Quantizations and global hypoellipticity for pseudodifferential operators of infinite order in classes of ultradifferentiable functions

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Abstract

We study the change of quantization for a class of global pseudodifferential operators of infinite order in the setting of ultradifferentiable functions of Beurling type. The composition of different quantizations as well as the transpose of a quantization are also analysed, with applications to the Weyl calculus. We also compare global ω-hypoellipticity and global ω-regularity of these classes of pseudodifferential operators.

1 Introduction

In the present paper, we deal with the change of quantization in the class of *global* pseudodifferential operators introduced by Jornet and the author in [1]. The symbols are of *infinite* order with exponential growth in all the variables, in contrast to the approach of Zanghirati [26] and Fernández, Galbis, and Jornet [16], who treat pseudodifferential operators of infinite order in the *local* sense and infinite order only in the last variable, for Gevrey classes and for classes of ultradifferentiable functions of the Beurling type in the sense of Braun, Meise, and Taylor [9]. In [1, 16], the composition of two operators is given in terms of a suitable symbolic calculus. On the other hand, Prangoski [24] studies pseudodifferential operators of *global* type and infinite order for ultradifferentiable classes of Beurling and Roumieu type in the sense of Komatsu. We refer also to [10, 11, 13, 22] and the references therein to find other papers discussing pseudodifferential operators defined in global classes (especially Gelfand-Shilov classes).

The appropriate setting in the present paper and in [1] is the space of (non-quasianalytic) global ultradifferentiable functions defined by Björck [2], characterized as those $f \in \mathcal{S}(\mathbb{R}^d)$, i.e. in the Schwartz class, such that for all $\lambda > 0$ and all $\alpha \in \mathbb{N}_0^d$ both

$$\sup_{x\in\mathbb{R}^d} e^{\lambda\omega(x)}|\partial^\alpha f(x)| \qquad \text{and} \qquad \sup_{\xi\in\mathbb{R}^d} e^{\lambda\omega(\xi)}|\partial^\alpha \widehat{f}(\xi)|$$

are finite, ω denoting a (non-quasianalytic) weight function in the sense of [9]. These spaces are always contained in the Schwartz class, and they equal the Schwartz class for the case $\omega(t) = \log(1+t)$, $t > 0$, not considered in our setting.

The notion of *hypoellipticity* comes from the problem of determining whether a distribution solution to the partial differential equation $Pu = f$ is a classical solution or not. The authors in [16] provide adequate conditions for the construction of a (left) *parametrix* for their symbols, which guarantee the hypoellipticity in the desired class in [15]. For the operators defined in [24], the corresponding construction of parametrices is done in Cappiello, Pilipović, and Prangoski [12].

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Here, we develop the method of the parametrix in Section 5 for the class of operators introduced in [1], but also for every quantization of the pseudodifferential operator. In particular, we obtain a sufficient condition for any quantization of a pseudodifferential operator to be ω-regular in the sense of Shubin [25] (see the definition of ω-regularity at the beginning of Section 5). In a forthcoming paper, the global parametrix method presented here will be used to define a suitable Weyl wave front set for $\mathcal{S}'_\omega(\mathbb{R}^d)$ and complete the characterization of global wave front sets given in [5].

As we mention at the beginning, one of the goals of the present paper is to extend the results in [1] by adapting them for a valid change of quantization for these symbols (see Sections 3 and 4). Namely, we follow the ideas for the change of quantization set within the framework of global symbol classes of Shubin [25, §23]. In [24] it is considered the change of quantization and its corresponding symbolic calculus for classes in the sense of Komatsu [19], also in the Roumieu setting. Nonetheless, as pointed out in [1], whenever the weight ω is under the mild condition

$$\exists H > 1 : 2\omega(t) \leq \omega(Ht) + H, \qquad t > 0,$$

the classes of ultradifferentiable functions are equally defined either by weights as in [9] or by sequences as in [19] (see Bonet, Meise, and Melikhov [7]). Thus, if the weight sequence $(M_p)_p$ satisfies only stability under ultradifferential operators, as assumed in [24], our classes of symbols (and amplitudes) might not coincide with the ones defined in [24]. It turns out that, even only in the Beurling setting, we are discussing different cases.

Finally, in Section 6, inspired by Boggiatto, Buzano, and Rodino [3], we show that some ω-hypoelliptic symbols are stable under change of quantization and we compare the notions of ω-regularity and ω-hypoellipticity following the ideas of [4].

2 Preliminaries

We begin with some notation on multi-indices. Throughout the text we will denote by $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ a multi-index of dimension d. The length of α is $|\alpha| = \alpha_1 + \cdots + \alpha_d$. For two multi-indices α and β we write $\beta \leq \alpha$ for $\beta_j \leq \alpha_j$, when $j = 1, \ldots, d$. Moreover, $\alpha! = \alpha_1! \cdots \alpha_d!$ and if $\beta \leq \alpha$, then $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}$. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we have $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. We write $\partial^\alpha = \big(\frac{\partial}{\partial x_1}\big)^{\alpha_1} \ldots \big(\frac{\partial}{\partial x_d}\big)^{\alpha_d}$ and we set

$$D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_d}^{\alpha_d},$$

where $D_{x_j}^{\alpha_j} = (-i)^{|\alpha_j|}\big(\frac{\partial}{\partial x_j}\big)^{\alpha_j}$, $j = 1, \ldots, d$.

In our setting we work with weight functions as the ones defined by Braun, Meise, and Taylor [9].

