F}_n(X)$ was defined in 2004 [27]. Subsequently, the space $\mathcal{C}_n(X)/\mathcal{F}_1(X)$ was studied [26]. In 2010 [4], the first named author of this paper defined the space $\mathcal{F}_n(X)/\mathcal{F}_1(X)$ which is denoted by $\mathcal{SF}_n(X)$ and is called the n-fold symmetric product suspension of the continuum X. Some topological properties of $\mathcal{SF}_n(X)$ are studied in [4, 6]. Finally, in 2013 [14], the space $\mathcal{F}_n(X)/\mathcal{F}_m(X)$ is defined $(1 \leq m < n)$ and is denoted by $\mathcal{SF}^n_m(X)$. In [14] are studied several properties of this quotient space. Note that when $m = 1$, $\mathcal{SF}^n_m(X) = \mathcal{SF}_n(X)$.

A map (continuous surjection) $f\colon X \to X$, where X is a continuum, induces a map on the hyperspace 2^X, denoted by $2^f\colon 2^X \to 2^X$ and defined by $2^f(A) = f(A)$, for each $A \in 2^X$. If n is a positive integer, the induced map to the hyperspace $\mathcal{C}_n(X)$ is the restriction of 2^f to $\mathcal{C}_n(X)$, and is denoted by $\mathcal{C}_n(f)$ and the induced map to the hyperspace $\mathcal{F}_n(X)$ is simply the restriction of 2^f to $\mathcal{F}_n(X)$ which is denoted by $\mathcal{F}_n(f)$. This last map, $\mathcal{F}_n(f)$, induces a map on the space $\mathcal{SF}_n(X)$ which is denoted by $\mathcal{SF}_n(f) : \mathcal{SF}_n(X) \to \mathcal{SF}_n(X)$ [5, 7]. Thus, the dynamical system (X, f) induces the dynamical systems $(2^X, 2^f)$, $(\mathcal{C}_n(X), \mathcal{C}_n(f))$, $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ and $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$.

A line of research consists of analyzing the relationships between the dynamical system (X, f) (individual dynamic) and the dynamical systems on the hyperspaces $(2^X, 2^f)$, $(\mathcal{C}_n(X), \mathcal{C}_n(f))$, $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ and $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ (collective dynamic). In 1975 [9], the study of this line of research began, and nowadays there are a lot of results in the literature, for instance in [1, 3, 12, 13, 17, 18, 19, 25, 31, 32, 34]. It is important to note that recently, in 2016 [8], the relationships between the dynamical systems (X, f), $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ and $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ were investigated.

Let n and m be two integers such that $n > m \geq 1$ and let X be a continuum. Note that the function $\mathcal{F}_n(f)$ induces another map on the space $\mathcal{SF}_m^n(X)$ which is denoted by $\mathcal{SF}_m^n(f) : \mathcal{SF}_m^n(X) \to \mathcal{SF}_m^n(X)$ [15]. In this paper, we introduce the dynamical system $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ and we investigate some relationships between the dynamical systems (X, f), $(\mathcal{F}_n(X), \mathcal{F}_n(f))$, $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ and $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ when these systems are: exact, mixing, weakly mixing, transitive, totally transitive, strongly transitive, chaotic, irreducible, feebly open and turbulent.

This paper is organized as follows: In Section 2, we recall basic definitions and we introduce some notations. In Section 3, we present properties related with the transitivity of the dynamical systems (X, f), $(\mathcal{F}_n(X), \mathcal{F}_n(f))$, $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ and $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$, namely: exact, mixing, weakly mixing, transitive, totally transitive, strongly transitive and chaotic. Finally, in Section 4, we review others properties of these dynamical systems, namely: irreducible, feebly open and turbulent.

2. Preliminaries

The symbols $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of positive integers, rational numbers, real numbers and complex numbers, respectively. A *continuum* is a nonempty compact connected metric space. A continuum is said to be *non-degenerate* if it has more than one point. A *subcontinuum* of a space X is a continuum contained in X. Given a continuum X, a point $a \in X$ and $\epsilon > 0$, $V_\epsilon(a)$ denotes the open ball with center a and radius ϵ. A *map* is a continuous function. We denote by Id_X the identity map on the continuum X.

Given a continuum X and a positive integer n, we consider the hyperspaces of X, $2^X = \{A \subseteq X \mid A \text{ is closed and nonempty}\}$ and $\mathcal{F}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ points}\}$. We topologize these sets with the Hausdorff metric [30, (0.1)]. The hyperspace $\mathcal{F}_n(X)$ is the *n-fold symmetric product* of X [11].

Given a finite collection $U_1, U_2, \ldots, U_m$ of nonempty subsets of X, with $\langle U_1, U_2, \ldots, U_m \rangle$ we denote the following subset of 2^X:

$$\left\{ A \in 2^X \mid A \subseteq \bigcup_{i=1}^{m} U_i \text{ and } A \cap U_i \neq \varnothing, \text{ for each } i \in \{1, 2, \ldots, m\} \right\}.$$

The family:

$$\{\langle U_1, U_2, \ldots, U_l \rangle \mid l \in \mathbb{N} \text{ and } U_1, U_2, \ldots, U_l \text{ are open subsets of } X\}$$

forms a basis for a topology on 2^X called the *Vietoris topology* [30, (0.11)]. It is well known that the Vietoris topology and the topology induced by the Hausdorff metric coincide [30, (0.13)]. For those who are interested in learning more about these topics can see [20, 28, 30].

Notation 2.1. Let X be a continuum, let n and m be positive integers, and let $U_1, U_2, \ldots, U_m$ be a finite family of open subsets of X. By $\langle U_1, U_2, \ldots, U_m\rangle_n$ we denote the intersection $\langle U_1, U_2, \ldots, U_m\rangle \cap \mathcal{F}_n(X)$.

Given two integers n and m such that $n > m \geq 1$, $\mathcal{SF}^n_m(X)$ denotes the quotient space $\mathcal{F}_n(X)/\mathcal{F}_m(X)$ obtained by shrinking $\mathcal{F}_m(X)$ to a point in $\mathcal{F}_n(X)$, with the quotient topology [15]. Here, we denote the quotient map by $q_m : \mathcal{F}_n(X) \to \mathcal{SF}^n_m(X)$ and $q_m(\mathcal{F}_m(X))$ by F^m_X. Thus:

$$\mathcal{SF}^n_m(X) = \{\{A\} \mid A \in \mathcal{F}_n(X) \setminus \mathcal{F}_m(X)\} \cup \{F^m_X\}.$$

Note that, if $m = 1$, then $\mathcal{SF}^n_1(X) = \mathcal{SF}_n(X)$ (see [4]).

Remark 2.2. The space $\mathcal{SF}^n_m(X) \setminus \{F^m_X\}$ is homeomorphic to $\mathcal{F}_n(X) \setminus \mathcal{F}_m(X)$, using the appropriate restriction of q_m.

