APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 22, no. 2 (2021), 355-366
doi:10.4995/agt.2021.14868

Further aspects of $\mathcal{I}^{\mathcal{K}}$-convergence in topological spaces

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Communicated by D. Georgiou

Abstract

In this paper, we obtain some results on the relationships between different ideal convergence modes namely, $\mathcal{I}^{\mathcal{K}}$, $\mathcal{I}^{\mathcal{K}^*}$, $\mathcal{I}$, $\mathcal{K}$, $\mathcal{I}\cup\mathcal{K}$ and $(\mathcal{I}\cup\mathcal{K})^*$. We introduce a topological space namely $\mathcal{I}^{\mathcal{K}}$-sequential space and show that the class of $\mathcal{I}^{\mathcal{K}}$-sequential spaces contain the sequential spaces. Further $\mathcal{I}^{\mathcal{K}}$-notions of cluster points and limit points of a function are also introduced here. For a given sequence in a topological space X, we characterize the set of $\mathcal{I}^{\mathcal{K}}$-cluster points of the sequence as closed subsets of X.

2010 MSC: *54A20; 40A05; 40A35.*

Keywords: *$\mathcal{I}$-convergence; $\mathcal{I}^{\mathcal{K}}$-convergence; $\mathcal{I}^{\mathcal{K}^*}$-convergence; $\mathcal{I}^{\mathcal{K}}$-sequential space; $\mathcal{I}^{\mathcal{K}}$-cluster point.*

1. Introduction

For basic general topological terminologies and results we refer to [5]. The ideal convergence of a sequence of real numbers was introduced by Kostyrko et al. [11], as a natural generalization of existing convergence notions such as usual convergence [5], statistical convergence [4]. It was further introduced in arbitrary topological spaces accordingly for sequences [3] and nets [2] by Das et al. The main goal of this article is to study $\mathcal{I}^{\mathcal{K}}$-convergence which arose as a generalization of a type of ideal convergence. In this continuation we begin with a prior mentioning of ideals and ideal convergence in topological spaces.

Received 01 January 2020 – Accepted 26 March 2021

An ideal $\mathcal{I}$ on a arbitrary set $\mathcal{S}$ is a family $\mathcal{I} \subset 2^S$ (the power set of S) that is closed under finite unions and taking subsets. Fin and $\mathcal{I}_0$ are two basic ideals on ω, the set of all natural numbers, defined as Fin:= collection of all finite subsets of ω and $\mathcal{I}_0$:= subsets of ω with density 0, we say $A(\subset \omega) \in \mathcal{I}_0$ if and only if $\lim sup_{n\to\infty} \frac{|A\cap\{1,2,...,n\}|}{n} = 0$. For an ideal $\mathcal{I}$ in $P(\omega)$, we have two additional subsets of $P(\omega)$ namely $\mathcal{I}^\star$ and $\mathcal{I}^+$, where $\mathcal{I}^\star := \{A \subset \omega : A^c \in \mathcal{I}\}$, the filter dual of $\mathcal{I}$ and $\mathcal{I}^+$:= collection of all subsets not in $\mathcal{I}$. Clearly, $\mathcal{I}^\star \subseteq \mathcal{I}^+$. A sequence $x = \{x_n\}_{n\in\omega}$ is said to be $\mathcal{I}$-convergent [3] to ξ, denoted by $x_n \to_{\mathcal{I}} \xi$, if $\{n : x_n \notin U\} \in \mathcal{I}$, for all neighborhood U of ξ. A sequence $x = \{x_n\}_{n\in\omega}$ of elements of X is said to be $\mathcal{I}^\star$-convergent to ξ if there exists a set $M := \{m_1 < m_2 < ... < m_k < ...\} \in \mathcal{I}^\star$ such that $\lim_{k\to\infty} x_{m_k} = \xi$. Lahiri and Das [3] found an equivalence between $\mathcal{I}$ and $\mathcal{I}^\star$-convergences under certain assumptions.

In 2011, Macaj and Sleziak [6] introduced the $\mathcal{I}^{\mathcal{K}}$-convergence of function in a topological space, which was derived from $\mathcal{I}^*$-convergence [3] by simply replacing Fin by an arbitrary ideal $\mathcal{K}$. Interestingly, $\mathcal{I}^{\mathcal{K}}$-convergence arose as an independent mode of convergence. Comparisons of $\mathcal{I}^{\mathcal{K}}$-convergence with $\mathcal{I}$-convergence [11] can be found in [1, 6, 8]. A few articles for example [9, 7] contributed to the study of $\mathcal{I}^{\mathcal{K}}$-convergence of sequence of functions. Some of the definitions and results of [3, 6] that are used in subsequent sections are listed below. Here X is a topological space and S is a set where ideals are defined.

We say that a function $f : S \to X$ is $\mathcal{I}^{\mathcal{M}}$-convergent to a point $x \in X$ if $\exists M \in \mathcal{I}^*$ such that the function $g : S \to X$ given by

$$g(s) = \begin{cases} f(s), & s \in M \\ x, & s \notin M \end{cases}$$

is $\mathcal{M}$-convergent to x, where $\mathcal{M}$ is a convergence mode via ideal.

If $\mathcal{M} = \mathcal{K}^*$, then $f : S \to X$ is said to be $\mathcal{I}^{\mathcal{K}^*}$-convergent [6] to a point $x \in X$. Also, if $\mathcal{M} = \mathcal{K}$, then $f : S \to X$ is said to be $\mathcal{I}^{\mathcal{K}}$-convergent [6] to a point $x \in X$. In particular, if X is a discrete space, our immediate observation is that only the $\mathcal{I}$-constant functions are $\mathcal{I}$-convergent, for a given ideal $\mathcal{I}$, $f : S \to X$ is an $\mathcal{I}$-constant function if it attains a constant value except for a set in $\mathcal{I}$. It follows that $\mathcal{I}$ and $\mathcal{I}^*$ convergence coincide for X. Thus, $\mathcal{I}^{\mathcal{K}}$ and $\mathcal{I}^{\mathcal{K}^*}$-convergence modes also coincide on discrete spaces.

