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Extended convergence for a fifth order novel scheme free from derivatives

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Abstract

The objective in this paper is the expansion of the utilization for a fifth convergence order scheme without derivatives for finding solutions of of Banach space valued equations. Conditions of the first order divided difference of the operator involved are only imposed. In this way the use of the scheme is expanded, since in earlier articles the derivatives until order four that do appear in the iterative method are required for setting convergence. Our technique also provides bounds on error distances as well as information about the location of the solutions not given in earlier works. Experiments with concrete problems complete this study.

KEYWORDS:
Derivative free scheme; convergence analysis; divided difference; Banach space.

1 | INTRODUCTION

A plethora of applications from Mathematics as well as scientific disciplines require finding a simple solution x_* of nonlinear equation

$$F(x) = 0, \tag{1}$$

where for $\mathbb{X}$ being a Banach space and $\Delta \subset \mathbb{X}$ standing for an open set, $F : \Delta \to \mathbb{X}$ denotes a continuous operator. To solve $F(x) = 0$, we study the local convergence of the following multi-step method defined for $\sigma = 0, 1, 2, \dots$ as

$$\begin{aligned} u_\sigma &= x_\sigma + \beta F(x_\sigma),\ \ v_\sigma = x_\sigma - \beta F(x_\sigma),\\ y_\sigma &= x_\sigma - A_\sigma^{-1} F(x_\sigma)\\ z_\sigma &= y_\sigma - \frac{9}{5} A_\sigma^{-1} F(y_\sigma)\\ h_\sigma &= y_\sigma - \frac{16}{5} A_\sigma^{-1} F(y_\sigma)\\ x_{\sigma+1} &= z_\sigma - \frac{1}{5} A_\sigma^{-1} F(h_\sigma) \end{aligned} \tag{2}$$

where $A_\sigma = [u_\sigma, v_\sigma; F],\ A : \Delta \times \Delta \to \ell(\mathbb{X}, \mathbb{X})$ and $\beta \in \mathbb{R}$. Here, $\ell(\mathbb{X}, \mathbb{X})$ denotes the space of bounded linear operators from $\mathbb{X}$ to $\mathbb{X}$. In the local convergence, we use information about the solution to determine a radius of convergence and estimates on $x_\sigma - x_*$. The conclusions were obtained for the special case when $\mathbb{X} = \mathbb{Y} = \mathbb{R}^i$. It is a fifth order scheme using up to the fourth order derivatives in the local convergence order[1]. But, it is important to note that the scheme (2) is derivative free,[2,3,4]. So, these hypotheses restrict the applicability of the methods,[5]. Let us consider a motivational example. We assume the following function F on $\mathbb{X} = \mathbb{R}$ and $\mathbb{D} = [-\frac{1}{2}, \frac{3}{2}]$ such as:

$$F(\kappa) = \begin{cases} \kappa^3 \ln \kappa^2 + \kappa^5 - \kappa^4, & \kappa \neq 0 \\ 0, & \kappa = 0 \end{cases}, \tag{3}$$

that leads to

$$F'(\kappa) = 3\kappa^2 \ln \kappa^2 + 5\kappa^4 - 4\kappa^3 + 2\kappa^2,$$
$$F''(\kappa) = 6\kappa \ln \kappa^2 + 20\kappa^3 - 12\kappa^2 + 10\kappa,$$
$$F'''(\kappa) = 6 \ln \kappa^2 + 60\kappa^2 - 12\kappa + 22.$$

Notice that $F'(\kappa)$, $F''(\kappa)$ are defined at $\kappa = 0$ but $F'''(\kappa)$ is not. Therefore, results requiring the existence of $F'''(\kappa)$ or higher cannot be applied for studying the convergence of (2). Moreover, no computable error bounds $\|x_\sigma - x_*\|$, or the lipschitz continuity of $F''(\kappa)$, or any information regarding the uniqueness of the solution are provided using Lipschitz-type functions. Furthermore, the convergence criteria can not be compared, since they are based on different hypotheses. The novelty of our paper lies in the fact that we address all these problems by using only the first derivative. Moreover, we rely on the computational order of convergence (COC) or approximated computational order of convergence ($ACOC$) to determine the convergence order not requiring derivatives of order higher than one (please see [6,7]). The new technique uses the same set of conditions for the three methods. Furthermore, it can also be used to extend the applicability of other methods along the same lines, [8].