Let n be a positive integer and let X be a continuum. If $f : X \to X$ is a map, we consider the *induced map* of f on the n-fold symmetric product of X, $\mathcal{F}_n(f) : \mathcal{F}_n(X) \to \mathcal{F}_n(X)$, defined by $\mathcal{F}_n(f)(A) = f(A)$, for all $A \in \mathcal{F}_n(X)$ [28, 1.8.23]. Also, given two integers n and m such that $n > m \geq 1$, we consider the function $\mathcal{SF}^n_m(f) : \mathcal{SF}^n_m(X) \to \mathcal{SF}^n_m(X)$ given by:

$$\mathcal{SF}^n_m(f)(\chi) = \begin{cases} q_m(\mathcal{F}_n(f)(q_m^{-1}(\chi))), & \text{if } \chi \neq F^m_X; \\ F^m_X, & \text{if } \chi = F^m_X, \end{cases}$$

for each $\chi \in \mathcal{SF}^n_m(X)$.

Note that, by [16, 4.3, p. 126], $\mathcal{SF}^n_m(f)$ is continuous. Moreover, diagram 1 is commutative, that is, $q_m \circ \mathcal{F}_n(f) = \mathcal{SF}^n_m(f) \circ q_m$.

$$\begin{tikzcd}
\mathcal{F}_n(X) \arrow[r, "\mathcal{F}_n(f)"] \arrow[d, "q_m"'] & \mathcal{F}_n(X) \arrow[d, "q_m"] \\
\mathcal{SF}^n_m(X) \arrow[r, "\mathcal{SF}^n_m(f)"'] & \mathcal{SF}^n_m(X)
\end{tikzcd}$$

Diagram 1

Note that if $m = 1$, then $\mathcal{SF}^n_1(f) = \mathcal{SF}_n(f)$ (see [5]).

Now, by diagram $(*)$ from [8, p. 457] and diagram 1, the maps $\mathcal{SF}_n(f)$ and $\mathcal{SF}^n_m(f)$ are related under the diagram 2, where $q \colon \mathcal{F}_n(X) \to \mathcal{SF}_n(X)$ is the quotient map.

$$\begin{tikzcd} \mathcal{SF}_n(X) \arrow[r, "\mathcal{SF}_n(f)"] & \mathcal{SF}_n(X) \\ \mathcal{F}_n(X) \arrow[u, "q"] \arrow[r, "\mathcal{F}_n(f)"] \arrow[d, "q_m"'] & \mathcal{F}_n(X) \arrow[u, "q"'] \arrow[d, "q_m"] \\ \mathcal{SF}_m^n(X) \arrow[r, "\mathcal{SF}_m^n(f)"] & \mathcal{SF}_m^n(X) \end{tikzcd}$$

Diagram 2

Observe that $\mathcal{SF}_n(f) \circ q = q \circ \mathcal{F}_n(f)$.

On the other hand, in this paper a *dynamical system* is a pair (X, f), where X is a nondegenerate continuum and $f : X \to X$ is a map. Given a dynamical system (X, f), define $f^0 = Id_X$ and for each $k \in \mathbb{N}$, let $f^k = f \circ f^{k-1}$. A point $p \in X$ is a *periodic point* in (X, f) provided that there exists $k \in \mathbb{N}$ such that $f^k(p) = p$. The set of periodic points of (X, f) is denoted by $\mathrm{Per}(f)$. Given $x \in X$, the *orbit of x under f* is the set $\mathcal{O}(x, f) = \{f^k(x) \mid k \in \mathbb{N} \cup \{0\}\}$. Finally, a subset K of X is said to be *invariant under f* if $f(K) = K$.

Let (X, f) be a dynamical system. We say that (X, f) is:

(1) *exact* if for each nonempty open subset U of X, there exists $k \in \mathbb{N}$ such that $f^k(U) = X$;
(2) *mixing* if for every pair of nonempty open subsets U and V of X, there exists $N \in \mathbb{N}$ such that $f^k(U) \cap V \neq \varnothing$, for every $k \geq N$;
(3) *weakly mixing* if for all nonempty open subsets U_1, U_2, V_1 and V_2 of X, there exists $k \in \mathbb{N}$ such that $f^k(U_i) \cap V_i \neq \varnothing$, for each $i \in \{1, 2\}$;
(4) *transitive* if for every pair of nonempty open subsets U and V of X, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \varnothing$;
(5) *totally transitive* if (X, f^s) is transitive, for all $s \in \mathbb{N}$;
(6) *strongly transitive* if for each nonempty open subset U of X, there exists $s \in \mathbb{N}$ such that $X = \bigcup_{k=0}^{s} f^k(U)$;
(7) *chaotic* if it is transitive and $\mathrm{Per}(f)$ is dense in X;
(8) *irreducible* if the only closed subset $A \subseteq X$ for which $f(A) = X$ is $A = X$;
(9) *feebly open* (or *semi-open*) if for every nonempty open subset U of X, there is a nonempty open subset V of X such that $V \subseteq f(U)$;
(10) *turbulent* if there are compact nondegenerate subsets C and K of X such that $C \cap K$ has at most a point and $K \cup C \subseteq f(K) \cap f(C)$.

Inclusions between some classes of dynamical systems, which are considered here, are showed in diagram 3. An arrow means inclusion; this is, the class of dynamical system above is contained in the class of dynamical system below. For some of these inclusions see, for instance, [21, 22].

$$
\begin{tikzcd}
\text{Exact} \arrow[r] & \text{Mixing} \arrow[d] & & \\
& \text{Weakly mixing} \arrow[d] & & \\
& \text{Totally transitive} \arrow[d] & \text{Strongly transitive} \arrow[dl] & \text{Irreducible} \arrow[d] \arrow[ddll] \\
\text{Chaotic} \arrow[r] & \text{Transitive} \arrow[d] & & \text{Feebly open} \\
& \text{Surjective} & &
\end{tikzcd}
$$

Diagram 3

By diagram 3 and [5, Theorem 3.2], we have the following result (compare with [8, Lemma 2.3]):

Lemma 2.3. *Let (X, f) be a dynamical system and n and m be integers such that $n > m \geq 1$. Let $\mathcal{N}$ be one of the following classes of dynamical systems: exact, mixing, weakly mixing, transitive, totally transitive, strongly transitive, chaotic, and irreducible. If $(X, f) \in \mathcal{N}$, then $f, \mathcal{F}_n(f)$, $\mathcal{SF}_n(f)$ and $\mathcal{SF}^n_m(f)$ are surjective.*

Let n be an integer greater than or equal to two and let (X, f) be a dynamical system. Observe that $\mathcal{F}_1(X)$ is a subcontinuum of $\mathcal{F}_n(X)$ such that $\mathcal{F}_1(X)$ is invariant under $\mathcal{F}_n(f)$. In Section 4 of [8] the authors defined and studied the dynamical system $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$. Similarly, given an integer m such that $n > m \geq 1$, $\mathcal{F}_m(X)$ is also an invariant subcontinuum of $\mathcal{F}_n(X)$ under $\mathcal{F}_n(f)$. Thus, by [8, Remark 3.1], we can define the dynamical system $(\mathcal{SF}^n_m(X), \mathcal{SF}^n_m(f))$.

3. Dynamical properties related to transitivity of $(\mathcal{SF}^n_m(X), \mathcal{SF}^n_m(f))$

Arguing as in [8, Proposition 4.1] and considering diagram 1, we have the following result.

Proposition 3.1. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$. Then, for each $k, s \in \mathbb{N}$, the following holds:*

(a) $(\mathcal{F}_n(f))^k(A) = f^k(A)$, *for every* $A \in \mathcal{F}_n(X)$.
(b) $q_m \circ (\mathcal{F}_n(f))^k = (\mathcal{SF}^n_m(f))^k \circ q_m$.
(c) $((\mathcal{F}_n(f))^s)^k = (\mathcal{F}_n(f))^{sk}$.