Definition 2.1. *A* non-quasianalytic weight function $\omega : [0, +\infty[\to [0, +\infty[$ *is a continuous and increasing function which satisfies:*

(α) $\exists\ L \geq 1 :\ \omega(2t) \leq L(\omega(t) + 1),\ t \geq 0,$

(β) $\displaystyle\int_1^{+\infty} \frac{\omega(t)}{t^2} dt < +\infty,$

(γ) $\log(t) = o(\omega(t))$ *as* $t \to \infty$,

(δ) $\varphi_\omega : t \mapsto \omega(e^t)$ *is convex.*

We extend the weight function ω *to* $\mathbb{C}^d$ *in a radial way:* $\omega(z) = \omega(|z|)$, $z \in \mathbb{C}^d$, *where* $|z|$ *denotes the Euclidean norm.*

From Definition 2.1(α) we immediately have:

$$\omega(x+y) \leq L(\omega(x)+\omega(y)+1), \qquad x, y \in \mathbb{R}^d. \tag{2.1}$$

For $z \in \mathbb{C}^d$ we denote $\langle z\rangle := \sqrt{1+|z|^2}$. From (2.1) we have

$$\omega(\langle z\rangle) \leq \omega(1+|z|) \leq L\omega(z) + L(1+\omega(1)), \qquad z \in \mathbb{C}^d. \tag{2.2}$$

Definition 2.2. *Given a weight function ω, the* Young conjugate $\varphi^*_\omega : [0,\infty[\to [0,\infty[$ *of φ_ω is defined as*

$$\varphi^*_\omega(t) := \sup_{s\geq 0}\{st - \varphi_\omega(s)\}.$$

When the weight function ω is clear or irrelevant in the context, we simply denote φ_ω and φ^*_ω by φ and φ^*. From now on, we assume that $\omega|_{[0,1]} \equiv 0$, which implies that $\varphi^*(0) = 0$ (in particular, this gives that $\omega(1) = 0$ in formula (2.2)). Moreover, it is known that φ^* is convex, the function $\varphi^*(x)/x$ is increasing for $x > 0$ and $\varphi^{**} := (\varphi^*)^* = \varphi$ (see [9]). From [18, Remark 2.8(c)] is not difficult to see (cf. [6, Lemma A.1]):

Proposition 2.3. *If a weight function ω satisfies $\omega(t) = o(t^a)$ as $t \to +\infty$ for some $0 < a \leq 1$, then for every $B > 0$ and $\lambda > 0$ there exists $C > 0$ such that*

$$B^n n! \leq Ce^{a\lambda\varphi^*(\frac{n}{\lambda})}, \qquad n \in \mathbb{N}_0.$$

The following result can be found in [9].

Lemma 2.4. *(1) Let $L > 0$ be such that $\omega(et) \leq L(\omega(t)+1)$. Then*

$$\lambda L^n\varphi^*\big(\frac{y}{\lambda L^n}\big) + ny \leq \lambda\varphi^*\big(\frac{y}{\lambda}\big) + \lambda\sum_{j=1}^{n} L^j \tag{2.3}$$

for every $y \geq 0$, $\lambda > 0$, $n \in \mathbb{N}$.

(2) For all $s, t, \lambda > 0$, we have

$$2\lambda\varphi^*\big(\frac{s+t}{2\lambda}\big) \leq \lambda\varphi^*\big(\frac{s}{\lambda}\big) + \lambda\varphi^*\big(\frac{t}{\lambda}\big) \leq \lambda\varphi^*\big(\frac{s+t}{\lambda}\big). \tag{2.4}$$

We will consider without losing generality with no explicit mention that the constant $L \geq 1$ that comes from Definition 2.1(α) fulfils the condition of Lemma 2.4. For more results involving φ^* see, for instance, [1, 9, 16] and [6, Lemma A.1].

We deal with a class of global ultradifferentiable functions, which extends the classical Schwartz class with the use of weight functions. It was introduced by Björck [2], but only considering a subadditive weight function ω (so the following definition is slightly more general than the given by Björck).

Definition 2.5. *For a weight ω as in Definition 2.1 we define $\mathcal{S}_\omega(\mathbb{R}^d)$ as the set of all $u \in L^1(\mathbb{R}^d)$ such that (u and its Fourier transform $\widehat{u}$ belong to $C^\infty(\mathbb{R}^d)$ and)*

(i) for each $\lambda > 0$ and $\alpha \in \mathbb{N}^d_0$, $\displaystyle\sup_{x\in\mathbb{R}^d} e^{\lambda\omega(x)}|D^\alpha u(x)| < +\infty,$

(ii) for each $\lambda > 0$ and $\alpha \in \mathbb{N}^d_0$, $\displaystyle\sup_{\xi\in\mathbb{R}^d} e^{\lambda\omega(\xi)}|D^\alpha \widehat{u}(\xi)| < +\infty.$

The corresponding strong dual is denoted by $\mathcal{S}'_\omega(\mathbb{R}^d)$ *and is the set of all the linear and continuous functionals* $u : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathbb{C}$. *We say that an element of* $\mathcal{S}'_\omega(\mathbb{R}^d)$ *is an* ω-temperate ultradistribution.

The space $\mathcal{S}_\omega(\mathbb{R}^d)$ has been studied for different purposes by many authors. We refer, for instance, to [4, 6, 17] for some examples of publications that treat different problems in the setting of the class $\mathcal{S}_\omega(\mathbb{R}^d)$. We recall here [1, Lemma 2.11], which will be useful below.