2 | CONVERGENCE

It is convenient to define functions on the real line given $\beta \in \mathbb{R}, \alpha \geq 0, \delta \geq 0$ and $\gamma \geq 0$. Set $I = [0, \infty)$. Consider the existence of a continuous and increasing function $\Omega_0 : I \times I \to I$ such that

$$\Omega_0(\gamma\zeta, \delta\zeta) = 1, \tag{4}$$

has a smallest positive solution r, with $\Omega_0(0, 0) = 0$.

Let $\Omega : I_0 \times I_0 \to I$ be a continuous and increasing function with $\Omega(0, 0) = 0$, and $I_0 = [0, r)$. Define functions $\phi_m,\ \psi_m, m = 1, 2, 3, 4$ to be on the interval I_0 by

$$\phi_1(\zeta) = \frac{\Omega\Big((1+\gamma)\zeta, \delta\zeta\Big)}{1 - \Omega_0(\gamma\zeta, \delta\zeta)},$$
$$\phi_2(\zeta) = \frac{\Big[\Omega\Big(\big(\gamma + \phi_1(\zeta)\big)\zeta, \delta\zeta\Big) + \frac{4}{5}\alpha\Big]\phi_1(\zeta)}{1 - \Omega_0(\gamma\zeta, \delta\zeta)},$$
$$\phi_3(\zeta) = \phi_2(\zeta) + \frac{16\alpha\phi_1(\zeta)}{5\Big(1 - \Omega_0(\gamma\zeta, \delta\zeta)\Big)},$$
$$\phi_4(\zeta) = \phi_2(\zeta) + \frac{\alpha\phi_3(\zeta)}{5\Big(1 - \Omega_0(\gamma\zeta, \delta\zeta)\Big)},$$

and

$$\psi_m(\zeta) = \phi_m(\zeta) - 1.$$

Using these definitions, we have $\psi_m(0) = -1$ and $\psi_m(\zeta) \to \infty$ as $\zeta \to r^-$. Denote by R_m the least solution of equations $\psi_m(\zeta) = 0$, $m = 1, 2, 3, 4$ in the interval $(0, r)$ assured to exist by the intermediate value theorem. Define a radius of convergence R by

$$R = \min\{R_m\}. \tag{5}$$

Then, if $\zeta \in [0, R)$

$$0 \leq \Omega_0(\gamma\zeta, \delta\zeta) < 1, \tag{6}$$

and

$$0 \leq \phi_m(\zeta) < 1. \tag{7}$$

Set $U(x_*, \rho) = \Big\{x \in \Delta : \|x - x_*\| < \rho\Big\}$ and let $\bar{U}(x_*, \rho)$ stand for the closure of $U(x_*, \rho)$.

Let us introduce conditions (C):

(C_1) $F : \Delta \to \mathbb{X}$, is Fréchet differentiable at a simple solution x_* of equation $F(x_*) = 0$ with $[\cdot,\cdot; F] : \mathbb{D} \times \mathbb{D} \to \ell(\mathbb{X}, \mathbb{X})$ a standard divided difference of order one, and for $\beta \in \mathbb{R}$

$$\left\|F'(x_*)^{-1}[x, x_*; F]\right\| \leq \alpha,$$
$$\|I + \beta[x, x_*; F]\| \leq \gamma$$
and
$$\|I - \beta[x, x_*; F]\| \leq \delta$$

holds for all $x \in \Delta$ and some $\gamma, \delta \geq 0$.