(d) $q_m \circ ((\mathcal{F}_n(f))^s)^k = ((\mathcal{SF}_m^n(f))^s)^k \circ q_m$.

Let (X, d) be a continuum and let $f : X \to X$ be a map. Recall that f is an *isometry* if $d(x, y) = d(f(x), f(y))$, for each $x, y \in X$.

Theorem 3.2. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$. If f is an isometry, then the dynamical system $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is not transitive.*

Proof. Suppose that f is an isometry and that $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is transitive. Let $x_1, x_2, \ldots, x_{m+1} \in X$ be such that $x_i \neq x_j$, for each $i, j \in \{1, \ldots, m+1\}$ with $i \neq j$. Let $r = \min\{d(x_i, x_j) \colon i, j \in \{1, \ldots, m+1\}, i \neq j\}$, where d is the metric of X. For each $i \in \{1, \ldots, m+1\}$, we put $U_i = V_{\frac{r}{4}}(x_i)$. Observe that $U_1, \ldots, U_{m+1}$ are nonempty open subsets of X such that $x_i \in U_i$, for each $i \in \{1, \ldots, m+1\}$ and $U_i \cap U_j = \varnothing$, for each $i, j \in \{1, \ldots, m+1\}$ with $i \neq j$. Moreover, we consider $V_1, \ldots, V_{m+1}$ nonempty open subsets of X such that $\bigcup_{i=1}^{m+1} V_i \subseteq U_1$ and $V_i \cap V_j = \varnothing$, for each $i, j \in \{1, \ldots, m+1\}$ with $i \neq j$. It follows that $\langle U_1, \ldots, U_{m+1}\rangle_n$ is a nonempty open subset of $\mathcal{F}_n(X)$ such that $\langle U_1, \ldots, U_{m+1}\rangle_n \cap \mathcal{F}_m(X) = \varnothing$ and $\langle V_1, \ldots, V_{m+1}\rangle_n \cap \mathcal{F}_m(X) = \varnothing$. By remark 2.2, we have that $q_m(\langle U_1, \ldots, U_{m+1}\rangle_n)$ and $q_m(\langle V_1, \ldots, V_{m+1}\rangle_n)$ are nonempty open subsets of $\mathcal{SF}_m^n(X)$. Since $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is transitive, there exists $k \in \mathbb{N}$ such that $(\mathcal{SF}_m^n(f))^k(q_m(\langle U_1, \ldots, U_{m+1}\rangle_n)) \cap q_m(\langle V_1, \ldots, V_{m+1}\rangle_n) \neq \varnothing$. Hence, by proposition 3.1-(b), we obtain that:

$$q_m((\mathcal{F}_n(f))^k(\langle U_1, \ldots, U_{m+1}\rangle_n)) \cap q_m(\langle V_1, \ldots, V_{m+1}\rangle_n) \neq \varnothing.$$

Let $B \in (\mathcal{F}_n(f))^k(\langle U_1, \ldots, U_{m+1}\rangle_n)$ with $q_m(B) \in q_m(\langle V_1, \ldots, V_{m+1}\rangle_n)$. We consider an element $A \in \langle V_1, \ldots, V_{m+1}\rangle_n$ such that $q_m(A) = q_m(B)$. By remark 2.2, we have that $A = B$. Let $C \in \langle U_1, \ldots, U_{m+1}\rangle_n$ be such that $(\mathcal{F}_n(f))^k(C) = B$. Thus, $(\mathcal{F}_n(f))^k(C) = A$. By proposition 3.1-(a), $f^k(C) = A$. Let $c_1 \in C \cap U_1$ and let $c_2 \in C \cap U_2$. Hence, $d(x_1, x_2) \leq d(x_1, c_1) + d(c_1, c_2) + d(c_2, x_2) < \frac{r}{2} + d(c_1, c_2)$. This implies that $\frac{r}{2} < d(c_1, c_2)$. On the other hand, $f^k(c_1), f^k(c_2) \in f^k(C) \subseteq \bigcup_{i=1}^{m+1} V_i \subseteq U_1$. Thus, $d(f^k(c_1), f^k(c_2)) \leq \frac{r}{2}$. In consequence, $d(f^k(c_1), f^k(c_2)) < d(c_1, c_2)$, which is a contradiction to [8, Remark 4.2]. Therefore, we conclude that $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is not transitive. □

The proof of the following result is obtained from theorem 3.2 and diagram 3.

Theorem 3.3. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$. Let $\mathcal{N}$ be one of the following classes of dynamical systems: exact, mixing, weakly mixing, transitive, totally transitive, strongly transitive, and chaotic. If f is an isometry, then $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f)) \notin \mathcal{N}$.*

We recall that $S^1 = \{e^{2\pi i\theta} \in \mathbb{C} \mid \theta \in [0, 1]\}$.

Example 3.4. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $r : S^1 \to S^1$ be the map defined by $r(e^{2\pi i\theta}) = e^{2\pi i(\theta+\alpha)}$, for each $\theta \in [0, 1]$. Note that r is an isometry. Let $\mathcal{N}$ be one of the following classes of dynamical systems: exact, mixing, weakly mixing, transitive, totally transitive, strongly transitive, and chaotic. By theorem 3.3, we obtain that $(\mathcal{SF}_m^n(S^1), \mathcal{SF}_m^n(r)) \notin \mathcal{N}$. On the other hand, we have that the dynamical system (S^1, r) is transitive, totally transitive, and strongly transitive (see [33, p. 261]).

Theorem 3.5. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$. Then the following are equivalent:*

(1) (X, f) *is exact;*
(2) $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ *is exact;*
(3) $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ *is exact;*
(4) $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ *is exact.*

Proof. By [8, Theorem 4.7], we have that (1), (2) and (3) are equivalent. Now, if $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ is exact, then by, [8, Theorem 3.4], we obtain that $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is exact. That is, (2) implies (4). Therefore, for complete the proof it is enough to prove that (4) implies (1).

Suppose that $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is exact, we prove that (X, f) is exact. Let U be a nonempty open subset of X. We see that $f^k(U) = X$, for some $k \in \mathbb{N}$. We take $U_1, \ldots, U_{m+1}$ nonempty open subsets of X such that $\bigcup_{i=1}^{m+1} U_i \subseteq U$ and $U_i \cap U_j = \varnothing$, for each $i, j \in \{1, \ldots, m+1\}$ with $i \neq j$. Note that $\langle U_1, U_2, \ldots, U_{m+1}\rangle_n$ is a nonempty open subset of $\mathcal{F}_n(X)$, and moreover $\langle U_1, U_2, \ldots, U_{m+1}\rangle_n \cap \mathcal{F}_m(X) = \varnothing$. Hence, by remark 2.2, we obtain that $q_m(\langle U_1, U_2, \ldots, U_{m+1}\rangle_n)$ is a nonempty open subset of $\mathcal{SF}_m^n(X)$. Note that $F_X^m \notin q_m(\langle U_1, U_2, \ldots, U_{m+1}\rangle_n)$. Thus, by the assumption, there exists $k \in \mathbb{N}$ such that:

$$(\mathcal{SF}_m^n(f))^k(q_m(\langle U_1, U_2, \ldots, U_{m+1}\rangle_n)) = \mathcal{SF}_m^n(X).$$

