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Article

Classifying Topologies through $\mathfrak{G}$-Bases

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Abstract: We classify several topological properties of a Tychonoff space X by means of certain locally convex topologies $\mathcal{T}$ with a $\mathfrak{G}$-base located between the pointwise topology τ_p and the bounded-open topology τ_b of the real-valued continuous function space $C(X)$.

Keywords: compact resolution; σ-compact space; hemicompact space; $\mathfrak{G}$-base

MSC: 54C; 54D; 54E; 46A; 46B

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1. Introduction

Unless otherwise stated, X stands for an infinite Tychonoff space. We denote by $C_p(X)$ the linear space $C(X)$ of real-valued continuous functions on X equipped with the pointwise topology τ_p. The topological dual of $C_p(X)$ is denoted by $L(X)$, or by $L_p(X)$ when provided with the weak* topology. We represent by $C_k(X)$ the space $C(X)$ equipped with the compact-open topology τ_k. A family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of a set X is called a *resolution* for X if it covers X and verifies that $A_\alpha \subseteq A_\beta$ for $\alpha \leq \beta$. Let us recall that a Hausdorff topological space X is called a *μ-space* if the closure of each functionally bounded set in X is compact. Every realcompact space is a μ-space, but the converse is not true. If E is a locally convex space, the *bidual* E'' of E is the topological dual of the strong dual E'_β of E. By the strong dual E'_β of E, we mean the dual E' of E equipped with the strong topology $\beta(E', E)$ of the uniform convergence on the bounded sets in E. We denote by τ_w and τ_b the weak locally convex topology of $C_k(X)$ and the bounded-open topology of $C(X)$, respectively. If A is a nonempty set in a real linear space L, we represent by $\operatorname{abx}(A)$ the (real) absolutely convex hull of A. The linear subspace of $C(X)$ consisting of those bounded functions is denoted by $C^b(X)$.

A base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of (absolutely convex) neighborhoods of the origin in a locally convex space E such that $U_\beta \subseteq U_\alpha$ if $\alpha \leq \beta$ is called a *$\mathfrak{G}$-base*. Let us mention that the notion of a $\mathfrak{G}$-base, originally introduced in the realm of locally convex spaces [1], has been extended to topological groups and general topological spaces, sometimes under the name of *ω^ω-base*, by some authors (see [2,3]). However, in this paper we keep the original name. Trivially, if E is a metrizable locally convex space with a decreasing base $\{V_n : n \in \mathbb{N}\}$ of locally convex neighborhoods of the origin, the family $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, where $U_\alpha = V_{\alpha(1)}$ for all $\alpha \in \mathbb{N}^{\mathbb{N}}$ is a $\mathfrak{G}$-base for E. The converse is not true, i.e., a locally convex space with a $\mathfrak{G}$-base need not be metrizable (see p. 107, [4]).

A locally convex space E with a $\mathfrak{G}$-base has metrizable compact sets, since, in this case, the weak dual $(E', \sigma(E', E))$ is quasi-Suslin [5], hence trans-separable [6] (properties not defined in this paper can be found in [7–10]). Therefore, if there is a locally convex topology $\mathcal{T}$ on $C(X)$ with a $\mathfrak{G}$-base such that $\tau_p \leq \mathcal{T} \leq \tau_b$, every completely regular topology τ on $C(X)$ such that $\mathcal{T} \leq \tau$ is angelic and has metrizable compact sets. Research on $\mathfrak{G}$-bases and their generalizations remain active since [1]. For recent results on this topic see [3,11–14]. Research on the existence of locally convex topologies on $C(X)$ stronger than the pointwise topology has been studied in [15,16]. In this paper, we enlarge the classification of topological properties on X provided in (Theorem 3.1, [15]).

2. A Preliminary Result

A resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ for X is *functionally bounded* if it consists of functionally bounded sets A_α in X. A space X is said to be *strictly angelic* [17] if it is angelic and all separable compact subsets of X are first countable. A classification of topologies on X by locally convex topologies on $C_p(X)$ is provided in [15] (see also (Theorem 98, [4])).