(C_2) For each $x, y \in \Delta$, we have

$$\left\|F'(x_*)^{-1}\Big([x, x_*; F] - F'(x_*)\Big)\right\| \leq \Omega_0(\|x - x_*\|, \|y - x_*\|).$$

Set $\Delta_0 = \Delta \cap U(x_*, r)$.

(C_3) For each $x, y, z \in \Delta_0$, we get

$$\left\|F'(x_*)^{-1}\Big([x, y; F] - [z, x_*; F]\Big)\right\| \leq \Omega(\|x - x_*\|, \|y - x_*\|).$$

(C_4) The ball $\bar{U}(x_*, \tilde{R}) \subset \Delta$, where R is defined in (5), and r exists, is given in (4) and $\tilde{R} = \max\{R, \gamma R, \delta R\}$.

(C_5) There exists $R_* \geq R$ such that

$$\Omega_0(0, R_*) < 1 \text{ or } \Omega_0(R_*, 0) < 1.$$

Set $\Delta_1 = \Delta \cap U^*(x_*, R_*)$.

Next, we base the convergence analysis on the C conditions, and the developed notations.

Theorem 1. Suppose the (C) conditions are satisfied. Then, if we choose $x_0 \in U(x_*, R) - \{x_*\}$, the following assertions hold:

$$\{x_\sigma\} \subset \Delta, \tag{8}$$

$$\lim_{\sigma\to\infty} x_\sigma = x_*, \tag{9}$$

$$\|y_\sigma - x_*\| \leq \phi_1(\|x_\sigma - x_*\|)\|x_\sigma - x_*\| \leq \|x_\sigma - x_*\| < r, \tag{10}$$

$$\|z_\sigma - x_*\| \leq \phi_2(\|x_\sigma - x_*\|)\|x_\sigma - x_*\| \leq \|x_\sigma - x_*\|, \tag{11}$$

$$\|h_\sigma - x_*\| \leq \phi_3(\|x_\sigma - x_*\|)\|x_\sigma - x_*\| \leq \|x_\sigma - x_*\|, \tag{12}$$

$$\|x_{\sigma+1} - x_*\| \leq \phi_4(\|x_\sigma - x_*\|)\|x_\sigma - x_*\| \leq \|x_\sigma - x_*\|, \tag{13}$$

and the only solution of equation $F(x) = 0$ in the set Δ_1 given below (C_4) is x_*.

Proof. By conditions $x_0 \in U(x_*, R) - \{x_*\}$, $F(x_0) = F(x_0) - F(x_*) = [x_0, x_*; F](x_0 - x_*)$ the definition of divided difference, first, second and third condition in (C_1), we have

$$\begin{aligned}
\|u_0 - x_*\| &= \|x_0 - x_* + \beta F(x_0)\| \\
&= \|(I + \beta[x_0, x_*; F])(x_0 - x_*)\| \\
&\leq \|I + \beta[x_0, x_*; F]\|\|x_0 - x_*\| \\
&\leq \gamma\|x_0 - x_*\| \leq \gamma R, \\
\|v_0 - x_*\| &= \|x_0 - x_* - \beta F(x_0)\| \\
&= \|(I - \beta[x_0, x_*; F])(x_0 - x_*)\| \\
&\leq \delta\|x_0 - x_*\| \leq \delta R,
\end{aligned}$$

and

$$\left\|F'(x_*)^{-1}[x_0, x_*; F]\right\| \leq \alpha,$$

respectively.

By adopting (5), (6) and (C_2), we get

$$\begin{aligned}\left\|F'(x_*)^{-1}\Big(A_0 - F'(x_*)\Big)\right\| &\leq \Omega_0(\|u_0 - x_*\|, \|v - x_*\|)\\ &\leq \Omega_0(\gamma\|x_0 - x_*\|, \delta\|x_0 - x_*\|) \leq \Omega_0(\gamma R, \delta R) < 1,\end{aligned}$$

which together with a Lemma by Banach operators that are invertible[9,10,11], leads to A_0 is invertible,

$$\left\|A_0^{-1}F'(x_*)\right\| \leq \frac{1}{\Omega_0(\gamma\|x_0 - x_*\|, \delta\|x_0 - x_*\|)}, \tag{14}$$

and y_0, z_0, h_0, x_1 exist by C_1 condition for method (2) for $\sigma = 0$.