In consequence, by part (b) from proposition 3.1, we have that:

$$q_m((\mathcal{F}_n(f))^k(\langle U_1, U_2, \ldots, U_{m+1}\rangle_n)) = \mathcal{SF}_m^n(X).$$

Let $x \in X$. We take $y_1, y_2, \ldots, y_m \in X \setminus \{x\}$ such that $y_i \neq y_j$, for each $i, j \in \{1, 2, \ldots, m\}$ with $i \neq j$, and we define $A = \{x, y_1, \ldots, y_m\}$. Note that $A \in \mathcal{F}_n(X) \setminus \mathcal{F}_m(X)$. Thus, $q_m(A) \neq F_X^m$. Since $q_m(A) \in \mathcal{SF}_m^n(X)$, there exists $B \in (\mathcal{F}_n(f))^k(\langle U_1, U_2, \ldots, U_{m+1}\rangle_n)$ such that $q_m(B) = q_m(A)$. Hence, by remark 2.2, we have that $B = A$. Let $C \in \langle U_1, U_2, \ldots, U_{m+1}\rangle_n$ be such that $(\mathcal{F}_n(f))^k(C) = B$. By proposition 3.1-(a), we deduce that $f^k(C) = B$. Since $A = B$ and $C \subseteq U$, it follows that $A \subseteq f^k(U)$. Hence, $x \in f^k(U)$. Thus, $X \subseteq f^k(U)$. This implies that (X, f) is exact. □

As a consequence from theorem 3.5 and diagram 3, we have the following result.

Corollary 3.6. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$. If (X, f) is exact, then $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is mixing, weakly mixing, totally transitive and transitive.*

Corollary 3.7. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$. If f is an isometry, then (X, f) is not exact.*

Proof. Suppose that f is an isometry. If the dynamical system (X, f) is exact, then, by theorem 3.5, the dynamical system $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is exact. However, by theorem 3.3, we know that the dynamical system $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is not exact. Therefore, the dynamical system (X, f) is not exact. □

Theorem 3.8. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$. Then the following are equivalent:*

(1) (X, f) is mixing;
(2) $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ is mixing;
(3) $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ is mixing;
(4) $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is mixing.

Proof. From [8, Theorem 4.9], it follows that (1), (2) and (3) are equivalent. Now, if the system $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ is mixing, then by [8, Theorem 3.4], we have that the system $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is mixing. Thus, (2) implies (4). Finally, we prove that (4) implies (1).

Suppose that $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is mixing, we prove that (X, f) is mixing. For this end, let U and V be nonempty open subsets of X. We see that there exists $N \in \mathbb{N}$ such that $f^k(U) \cap V \neq \varnothing$, for every $k \geq N$. We consider nonempty open subsets $U_1, U_2, \ldots, U_{m+1}$ and $V_1, V_2, \ldots, V_{m+1}$ of X such that $\bigcup_{i=1}^{m+1} U_i \subseteq U$, $\bigcup_{i=1}^{m+1} V_i \subseteq V$, $U_i \cap U_j = \varnothing$ for each $i, j \in \{1, 2, \ldots, m+1\}$ with $i \neq j$, and $V_i \cap V_j = \varnothing$ for each $i, j \in \{1, 2, \ldots, m+1\}$ with $i \neq j$. It follows that $\langle U_1, U_2, \ldots, U_{m+1}\rangle_n$ and $\langle V_1, V_2, \ldots, V_{m+1}\rangle_n$ are nonempty open subset of $\mathcal{F}_n(X)$ such that $\langle U_1, U_2, \ldots, U_{m+1}\rangle_n \cap \mathcal{F}_m(X) = \varnothing$ and $\langle V_1, V_2, \ldots, V_{m+1}\rangle_n \cap \mathcal{F}_m(X) = \varnothing$. Hence, by remark 2.2, we have that:

$$q_m(\langle U_1, U_2, \ldots, U_{m+1}\rangle_n) \quad \text{and} \quad q_m(\langle V_1, V_2, \ldots, V_{m+1}\rangle_n)$$

are open subsets of $\mathcal{SF}_m^n(X)$. Note that $F_X^m \notin q_m(\langle U_1, U_2, \ldots, U_{m+1}\rangle_n)$. Additionally, $F_X^m \notin q_m(\langle V_1, V_2, \ldots, V_{m+1}\rangle_n)$. Since $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is mixing, there exists $N \in \mathbb{N}$ such that for each $k \geq N$:

$$(\mathcal{SF}_m^n(f))^k(q_m(\langle U_1, U_2, \ldots, U_{m+1}\rangle_n)) \cap q_m(\langle V_1, V_2, \ldots, V_{m+1}\rangle_n) \neq \varnothing.$$

Fix $k \geq N$ and let $\chi \in q_m(\langle U_1, \ldots, U_{m+1}\rangle_n)$ satisfying $(\mathcal{SF}_m^n(f))^k(\chi) \in q_m(\langle V_1, \ldots, V_{m+1}\rangle_n)$. Let $A \in \langle U_1, U_2, \ldots, U_{m+1}\rangle_n$ such that $q_m(A) = \chi$ and let $B \in \langle V_1, V_2, \ldots, V_m\rangle_n$ such that $(\mathcal{SF}_m^n(f))^k(\chi) = q_m(B)$. Hence, we have that $(\mathcal{SF}_m^n(f))^k(q_m(A)) = q_m(B)$. By part (b) from proposition 3.1, we obtain that $q_m((\mathcal{F}_n(f))^k(A)) = q_m(B)$. From remark 2.2, it follows that $(\mathcal{F}_n(f))^k(A) = B$. Again, by part (a) from proposition 3.1, we deduce that $f^k(A) = B$. We take $a \in A \cap U_1$. This implies that $f^k(a) \in f^k(A) \cap f^k(U)$. Moreover, $f^k(a) \in B \cap f^k(U)$. Since $B \subseteq V$, we have that $f^k(U) \cap V \neq \varnothing$. In consequence, (X, f) is mixing. □

Using theorem 3.8 and diagram 3, we deduce the following result.

Corollary 3.9. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$. If (X, f) is mixing, then $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is weakly mixing, totally transitive and transitive.*

The proof of the following result is similar to the proof of the corollary 3.7.

Corollary 3.10. *Let (X, f) be a dynamical system. If f is an isometry, then (X, f) is not mixing.*

Theorem 3.11. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$. Consider the following statements:*
(1) (X, f) is transitive;
(2) $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ is transitive;
(3) $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ is transitive;
(4) $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is transitive.
Then (2), (3) and (4) are equivalent, (2) implies (1), (3) implies (1), (4) implies (1), (1) does not imply (2), (1) does not imply (3), and (1) does not imply (4).

Proof. By [8, Theorem 4.10], we have that (2) and (3) are equivalent, (2) implies (1), (3) implies (1), (1) does not imply (2) and (1) does not imply (3). Now, if $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ is transitive, then by [8, Theorem 3.4], $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is transitive. Hence, we have that (2) implies (4).