Theorem 1 (Ferrando–Gabriyelyan–Kąkol (Theorem 3.1, [15]))**.** *If X is a Tychonoff space, the following properties hold*

1. *There exists a metrizable locally convex topology $\mathcal{T}$ on $C(X)$ such that $\tau_p \leq \mathcal{T} \leq \tau_k$ if and only if X is a σ-compact space.*
2. *There exists a metrizable locally convex topology $\mathcal{T}$ on $C(X)$ such that $\tau_k \leq \mathcal{T} \leq \tau_b$ if and only if there is an increasing sequence $\{A_n : n \in \mathbb{N}\}$ of functionally bounded subsets of X swallowing the compact sets of X.*
3. *There exists a metrizable locally convex topology $\mathcal{T}$ on $C(X)$ such that $\tau_p \leq \mathcal{T} \leq \tau_b$ if and only if there is an increasing sequence $\{A_n : n \in \mathbb{N}\}$ of functionally bounded subsets of X covering X, or equivalently, if and only if υX is σ-compact.*
4. *There is a metrizable locally convex topology $\mathcal{T}$ on $C(X)$ such that $\tau_p \leq \mathcal{T} \leq \tau_w$ if and only if X is countable.*
5. *There is a locally convex topology $\mathcal{T}$ on $C(X)$ with a $\mathfrak{G}$-base such that $\tau_p \leq \mathcal{T} \leq \tau_k$ if and only if X has a compact resolution.*
6. *There is a locally convex topology $\mathcal{T}$ on $C(X)$ with a $\mathfrak{G}$-base such that $\tau_k \leq \mathcal{T} \leq \tau_b$ if and only if X has a functionally bounded resolution swallowing the compact sets.*
7. *There exists a locally convex topology $\mathcal{T}$ on $C(X)$ with a $\mathfrak{G}$-base such that $\tau_p \leq \mathcal{T} \leq \tau_b$ if and only if X has a functionally bounded resolution. Equivalently, if and only if υX is K-analytic. In this case $(C(X), \tau_b)$ is strictly angelic.*
8. *There is a locally convex topology $\mathcal{T}$ on $C(X)$ with a $\mathfrak{G}$-base such that $\tau_p \leq \mathcal{T} \leq \tau_w$ if and only if X is countable.*

3. A More Complete Classification

To enlarge the previous classification, we need the following result.

Lemma 1. *Let X be completely regular. If Q is a metrizable and compact subspace of X, there exists a continuous linear extender map $\varphi : C_k(Q) \to C_k(X)$, i.e., such that $\varphi(f)|_Q = f$ for every $f \in C(Q)$.*

Proof. Since Q is (homeomorphic to) a metrizable compact subspace of the Stone–Čech compactification βX of X, according to [18] (see also (Proposition 4.1, [19])) there exists a continuous linear map $\psi : C_p(Q) \to C_p(\beta X)$ such that $\psi(f)|_Q = f$ which embeds $C_p(Q)$ in $C_p(\beta X)$ (as a closed subspace). If $\phi : C_p(\beta X) \to C_p^b(X)$ is the restriction map $\phi(g) = g|_X$, it turns out that $\varphi := \phi \circ \psi$ is a continuous linear map from $C_p(Q)$ into $C_p^b(X)$ such that $\varphi(f)|_Q = \phi(\psi(f))|_Q = \psi(f)|_Q = f$ for every $f \in C(Q)$.

Hence, $\varphi : C(Q) \to C^b(X)$ has a closed graph when both spaces are regarded as Banach spaces. This implies in particular that $\varphi : C_k(Q) \to C_k^b(X)$ is continuous, since the supremum norm topology on $C^b(X)$ is stronger than the relative compact-open topology of $C_k(X)$. Thus, $\varphi : C_k(Q) \to C_k(X)$ is a continuous linear extender map, as stated. □

Theorem 2. *If X is a Tychonoff space, the following properties hold*

1. *The compact-open topology $\mathcal{T} = \tau_k$ on $C(X)$ is metrizable if and only if X is a hemicompact space* (Arens' theorem *(Theorem 7, [20])* , see also *(Theorem 2.5, [21])*).