Using (2), (5), (7) (for $\sigma = 1$), (C_3) and (14), we obtain

$$\begin{aligned}\|y_0 - x_*\| &= \left\|x_0 - x_* - A_0^{-1}F(x_0)\right\|\\ &= \left\|\Big(A_0^{-1}F'(x_*)\Big)F'(x_*)^{-1}\Big(A_0 - [x_0, x_*; F]\Big)(x_0 - x_*)\right\|\\ &\leq \left\|A_0^{-1}F'(x_*)\right\|\left\|F'(x_*)^{-1}\Big(A_0 - [x_0, x_*; F]\Big)\right\|\|x_0 - x_*\|\\ &\leq \frac{\Omega\Big(\|(u_0 - x_*) + (x_* - v_0)\|, \|v_0 - x_*\|\Big)\|x_0 - x_*\|}{1 - \Omega_0(\gamma\|x_0 - x_*\|, \delta\|x_0 - x_*\|)}\\ &\leq \frac{\Omega\Big((1+\gamma)\|x_0 - x_*\|, \delta\|x_0 - x_*\|\Big)\|x_0 - x_*\|}{1 - \Omega_0(\gamma\|x_0 - x_*\|, \delta\|x_0 - x_*\|)}\\ &= \phi_1(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < R.\end{aligned} \tag{15}$$

Then, using the second step of method (2), (14) and (15), we have

$$\begin{aligned}\|z_0 - x_*\| &= \left\|\Big(y_0 - x_* - A_0^{-1}F(y_0)\Big) - \frac{4}{5}A_0^{-1}F(y_0)\right\|\\ &\leq \left\|y_0 - x_* - A_0^{-1}F(y_0)\right\| + \frac{4}{5}\left\|A_0^{-1}F'(x_*)\right\|\left\|F'(x_*)^{-1}F(y_0)\right\|\\ &\leq \left\|A_0^{-1}F'(x_*)\right\|\left\|F'(x_*)^{-1}\Big(A_0 - [y_0, x_*; F]\Big)\right\|\|y_0 - x_*\| + \frac{4}{5}\left\|A_0^{-1}F'(x_*)\right\|\left\|F'(x_*)^{-1}F(y_0)\right\|\\ &\leq \frac{\left[\Omega\Big(\|(u_0 - x_*) + (x_* - y_0)\|, \|v_0 - x_*\|\Big) + \frac{4}{5}\alpha\right]\|y_0 - x_*\|}{1 - \Omega_0(\gamma\|x_0 - x_*\|, \delta\|x_0 - x_*\|)}\\ &\leq \frac{\left[\Omega\Big(\|x_0 - x_*\|, \|x_0 - x_*\|\Big) + \frac{4}{5}\alpha\right]\phi_1(\|x_0 - x_*\|)\|x_0 - x_*\|}{1 - \Omega_0(\gamma\|x_0 - x_*\|, \delta\|x_0 - x_*\|)}\\ &= \phi_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|.\end{aligned} \tag{16}$$

Moreover, in the view of third substep of method (2), (14), (15) and the triangle inequality, we obtain

$$\begin{aligned}\|h_0 - x_*\| &= \left\|(z_0 - x_*) - \frac{16}{5}A_0^{-1}F(y_0)\right\|\\ &\leq \|z_0 - x_*\| + \frac{16}{5}\left\|A_0^{-1}F'(x_*)\right\|\left\|F'(x_*)^{-1}F(y_0)\right\|\\ &\leq \left[\phi_2(\|x_0 - x_*\|) + \frac{16\alpha\phi_1(\|x_0 - x_*\|)}{5\Big(1 - \Omega_0(\gamma\|x_0 - x_*\|, \delta\|x_0 - x_*\|)\Big)}\right]\|x_0 - x_*\|\\ &= \phi_3(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|,\end{aligned} \tag{17}$$