Finally, suppose that $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is transitive. We prove that the system $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ is transitive. Let Γ and Λ be nonempty open subsets of $\mathcal{SF}_n(X)$. Since $q^{-1}(\Gamma)$ and $q^{-1}(\Lambda)$ are nonempty open subsets of $\mathcal{F}_n(X)$, then by [19, Lemma 4.2], there exist nonempty open subsets $U_1, U_2, \ldots, U_n$ and $V_1, V_2, \ldots, V_n$ of X such that:

$$\langle U_1, U_2, \ldots, U_n\rangle_n \subseteq q^{-1}(\Gamma) \quad \text{and } \langle V_1, V_2, \ldots, V_n\rangle_n \subseteq q^{-1}(\Lambda).$$

We take, for each $i \in \{1, 2, \ldots, n\}$, a nonempty open subset W_i of X such that $W_i \subseteq U_i$ and for each $i, j \in \{1, 2, \ldots, n\}$, $W_i \cap W_j = \varnothing$ with $i \neq j$. Also, for each $i \in \{1, 2, \ldots, n\}$, let O_i be a nonempty open subset of X such that $O_i \subseteq V_i$ and for each $i, j \in \{1, 2, \ldots, n\}$, $O_i \cap O_j = \varnothing$ with $i \neq j$. It follows that $\langle U_1, U_2, \ldots, U_n\rangle_n$ and $\langle V_1, V_2, \ldots, V_n\rangle_n$ are nonempty open subsets of $\mathcal{F}_n(X)$ such that:

$$\langle W_1, W_2, \ldots, W_n\rangle_n \subseteq \langle U_1, U_2, \ldots, U_n\rangle_n \subseteq q^{-1}(\Gamma)$$

and

$$\langle O_1, O_2, \ldots, O_n\rangle_n \subseteq \langle V_1, V_2, \ldots, V_n\rangle_n \subseteq q^{-1}(\Lambda).$$

Moreover, $\langle W_1, W_2, \ldots, W_n\rangle_n \cap \mathcal{F}_m(X) = \varnothing$ and $\langle O_1, O_2, \ldots, O_n\rangle_n \cap \mathcal{F}_m(X) = \varnothing$. Hence, by remark 2.2, we have that:

$$q_m(\langle W_1, W_2, \ldots, W_n\rangle_n) \quad \text{and} \quad q_m(\langle O_1, O_2, \ldots, O_n\rangle_n)$$

are nonempty open subsets of $\mathcal{SF}_m^n(X)$. Note that $F_X^m \notin q_m(\langle W_1, \ldots, W_n\rangle_n)$ and $F_X^m \notin q_m(\langle O_1, \ldots, O_n\rangle_n)$. Because $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is transitive, there exists $k \in \mathbb{N}$ such that:

$$(\mathcal{SF}_m^n(f))^k(q_m(\langle W_1, W_2, \ldots, W_n\rangle_n)) \cap q_m(\langle O_1, O_2, \ldots, O_n\rangle_n) \neq \varnothing.$$

As a consequence of proposition 3.1-(d), it follows that:

$$q_m((\mathcal{F}_n(f))^k(\langle W_1, W_2, \ldots, W_n\rangle_n)) \cap q_m(\langle O_1, O_2, \ldots, O_n\rangle_n) \neq \varnothing.$$

Let $B \in (\mathcal{F}_n(f))^k(\langle W_1, W_2, \ldots, W_n\rangle_n)$ with $q_m(B) \in q_m(\langle O_1, O_2, \ldots, O_n\rangle_n)$. Hence, we consider $A \in \langle O_1, O_2, \ldots, O_n\rangle_n$ such that $q_m(A) = q_m(B)$. By remark 2.2, we obtain that $A = B$. Thus, it follows that:

$$(\mathcal{F}_n(f))^k(\langle W_1, W_2, \ldots, W_n\rangle_n) \cap \langle O_1, O_2, \ldots, O_n\rangle_n \neq \varnothing.$$

Hence, there is an element $C \in \langle W_1, W_2, \ldots, W_n\rangle_n$ such that $(\mathcal{F}_n(f))^k(C) \in \langle O_1, O_2, \ldots, O_n\rangle_n$. Then, $q(C) \in q(\langle W_1, W_2, \ldots, W_n\rangle_n)$ and $q((\mathcal{F}_n(f))^k(C)) \in q(\langle O_1, O_2, \ldots, O_n\rangle_n)$. Moreover, since $q \circ \mathcal{F}_n(f) = \mathcal{SF}_n(f) \circ q$, we obtain that $(\mathcal{SF}_n(f))^k(q(C)) \in q(\langle O_1, O_2, \ldots, O_n\rangle_n)$. Also, observe that:

$$q(\langle W_1, \ldots, W_n\rangle_n) \subseteq q(q^{-1}(\Gamma)) \subseteq \Gamma \text{ and } q(\langle O_1, O_2, \ldots, O_n\rangle_n) \subseteq q(q^{-1}(\Lambda)) \subseteq \Lambda.$$

Hence, $(\mathcal{SF}_n(f))^k(\Gamma) \cap \Lambda \neq \varnothing$. In consequence, $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ is transitive. Since (2) and (4) are equivalent, we obtain that (4) implies (1).

By example 3.4, we note that (1) does not imply (4). □

As a consequence of diagram 3 and theorem 3.11, we have the next result:

Corollary 3.12. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$. If $(\mathcal{SF}^n_m(X), \mathcal{SF}^n_m(f))$ is strongly transitive, then the system $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ is transitive.*

Theorem 3.13. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$. Then the following are equivalent:*

(1) (X, f) is weakly mixing;
(2) $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ is weakly mixing;
(3) $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ is transitive;
(4) $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ is weakly mixing;
(5) $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ is transitive;
(6) $(\mathcal{SF}^n_m(X), \mathcal{SF}^n_m(f))$ is weakly mixing;
(7) $(\mathcal{SF}^n_m(X), \mathcal{SF}^n_m(f))$ is transitive.

Proof. By [8, Theorem 4.11], we have that (1), (2), (3), (4) and (5) are equivalent. On the other hand, by theorem 3.11, we have that (5) and (7) are equivalent. It follows from diagram 3 that (6) implies (7). Now, if $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ is weakly mixing, then by [8, Theorem 3.4], $(\mathcal{SF}^n_m(X), \mathcal{SF}^n_m(f))$ is weakly mixing. Hence, we have that (2) implies (6). Thus, (7) implies (6). Therefore, (6) and (7) are equivalent. □

The proof of the corollary 3.14 is similar to the proof of the corollary 3.7.

Corollary 3.14. *Let (X, f) be a dynamical system. If f is an isometry, then (X, f) is not weakly mixing.*

Moreover, by corollary 3.12 and theorem 3.13, we obtain:

Corollary 3.15. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$. If $(\mathcal{SF}^n_m(X), \mathcal{SF}^n_m(f))$ is strongly transitive, then the system $(\mathcal{SF}^n_m(X), \mathcal{SF}^n_m(f))$ is weakly mixing.*

Theorem 3.16. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$. Consider the following statements:*

(1) (X, f) is totally transitive;
(2) $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ is totally transitive;
(3) $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ is totally transitive;
(4) $(\mathcal{SF}^n_m(X), \mathcal{SF}^n_m(f))$ is totally transitive.

Then (2), (3) and (4) are equivalent, (2) implies (1), (3) implies (1), (4) implies (1), (1) does not imply (2), (1) does not imply (3) and (1) does not imply (4).