Lemma 2.6. *If* $f \in \mathcal{S}(\mathbb{R}^d)$, *then* $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ *if and only if for every* $\lambda, \mu > 0$ *there is* $C_{\lambda,\mu} > 0$ *such that for all* $\alpha \in \mathbb{N}_0^d$ *and* $x \in \mathbb{R}^d$, *we have*

$$|D^\alpha f(x)| \leq C_{\lambda,\mu} e^{\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)} e^{-\mu\omega(x)}.$$

From now on, m denotes a real number and $0 < \rho \leq 1$. In the following, we consider global symbols and global amplitudes of infinite order defined very similarly to the ones in [1, Definitions 3.1 and 3.2]. The unique difference is the factor $e^{m\omega(x,\xi)}$ in the case of symbols and $e^{m\omega(x,y,\xi)}$ in the case of amplitudes, which are more suitable for our purposes. We observe that these definitions are *equivalent* to those in [1]. In fact, when considering symbols for example, it is enough to use that there exist $A, B > 0$ such that $A(\omega(x)+\omega(\xi)) \leq \omega(x,\xi) \leq B(\omega(x)+\omega(\xi)+1)$ for every $x, \xi \in \mathbb{R}^d$.

Definition 2.7. *A* global symbol *(of order m) in* $\mathrm{GS}_\rho^{m,\omega}$ *is a function* $p(x,\xi) \in C^\infty(\mathbb{R}^{2d})$ *such that for all* $n \in \mathbb{N}$ *there exists* $C_n > 0$ *with*

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq C_n \langle (x,\xi)\rangle^{-\rho|\alpha+\beta|} e^{n\rho\varphi^*\left(\frac{|\alpha+\beta|}{n}\right)} e^{m\omega(x,\xi)},$$

for all $\alpha, \beta \in \mathbb{N}_0^d$ *and* $x, \xi \in \mathbb{R}^d$.

Definition 2.8. *A* global amplitude *(of order m) in* $\mathrm{GA}_\rho^{m,\omega}$ *is a function* $a(x,y,\xi) \in C^\infty(\mathbb{R}^{3d})$ *such that for all* $n \in \mathbb{N}$ *there exists* $C_n > 0$ *with*

$$|D_x^\alpha D_y^\beta D_\xi^\gamma a(x,y,\xi)| \leq C_n \frac{\langle x-y\rangle^{\rho|\alpha+\beta+\gamma|}}{\langle (x,y,\xi)\rangle^{\rho|\alpha+\beta+\gamma|}} e^{n\rho\varphi^*\left(\frac{|\alpha+\beta+\gamma|}{n}\right)} e^{m\omega(x,y,\xi)},$$

for all $\alpha, \beta, \gamma \in \mathbb{N}_0^d$ *and* $x, y, \xi \in \mathbb{R}^d$.

In [1] we introduce global pseudodifferential operators on $\mathcal{S}_\omega(\mathbb{R}^d)$ by means of oscillatory integrals for global amplitudes as in Definition 2.8 (see [1, Proposition 3.3]). It turns out that the action of a pseudodifferential operator on a function in $\mathcal{S}_\omega(\mathbb{R}^d)$ can be written as an iterated integral [1, Theorem 3.7] and it is continuous and linear from $\mathcal{S}_\omega(\mathbb{R}^d)$ into itself. In fact, we use these properties to state the following definition:

Definition 2.9. *Given a global amplitude* $a(x,y,\xi) \in \mathrm{GA}_\rho^{m,\omega}$ *(as in Definition 2.8), we define the associated* global pseudodifferential operator $A : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ *by*

$$A(f)(x) := \int \Big(\int e^{i(x-y)\cdot\xi} a(x,y,\xi) f(y)dy \Big) d\xi, \qquad f \in \mathcal{S}_\omega(\mathbb{R}^d).$$

Moreover, this operator can be extended linearly and continuously to an operator $\widetilde{A}$ from $\mathcal{S}'_\omega(\mathbb{R}^d)$ into $\mathcal{S}'_\omega(\mathbb{R}^d)$ [1, Proposition 3.10].

At some stages we need classes of ultradifferentiable functions defined in the local sense; we refer the reader to [9, 16] for a theory of pseudodifferential operators of infinite order when defined in local spaces. Let ω be a weight function. For an open set $\Omega \subset \mathbb{R}^d$, we define the space of *ultradifferentiable functions of Beurling type* in Ω as

$$\mathcal{E}_\omega(\Omega) := \big\{ f \in C^\infty(\Omega) : |f|_{K,\lambda} < \infty \text{ for every } \lambda > 0, \text{ and every } K \subset \Omega \text{ compact} \big\},$$

where

$$|f|_{K,\lambda} := \sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in K} |D^\alpha f(x)| e^{-\lambda \varphi^*\left(\frac{|\alpha|}{\lambda}\right)}.$$

We endow such space with the Fréchet topology given by the sequence of seminorms $|f|_{K_n,n}$, where $(K_n)_n$ is any compact exhaustion of Ω and $n \in \mathbb{N}$. The strong dual of $\mathcal{E}_\omega(\Omega)$ is the space of compactly supported ultradistributions of Beurling type and is denoted by $\mathcal{E}'_\omega(\Omega)$.