2. *The bounded-open topology $\mathcal{T} = \tau_b$ on $C(X)$ is metrizable if and only if there is an increasing sequence $\{A_n : n \in \mathbb{N}\}$ of functionally bounded subsets of X swallowing the functionally bounded sets of X.*
3. *The weak topology $\mathcal{T} = \tau_w$ on $C(X)$ is metrizable if and only if X is countable and compact sets in X are finite.*
4. *The pointwise topology $\mathcal{T} = \tau_p$ on $C(X)$ is metrizable if and only X is countable.*
5. *The compact-open topology $\mathcal{T} = \tau_k$ on $C(X)$ has a $\mathfrak{G}$-base if and only if X has a compact resolution that swallows the compact sets (Theorem 2, [22]).*
6. *The bounded-open topology $\mathcal{T} = \tau_b$ on $C(X)$ has a $\mathfrak{G}$-base if and only if X has a functionally bounded resolution that swallows the functionally bounded sets.*
7. *The weak topology $\mathcal{T} = \tau_w$ on $C(X)$ has a $\mathfrak{G}$-base if and only if X is countable and compact sets in X are finite.*
8. *The pointwise topology $\mathcal{T} = \tau_p$ on $C(X)$ has a $\mathfrak{G}$-base if and only X is countable (Corollary 15.2, [7]).*

Proof. The proof of statements (2) and (6) is similar (for (6) see (Theorem 12, [23])).

Let us prove statement (3). If $C_k(X)$ is weakly metrizable, the weakly bounded sets are metrizable, so X is countable by (Theorem 2.3, [21]). This fact also follows from statement (4) of Theorem 1. We claim that, in addition, the compact sets of X are finite. Otherwise, there exists an infinite compact set Q in X. However, since X is countable, Q is metrizable. By Lemma 1 there is a continuous linear extender φ from $C_k(Q)$ into $C_k(X)$, i.e., such that $\varphi(f)|_Q = f$ for every $f \in C(Q)$. If $\varphi(f_d) \to g$ in $C_k(X)$, given $\epsilon > 0$ there is $h \in D$ with $\sup_{x\in Q}|\varphi(f_d)(x) - g(x)| < \epsilon$ for every $d \geq h$, so $\sup_{x\in Q}|f_d(x) - g(x)| < \epsilon$ for $d \geq h$. Thus, $f_d \to f := g|_Q$ in $C_k(Q)$, and hence $\varphi(f_d) \to \varphi(f)$ in $C_k(X)$, i.e., $g = \varphi(f)$, which means that φ embeds the Banach space $C_k(Q)$ in $C_k(X)$ as a closed subspace. Since the weak topology is inherited by linear subspaces, the space $C_w(Q)$ is linearly homeomorphic to a subspace of $C_w(X)$. In other words, the Banach space $C_k(Q)$ is weakly metrizable. As the weak topology of a Banach space is metrizable if and only if it is finite-dimensional, it turns out that Q must be finite, a contradiction. Conversely, if X is both countable and has finite compact sets, the former statement guarantees that $C_p(X)$ is metrizable whereas the latter implies that $C_p(X) = C_k(X)$ coincides with $C_w(X)$. Hence, the weak topology τ_w is metrizable.

The proof of statement (7) is similar to that of statement (3). The only difference is that the weak topology of the Banach space $C_k(Q)$ now carries a $\mathfrak{G}$-base. However, if a locally convex space in its weak topology has a $\mathfrak{G}$-base, it is countable-dimensional (Proposition 11.2, [7]). Thus, $C(Q)$ must be countable-dimensional, so finite-dimensional by the Baire category theorem. This ensures that the compact set Q must be finite. □