Lemma 1.1 ([6, Lemma 2.1])**.** *If $\mathcal{I}$ and $\mathcal{K}$ are two ideals on a set S and $f : S \to X$ is a function such that $\mathcal{K} - \lim f = x$, then $\mathcal{I}^{\mathcal{K}} - \lim f = x$.*

Remark 1.2. We say two ideals $\mathcal{I}$ and $\mathcal{K}$ satisfy ideality condition if $\mathcal{I} \cup \mathcal{K}$ is an proper ideal [10]. Again, $\mathcal{I}$ and $\mathcal{K}$ satisfy ideality condition if and only if $S \neq I \cup K$, for all $I \in \mathcal{I}$, $K \in \mathcal{K}$.

The main results of this article are divided into 3 sections. Section 2 is devoted to a comparative study of different convergence modes for example $\mathcal{I}^{\mathcal{K}}$, $\mathcal{I}^{\mathcal{K}^*}$, $\mathcal{I}$, $\mathcal{K}$, $\mathcal{I} \cup \mathcal{K}$, $(\mathcal{I} \cup \mathcal{K})^*$ etc. We justify the existence of an ideal $\mathcal{J}$, such that the behavior of $\mathcal{I}^{\mathcal{K}}$ and $\mathcal{J}$-convergence coincides in Hausdorff spaces. Then in section 3, we introduce $\mathcal{I}^{\mathcal{K}}$-sequential space and study its properties. In Section 4 we basically define $\mathcal{I}^{\mathcal{K}}$-cluster point and $\mathcal{I}^{\mathcal{K}}$-limit point of a function in a topological space. Here we observe that the ideality condition of $\mathcal{I}$ and $\mathcal{K}$ in $\mathcal{I}^{\mathcal{K}}$-convergence allows to get some effective conclusions. Moreover, we characterize the set of $\mathcal{I}^{\mathcal{K}}$-cluster points of a function as closed sets.

Throughout this paper we focus on the proper ideals [10] containing Fin ($S \notin \mathcal{I}$).

2. $\mathcal{I}^{\mathcal{K}}$-convergence and several comparisons

In this section, we study some more relations among different convergence modes $\mathcal{I}^{\mathcal{K}}$, $\mathcal{I}^{\mathcal{K}^*}$, $\mathcal{I}\cup\mathcal{K}$, $(\mathcal{I}\cup\mathcal{K})^*$ etc. We mainly focus on $\mathcal{I}^{\mathcal{K}}$-convergence where $\mathcal{I}\cup\mathcal{K}$ forms an ideal.

Proposition 2.1. *Let X be a topological space and $f : S \to X$ be a function. Let $\mathcal{I}, \mathcal{K}$ be two ideals on S such that $\mathcal{I}\cup\mathcal{K}$ is an ideal. Then*

(i) $\mathcal{I}^{\mathcal{K}^*} - \lim f = x$ *if and only if* $(\mathcal{I}\cup\mathcal{K})^* - \lim f = x$.

(ii) $\mathcal{I}^{\mathcal{K}} - \lim f = x$ *implies* $\mathcal{I}\cup\mathcal{K} - \lim f = x$.

Proof. (i) Let $f : S \to X$ be $\mathcal{I}^{\mathcal{K}^*}$-convergent to x. So, there exists a set $M \in \mathcal{I}^*$ for which the function $g : S \to X$ such that

$$g(s) = \begin{cases} f(s), & s \in M \\ x, & s \notin M \end{cases}$$

is $\mathcal{K}^*$-convergent to x. So, there further exists a set $N \in \mathcal{K}^*$ for which we can consider the function $h : S \to X$ such that

$$h(s) = \begin{cases} f(s), & s \in M,\ s \in N \\ x, & s \notin M \text{ or } s \notin N \end{cases}$$

is Fin-convergent to x. Now, Let $K = N^{\complement} \in \mathcal{K}$, $I = M^{\complement} \in \mathcal{I}$ (say). Then

$$h(s) = \begin{cases} f(s), & s \in (I\cup K)^{\complement} \\ x, & s \notin (I\cup K)^{\complement}. \end{cases}$$

In essence, we can conclude f is $(\mathcal{I}\cup\mathcal{K})^*$-convergent to x.

Conversely, the function $f : S \to X$ is $(\mathcal{I}\cup\mathcal{K})^*$-convergent to x. So, there exists a set $P = (I\cup K)^{\complement} \in (\mathcal{I}\cup\mathcal{K})^*$ for which the function $h : S \to X$ such that

$$h(s) = \begin{cases} f(s), & s \in P \\ x, & s \notin P \end{cases}$$

$$h(s) = \begin{cases} f(s), & s \in (I\cup K)^{\complement} \\ x, & s \notin (I\cup K)^{\complement} \end{cases}$$

is Fin-convergent to x. Lets consider the function $g : S \to X$ defined as

$$g(s) = \begin{cases} f(s), & s \in I^{\complement} \\ x, & s \notin I^{\complement} \end{cases}$$

for which the function $h : S \to X$ such that

$$h(s) = \begin{cases} f(s), & s \in I^{\complement},\ s \in K^{\complement} \\ x, & s \notin (I \cup K)^{\complement} \end{cases}$$

is Fin-convergent to x.
Consequently, f is $\mathcal{I}^{\mathcal{K}^*}$-convergent to x.