The space of *ultradifferentiable functions of Beurling type with compact support* in Ω is denoted by $\mathcal{D}_\omega(\Omega)$, and it is the space of those functions $f \in \mathcal{E}_\omega(\Omega)$ such that its support, denoted by $\operatorname{supp} f$, is compact in Ω. Its corresponding dual space is denoted by $\mathcal{D}'_\omega(\Omega)$ and it is called the space of *ultradistributions of Beurling type* in Ω. The following continuous embeddings hold:

$$\mathcal{E}'_\omega(\mathbb{R}^d) \subseteq \mathcal{S}'_\omega(\mathbb{R}^d) \subseteq \mathcal{D}'_\omega(\mathbb{R}^d).$$

We recall that the space $\mathcal{S}_\omega(\mathbb{R}^d)$, as well as its strong dual $\mathcal{S}'_\omega(\mathbb{R}^d)$, are stable under Fourier transform (see, for instance [2]).

Since the global amplitudes have exponential growth in all the variables, it becomes useful a particular kind of integration by parts to understand the behaviour of a pseudodifferential operator in this setting. Following [24], but with a different point of view, we use in [1] entire functions with prescribed exponential growth in terms of a weight function ω. The existence of this type of entire functions was proven by Braun [8] and Langenbruch [20]. In several variables we have a similar result:

Theorem 2.10 ([1], Theorem 2.16)**.** *Let $\omega : [0,\infty[\to [0,\infty[$ be a continuous and increasing function satisfying the conditions (α), (γ), and (δ) of Definition 2.1. Then there are a function $G \in \mathcal{H}(\mathbb{C}^d)$ and some constants $C_1, C_2, C_3, C_4 > 0$ such that*

i') $\log|G(z)| \leq \omega(z) + C_1, \quad z \in \mathbb{C}^d;$

ii') $\log|G(z)| \geq C_2\omega(z) - C_4, \quad z \in \widetilde{U} := \{z \in \mathbb{C}^d : |\operatorname{Im}(z)| \leq C_3(|\operatorname{Re}(z)| + 1)\}.$

We also need the notion of ω-*ultradifferential operator* with constant coefficients. Let G be an entire function in $\mathbb{C}^d$ with $\log|G| = O(\omega)$. For $\varphi \in \mathcal{E}_\omega(\mathbb{R}^d)$, the map $T_G : \mathcal{E}_\omega(\mathbb{R}^d) \to \mathbb{C}$ given by

$$T_G(\varphi) := \sum_{\alpha \in \mathbb{N}_0^d} \frac{D^\alpha G(0)}{\alpha!} D^\alpha \varphi(0)$$

defines an ultradistribution $T_G \in \mathcal{E}'_\omega(\mathbb{R}^d)$ with support equal to $\{0\}$. The convolution operator $G(D) : \mathcal{D}'_\omega(\mathbb{R}^d) \to \mathcal{D}'_\omega(\mathbb{R}^d)$ defined by $G(D)(\mu) = T_G * \mu$ is said to be an ultradifferential operator of ω-class.

Proposition 2.11. *Let G be the entire function given in Theorem 2.10 and $n \in \mathbb{N}$. If*

$$G^n(z) = \sum_{\alpha \in \mathbb{N}_0^n} b_\alpha z^\alpha, \qquad z \in \mathbb{C}^d$$

denotes the n-th power of G, then there exist $C, K > 0$ such that

$$|b_\alpha| \leq e^{nC} e^{-nC\varphi^*\left(\frac{|\alpha|}{nC}\right)}, \quad \alpha \in \mathbb{N}_0^d;$$
$$|G^n(\xi)| \geq C^{-n} e^{nK\omega(\xi)}, \quad \xi \in \mathbb{R}^d.$$

The following result characterizes those operators whose kernel is a function in $\mathcal{S}_\omega(\mathbb{R}^d)$. These operators are fundamental to understand the symbolic calculus. The proof is standard.

Proposition 2.12. *Let $T : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ be a pseudodifferential operator. The following assertions are equivalent:*

(1) *T has a linear and continuous extension $\widetilde{T} : \mathcal{S}'_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$.*

(2) *There exists $K(x,y) \in \mathcal{S}_\omega(\mathbb{R}^{2d})$ such that*

$$(T\varphi)(x) = \int K(x,y)\varphi(y)dy, \qquad \varphi \in \mathcal{S}_\omega(\mathbb{R}^d).$$

Any operator $T : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ which satisfies (1) or (2) of Proposition 2.12 is called *ω-regularizing.*

3 Symbolic calculus for quantizations

We generalize the symbolic calculus developed in [1] for quantizations.

Definition 3.1. *We define $\mathrm{FGS}^{m,\omega}_\rho$ to be the set of all formal sums $\sum_{j\in\mathbb{N}_0} a_j(x,\xi)$ such that $a_j(x,\xi) \in C^\infty(\mathbb{R}^{2d})$ and there is $R \geq 1$ such that for every $n \in \mathbb{N}$ there exists $C_n > 0$ with*

$$|D^\alpha_x D^\beta_\xi a_j(x,\xi)| \leq C_n \langle (x,\xi)\rangle^{-\rho(|\alpha+\beta|+j)} e^{n\rho\varphi^*\left(\frac{|\alpha+\beta|+j}{n}\right)} e^{m\omega(x,\xi)},$$

for each $j \in \mathbb{N}_0$, $\alpha, \beta \in \mathbb{N}^d_0$, and $\log\left(\frac{\langle (x,\xi)\rangle}{R}\right) \geq \frac{n}{j}\varphi^*\left(\frac{j}{n}\right)$.