Furthermore, by adopting (14) (16), (17) and the triangle inequality, we yield

$$\begin{aligned}\|x_1 - x_*\| &= \left\| z_0 - x_* - \frac{1}{5} A_0^{-1} F(h_0) \right\| \\ &\le \|z_0 - x_*\| + \frac{1}{5} \left\| A_0^{-1} F'(x_*) \right\| \left\| F'(x_*)^{-1} F(h_0) \right\| \\ &\le \left[\phi_2(\|x_0 - x_*\|) + \frac{\alpha \phi_3(\|x_0 - x_*\|)}{5\Big(1 - \Omega_0(\gamma\|x_0 - x_*\|, \delta\|x_0 - x_*\|)\Big)} \right] \|x_0 - x_*\| \\ &= \phi_4(\|x_0 - x_*\|)\|x_0 - x_*\| \le \|x_0 - x_*\|,\end{aligned} \tag{18}$$

deducing that $y_0, z_0, h_0, x_1 \in U(x_*, R)$ and (10)–(13) hold for $\sigma = 0$. Next, by the inequation

$$\|x_{k+1} - x_*\| \le p\|x_k - x_*\| < R, \tag{19}$$

where $p = \phi_4(\|x_0 - x_*\|) \in [0, 1)$, we obtain $\lim_{k\to\infty} x_k = x_*$, with $x_{k+1} \in U(x_*, R)$.

Uniqueness of the solution:

It remains to show the uniqueness of the x_* in the set Δ_1. For this purpose, let us assume another solution $y_* \in \Delta_1$ with $F(y_*) = 0$. Set $T = [x_*, y_*; F]$. In view of (C_2) and (C_5)

$$\left\| F'(x_*)^{-1}\Big(T - F'(x_*)\Big) \right\| \le \Omega_0(0, \|x_* - y_*\|) \le \Omega_0(0, R) < 1,$$

so T is invertible. Finally, we get $x_* = y_*$ from the approximation $0 = F(y_*) - F(x_*) = T(y_* - x_*)$. □

Remark 1. The proposed technique can be potentially applicable to the analysis of synchronization manifold in control systems, see e.g., Fixed-time group consensus for multi-agent systems with nonlinear dynamics and uncertainties, kinematic synthesis problem for steering and prominent 2D Bratu problem (details can be found in [12,13,14]).

3 | NUMERICAL EXAMPLES

The theoretical results developed in the previous sections are illustrated numerically in this section. We consider two real life problems and two standard nonlinear problems that are illustrated in examples 3.1–3.3. The results are listed in Tables 1, 3 and 4. Additionally, we obtain the COC [7], approximated by means of

$$\xi = \frac{\ln \frac{\|x_{\sigma+1} - x_*\|}{|x_\sigma - x_*\|}}{\ln \frac{\|x_\sigma - x_*\|}{\|x_{\sigma-1} - x_*\|}}, \quad \text{for } \sigma = 1, 2, \dots \tag{20}$$

or $ACOC$ [10] by:

$$\xi^* = \frac{\ln \frac{\|x_{\sigma+1} - x_\sigma\|}{\|x_\sigma - x_{\sigma-1}\|}}{\ln \frac{\|x_\sigma - x_{\sigma-1}\|}{\|x_{\sigma-1} - x_{\sigma-2}\|}}, \quad \text{for } \sigma = 2, 3, \dots \tag{21}$$

We adopt $\epsilon = 10^{-200}$ as the error tolerance. The terminating criteria to solve nonlinear system or scalar equation are: (i) $\|x_{\sigma+1} - x_\sigma\| < \epsilon$, and (ii) $\|F(x_\sigma)\| < \epsilon$.