(ii) Let $f : S \to X$ be $\mathcal{I}^{\mathcal{K}}$-convergent to x. So, there exists a set $M \in \mathcal{I}^*$ for which the function $g : S \to X$ such that

$$g(s) = \begin{cases} f(s), & s \in M \\ x, & s \notin M \end{cases}$$

is $\mathcal{K}$-convergent to x. Then for each $\mathcal{U}_x$, neighborhood of x, we have $\{s : g(s) \notin \mathcal{U}_x\} \in \mathcal{K}$. Accordingly, the set given by $\{s : f(s) \notin \mathcal{U}_x, s \in M\} \in \mathcal{K}$. Further $\{s : f(s) \notin \mathcal{U}_x\} \subseteq \{s : f(s) \notin \mathcal{U}_x, s \in M\} \cup \{s : s \notin M\}$. Hence, $\{s : f(s) \notin \mathcal{U}_x\} \in \mathcal{I} \cup \mathcal{K}$.

□

Following are immediate corollaries of the above proposition provided $\mathcal{I} \cup \mathcal{K}$ is an ideal.

Corollary 2.2. *$\mathcal{I}^{\mathcal{K}^*}$-convergence implies $\mathcal{I} -$ convergence.*

Corollary 2.3. *$\mathcal{I}^{\mathcal{K}^*}$-convergence implies $\mathcal{K} -$ convergence.*

Following results in [1] are corollaries of the above proposition.

Corollary 2.4. *$\mathcal{I}^{\mathcal{K}}$-convergence implies $\mathcal{I} -$ convergence provided $\mathcal{K} \subseteq \mathcal{I}$.*

Corollary 2.5. *$\mathcal{I}^{\mathcal{K}}$-convergence implies $\mathcal{K} -$ convergence provided $\mathcal{I} \subseteq \mathcal{K}$.*

Following diagram shows the connections between different convergence modes.

$$\mathcal{I} \cup \mathcal{K} \leftarrow \mathcal{I}^{\mathcal{K}} \leftarrow \mathcal{I}^* \to (\mathcal{I} \cup \mathcal{K})^* \equiv \mathcal{I}^{\mathcal{K}^*} \to \mathcal{I}^{\mathcal{K}^{\mathcal{J}}}$$

In this segment we are interested to find whether there exists an ideal $\mathcal{J}$ such that the behavior of $\mathcal{I}^{\mathcal{K}}$ and $\mathcal{J}$-convergence coincides. Recalling that a filter-base is a non empty collection closed under finite intersection, we have the following result for a given function f in X by taking an ideal-base to be complement of a filter-base.

Lemma 2.6. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals on S satisfying ideality condition. $f : S \to X$ be a function on a topological space X. If $\mathcal{J}$ = ideal generated by $(\mathcal{K} \cup J)$, for any $J \in \mathcal{I}$. Then f is $\mathcal{J}$-convergent to $x \implies f$ is $\mathcal{I}^{\mathcal{K}}$-convergent to x.*

Proof. Let f be $\mathcal{J}$-convergent to x, where $\mathcal{J}=$ ideal generated by the ideal base $(\mathcal{K} \cup J)$, for any $J \in \mathcal{I}$. Now for $J = M^c$, consider the function $g : S \to X$ defined as

$$g(s) = \begin{cases} f(s), & s \in M \\ x, & s \notin M. \end{cases}$$

Then, for any open set V containing x, we have

$$\begin{aligned} \{s \in S : g(s) \notin V\} &= \{s \in S : f(s) \notin V, s \in M\} \\ &\subseteq \{s \in S : f(s) \notin V\} \setminus \{s \in S : s \notin M\}. \end{aligned}$$

Since, f be $\mathcal{J}$-convergent to x, that implies $\{s \in S : f(s) \notin V\} \in \mathcal{J}$. Therefore, there exists $K \in \mathcal{K}$ such that $\{s \in S : f(s) \notin V\} \setminus J \subseteq (K \cup J) \setminus J \in \mathcal{K}$. Subsequently, g is $\mathcal{K} -$ *convergent* to x. Hence, f is $\mathcal{I}^{\mathcal{K}}$-convergent to x. □

Theorem 2.7 ([1, Theorem 3.1]). *In a Hausdorff space X, each function $f : S \to X$ possess a unique $\mathcal{I}^{\mathcal{K}}$-limit provided $\mathcal{I} \cup \mathcal{K}$ is an ideal.*

Theorem 2.8. *Let X be a Hausdorff Space. Let $f : S \to X$ be $\mathcal{I}^{\mathcal{K}}$-convergent to x. Then $\exists$ an ideal $\mathcal{J}$ such that $x \in X$ is an $\mathcal{I}^{\mathcal{K}}$-limit of the function f if and only if x is also a $\mathcal{J}$-limit of f provided $\mathcal{I} \cup \mathcal{K}$ is an ideal.*

Proof. Let $f : S \to X$ is $\mathcal{I}^{\mathcal{K}}$-convergent to x. So, there exists a set $M \in \mathcal{I}^*$ such that $g : S \to X$ with

$$g(s) = \begin{cases} f(s), & s \in M \\ x, & s \notin M \end{cases}$$

is $\mathcal{K}$-convergent to x. Consequently, for each neigfhborhood $\mathcal{U}_x$ of x. We have

$$\{s \in S : g(s) \notin \mathcal{U}_x\} \in \mathcal{K}.$$

$$\implies \{s \in S : f(s) \notin \mathcal{U}_x, s \in M\} \in \mathcal{K}.$$

Now, let $J = M^c$ and $(\mathcal{K} \cup J)$ is an ideal base provided $(\mathcal{I} \cup \mathcal{K})$ is an ideal. Now we consider $\mathcal{J}$, the ideal generated by $(\mathcal{K} \cup J)$. Then

$$\{s \in S : f(s) \notin \mathcal{U}_x\} \subseteq \{s \in S : f(s) \notin \mathcal{U}_x, s \in M\} \cup \{s \in S : s \notin M\}.$$