Definition 3.2. *Two formal sums $\sum a_j$ and $\sum b_j$ in $\mathrm{FGS}^{m,\omega}_\rho$ are said to be* equivalent, *denoted by $\sum a_j \sim \sum b_j$, if there is $R \geq 1$ such that for each $n \in \mathbb{N}$ there exist $C_n > 0$ and $N_n \in \mathbb{N}$ with*

$$\Big| D^\alpha_x D^\beta_\xi \sum_{j<N} (a_j - b_j)\Big| \leq C_n \langle (x,\xi)\rangle^{-\rho(|\alpha+\beta|+N)} e^{n\rho\varphi^*\left(\frac{|\alpha+\beta|+N}{n}\right)} e^{m\omega(x,\xi)},$$

for every $N \geq N_n$, $\alpha, \beta \in \mathbb{N}^d_0$, and $\log\left(\frac{\langle (x,\xi)\rangle}{R}\right) \geq \frac{n}{N}\varphi^*\left(\frac{N}{n}\right)$.

The following construction has been carried out in [1] following the lines of [16, Theorem 3.7]. Let $\Phi \in \mathcal{D}_\sigma(\mathbb{R}^{2d})$, where σ is a weight function which satisfies $\omega(t^{1/\rho}) = O(\sigma(t))$, as $t \to +\infty$, and

$$|\Phi(t)| \leq 1, \qquad \Phi(t) = 1 \text{ if } |t| \leq 2, \qquad \Phi(t) = 0 \text{ if } |t| \geq 3. \tag{3.1}$$

Let $(j_n)_n$ be a sequence of natural numbers such that $j_n/n \to \infty$ as n tends to infinity. For each $j_n \leq j < j_{n+1}$, we set

$$\varphi_j(x,\xi) := 1 - \Phi\Big(\frac{(x,\xi)}{A_{n,j}}\Big), \qquad A_{n,j} = Re^{\frac{n}{j}\varphi^*_\omega\left(\frac{j}{n}\right)}, \tag{3.2}$$

where $R \geq 1$ is the constant which appears in Definition 3.1. It is understood that $\varphi_0 = 1$. We have shown in [1] that $\varphi_j \in \mathrm{GS}^{0,\omega}_\rho$. Moreover, if $\sum_j a_j \in \mathrm{FGS}^{m,\omega}_\rho$ then, by [1, Theorem 4.6],

$$a(x,\xi) := \sum_{j=0}^{\infty} \varphi_j(x,\xi) a_j(x,\xi)$$

is a global symbol in $\mathrm{GS}_\rho^{m,\omega}$, equivalent to $\sum_j a_j$ in $\mathrm{FGS}_\rho^{m,\omega}$.

Now, we extend some results in [1] for quantizations. In what follows, τ stands for a real number. Let $k \in \mathbb{N}_0$ denote the minimum natural number satisfying

$$|\tau| + |1-\tau| \leq 2^k. \tag{3.3}$$

Furthermore, for any $m \in \mathbb{R}$ we denote

$$m' = mL^k, \tag{3.4}$$

where $L \geq 1$ is the constant of Lemma 2.4. We observe that $m' = m$ if and only if $0 \leq \tau \leq 1$.

Lemma 3.3. *If $b(x,\xi) \in \mathrm{GS}_\rho^{m,\omega}$ and $\tau \in \mathbb{R}$, then*

$$a(x,y,\xi) := b((1-\tau)x + \tau y, \xi)$$

is a global amplitude in $\mathrm{GA}_\rho^{\max\{0,m'\},\omega}$.

Proof. The following inequality is easy to check:

$$\langle (x,y,\xi)\rangle \leq \sqrt{6}\langle \tau\rangle \langle x-y\rangle \langle ((1-\tau)x+\tau y,\xi)\rangle, \qquad x,y,\xi \in \mathbb{R}^d,\ \tau \in \mathbb{R}.$$

We take $\widetilde{p} \in \mathbb{N}$ such that $\max\{|1-\tau|, |\tau|, (\sqrt{6}\langle\tau\rangle)^\rho\} \leq e^{\rho\widetilde{p}}$. By assumption, for all $\lambda > 0$ there exists $C_\lambda > 0$ such that ($L \geq 1$ is the constant of Lemma 2.4)

$$\begin{aligned}|D_x^\alpha D_y^\beta D_\xi^\gamma a(x,y,\xi)| \leq |1-\tau|^{|\alpha|}|\tau|^{|\beta|} C_\lambda \langle ((1-\tau)x+\tau y,\xi)\rangle^{-\rho|\alpha+\beta+\gamma|} \times \\ \times\, e^{\lambda L^{2\widetilde{p}}\rho\varphi^*\left(\frac{|\alpha+\beta+\gamma|}{\lambda L^{2\widetilde{p}}}\right)} e^{m\omega((1-\tau)x+\tau y,\xi)}.\end{aligned}$$

The choice of $\widetilde{p}$ gives $|1-\tau|^{|\alpha|}|\tau|^{|\beta|}(\sqrt{6}\langle\tau\rangle)^{\rho|\alpha+\beta+\gamma|} \leq e^{2\widetilde{p}\rho|\alpha+\beta+\gamma|}$. Then, by (2.3), we get

$$\left[e^{2\widetilde{p}|\alpha+\beta+\gamma|} e^{\lambda L^{2\widetilde{p}}\varphi^*\left(\frac{|\alpha+\beta+\gamma|}{\lambda L^{2\widetilde{p}}}\right)}\right]^\rho \leq e^{\lambda\rho\varphi^*\left(\frac{|\alpha+\beta+\gamma|}{\lambda}\right)} e^{\lambda\rho\sum_{j=1}^{2\widetilde{p}} L^j}.$$

Finally, since ω is radial and increasing, applying k times property (α) of the weight function ω, we get, for $m \geq 0$,

$$e^{m\omega((1-\tau)x+\tau y,\xi)} \leq e^{m\omega(2^k(x,y,\xi))} \leq e^{m'\omega(x,y,\xi)} e^{mL^k + mL^{k-1} + \cdots + mL}. \tag{3.5}$$