Remark 1. *In (Theorem 2.3, [21]), it is shown that if X is countable, the bounded sets of $C_k(X)$ are weakly metrizable. According to statement (3) of the previous theorem, if X contains an infinite compact set, then $C_k(X)$ cannot be weakly metrizable. Thus, if X is countable and contains an infinite compact set, then $C_w(X)$ is not metrizable but has metrizable bounded sets.*

Example 1. *As $\mathbb{Q}$ is a countable space with infinite compact sets which is not hemicompact, neither τ_k nor τ_w are metrizable, the former statement by Arens' theorem and the latter by the previous remark, but τ_p is metrizable. Of course, τ_w has no $\mathfrak{G}$-base. In fact, since $\mathbb{Q}$ is not a Polish space, Christensen's theorem (Theorem 94, [4]) prevents $\mathbb{Q}$ from having a compact resolution that swallows the compact sets. This also implies that $C_k(\mathbb{Q})$ has no $\mathfrak{G}$-base, by statement (5) of Theorem 2.*

Example 2. *Let $\mathbb{N}$ be equipped with the discrete topology and choose $p \in \beta\mathbb{N} \setminus \mathbb{N}$. Then, $X := \mathbb{N} \cup \{p\}$ provided with the relative topology of $\beta\mathbb{N}$ is not discrete but has finite compact sets. Thus, the weak topology τ_w on $C(X)$ is metrizable. In fact, clearly, $\tau_p = \tau_k$, although $C_p(X) \neq \mathbb{R}^X$.*

Example 3. *$\mathbb{R}$ is an uncountable hemicompact space, hence τ_k is metrizable, but τ_w and τ_p are not.*

Example 4. *If K is a countable infinite compact set, obviously both τ_p and τ_k are metrizable, but τ_w is not.*

Example 5. *The Sorgenfrey line $\mathbb{S}$ is a Lindelöf space which is not σ-compact, so $\tau_k = \tau_b$ and by statement (1) of Theorem 1 there is no metrizable, locally convex topology $\mathcal{T}$ on $C(\mathbb{S})$ such that $\tau_p \leq \mathcal{T} \leq \tau_k$.*

Example 6. *The space $C_k(\mathbb{N}^{\mathbb{N}})$ has a $\mathfrak{G}$-base by statement (5) of Theorem 2, but there is no metrizable topology $\mathcal{T}$ such that $\tau_p \leq \mathcal{T} \leq \tau_k$, since $\mathbb{N}^{\mathbb{N}}$ is not σ-compact.*

4. The Interval $\tau_w \leq \mathcal{T} \leq \tau_k$

In this section, we deal with the interval $\tau_w \leq \mathcal{T} \leq \tau_k$. Before stating our main result for this case, we need to establish two auxiliary results concerning the weak* dual $L_p(X)$ of the space $C_p(X)$. We regard (the canonical homeomorphic copy $\delta(X)$ of) X in $L_p(X)$ as a Hamel basis of $L(X)$, and denote by δ_x the image of $x \in X$ in $\delta(X)$. If $f \in C(X)$ and $u \in L(X)$, we write $\langle f, u \rangle$ to represent the action of the linear functional u on f, in particular $\langle f, \delta_x \rangle = f(x)$.

Lemma 2. *Let E denote the dual of $C_k(X)$. If X is a μ-space, then E coincides with the bidual of $L_p(X)$.*

Proof. If X is a μ-space, then $C_k(X)$ is the strong dual of $L_p(X)$ (see (Lemma 2.2, [24])). Therefore, $C_k(X) = L(X)'_\beta$ and hence, E coincides with the bidual $L_p(X)''$ of $L_p(X)$. □

Lemma 3. *Let X be a μ-space. If A is a bounded set in $L_p(X)$, there are a compact set K_A in X and a real number ϵ_A with $0 < \epsilon_A < 1$ such that $A \subseteq 2\,\epsilon_A^{-1}\mathrm{abx}(K_A)$.*