Therefore, $\{s \in S : f(s) \notin \mathcal{U}_x\} \in (\mathcal{K} \cup J)$.
Converse part of the proof is immediate by lemma 2.6. □

The following arrow diagram exhibit the equivalence shown in theorem 2.8.

$$\mathcal{K} \xrightarrow{for\ any\ J \in \mathcal{I}} \mathcal{J} \to \mathcal{I}^{\mathcal{K}} \xrightarrow{fixed\ J \in \mathcal{I}} \mathcal{J} \to \mathcal{I} \cup \mathcal{K}$$

Comprehensively, we may ask the following question.

Problem. Whether there exists an ideal $\mathcal{J}$ for $\mathcal{I}^{\mathcal{K}}$-convergence in a given non-Hausdorff topological space X such that $\mathcal{I}^{\mathcal{K}} \equiv \mathcal{J}$-convergence?

3. $\mathcal{I}^{\mathcal{K}}$-sequential space

Recently, $\mathcal{I}$-sequential space were defined by S.K. Pal [12] for an ideal $\mathcal{I}$ on ω. An equivalent definition was suggested by Zhou et al. [13] and further obtain that class of $\mathcal{I}$-sequential spaces includes sequential spaces [5].

First, recall the notion of $\mathcal{I}$-sequential spaces. Let X be a topological space and $O \subseteq X$ is $\mathcal{I}$-open if no sequence in $X \setminus O$ has an $\mathcal{I}$-limit in O. Equivalently, for each sequence $\{x_n : n \in \omega\} \subseteq X \setminus O$ with $\mathcal{I}-\lim x_n = x \in X$, then $x \in X \setminus O$. Now X is said to be an $\mathcal{I}$-sequential space if and only if each $\mathcal{I}$-open subset of X is open.

Here we introduce a topological space namely $\mathcal{I}^{\mathcal{K}}$-sequential space for given ideals $\mathcal{I}$ and $\mathcal{K}$ on ω.

Definition 3.1. Let X be a topological space and $O, A \subseteq X$. Then

(1) O is said to be $\mathcal{I}^{\mathcal{K}}$-open if no sequence in $X \setminus O$ has an $\mathcal{I}^{\mathcal{K}}$-limit in O. Otherwise, for each sequence $\{x_n : n \in \omega\} \subseteq X \setminus O$ with $\mathcal{I}^{\mathcal{K}} - \lim x_n = x \in X$, then $x \in X \setminus O$.

(2) A subset $F \subseteq X$ is said to be $\mathcal{I}^{\mathcal{K}}$-closed if $X \setminus A$ is $\mathcal{I}^{\mathcal{K}}$-open in X.

Remark 3.2. The following are obvious for a topological space X and ideals $\mathcal{I}$ and $\mathcal{K}$ on ω.

1. Each open(closed) set of X is $\mathcal{I}^{\mathcal{K}}$-open(closed).
2. If A and B are $\mathcal{I}^{\mathcal{K}}$-open (closed), then $A \cup B$ is $\mathcal{I}^{\mathcal{K}}$-open (closed).
3. A topological space X is said to be an $\mathcal{I}^{\mathcal{K}}$-sequential space if and only if each $\mathcal{I}^{\mathcal{K}}$-open set of X is open.

for $\mathcal{I} = \mathcal{K}$, each $\mathcal{I}^{\mathcal{K}}$-sequential space coincides with a $\mathcal{I}$-sequential space.

Lemma 3.3. *Let $\mathcal{M}_1, \mathcal{M}_2$ be two convergence modes in a topological space X such that $\mathcal{M}_1$-convergence implies $\mathcal{M}_2$-convergence. Then $O \subseteq X$ is $\mathcal{M}_2$-open implies that O is $\mathcal{M}_1$-open.*

Proof. Let O be not $\mathcal{M}_1$-open in X, then $\exists\{x_n\}$ in $(X \setminus O)$ which is $\mathcal{M}_1$-convergent in X. So, $\{x_n\}$ is $(X \setminus O)$ is $\mathcal{M}_2$-convergent in X and hence O is not $\mathcal{M}_2$-open. □

Corollary 3.4. *Let $\mathcal{M}_1, \mathcal{M}_2$ be two convergence modes in X such that $\mathcal{M}_1$-convergence implies $\mathcal{M}_2$-convergence in X. Then X is a $\mathcal{M}_1$-sequential space implies that X is $\mathcal{M}_2$-sequential space.*

The following is an example of a topological space which is not $\mathcal{I}^{\mathcal{K}}$-sequential space.

Example 3.5. Let $S = [a, b]$ be a closed interval with the countable complement topology τ_{cc}, where a, $b \in \mathbb{R}$. Let A be any subset of S and x_n be a sequence in A, $\mathcal{I}^{\mathcal{K}}$-convergent to x, provided $\mathcal{I}, \mathcal{K}$ and $\mathcal{I} \cup \mathcal{K}$ is an ideal i.e, $\mathcal{I} \cup \mathcal{K} - \lim x_n = x$. Consider the neighborhood U of x, be the complement of the set $\{x_n : x_n \neq x\}$ in S. Then $x_n = x$ for all n except for a set in the ideal $\mathcal{I} \cup \mathcal{K}$. Therefore, a sequence in any set A can only $\mathcal{I} \cup \mathcal{K}$-convergent to an element of A i.e C is $\mathcal{I}\cup\mathcal{K}$-open. Thus every subset of C is $\mathcal{I}^{\mathcal{K}}$-sequentially open. But not every subset of S is open. Hence $([a,b],\tau_{cc})$ is not $\mathcal{I}^{\mathcal{K}}$-sequential.