Proof. By [8, Theorem 4.12], we have that (2) and (3) are equivalent, (3) implies (1), (2) implies (1), (1) does not imply (2) and (1) does not imply (3). Now, if the system $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ is totally transitive, then, by [8, Theorem 3.4], we have that the system $(\mathcal{SF}^n_m(X), \mathcal{SF}^n_m(f))$ is totally transitive. That is, (2) implies (4). In consequence (3) implies (4).

Now, we prove that (4) implies (3). Suppose that $(\mathcal{SF}^n_m(X), \mathcal{SF}^n_m(f))$ is totally transitive, we prove that $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ is totally transitive. For this end, let $s \in \mathbb{N}$. We see that $(\mathcal{SF}_n(X), (\mathcal{SF}_n(f))^s)$ is transitive. Let Γ and Λ be nonempty open subsets of $\mathcal{SF}_n(X)$. Since q is continuous, $q^{-1}(\Gamma)$ and $q^{-1}(\Lambda)$ are nonempty open subsets of $\mathcal{F}_n(X)$. Applying [19, Lemma 4.2], we can take nonempty open subsets $U_1, U_2, \ldots, U_n$ and $V_1, V_2, \ldots, V_n$ of X such that $\langle U_1, U_2, \ldots, U_n\rangle_n \subseteq q^{-1}(\Gamma)$ and $\langle V_1, V_2, \ldots, V_n\rangle_n \subseteq q^{-1}(\Lambda)$. Hence, for every $i \in \{1, 2, \ldots, n\}$, we consider a nonempty open subset W_i of X such that $W_i \subseteq U_i$ and for each $i, j \in \{1, 2, \ldots, n\}$, $W_i \cap W_j = \varnothing$, when $i \neq j$. Moreover, for every $i \in \{1, 2, \ldots, n\}$, let O_i be a nonempty open subset of X such that $O_i \subseteq V_i$ and for each $i, j \in \{1, 2, \ldots, n\}$, $O_i \cap O_j = \varnothing$, when $i \neq j$. Observe that $\langle U_1, U_2, \ldots, U_n\rangle_n$ and $\langle V_1, V_2, \ldots, V_n\rangle_n$ are nonempty open subsets of $\mathcal{F}_n(X)$ with $\langle W_1, W_2, \ldots, W_n\rangle_n \subseteq \langle U_1, U_2, \ldots, U_n\rangle_n \subseteq q^{-1}(\Gamma)$ and $\langle O_1, O_2, \ldots, O_n\rangle_n \subseteq \langle V_1, V_2, \ldots, V_n\rangle_n \subseteq q^{-1}(\Lambda)$. Moreover, $\langle W_1, \ldots, W_n\rangle_n \cap \mathcal{F}_m(X) = \varnothing$ and $\langle O_1, O_2, \ldots, O_n\rangle_n \cap \mathcal{F}_m(X) = \varnothing$. Hence, by remark 2.2, we have that $q_m(\langle W_1, W_2, \ldots, W_n\rangle_n)$ and $q_m(\langle O_1, O_2, \ldots, O_n\rangle_n)$ are nonempty open subsets of $\mathcal{SF}^n_m(X)$. Note that:

$$F^m_X \notin q_m(\langle W_1, \ldots, W_n\rangle_n) \text{ and } F^m_X \notin q_m(\langle O_1, O_2, \ldots, O_n\rangle_n).$$

Since $(\mathcal{SF}^n_m(X), \mathcal{SF}^n_m(f))$ is totally transitive, $(\mathcal{SF}^n_m(X), (\mathcal{SF}^n_m(f))^s)$ is transitive. It follows that there exists $k \in \mathbb{N}$ such that:

$$((\mathcal{SF}^n_m(f))^s)^k(q_m(\langle W_1, W_2, \ldots, W_n\rangle_n)) \cap q_m(\langle O_1, O_2, \ldots, O_n\rangle_n) \neq \varnothing.$$

Using proposition 3.1-(d), we obtain that:

$$q_m(((\mathcal{F}_n(f))^s)^k(\langle W_1, W_2, \ldots, W_n\rangle_n)) \cap q_m(\langle O_1, O_2, \ldots, O_n\rangle_n) \neq \varnothing.$$

By remark 2.2, we have that:

$$((\mathcal{F}_n(f))^s)^k(\langle W_1, W_2, \ldots, W_n\rangle_n) \cap \langle O_1, O_2, \ldots, O_n\rangle_n \neq \varnothing.$$

In consequence, there exists $C \in \langle W_1, W_2, \ldots, W_n\rangle_n$ such that $((\mathcal{F}_n(f))^s)^k(C) \in \langle O_1, O_2, \ldots, O_n\rangle_n$. Then $q(C) \in q(\langle W_1, W_2, \ldots, W_n\rangle_n)$ and $q(((\mathcal{F}_n(f))^s)^k(C)) \in q(\langle O_1, O_2, \ldots, O_n\rangle_n)$. Since $q \circ \mathcal{F}_n(f) = \mathcal{SF}_n(f) \circ q$, we obtain that:

$$((\mathcal{SF}_n(f))^s)^k(q(C))) \in q(\langle O_1, O_2, \ldots, O_n\rangle_n).$$

Moreover, we note that $q(\langle W_1, W_2, \ldots, W_n\rangle_n) \subseteq \Gamma$ and $q(\langle O_1, O_2, \ldots, O_n\rangle_n) \subseteq \Lambda$. Hence, $q(C) \in \Gamma$ and $((\mathcal{SF}_n(f))^s)^k(q(C))) \in \Lambda$. Thus, it follows that $((\mathcal{SF}_n(f))^s)^k(\Gamma) \cap \Lambda \neq \varnothing$. Therefore, (3) and (4) are equivalent. In consequence, (4) implies (2), (4) implies (1), and (1) does not imply (4). $\square$

Theorem 3.17. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$. Consider the following statements:*

(1) (X, f) is strongly transitive;
(2) $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ is strongly transitive;
(3) $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ is strongly transitive;
(4) $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is strongly transitive.

Then (2) implies (1), (2) implies (3), (2) implies (4), (3) implies (1), (4) implies (1), (1) does not imply (2), (1) does not imply (3), and (1) does not imply (4).

Proof. By [8, Theorem 4.13], we have that (2) implies (1), (2) implies (3), (3) implies (1), (1) does not imply (2) and (1) does not imply (3). On the other hand, if $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ is strongly transitive, then, by [8, Theorem 3.4], we have that $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is strongly transitive. Hence, (2) implies (4). Also, by example 3.4, we have that (1) does not implies (4).