□

Corollary 3.4. *Let φ_j be the function in (3.2). For all $\lambda > 0$ there exists $C_\lambda > 0$ such that*

$$|D_x^\alpha D_y^\beta D_\xi^\gamma \varphi_j((1-\tau)x+\tau y,\xi)| \leq C_\lambda \langle ((1-\tau)x+\tau y,\xi)\rangle^{-\rho|\alpha+\beta+\gamma|} e^{\lambda\rho\varphi^*\left(\frac{|\alpha+\beta+\gamma|}{\lambda}\right)},$$

for every $\alpha,\beta,\gamma \in \mathbb{N}_0^d$ and $x,y,\xi \in \mathbb{R}^d$. Hence $\varphi_j((1-\tau)x+\tau y,\xi) \in \mathrm{GA}_\rho^{0,\omega}$ for all $\tau \in \mathbb{R}$.

Here, we generalize [1, Lemma 4.7] to readapt it to our context.

Lemma 3.5. *Let $a(x,y,\xi)$ be an amplitude in $\mathrm{GA}_\rho^{m,\omega}$ and let A be the corresponding pseudodifferential operator. For each $u \in \mathcal{S}_\omega(\mathbb{R}^d)$,*

$$A(u) = \mathcal{S}_\omega(\mathbb{R}^d) - \sum_{j=0}^\infty A_j(u),$$

where A_j is the pseudodifferential operator defined by the amplitude

$$(\varphi_j - \varphi_{j+1})((1-\tau)x+\tau y,\xi) a(x,y,\xi), \quad j \in \mathbb{N}_0.$$

Proof. By Corollary 3.4, $(\varphi_j - \varphi_{j+1})((1-\tau)x+\tau y,\xi)a(x,y,\xi) \in \mathrm{GA}_\rho^{m,\omega}$. Since $A_{n,N+1} \to \infty$ as $N \to \infty$, proceeding as in [1, Proposition 3.3], one can show that, for each $u \in \mathcal{S}_\omega(\mathbb{R}^d)$,

$$\sum_{j=0}^{\infty} A_j(u)(x) = \mathcal{S}_\omega(\mathbb{R}^d) - \lim_{N\to\infty} \iint e^{i(x-y)\cdot\xi}\big(1-\varphi_{N+1}((1-\tau)x+\tau y,\xi)\big)a(x,y,\xi)u(y)dyd\xi.$$

We show that this limit is, for all $\tau \in \mathbb{R}$, equal to A in $L(\mathcal{S}_\omega(\mathbb{R}^d), \mathcal{S}'_\omega(\mathbb{R}^d))$. We recall that

$$(1-\varphi_{N+1})((1-\tau)x+\tau y,\xi) = \Phi\Big(\frac{((1-\tau)x+\tau y,\xi)}{A_{n,N+1}}\Big)$$

and $\Phi(0) = 1$, being $\Phi \in \mathcal{D}_\sigma(\mathbb{R}^{2d})$ the function in (3.1) with $\omega(t^{1/\rho}) = O(\sigma(t))$, $t \to \infty$. We claim that for each $f, g \in \mathcal{S}_\omega(\mathbb{R}^d)$,

$$\iiint e^{i(x-y)\cdot\xi}\Big(\Phi\Big(\frac{(1-\tau)x+\tau y,\xi}{k}\Big)-1\Big)a(x,y,\xi)f(y)g(x)dyd\xi dx \to 0 \tag{3.6}$$

as $k \to \infty$. We use the following identity to integrate by parts with the ultradifferential operator $G(D)$ associated to the entire function in Proposition 2.11:

$$e^{i(x-y)\cdot\xi} = \frac{1}{G^s(\xi)}G^s(-D_y)e^{i(x-y)\cdot\xi}, \tag{3.7}$$

for some power $s \in \mathbb{N}$ that we will determine later. Then, the integrand in the left-hand side of (3.6) equals

$$\begin{aligned}
&e^{i(x-y)\cdot\xi}\frac{1}{G^s(\xi)}G^s(D_y)\Big(\Big(\Phi\Big(\frac{(1-\tau)x+\tau y,\xi}{k}\Big)-1\Big)a(x,y,\xi)f(y)g(x)\Big)\\
&\quad = e^{i(x-y)\cdot\xi}\frac{1}{G^s(\xi)}\sum_{\eta\in\mathbb{N}_0^d} b_\eta \sum_{\eta_1+\eta_2+\eta_3=\eta}\frac{\eta!}{\eta_1!\eta_2!\eta_3!}\big(\frac{\tau}{k}\big)^{|\eta_1|}D_y^{\eta_1}\Big(\Phi\Big(\frac{(1-\tau)x+\tau y,\xi}{k}\Big)-1\Big)\times\\
&\qquad \times D_y^{\eta_2}a(x,y,\xi)D_y^{\eta_3}f(y)g(x).
\end{aligned}$$

Therefore, the integral in (3.6) is equal to

$$\begin{aligned}
&\sum_{\eta\in\mathbb{N}_0^d} b_\eta \sum_{\eta_1+\eta_2+\eta_3=\eta}\frac{\eta!}{\eta_1!\eta_2!\eta_3!}\big(\frac{\tau}{k}\big)^{|\eta_1|}\iiint e^{i(x-y)\cdot\xi}\frac{1}{G^s(\xi)}\times\\
&\qquad \times D_y^{\eta_1}\Big(\Phi\Big(\frac{(1-\tau)x+\tau y,\xi}{k}\Big)-1\Big)D_y^{\eta_2}a(x,y,\xi)D_y^{\eta_3}f(y)g(x)dyd\xi dx.
\end{aligned}$$