Proposition 3.6. *Let X be a topological space and $\mathcal{I}_1,\mathcal{I}_2,\mathcal{K}_1,\mathcal{K}_2$ be ideals on S. Then the following implications hold:*

1. *For $\mathcal{K}_1\subseteq\mathcal{K}_2$ whenever $U\subseteq X$ is $\mathcal{I}^{\mathcal{K}_2}$-open, then it is $\mathcal{I}^{\mathcal{K}_1}$-open.*
2. *For $\mathcal{I}_1\subseteq\mathcal{I}_2$ whenever $U\subseteq X$ is $\mathcal{I}_2^{\mathcal{K}}$-open, then it is $\mathcal{I}_1^{\mathcal{K}}$-open.*

Proof. Let $f:S\to X$ be a function. Then by Proposition 3.6 in [6],

$$\mathcal{I}_1^{\mathcal{K}}-\lim f=x \implies \mathcal{I}_2^{\mathcal{K}}-\lim f=x.$$
$$\mathcal{I}^{\mathcal{K}_1}-\lim f=x \implies \mathcal{I}^{\mathcal{K}_2}-\lim f=x.$$

By lemma 3.3 we have the required results correspondingly. □

Corollary 3.7. *For X be topological space and $\mathcal{I}_1,\mathcal{I}_2,\mathcal{K}_1,\mathcal{K}_2$ be ideals on ω where $\mathcal{I}_1\subseteq\mathcal{I}_2$, $\mathcal{K}_1\subseteq\mathcal{K}_2$. Then the following observations are valid:*

1. *If X is $\mathcal{I}_1^{\mathcal{K}}$-sequential, then it is $\mathcal{I}_2^{\mathcal{K}}$-sequential.*
2. *If X is $\mathcal{I}^{\mathcal{K}_1}$-sequential, then it is $\mathcal{I}^{\mathcal{K}_2}$-sequential.*

Theorem 3.8. *In a topological space X, if O is open then O is $\mathcal{I}^{\mathcal{K}}$-open.*

Proof. Let O be open and $\{x_n\}$ be a sequence in $X\setminus O$. Let $y\in O$. Then there is a neighborhood U of y which contained in O. Hence U can not contain any term of $\{x_n\}$. So y is not an $\mathcal{I}^{\mathcal{K}}$-limit of the sequence and O is $\mathcal{I}^{\mathcal{K}}$-open. □

Theorem 3.9. *In a metric space X, the notions of open and $\mathcal{I}^{\mathcal{K}}$-open coincide.*

Proof. Forward implication is obvious from Theorem 3.8.
Conversely, Let O be not open i.e., $\exists y\in O$ such that for all neighborhood of y intersect $(X\setminus O)$. Let $x_n\in(X\setminus O)\cap B(y,\frac{1}{n+1})$. Then $x_n\to y$. Hence x_n is $\mathcal{I}^{\mathcal{K}}$-convergent to y. Thus O is not $\mathcal{I}^{\mathcal{K}}$-open. □

Theorem 3.10. *Every first countable space is $\mathcal{I}^{\mathcal{K}}$-sequential space.*

Proof. We need to prove the reverse implication.
If $A\subset X$ be not open. Then $\exists y\in A$ such that every neighborhood of y intersects $X\setminus A$. Let $\{U_n:n\in\omega\}$ be a decreasing countable basis at y (say). Consider $x_n\in(X\setminus A)\cap U_n$. Then for each neighborhood V of y, $\exists n\in\omega$ with $U_n\subset V$. So, $x_m\in V,\forall m\geq n$ i.e $x_n\to y$. Hence $\mathcal{K}-\lim x_n=y$. Therefore, A is not $\mathcal{I}^{\mathcal{K}}$-open. □

The following theorem about continuous mapping was also proved by Banerjee et al. [1]. However, we have given here an alternative approach to prove.

Theorem 3.11. *Every continuous function preserves $\mathcal{I}^{\mathcal{K}}$-convergence.*

Proof. Let X and Y be two topological spaces and $c:X\to Y$ be a continuous function. Let $f:S\to X$ be $\mathcal{I}^{\mathcal{K}}$-convergent. So $\exists M\in\mathcal{I}^*$ such that $g:S\to X$ given by

$$g(s)= \begin{cases} f(s), & s \in M \\ x, & s \notin M \end{cases}$$

is $\mathcal{K}$-convergent to x. Now the function $c \circ f : S \to Y$, the image function on Y is $\mathcal{K}$-convergent to x by Theorem 3 in [3]. Hence $c \circ f$ is $\mathcal{I}^{\mathcal{K}}$-convergent. □

We now recall the definition of a quotient space. Let $(X, \sim)$ be a topological space with an equivalence relation $\sim$ on X. Consider the projection mapping $\prod : X \to X/\sim$ (the set of equivalence classes) and taking $A \subset X/\sim$ to be open if and only if $\prod^{-1}(A)$ is open in X, we have the quotient space $X/\sim$ induced by $\sim$ on X.