Proof. Since X is a μ-space, if A is a bounded set in $L_p(X)$, then A^0 is a neighborhood of the origin in $C_k(X)$, so there is $0 < \epsilon_A < 1$ and a compact set K_A in X such that

$$U_A := \{f \in C(X) : \sup_{x \in K_A} |f(x)| \leq \epsilon_A\} \subseteq A^0$$

which implies that $A \subseteq A^{00} \subseteq U_A^0$, where the bipolar of A is taken in $L(X)$ as well as the polar of U_A. The fact that $U_A = \epsilon_A K_A^0$ yields $\epsilon_A A \subseteq K_A^{00}$ so that $\epsilon_A A \subseteq \overline{\mathrm{abx}(K_A)}^{L_p(X)}$. We claim that $\overline{\mathrm{abx}(K_A)}^{L_p(X)} \subseteq 2\mathrm{abx}(K_A)$.

Assume that $u \in \overline{\mathrm{abx}(K_A)}^{L_p(X)} = \mathrm{abx}(K_A)^{00}$ with $u \neq \mathbf{0}$. Since X is a Hamel basis of $L(X)$, we have that $u = \sum_{i=1}^m a_i\, \delta_{x_i}$, with $a_i \neq 0$ for $1 \leq i \leq m$. First, we show that the support of u is contained in K_A, so that $x_i \in K_A$ for $1 \leq i \leq m$. Indeed if $x_k \notin K_A$ for some $k \in \{1, \ldots, m\}$, there is $f \in C(X)$ such that $f(x_k) = 2a_k^{-1}$ and $f(z) = 0$ for every $z \in K_A \cup \{x_i : 1 \leq i \leq m,\ i \neq k\}$. Note that $f \in \mathrm{abx}(K_A)^0$ since if $v = \sum_{i=1}^n b_i\, \delta_{y_i} \in \mathrm{abx}(K_A)$ with $\sum_{i=1}^n |b_i| \leq 1$ and $y_i \in K_A$ for $1 \leq i \leq n$, then $\langle f, v \rangle = \sum_{i=1}^n b_i\, f(y_i) = 0$. As $\langle f, u \rangle = 2$, this contradicts the fact that $u \in \mathrm{abx}(K_A)^{00}$. Thus, $x_i \in K_A$ for $1 \leq i \leq m$.

Now, we show that $\sum_{i=1}^m |a_i| \leq 2$, so that $u \in 2\mathrm{abx}(K_A)$. Since the support of u is finite and is contained in K_A, there are pairwise disjoint open sets $\{U_1, \ldots, U_m\}$ in K_A with $x_i \in U_i$, so there are compactly supported continuous functions $\{f_1, \ldots, f_m\}$ such that $\{x_i\} \prec f_i \prec U_i$ for $1 \leq i \leq m$. This means that $f_i \in C(X)$, f_i has compact support, $0 \leq f_i \leq 1$, $\mathrm{supp}\, f_i \subseteq U_i$ and $f(x_i) = 1$ for $1 \leq i \leq m$. For $1 \leq i \leq m$, set $h_i = \mathrm{sgn}(a_i)\, f_i$, where $\mathrm{sgn}(a_i) \in \{-1, 1\}$ denotes the sign of the real number a_i, and define $h := \sum_{i=1}^m h_i$. Clearly $h \in C(X)$, $h(x_i) = \mathrm{sgn}(a_i)$, and $|h(x)| \leq 1$ for every $x \in X$, because of the supports of the functions f_i are pairwise disjoint.

Since $u \in \overline{\mathrm{abx}(K_A)}^{L_p(X)}$ and $h \in C(X)$ there exists $v = \sum_{i=1}^{k} c_i\, \delta_{z_i} \in \mathrm{abx}(K_A)$ with $\sum_{i=1}^{k} |c_i| \leq 1$ and $z_i \in K_A$ for $1 \leq i \leq k$ such that $|\langle h, u - v \rangle| \leq 1$. Consequently,

$$\left| \sum_{i=1}^{m} a_i h(x_i) - \sum_{j=1}^{k} c_j h(z_j) \right| \leq 1,$$

and due to the fact that $v \in \mathrm{abx}(K_A)$ and $a_i h(x_i) = |a_i|$ it follows that