We prove that (4) implies (1). Suppose that $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is strongly transitive. Let U be a nonempty open subset of X. Let $U_1, \ldots, U_n$ be nonempty open subsets of X such that $\bigcup_{i=1}^n U_i \subseteq U$ and $U_i \cap U_j = \varnothing$ for each $i, j \in \{1, \ldots, n\}$ with $i \neq j$. It follows that $\langle U_1, \ldots, U_n\rangle_n$ is a nonempty open subset of $\mathcal{F}_n(X)$ such that $\langle U_1, \ldots, U_n\rangle_n \cap \mathcal{F}_m(X) = \varnothing$. Using remark 2.2, we obtain that $q_m(\langle U_1, \ldots, U_n\rangle_n)$ is a nonempty open subset of $\mathcal{SF}_m^n(X)$. Note that $F_X^m \notin q_m(\langle U_1, \ldots, U_n\rangle_n)$. Considering that $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is strongly transitive, we have that $\mathcal{SF}_m^n(X) = \bigcup_{k=0}^s (\mathcal{SF}_m^n(f))^k(q_m(\langle U_1, \ldots, U_n\rangle_n))$, for some $s \in \mathbb{N}$. As a consequence from proposition 3.1-(b), it follows that:

$$\mathcal{SF}_m^n(X) = \bigcup_{k=0}^s q_m((\mathcal{F}_n(f))^k(\langle U_1, \ldots, U_n\rangle_n)).$$

Finally, we see that $X = \bigcup_{k=0}^s f^k(U)$. Let $x \in X$. We take $y_1, \ldots, y_m \in X \setminus \{x\}$ such that $y_i \neq y_j$ for each $i, j \in \{1, \ldots, m\}$ with $i \neq j$. Let $A = \{x, y_1, \ldots, y_m\}$. We have that $A \in \mathcal{F}_n(X) \setminus \mathcal{F}_m(X)$. In consequence, $q_m(A) \in \mathcal{SF}_m^n(X) \setminus \{F_X^m\}$. This implies that there exists $j \in \{0, 1, \ldots, s\}$ such that $q_m(A) \in q_m((\mathcal{F}_n(f))^j(\langle U_1, \ldots U_n\rangle_n))$. Hence, there exists $B \in (\mathcal{F}_n(f))^j(\langle U_1, \ldots, U_n\rangle_n)$ such that $q_m(B) = q_m(A)$. Note that, by remark 2.2, $A = B$. Observe that there exists $C \in \langle U_1, \ldots, U_n \rangle_n$ such that $(\mathcal{F}_n(f))^j(C) = B$. Thus, by proposition 3.1-(a), $f^j(C) = B$. Moreover, since $C \subseteq U$, it follows that $f^j(C) \subseteq f^j(U)$. Then, $A \subseteq f^j(U)$. In consequence, $x \in f^j(U)$. Thus, $X \subseteq \bigcup_{k=0}^{s} f^k(U)$. Hence, (X, f) is strongly transitive. □

We have the following questions (compare with [8, Question 4.1]).

Questions 3.18. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$.*

- *(i) If $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is strongly transitive, then is $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ strongly transitive?*
- *(ii) If $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is strongly transitive, then is $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ strongly transitive?*
- *(iii) If $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ is strongly transitive, then is $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ strongly transitive?*

In order to prove the theorem 3.20, we have the next result.

Lemma 3.19. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$. Suppose that f is a surjective map. Then the following are equivalent:*

(1) $Per(f)$ is dense in X;
(2) $Per(\mathcal{F}_n(f))$ is dense in $\mathcal{F}_n(X)$;
(3) $Per(\mathcal{SF}_n(f))$ is dense in $\mathcal{SF}_n(X)$;
(4) $Per(\mathcal{SF}_m^n(f))$ is dense in $\mathcal{SF}_m^n(X)$.

Proof. By [8, Theorem 4.16], we have that (1), (2) and (3) are equivalent. Now, by [8, Lemma 3.3], we have that (2) implies (4). Therefore, for complete the proof it is enough to prove that (4) implies (2).

Suppose that $\mathrm{Per}(\mathcal{SF}_m^n(f))$ is dense in $\mathcal{SF}_m^n(X)$, we prove that $\mathrm{Per}(\mathcal{F}_n(f))$ is dense in $\mathcal{F}_n(X)$. For this end, let $\mathcal{U}$ be a nonempty open subset of $\mathcal{F}_n(X)$. By [19, Lemma 4.2], there exist nonempty open subsets $U_1, U_2, \ldots, U_n$ of X such that $\langle U_1, U_2, \ldots, U_n \rangle_n \subseteq \mathcal{U}$. For each $i \in \{1, 2, \ldots, n\}$, let W_i be a nonempty open subset of X such that $W_i \subseteq U_i$ and for each $i, j \in \{1, 2, \ldots, n\}$, $W_i \cap W_j \neq \varnothing$, if $i \neq j$. It follows that $\langle W_1, W_2, \ldots, W_n \rangle_n$ is a nonempty open subset of $\mathcal{F}_n(X)$ such that $\langle W_1, W_2, \ldots, W_n \rangle_n \subseteq \langle U_1, U_2, \ldots, U_n \rangle_n \subseteq \mathcal{U}$ and $\langle W_1, W_2, \ldots, W_n \rangle_n \cap \mathcal{F}_m(X) = \varnothing$. Hence, by remark 2.2, we have that $q_m(\langle W_1, W_2, \ldots, W_n \rangle_n)$ is a nonempty open subset of $\mathcal{SF}_m^n(X)$. Observe that $F_X^m \notin q_m(\langle W_1, W_2, \ldots, W_n \rangle_n)$. Thus, by hypothesis, we obtain that $q_m(\langle W_1, W_2, \ldots, W_n \rangle_n) \cap \mathrm{Per}(\mathcal{SF}_m^n(f)) \neq \varnothing$. In consequence, there exist $A \in \langle W_1, W_2, \ldots, W_n \rangle_n$ and $k \in \mathbb{N}$ such that $(\mathcal{SF}_m^n(f))^k(q_m(A)) = q_m(A)$. This implies, by proposition 3.1-(b) that $q_m((\mathcal{F}_n(f))^k(A)) = q_m(A)$. Furthermore, by remark 2.2, we have that $(\mathcal{F}_n(f))^k(A) = A$. Therefore, there exist $A \in \mathcal{U}$ and $k \in \mathbb{N}$ such that $(\mathcal{F}_n(f))^k(A) = A$. Hence, $\mathrm{Per}(\mathcal{F}_n(f))$ is dense in $\mathcal{F}_n(X)$. □

Theorem 3.20. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$. Then the next propositions are equivalent:*

(1) (X, f) *is weakly mixing and chaotic;*
(2) $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ *is chaotic;*
(3) $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ *is chaotic;*
(4) $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ *is chaotic.*

Proof. By [8, Theorem 4.17], we have that (1), (2) and (3) are equivalent. Now, if $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ is chaotic, then, by [8, Theorem 3.4], we have that $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is chaotic. Thus, (2) implies (4). As a consequence of lemma 3.19 and theorem 3.11, we conclude that (4) implies (2). □

Arguing as in corollary 3.7, we obtain the following result.

Corollary 3.21. *Let* (X, f) *be a dynamical system. If* f *is an isometry, then* (X, f) *is not chaotic or* (X, f) *is not weakly mixing.*

4. Other dynamical properties of $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$

In this section we study irreducible, feebly open and turbulent dynamical systems.

Theorem 4.1. *Let* (X, f) *be a dynamical system and let* $n, m \in \mathbb{N}$ *be such that* $n > m \geq 1$. *Consider the following statements:*
(1) (X, f) *is irreducible;*
(2) $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ *is irreducible;*
(3) $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ *is irreducible;*
(4) $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ *is irreducible.*
Then (2) implies (1), (3) implies (1) and (4) implies (1).

Proof. By [8, Theorem 5.1], we obtain that (2) implies (1) and (3) implies (1). Therefore, it is enough to prove that (4) implies (1).