Theorem 3.12. *Every quotient space of an $\mathcal{I}^{\mathcal{K}}$-sequential space X is $\mathcal{I}^{\mathcal{K}}$-sequential.*

Proof. Let $A \subset X/\sim$ be not open. Let $X/\sim$ is a quotient space with an equivalence relation $\sim$, $\prod^{-1}(A)$ is not open in X i.e., $\exists$ a sequence $\{x_n\}$ in $X \setminus \prod^{-1}(A)$ which is $\mathcal{I}^{\mathcal{K}}$-convergent to $y \in \prod^{-1}(A)$. Also $\prod$ is continuous, hence preserves $\mathcal{I}^{\mathcal{K}}$-convergence by Theorem 3.11. Therefore, $\prod(x_n) \in (X/\sim) \setminus A$ with $\mathcal{I}^{\mathcal{K}}$-limit $\prod(y) \in A$. So, A is not $\mathcal{I}^{\mathcal{K}}$-open i.e., $X/\sim$ is $\mathcal{I}^{\mathcal{K}}$-sequential. □

Following result is immediate via Proposition 3.4.

Theorem 3.13. *Every sequential space is an $\mathcal{I}^{\mathcal{K}}$-sequential space.*

Recall that a topological space X is said to be of countable tightness, if for $A \subseteq X$ and $x \in \bar{A}$, then $x \in \bar{C}$ for some countable subset $C \subseteq A$. Every sequential and $\mathcal{I}$-sequential space is of countable tightness [13].

Proposition 3.14. *Every $\mathcal{I}^{\mathcal{K}}$-sequential space X is of countable tightness.*

Proof. Let X be an $\mathcal{I}^{\mathcal{K}}$-sequential space and $A \subseteq X$. Consider $[A]_\omega = \bigcup\{\overline{B} : B$ is a countable subset of $A\}$. Clearly, $A \subseteq [A]_\omega \subseteq \overline{A}$. We claim that, $[A]_\omega$ is $\mathcal{I}^{\mathcal{K}}$-closed in X. Consider $\{x_n\}$ be a sequence in $[A]_\omega$, $\mathcal{I}^{\mathcal{K}}$-convergent to $x \in X$. Since $x_n \in [A]_\omega$, then we can find a countable subset B of A such that $x_n \in \overline{B}$ for all $n \in \omega$. Since X be an $\mathcal{I}^{\mathcal{K}}$-sequential space, so $\overline{B}$ is $\mathcal{I}^{\mathcal{K}}$-closed, thus $x \in \overline{B} \subseteq [A]_\omega$, and further $[A]_\omega$ is $\mathcal{I}^{\mathcal{K}}$-closed in X.

Now, let X be an $\mathcal{I}^{\mathcal{K}}$-sequential space and A be a subset of X. Since the set $[A]_\omega$ is closed in X, and $[A]_\omega \subseteq \overline{A} \subseteq \overline{[A]_\omega}$, thus $\overline{A} = [A]_\omega$. If $x \in \overline{A}$, then $x \in [A]_\omega$, and further, there exists a countable subset C of A such that $x \in \overline{C}$, i.e., X is of countable tightness. □

Now we show that every $\mathcal{I}^{\mathcal{K}}$-sequential space is hereditary with respect to $\mathcal{I}^{\mathcal{K}}$-open ($\mathcal{I}^{\mathcal{K}}$-closed) subspaces. First we have the following lemma.

Lemma 3.15 ([13, Lemma 2.4])**.** *Let $\mathcal{I}$ be an ideal on ω and x_n, y_n be two sequences in a topological space X such that $\{n \in \omega : x_n \neq y_n\} \in \mathcal{I}$. Then $\mathcal{I} - \lim x_n = x$ if and only if $\mathcal{I} - \lim y_n = x$.*

Theorem 3.16. *If X is an $\mathcal{I}^{\mathcal{K}}$-sequential space then every $\mathcal{I}^{\mathcal{K}}$-open ($\mathcal{I}^{\mathcal{K}}$-closed) subspaces of X is $\mathcal{I}^{\mathcal{K}}$-sequential.*

Proof. Let X be an $\mathcal{I}^{\mathcal{K}}$-sequential space. Suppose that Y is an $\mathcal{I}^{\mathcal{K}}$-open subset of X. Then Y is also open in X. We anticipate Y to be $\mathcal{I}^{\mathcal{K}}$-sequential space. Consider U to be $\mathcal{I}^{\mathcal{K}}$-open in Y. Here Y is open, so we claim that U is open in X. Since X is $\mathcal{I}^{\mathcal{K}}$-sequential space, we need to show that U is $\mathcal{I}^{\mathcal{K}}$-open in X. Contra-positively, take U be not $\mathcal{I}^{\mathcal{K}}$-open in X. Then, $\exists\{x_n\}$ in $X \setminus U$ such that $\mathcal{I}^{\mathcal{K}} - \lim x_n = x$ $(\in U.)$ i.e. $\exists M \in \mathcal{I}^*$ such that $x_{n_k} \to_{\mathcal{K}} x$, where $n_k \in M$ and $x_{n_k} \in X \setminus U$. Now $\{n_k : x_{n_k} \notin Y\} \in \mathcal{K}$. For a point $y \in Y \setminus U$ (assume), Now Consider a sequence $\{y_n\}$ such that $y_n = x_n$ for $n \in M$ and $y_n = y_{n_k}$ for $n \notin M$ where $\{y_{n_k}\}$ is defined as $y_{n_k} = x_{n_k}$ for $x_{n_k} \in Y$ and $y_{n_k} = y$ for $x_{n_k} \notin Y$. Then by Lemma 3.15, $\{y_{n_k}\}$ is $\mathcal{K}$-convergent to x. Hence $\{y_n\}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to x. So U is not $\mathcal{I}^{\mathcal{K}}$-open in Y. That is a contradiction to our assumption.