The computations are performed with the package $Mathematica$ 9 with multiple precision arithmetic, [15,16]. The divided difference in all examples is given by $[x, y; F] = \int_0^1 F'\Big(y + \theta(x - y)\Big) d\theta$. We also consider $\beta = 0$ in all examples 3.1–3.3.

3.1 | Numerical Example 1

Let us consider the following system of nonlinear equations (chosen from Grau-Sánchez et al. [17])

$$F(x_1, x_2, \dots, x_\sigma) = \sum_{j=1, j\ne i}^{\sigma} x_j - e^{-x_i}, \quad 1 \le i \le \sigma. \tag{22}$$

We choose $\sigma = 10$ in order to check the theoretical results mentioned with a large size system. The obtained solution of this problem is

$$x_* = (0.5671433\ldots, 0.5671433\ldots, 0.5671433\ldots, \cdots, 0.5671433\ldots(10 times))^T.$$

Choose $\mathbb{X} = \mathbb{R}^{10}$ and $\Delta = U\left(x_*, \frac{1}{2}\right)$. Then, we get

$$\Omega_0(t,s) = \frac{t+s}{2}, \quad \Omega(t,s) = t+s, \quad \alpha = 7, \ \gamma = 1, \quad \text{and} \quad \delta = 1.$$

The obtained results can be observed in Table 1.

TABLE 1 Radii for Example 3.1

Method	R_1	R_2	R_3	R_4	R	x_0	σ	ξ
(2)	0.076923	0.053984	0.022085	0.279501	0.022085	$(0.55, 0.55, \cdots, 0.55(10 times))^T$	3	5.0000

3.2 | Numerical Example 2

The kinematic synthesis problem for steering[14,1], is given as

$$\begin{aligned}&\left[E_i\left(x_2\sin\left(\psi_i\right)-x_3\right)-F_i\left(x_2\sin\left(\varphi_i\right)-x_3\right)\right]^2+\left[F_i\left(x_2\cos\left(\varphi_i\right)+1\right)-F_i\left(x_2\cos\left(\psi_i\right)-1\right)\right]^2\\&-\left[x_1\left(x_2\sin\left(\psi_i\right)-x_3\right)\left(x_2\cos\left(\varphi_i\right)+1\right)-x_1\left(x_2\cos\left(\psi_i\right)-x_3\right)\left(x_2\sin\left(\varphi_i\right)-x_3\right)\right]^2=0,\ \text{for } i=1,2,3,\end{aligned}$$

where

$$E_i = -x_3x_2\left(\sin\left(\varphi_i\right)-\sin\left(\varphi_0\right)\right)-x_1\left(x_2\sin\left(\varphi_i\right)-x_3\right)+x_2\left(\cos\left(\varphi_i\right)-\cos\left(\varphi_0\right)\right),\ i=1,2,3$$

and

$$F_i = -x_3x_2\sin\left(\psi_i\right)+\left(-x_2\right)\cos\left(\psi_i\right)+\left(x_3-x_1\right)x_2\sin\left(\psi_0\right)+x_2\cos\left(\psi_0\right)+x_1x_3,\ i=1,2,3.$$

In Table 2, we present the values of ψ_i and φ_i (in radians).

The approximated solution is

$$x_* = (0.9051567\ldots,\ 0.6977417\ldots,\ 0.6508335\ldots)^T.$$

Choose $\mathbb{X} = \mathbb{R}^3$ and $\Delta = U\left(x_*, 1\right) \times U\left(x_*, 1\right) \times U\left(x_*, 1\right)$.

Then, we get

$$\Omega_0(t,s) = \frac{t+s}{4}, \quad \Omega(t,s) = \frac{t+s}{2} \quad \alpha = 11, \ \gamma = 1, \quad \text{and} \quad \delta = 1.$$

We provide the radii of convergence for Example 3.2 in Table 3.