$$\sum_{i=1}^{m} |a_i| = \left| \sum_{i=1}^{m} a_i h(x_i) \right| \leq 1 + \left| \sum_{j=1}^{k} c_j h(z_j) \right| \leq 1 + \sum_{j=1}^{k} |c_j| |h(z_j)| \leq 2$$

as required. □

If X is a μ-space and $\mathsf{Bound}(X)$ denotes the family of all bounded sets in $L_p(X)$, as a consequence of Lemma 2, the space E coincides with (see (23.2.(1), [8]))

$$E = \bigcup \{ \overline{A} : A \in \mathsf{Bound}(X) \}$$

where the closure is in E under the weak* topology $\sigma(E, C(X))$ of E. Hence, if $\mu \in E$ there is a bounded set A in $L_p(X)$ with $\mu \in \overline{A}$. However according to Lemma 3, given $A \in \mathsf{Bound}(X)$, there is a compact S_A in X together with some $n_A \in \mathbb{N}$ with $A \subseteq n_A \,\mathrm{abx}(S_A)$. Thus, if $\mathcal{B}(X)$ designates the family of all functionally bounded sets in X, then

$$E = \bigcup \{ n_A \,\overline{\mathrm{abx}(S_A)} : A \in \mathsf{Bound}(X) \} = \bigcup \{ n \,\overline{\mathrm{abx}(S)} : S \in \mathcal{B}(X),\, n \in \mathbb{N} \}$$

where the closures are in $\sigma(E, C(X))$. Therefore, the following property holds.

Remark 2. *If X is a μ-space and $\mathcal{K}(X)$ stands for the family of all compact sets in X, then*

$$E = \bigcup \left\{ n S^{00} : S \in \mathcal{K}(X),\, n \in \mathbb{N} \right\}$$

where E is the dual of $C_k(X)$.

Proof. Because of the bipolar theorem, one has $\overline{\mathrm{abx}(S)} = S^{00}$. □

This motivates the following definition.

Definition 1. *Let E be the dual of $C_k(X)$. We say that a countable family $\mathcal{F} = \{S_n : n \in \mathbb{N}\}$ of compact sets in X is a complete sequence if there exists a sequence $\{k_n : n \in \mathbb{N}\}$ of positive integers such that*

$$E = \bigcup\nolimits_{n \in \mathbb{N}} k_n S_n^{00}.$$

If X is a metrizable hemicompact space and $\mathcal{F} = \{S_n : n \in \mathbb{N}\}$ is a family of compact sets that swallows the compact sets in X, then $\mathcal{F}$ is a complete sequence. If $\{V_j : j \in \mathbb{N}\}$ is a base of neighborhoods of the origin in $C_k(X)$, there are by Lemma 3 a compact set S in X and $k_j \in \mathbb{N}$ such that $V_j^0 \subseteq k_j \,\mathrm{abx}(S)$. If $S \subseteq S_{n_j}$ then $V_j^0 \subseteq k_j S_{n_j}^{00}$ for every $j \in \mathbb{N}$. If we define $k_m = k_j$ if $m = n_j$ for some $j \in \mathbb{N}$ and $k_m = 1$ otherwise, then

$$E = \bigcup\nolimits_{j \in \mathbb{N}} V_j^0 = \bigcup\nolimits_{m \in \mathbb{N}} k_m S_m^{00}.$$

If $\langle F, E \rangle$ is a dual pair, a locally convex topology $\mathcal{T}$ for F is called a *polar topology* if $\mathcal{T}$ is the topology of the uniform convergence on the sets of a family $\mathcal{A}$ of $\sigma(E, F)$-bounded sets in E. Obviously, both the weak topology $\sigma(F, E)$ and the strong topology $\beta(F, E)$ are polar topologies for F. We are ready to establish our last classification result, which reads as follows.