Let Y be an $\mathcal{I}^{\mathcal{K}}$-closed subset of X. Then Y is closed in X. For any $\mathcal{I}^{\mathcal{K}}$-closed subset F of Y, it is sufficient to show that F is closed in X. Since X is an $\mathcal{I}^{\mathcal{K}}$-sequential space, it is enough to show that F is $\mathcal{I}^{\mathcal{K}}$-closed in X. Therefore, let $\{x_n : n \in \omega\}$ be an arbitrary sequence in F with $\mathcal{I}^{\mathcal{K}} - \lim x_n = x$ in X. We claim that $x \in F$. Indeed, since Y is closed, we have $x \in Y$, and then it is also clear that $x \in F$ since F is an $\mathcal{I}^{\mathcal{K}}$-closed subset of Y. □

Proposition 3.17. *The disjoint topological sum of any family of $\mathcal{I}^{\mathcal{K}}$-sequential spaces is $\mathcal{I}^{\mathcal{K}}$-sequential.*

Proof. Let $(X_\alpha)_{\alpha\in\Delta}$ be a family of $\mathcal{I}^{\mathcal{K}}$-sequential space and $X = \oplus_{\alpha\in\Delta} X_\alpha$. We claim that X is $\mathcal{I}^{\mathcal{K}}$-sequential space. Let F be $\mathcal{I}^{\mathcal{K}}$-closed in X. For each $\alpha \in \Delta$, X_α is closed in X i.e., X_α is $\mathcal{I}^{\mathcal{K}}$-closed in X. Hence, $F \cap X_\alpha$ is $\mathcal{I}^{\mathcal{K}}$-closed in X by Remark 3.2. As $(F \cap X_\alpha) \subseteq X_\alpha$ i.e. $F \cap X_\alpha$ is closed in X_α. Now F is closed in $X \equiv X \setminus F$ is open in $X \equiv \cup_\alpha(X_\alpha \setminus F)$ is open in X if and only if $X_\alpha \setminus F$ is open in $X_\alpha \equiv F \cap X_\alpha$ is closed in X_α. Hence F is closed in X. □

4. $\mathcal{I}^{\mathcal{K}}$-cluster point and $\mathcal{I}^{\mathcal{K}}$-limit point

The notions $\mathcal{I}$-cluster point and $\mathcal{I}$-limit point in a topological space X were defined by Das et al. [3] and also characterized $C_x(\mathcal{I})$, the collection of all $\mathcal{I}$-cluster points of a given sequence $x = \{x_n\}$ in X, as closed subsets of X (Theorem 10, [3]). Here we define $\mathcal{I}^{\mathcal{K}}$-notions of cluster point and limit points for a function in X.

For $\mathcal{I}^*$-convergence, $\mathcal{I} \cup Fin$ is an ideal, thereupon $\mathcal{I}$ and Fin satisfy ideality condition. Moreover we assume ideality condition of $\mathcal{I}$ and $\mathcal{K}$ in $\mathcal{I}^{\mathcal{K}}$-convergence to investigate some results.

Definition 4.1. Let $f : S \to X$ be a function and $\mathcal{I}$, $\mathcal{K}$ be two ideals on S. Then $x \in X$ is called an $\mathcal{I}^{\mathcal{K}}$-cluster point of f if there exists $M \in \mathcal{I}^*$ such that the function $g : S \to X$ defined by

$$\text{g(s)=} \begin{cases} f(s), & s \in M \\ x, & s \notin M \end{cases}$$

has a $\mathcal{K}$-cluster point x, i.e., $\{s \in S : g(s) \in U_x\} \notin \mathcal{K}$.

Definition 4.2. Let $f : S \to X$ be a function and $\mathcal{I}$, $\mathcal{K}$ be two ideals on S. Then $x \in X$ is called an $\mathcal{I}^{\mathcal{K}}$-limit point of f if there exists $M \in \mathcal{I}^*$ such that for the function $g : S \to X$ defined by

$$\text{g(s)}= \begin{cases} f(s), & s \in M \\ x, & s \notin M \end{cases}$$

has a $\mathcal{K}$-limit point x.

For $\mathcal{I} = \mathcal{K}$, we know the convergence modes $\mathcal{I}^{\mathcal{K}} \equiv \mathcal{I} \equiv \mathcal{K}$. Hence definitions 4.1 and 4.2 generalizes the definitions of $\mathcal{I}$ or $\mathcal{K}$-(limit point and cluster point) correspondingly. Again, for nets in a topological space $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points coincide [2]. Therefore, $\mathcal{I}^{\mathcal{K}}$-cluster points and $\mathcal{I}^{\mathcal{K}}$-limit points of nets also coincide.

Following the notation in [11], we denote the collection of all $\mathcal{I}^{\mathcal{K}}$-limit points and $\mathcal{I}^{\mathcal{K}}$-cluster points of a function f in a topological space X by $L_f(\mathcal{I}^{\mathcal{K}})$ and $C_f(\mathcal{I}^{\mathcal{K}})$ respectively. We observe that $C_f(\mathcal{I}^{\mathcal{K}}) \subseteq C_f(\mathcal{K})$ and $L_f(\mathcal{I}^{\mathcal{K}}) \subseteq L_f(\mathcal{K})$. We also observe that $L_f(\mathcal{I}^*) = L(\mathcal{I}^*)$, where $L(\mathcal{I}^*)$ denote the collection of $\mathcal{I}^*$-limits of f.