TABLE 2 Values of ψ_i and φ_i (in radians) for Example (3.2).

i	ψ_i	φ_i
0	1.3954170041747090114	1.7461756494150842271
1	1.7444828545735749268	2.0364691127919609051
2	2.0656234369405315689	2.2390977868265978920
3	2.4600678478912500533	2.4600678409809344550

TABLE 3 Radii for Example 3.2

Method	R_1	R_2	R_3	R_4	R	x_0	σ	ξ
(2)	0.10526	0.073747	0.030365	0.051422	0.030365	(0.88,0.67,0.63)	5	4.0000

3.3 | Numerical Example 3

We choose a prominent 2D Bratu problem[12,13], which is given by

$$\begin{aligned} &u_{xx} + u_{tt} + Ce^{u} = 0, \mathrm{o}n \\ &A : (x,t) \in 0 \le x \le 1, \quad 0 \le t \le 1, \\ &\mathrm{a}long\ boundary\ hypothesis\ u = 0\ \mathrm{o}n\ A. \end{aligned} \tag{23}$$

Let us assume that $\Theta_{i,j} = u(x_i, t_j)$ is a numerical result over the grid points of the mesh. In addition, we consider that τ_1 and τ_2 are the number of steps in the direction of x and t, respectively. Moreover, we choose that h and k are the respective step sizes in the direction of x and y, respectively. In order to find the solution of PDE (23), we adopt the following approach

$$u_{xx}(x_i, t_j) = \frac{\Theta_{i+1,j} - 2\Theta_{i,j} + \Theta_{i-1,j}}{h^2}, \ C = 0.1, \ t \in [0,1], \tag{24}$$

which further yields the succeeding SNE

$$\Theta_{i,j+1} + \Theta_{i,j-1} - \Theta_{i,j} + \Theta_{i+1,j} + \Theta_{i-1,j} + h^2 C \exp\left(\Theta_{i,j}\right) \quad i = 1,2,3,\dots,\tau_1, j = 1,2,3,\dots,\tau_2 \tag{25}$$

By choosing $\tau_1 = \tau_2 = 11, \ h = \frac{1}{11}$, and $C = 0.1$, we get a large SNE of order 100×100 which converges to the following required root

$$x_* = \begin{pmatrix} 0.0011\dots, 0.0018\dots, 0.0022\dots, 0.0025\dots, 0.0026\dots, 0.0026\dots, 0.0025\dots, 0.0022\dots, 0.0018\dots, 0.0011\dots \\ 0.0018\dots, 0.0030\dots, 0.0038\dots, 0.0043\dots, 0.0046\dots, 0.0046\dots, 0.0043\dots, 0.0038\dots, 0.0030\dots, 0.0018\dots, \\ 0.0022\dots, 0.0038\dots, 0.0049\dots, 0.0056\dots, 0.0059\dots, 0.0059\dots, 0.0056\dots, 0.0049\dots, 0.0038\dots, 0.0022\dots, \\ 0.0025\dots, 0.0043\dots, 0.0056\dots, 0.0064\dots, 0.0068\dots, 0.0068\dots, 0.0064\dots, 0.0056\dots, 0.0043\dots, 0.0025\dots, \\ 0.0026\dots, 0.0046\dots, 0.0059\dots, 0.0068\dots, 0.0072\dots, 0.0072\dots, 0.0068\dots, 0.0059\dots, 0.0046\dots, 0.0026\dots, \\ 0.0026\dots, 0.0046\dots, 0.0059\dots, 0.0068\dots, 0.0072\dots, 0.0072\dots, 0.0068\dots, 0.0059\dots, 0.0046\dots, 0.0026\dots, \\ 0.0025\dots, 0.0043\dots, 0.0056\dots, 0.0064\dots, 0.0068\dots, 0.0068\dots, 0.0064\dots, 0.0056\dots, 0.0043\dots, 0.0025\dots, \\ 0.0022\dots, 0.0038\dots, 0.0049\dots, 0.0056\dots, 0.0059\dots, 0.0059\dots, 0.0056\dots, 0.0049\dots, 0.0038\dots, 0.0022\dots, \\ 0.0018\dots, 0.0030\dots, 0.0038\dots, 0.0043\dots, 0.0046\dots, 0.0046\dots, 0.0043\dots, 0.0038\dots, 0.0030\dots, 0.0018\dots, \\ 0.0011\dots, 0.0018\dots, 0.0022\dots, 0.0025\dots, 0.0026\dots, 0.0026\dots, 0.0025\dots, 0.0022\dots, 0.0018\dots, 0.0011\dots \end{pmatrix}^T$$