Theorem 3. *Let X be a Tychonoff space. There is a metrizable polar topology $\mathcal{T}$ on $C(X)$ such that $\tau_w \le \mathcal{T} \le \tau_k$ if and only if X contains a complete sequence.*

Proof. Let us denote by E the dual of the locally convex space $C_k(X)$. Assume that $\mathcal{T}$ is a metrizable polar topology on $C(X)$ with a decreasing base $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of the origin enjoying that $\tau_w \le \mathcal{T} \le \tau_k$. By the first statement of Theorem 1, we know that X is a σ-compact space, so a μ-space.

As $\mathcal{T}$ is a polar topology, there exists a family $\mathcal{M}$ of bounded sets in $L_p(X)$ such that for each $n \in \mathbb{N}$, there are some $M_n \in \mathcal{M}$ with $U_n = M_n^0$. According to Lemma 3, there is a compact set S_n in X and some $k_n \in \mathbb{N}$ such that $M_n \subseteq k_n \operatorname{abx}(S_n)$, so that $S_n^0 \subseteq k_n U_n$. Set $\mathcal{F} := \{S_n : n \in \mathbb{N}\}$. Since $U_n^0 \subseteq k_n S_n^{00}$ for each $n \in \mathbb{N}$ and $\tau_w \le \mathcal{T}$, we get

$$\bigcup_{n\in\mathbb{N}} k_n S_n^{00} = \bigcup_{n\in\mathbb{N}} U_n^0 = E.$$

Thus, the sequence $\mathcal{F}$ is complete.

Assume conversely that there is a complete sequence $\mathcal{F} = \{S_n : n \in \mathbb{N}\}$ in X. We may suppose that $S_n \subseteq S_{n+1}$ as well as that $k_n \le k_{n+1}$. If not, replace $\mathcal{F}$ by the sequence $\mathcal{F}_1 := \{S'_n : n \in \mathbb{N}\}$ with $S'_n = \bigcup_{i=1}^n S_i$ and the sequence $\{k_n : n \in \mathbb{N}\}$ by $\{k'_n : n \in \mathbb{N}\}$ with $k'_n = \max_{1\le i\le n} k_i$. Since $S_n^{00} \subseteq S_n'^{00}$ for every $n \in \mathbb{N}$, if $\mathcal{F}$ is complete so is $\mathcal{F}_1$. Then, for each $n \in \mathbb{N}$, define $V_n = k_n^{-1} S_n^0$. Clearly, $V_{n+1} \subseteq V_n$ and $\{V_n : n \in \mathbb{N}\}$ is a base of neighborhoods of the origin of a metrizable locally convex topology $\mathcal{T}$ for $C(X)$. Observe that $\mathcal{T}$ is stronger than τ_w, since

$$\bigcup_{n\in\mathbb{N}} V_n^0 = \bigcup_{n\in\mathbb{N}} k_n S_n^{00} = E.$$

On the other hand, as $\mathcal{F} \subseteq \mathcal{K}(X)$, the family of all compact sets in X, we see that $\mathcal{T}$ is weaker than τ_k. $\square$

5. Conclusions

We enlarged the classification of some topological properties on X provided in (Theorem 3.1, [15]) by using specific locally convex topologies $\mathcal{T}$ with a $\mathfrak{G}$-base lying between the pointwise topology τ_p and the bounded-open topology τ_b of the real-valued continuous function space $C(X)$. Our main results are

Theorem 4. *Let X be a Tychonoff space.*

1. *The weak topology $\mathcal{T} = \tau_w$ on $C(X)$ is metrizable if and only if it has a $\mathfrak{G}$-base, and if and only if X is countable and compact sets in X are finite.*
2. *There is a metrizable polar topology $\mathcal{T}$ on $C(X)$ such that $\tau_w \le \mathcal{T} \le \tau_k$ if and only if X contains a complete sequence.*

In the second statement, we say that a countable family $\mathcal{F} = \{S_n : n \in \mathbb{N}\}$ of compact sets in X is a complete sequence if there exists a sequence $\{k_n : n \in \mathbb{N}\}$ of positive integers such that $\{k_n S_n^{00} : n \in \mathbb{N}\}$ covers the dual of $C_k(X)$.

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