Suppose that $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is irreducible and we prove that (X, f) is irreducible. We take a nonempty closed subset A of X with $f(A) = X$. We see that $A = X$. Note that $\langle A \rangle_n$ is a nonempty closed subset of $\mathcal{F}_n(X)$ such that $\mathcal{F}_n(f)(\langle A \rangle_n) = \mathcal{F}_n(X)$. Thus, $q_m(\mathcal{F}_n(f)(\langle A \rangle_n)) = \mathcal{SF}_m^n(X)$. Hence, by proposition 3.1-(b), we have that $\mathcal{SF}_n(f)(q_m(\langle A \rangle_n)) = \mathcal{SF}_m^n(X)$. Since $q_m(\langle A \rangle_n)$ is a nonempty closed subset of $\mathcal{SF}_m^n(X)$ and $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is irreducible, we have that $q_m(\langle A \rangle_n) = \mathcal{SF}_m^n(X)$. Now, let $x \in X$ and we consider $y_1, \ldots, y_m \in X \setminus \{x\}$ such that $y_i \neq y_j$ for each $i, j \in \{1, \ldots, m\}$ with $i \neq j$. Let $B = \{x, y_1, \ldots, y_m\}$. Clearly, $B \in \mathcal{F}_n(X) \setminus \mathcal{F}_m(X)$. Then, $q_m(B) \in \mathcal{SF}_m^n(X) \setminus \{F_X^m\}$. Considering that $q_m(B) \in \mathcal{SF}_m^n(X)$, there exists an element $C \in \langle A \rangle_n$ with $q_m(C) = q_m(B)$. Using remark 2.2, we obtain that $C = B$. Thus, $x \in A$. This implies that $X = A$. Therefore, (X, f) is irreducible. □

Questions 4.2. *Let* (X, f) *be a dynamical system and let* $n, m \in \mathbb{N}$ *be such that* $n > m \geq 1$.

(i) If (X, f) *is irreducible, then is* $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ *irreducible?*
(ii) If $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ *is irreducible, then is* $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ *irreducible?*

(iii) If $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ is irreducible, then is $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ irreducible?

Theorem 4.3. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$ with f a surjective map. Then the following propositions are equivalent:*

(1) (X, f) is feebly open;
(2) $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ is feebly open;
(3) $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ is feebly open;
(4) $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is feebly open.

Proof. By [7, Theorem 10.1], we deduce that (1), (2) and (3) are equivalent. Now, by theorem [7, Theorem 3.3], it follows that (2) and (4) are equivalent. □

Statement (1) in corollary 4.4 is a consequence of diagram 3 and theorem 4.3. Also, statement (2) in corollary 4.4 is a direct consequence of diagram 3.

Corollary 4.4. *Let (X, f) a dynamical system and n and m be integers such that $n > m \geq 1$. Then the following propositions hold:*

(1) If (X, f) is irreducible, then $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is feebly open.
(2) If $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is irreducible, then $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is feebly open.

Theorem 4.5. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$, where f is a surjective map. Consider the following statements:*

(1) (X, f) is turbulent;
(2) $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ is turbulent;
(3) $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ is turbulent;
(4) $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is turbulent.

Then (1) implies (2), (3) and (4).

Proof. By [8, Theorem 5.6], we have that (1) implies (2) and (3).

Now, suppose that (X, f) is turbulent. We see that $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is turbulent. Let K and C be nondegenerate compact subsets of X such that $K \cap C$ has at most one point and $K \cup C \subseteq f(K) \cap f(C)$. Observe that $\langle K \rangle_n$ and $\langle C \rangle_n$ are nondegenerate compact subsets of $\mathcal{F}_n(X)$. Let $\Lambda = q_m(\langle K \rangle_n)$ and $\Gamma = q_m(\langle C \rangle_n)$. This implies that Λ and Γ are nondegenerate compact subsets of $\mathcal{SF}_m^n(X)$. Next, we see that $\Lambda \cap \Gamma$ has at most one point. We have two cases:

Case (1): $K \cap C = \varnothing$. In this case, it follows that $\langle K \rangle_n \cap \langle C \rangle_n = \varnothing$. Moreover, since $\mathcal{F}_m(K) \subseteq \langle K \rangle_n$ and $\mathcal{F}_m(C) \subseteq \langle C \rangle_n$, we see that $F_X^m \in \Lambda \cap \Gamma$.

Case (2): $K \cap C = \{a\}$. In this case, we have that $\langle K \rangle_n \cap \langle C \rangle_n = \{\{a\}\}$. Thus, $F_X^m \in \Lambda \cap \Gamma$. Now, we suppose that $\chi \in (\Lambda \cap \Gamma) \setminus \{F_X^m\}$. Then, there exist $A \in \langle K \rangle_n \setminus \mathcal{F}_m(X)$ and $B \in \langle C \rangle_n \setminus \mathcal{F}_m(X)$ such that $q_m(A) = \chi = q_m(B)$. Using remark 2.2, we obtain that $A = B$. Hence, $A \subseteq K \cap C$. Thus, $K \cap C$ has at least two elements, which is a contradiction. Therefore, $\Lambda \cap \Gamma$ has at most one point.

We prove that $\Lambda \cup \Gamma \subseteq \mathcal{SF}_m^n(f)(\Lambda) \cap \mathcal{SF}_m^n(f)(\Gamma)$. For this end, we consider $\chi \in \Lambda \cup \Gamma$. It follows that, there exists $A \in \langle K \rangle_n \cup \langle C \rangle_n$ such that $q_m(A) = \chi$. This implies that $A \subseteq f(K) \cap f(C)$. Hence, $A \in \langle f(K) \cap f(C) \rangle_n$. In consequence, $q_m(A) \in q_m(\langle f(K) \rangle_n) \cap q_m(\langle f(C) \rangle_n)$. Since $q_m(A) = \chi$, we have that $\chi \in q_m(\mathcal{F}_n(f)(\langle K \rangle_n)) \cap q_m(\mathcal{F}_n(f)(\langle C \rangle_n))$. By part (b) from proposition 3.1, we obtain that $\chi \in \mathcal{SF}_m^n(f)(q_m(\langle K \rangle_n)) \cap \mathcal{SF}_m^n(f)(q_m(\langle C \rangle_n))$. Thus, $\chi \in \mathcal{SF}_m^n(f)(\Lambda) \cap \mathcal{SF}_m^n(f)(\Gamma)$. Then, $\Lambda \cup \Gamma \subseteq \mathcal{SF}_m^n(f)(\Lambda) \cap \mathcal{SF}_m^n(f)(\Gamma)$. Therefore, $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is turbulent. □

Finally, we have the following questions (compare with [8, Questions 5.7]).

Questions 4.6. *Let (X, f) be a dynamical system and let $n, m \in \mathbb{N}$ be such that $n > m \geq 1$.*

(i) If $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ is turbulent, then is $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ turbulent?
(ii) If $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ is turbulent, then is $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ turbulent?
(iii) If $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is turbulent, then is (X, f) turbulent?
(iv) If $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is turbulent, then is $(\mathcal{F}_n(X), \mathcal{F}_n(f))$ turbulent?
(v) If $(\mathcal{SF}_m^n(X), \mathcal{SF}_m^n(f))$ is turbulent, then is $(\mathcal{SF}_n(X), \mathcal{SF}_n(f))$ turbulent?

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