From Proposition 2.11, there are $C_1, C_2, C_3 > 0$ (depending only on G) such that for all $\eta \in \mathbb{N}_0^d$ and $\xi \in \mathbb{R}^d$ we have

$$|b_\eta| \le e^{sC_1}e^{-sC_1\varphi^*\left(\frac{|\eta|}{sC_1}\right)}, \qquad \Big|\frac{1}{G^s(\xi)}\Big| \le C_3^s e^{-sC_2\omega(\xi)}. \tag{3.8}$$

It follows from Definition 2.8 (see for example [1, Lemma 2.6]) that for all $\lambda > 0$ there exists $C_\lambda > 0$ such that ($L \ge 1$ is the constant of Lemma 2.4)

$$|D_y^{\eta_2}a(x,y,\xi)| \le C_\lambda e^{\lambda L^3\varphi^*\left(\frac{|\eta_2|}{\lambda L^3}\right)}e^{m\omega(x,y,\xi)}.$$

Since $f, g \in \mathcal{S}_\omega(\mathbb{R}^d)$, there exist $C'_{\lambda,m}, C_m > 0$ such that

$$|D_y^{\eta_3} f(y)| \le C'_{\lambda,m} e^{\lambda L^3 \varphi^*\left(\frac{|\eta_3|}{\lambda L^3}\right)} e^{-(mL+1)\omega(y)}, \qquad |g(x)| \le C_m e^{-(mL+1)\omega(x)}.$$

For $\eta_1 = 0$ we have $\Phi \equiv 1$ if $|((1-\tau)x + \tau y, \xi)| \le 2k$, and for $|\eta_1| > 0$ it follows that $D_y^{\eta_1}\Big(\Phi\Big(\frac{(1-\tau)x+\tau y,\xi}{k}\Big) - 1\Big) = D_y^{\eta_1}\Phi\Big(\frac{(1-\tau)x+\tau y,\xi}{k}\Big)$ is zero for $|((1-\tau)x + \tau y, \xi)| \le 2k$; therefore, we can assume that $|((1-\tau)x + \tau y, \xi)| > 2k$. In particular, we have

$$1 \le \frac{1}{2k}|((1-\tau)x + \tau y, \xi)| \le \frac{1}{k}(|1-\tau| + |\tau|)(|x|+1)(|y|+1)(|\xi|+1).$$

As $\Phi \in \mathcal{D}_\sigma(\mathbb{R}^{2d}) \subseteq \mathcal{D}_\omega(\mathbb{R}^{2d})$, there exists $C''_\lambda > 0$ such that

$$|\tau|^{|\eta_1|}\Big|D_y^{\eta_1}\Big(\Phi\Big(\frac{(1-\tau)x+\tau y,\xi}{k}\Big) - 1\Big)\Big| \le C''_\lambda e^{\lambda L^3 \varphi^*\left(\frac{|\eta_1|}{\lambda L^3}\right)}, \qquad \eta_1 \in \mathbb{N}_0^d.$$

For $m \ge 0$ (if $m < 0$, then $m\omega(x,y,\xi) < 0$), since

$$m\omega(x,y,\xi) \le mL\omega(x) + mL\omega(y) + mL\omega(\xi) + mL,$$

it is enough to take $s \in \mathbb{N}$ satisfying $sC_2 \ge mL + 1$ to get $e^{(-sC_2+mL)\omega(\xi)} \le e^{-\omega(\xi)}$, and therefore the integrals are convergent by condition (γ) of the weight ω. On the other hand, since $\sum \frac{\eta!}{\eta_1!\eta_2!\eta_3!} = 3^{|\eta|} \le e^{2|\eta|}$, by Lemma 2.4 we have

$$\sum_{\eta_1+\eta_2+\eta_3=\eta} \frac{\eta!}{\eta_1!\eta_2!\eta_3!} e^{\lambda L^3\varphi^*\left(\frac{|\eta_1|}{\lambda L^3}\right)} e^{\lambda L^3\varphi^*\left(\frac{|\eta_2|}{\lambda L^3}\right)} e^{\lambda L^3\varphi^*\left(\frac{|\eta_3|}{\lambda L^3}\right)} \le e^{\lambda L\varphi^*\left(\frac{|\eta|}{\lambda L}\right)} e^{\lambda L^2 + \lambda L^3}.$$

Now, the series

$$\sum_{\eta \in \mathbb{N}_0^d} e^{-sC_1\varphi^*\left(\frac{|\eta|}{sC_1}\right)} e^{\lambda L\varphi^*\left(\frac{|\eta|}{\lambda L}\right)}$$

converges provided $\lambda > sC_1$ (see [1, (3.5), (3.6)]). Thus, there exists $C > 0$ such that

$$\Big|\iiint e^{i(x-y)\cdot\xi}\Big(\Phi\Big(\frac{(1-\tau)x+\tau y,\xi}{k}\Big) - 1\Big)a(x,y,\xi)f(y)g(x)\,dy\,d\xi\,dx\Big| \le C\frac{1}{k} \to 0,$$

and hence (3.6) is satisfied. □

The next result is the corresponding extension of [1, Proposition 4.8].