Lemma 4.3. *If $\mathcal{I}$ and $\mathcal{K}$ be two ideal then $L_f(\mathcal{I}^{\mathcal{K}}) \subseteq C_f(\mathcal{I}^{\mathcal{K}})$.*

Proof. Since $L_f(\mathcal{K}) \subseteq C_f(\mathcal{K})$ for an ideal $\mathcal{K}$, hence the result is immediate. □

We have the following lemma provided the ideals $\mathcal{I}$ and $\mathcal{K}$ satisfy ideality condition.

Lemma 4.4. *$C_f(\mathcal{I} \cup \mathcal{K}) \subseteq C_f(\mathcal{I}^{\mathcal{K}})$.*

Proof. Let y be not a $\mathcal{I}^{\mathcal{K}}$-cluster point of $x = \{x_n\}_{n\in\omega}$. Then for all $M \in \mathcal{I}^*$ such that for the function $g : S \to X$ defined by

$$\text{g(s)}= \begin{cases} f(s), & s \in M \\ x, & s \notin M, \end{cases}$$

the set $\{s \in S : g(s) \in U_x\} \in \mathcal{K}$. Since $\{s : f(s) \in U_x\} \subseteq \{s : g(s) \in U_x\} \in \mathcal{K}$. i.e. $\{s : f(s) \in U_x\} \in \mathcal{I} \cup \mathcal{K}$. Hence y is not a $(\mathcal{I} \cup \mathcal{K})$-cluster point of x. □

Since above set inequalities signify the implication $\mathcal{K} \to \mathcal{I}^{\mathcal{K}} \to \mathcal{I} \cup \mathcal{K}$, We expect the following conclusion.

Conjecture 4.5. *$L_f(\mathcal{I} \cup \mathcal{K}) \subseteq L_f(\mathcal{I}^{\mathcal{K}})$.*

For sequential criteria in [11], we observe the following result.

Theorem 4.6. *Let $\mathcal{I}$, $\mathcal{K}$ be two ideals on ω and X be a topological space. Then*

(i) *For $x = \{x_n\}_{n\in\omega}$, a sequence in X; $C_x(\mathcal{I}^{\mathcal{K}})$ is a closed set.*
(ii) *If (X, τ) is closed hereditary separable and there exists a disjoint sequence of sets $\{P_n\}$ such that $P_n \subset \omega$, $P_n \notin \mathcal{I}, \mathcal{K}$ for all n, then for every non empty closed subset F of X, there exists a sequence x in X such that $F = C_x(\mathcal{I}^{\mathcal{K}})$ provided $\mathcal{I} \cup \mathcal{K}$ is an ideal.*

Proof. Consider the sequence $x = \{x_n\}$ in X and $\mathcal{I}$, $\mathcal{K}$ be the two ideals on ω.

(i) Let $y \in \overline{C_x(\mathcal{I}^{\mathcal{K}})}$; the derived set of $C_x(\mathcal{I}^{\mathcal{K}})$. Let U be an open set containing y. It is clear that $U \cap C_x(\mathcal{I}^{\mathcal{K}}) \neq \phi$. Let $p \in (U \cap C_x(\mathcal{I}^{\mathcal{K}}))$ i.e., $p \in U$ and $p \in C_x(\mathcal{I}^{\mathcal{K}})$. Now there exist a set $M \in \mathcal{I}^*$, such that $\{y_n\}_{n\in\omega}$ given by $y_n = x_n$ if $n \in M$ and p, otherwise; we have $\{n \in \omega : y_n \in U\} \notin \mathcal{K}$. Consider the sequence $\{z_n\}_{n\in\omega}$ given by $z_n = x_n$ if $n \in M$ and y, otherwise; then $\{n \in \omega : z_n \in U\} = \{n \in \omega : y_n \in U\} \notin \mathcal{K}$. Hence $y \in C_x(\mathcal{I}^{\mathcal{K}})$.

(ii) Being a closed subset of X, F is separable. Let $S = \{s_1, s_2, ...\} \subset F$ be a countable set such that $\overline{S} = F$. Consider $x_n = s_i$ for $n \in P_i$. Thus we have the subsequence $\{k_n\}$ of $\{n\}$ for which assume the sequence $x = \{x_{n_k}\}$. Let $y \in C_x(\mathcal{K})$ (taking $y \neq s_i$ otherwise if $y = s_i$ for some i, then y is eventually in F). We claim $C_x(\mathcal{K}) \subset F$. Let U be any open set containing y. Then $\{n : x_{n_k} \in U\} \notin \mathcal{K}$ and hence non empty i.e., $s_i \in U$ for some i. Therefore $F \cap U$ is non empty, So y is a limit point of F and closedness of F gives $y \in F$. Hence $C_x(\mathcal{K}) \subset F$. Further $C_x(\mathcal{I}^{\mathcal{K}}) \subseteq C_x(\mathcal{K}) \subset F$.
Conversely, for $a \in F$ and U be an open set containing a, then there exists $s_i \in S$ such that $s_i \in U$. Then $\{n : x_{n_k} \in U\} \supset P_i$ ($\notin \mathcal{K}$, $\mathcal{I}$). Thus $\{n : x_{n_k} \in U\} \notin (\mathcal{I} \cup \mathcal{K})$ i.e., $a \in C_x(\mathcal{I} \cup \mathcal{K})$. On the otherhand, by lemma 4.4, $C_f(\mathcal{I} \cup \mathcal{K}) \subseteq C_f(\mathcal{I}^{\mathcal{K}})$. So we get the reverse implication.

□

Remark 4.7. Theorem 4.6 generalizes Theorem 10 in [3], it follows by letting $\mathcal{I} = \mathcal{K}$ in the above theorem.

Acknowledgements. *The first author would like to thank the University Grants Comission (UGC) for awarding the junior research fellowship vide UGC-Ref. No.: 1115/(CSIR-UGC NET DEC. 2017), India.*

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