a the column vector. Choose $\mathbb{X} = \mathbb{R}^{100}$ and $\Delta = U\left(x_*, 0.006\right)$. Then, we have

$$\Omega_0(t,s) = \Omega(t,s) = 6(t+s), \quad \alpha = 12, \ \gamma = 1, \quad \text{and} \quad \delta = 1.$$

The obtained results are depicted in Table 4 with the following initial approximation (the column vector)

$$x_0 = \begin{pmatrix} 0.001113, 0.001812, 0.002260, 0.002530, 0.002657, 0.002657, 0.002530, 0.002260, 0.001812, 0.001113, \\ 0.001812, 0.003048, 0.003870, 0.004374, 0.004613, 0.004613, 0.004374, 0.003870, 0.003048, 0.001812, \\ 0.002260, 0.003870, 0.004968, 0.005652, 0.005979, 0.005979, 0.005652, 0.004968, 0.003870, 0.002260, \\ 0.002530, 0.004374, 0.005652, 0.006454, 0.006841, 0.006841, 0.006454, 0.005652, 0.004374, 0.002530, \\ 0.002657, 0.004613, 0.005979, 0.006841, 0.007257, 0.007257, 0.006841, 0.005979, 0.004613, 0.002657, \\ 0.002657, 0.004613, 0.005979, 0.006841, 0.007257, 0.007257, 0.006841, 0.005979, 0.004613, 0.002657, \\ 0.002530, 0.004374, 0.005652, 0.006454, 0.006841, 0.006841, 0.006454, 0.005652, 0.004374, 0.002530, \\ 0.002260, 0.003870, 0.004968, 0.005652, 0.005979, 0.005979, 0.005652, 0.004968, 0.003870, 0.002260, \\ 0.001812, 0.003048, 0.003870, 0.004374, 0.004613, 0.004613, 0.004374, 0.003870, 0.003048, 0.001812, \\ 0.001113, 0.0018125, 0.002260, 0.002530, 0.002657, 0.002657, 0.002530, 0.002260, 0.001812, 0.001113 \end{pmatrix}^T.$$

TABLE 4 Radii of convergence for Example (3.3)

Method	R_1	R_2	R_3	R_4	R	x_0	σ	ξ
(2)	0.033333	0.0050705	0.000080359	0.0846490	0.000080359	above this table	3	3.9742

4 | CONCLUSION

A novel fifth order scheme is developed for generating a sequence approximating x_*. The novelty of our work is that the new technique expands its applicability, since it only uses conditions on the divided difference of order one contrasting earlier articles requiring derivatives up to the order fourth. We presented estimates on $|x_\sigma - x_*|$ and results on the uniqueness of x_* based on our conditions. This was not done in the earlier articles, where expensive Taylor expansions were used to determine the convergence order even though such high order derivatives did not appear in method (2). Numerical experiments test our convergence results. Moreover, our results hold on a Banach space setting further extending the ones shown in[1] on $\mathbb{X} = \mathbb{R}^j$. As further works we suggest the technique to be utilized to expand the applicability of other schemes available in the literature by using inverses of linear operators along the same lines,[18,19,20,21].

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Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors declare no potential conflict of interests.

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