Lemma 3.6. *Let $\sum p_j \in \mathrm{FGS}_\rho^{m,\omega}$ and let $(C_n)_n, (C'_n)_n$ be the sequences of constants that appear in Definition 3.1 and in the estimate of the derivatives of φ_j in Corollary 3.4. We denote $D_n := C_{2nL^{\tilde{p}+1}}$ and $D'_n := C'_{nL^{\tilde{p}+1}}$, where $L \ge 1$ is the constant of Lemma 2.4 and $\tilde{p} \in \mathbb{N}_0$ is so that $3\langle\tau\rangle \le e^{\tilde{p}}$, for a fixed $\tau \in \mathbb{R}$. Consider $(j_n)_n$, $j_n \in \mathbb{N}$, such that $j_1 = 1$, $j_n < j_{n+1}$, $j_n/n \to \infty$ and*

$$D_{n+1}D'_{n+1}\sum_{j=j_{n+1}}^{\infty}(2R)^{-\rho j} \le \frac{D_nD'_n}{2}\sum_{j=j_n}^{j_{n+1}-1}(2R)^{-\rho j}, \quad n \in \mathbb{N},$$

and moreover,

$$\frac{n}{j}\varphi^*\Big(\frac{j}{n}\Big) \ge \max\{n, \log D_n, \log D'_n\}, \quad \text{for } j \ge j_n.$$

If

$$a(x,\xi) := \sum_{j=0}^{\infty} \varphi_j(x,\xi) p_j(x,\xi),$$

then the associated pseudodifferential operator A is the limit in $L(\mathcal{S}_\omega(\mathbb{R}^d), \mathcal{S}'_\omega(\mathbb{R}^d))$ of the sequence of operators $S_{N,\tau} : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$, where each $S_{N,\tau}$ is a pseudodifferential operator with amplitude

$$\sum_{j=0}^{N} (\varphi_j - \varphi_{j+1})((1-\tau)x + \tau y, \xi)(\sum_{l=0}^{j} p_l((1-\tau)x + \tau y, \xi)).$$

Proof. For each $j \in \mathbb{N}_0$, one can show that

$$(\varphi_j - \varphi_{j+1})((1-\tau)x + \tau y, \xi) \sum_{l=0}^{j} p_l((1-\tau)x + \tau y, \xi) = \sum_{l=0}^{j} ((\varphi_j - \varphi_{j+1}) p_l)((1-\tau)x + \tau y, \xi)$$

is a global amplitude in $\mathrm{GA}_\rho^{\max\{0,m'\},\omega}$, m' being set in (3.4). Hence, the function

$$\sum_{j=0}^{N} (\varphi_j - \varphi_{j+1}) \Big(\sum_{l=0}^{j} p_l \Big) = \sum_{j=0}^{N} \varphi_j p_j - \varphi_{N+1} \sum_{l=0}^{N} p_l$$

is a global amplitude in $\mathrm{GA}_\rho^{\max\{0,m'\},\omega}$.

Now, we prove that $S_{N,\tau} \to A$ in $L(\mathcal{S}_\omega(\mathbb{R}^d), \mathcal{S}'_\omega(\mathbb{R}^d))$ as $N \to \infty$. As in the proof of [1, Proposition 4.8], it is enough to show that, for any $f, g \in \mathcal{S}_\omega(\mathbb{R}^d)$, $\langle (S_{N,\tau} - A) f, g \rangle \to 0$ as $N \to \infty$. Note that A and $S_{N,\tau}$, $N = 1, 2, \ldots$ act continuously on $\mathcal{S}_\omega(\mathbb{R}^d)$. Thus

$$\begin{aligned} \langle (S_{N,\tau} - A) f, g \rangle &= \int (S_{N,\tau} - A) f(x) g(x) dx \\ &= \int \Big(\iint e^{i(x-y)\cdot\xi} \Big(\Big\{ \sum_{j=0}^{N} \varphi_j p_j - \varphi_{N+1} \sum_{l=0}^{N} p_l \Big\} - a \Big) f(y) dy d\xi \Big) g(x) dx \end{aligned}$$

for every $f, g \in \mathcal{S}_\omega(\mathbb{R}^d)$, where $\varphi_j, \varphi_N, p_j, p_l$, and a are evaluated at $((1-\tau)x + \tau y, \xi)$.

We show that, for each $f, g \in \mathcal{S}_\omega(\mathbb{R}^d)$,

$$a) \int \Big(\iint e^{i(x-y)\cdot\xi} \Big(\sum_{j=N+1}^{\infty} \varphi_j((1-\tau)x + \tau y, \xi) p_j((1-\tau)x + \tau y, \xi) \Big) f(y) dy d\xi \Big) g(x) dx \quad \text{and}$$

$$b) \int \Big(\iint e^{i(x-y)\cdot\xi} \Big(\varphi_{N+1}((1-\tau)x + \tau y, \xi) \sum_{l=0}^{N} p_l((1-\tau)x + \tau y, \xi) \Big) f(y) dy d\xi \Big) g(x) dx$$

tend to zero when $N \to \infty$.

Let us show that the integral in $a)$ goes to zero. We integrate by parts with formula (3.7) for some $s \in \mathbb{N}$ to be determined later. Then

$$\begin{aligned} & e^{i(x-y)\cdot\xi} \frac{1}{G^s(\xi)} G^s(D_y) \Big(\sum_{j=N+1}^{\infty} \varphi_j \cdot p_j \cdot f(y) \Big) \\ & \quad = e^{i(x-y)\cdot\xi} \frac{1}{G^s(\xi)} \sum_{\eta \in \mathbb{N}_0^d} b_\eta \sum_{\eta_1+\eta_2+\eta_3=\eta} \frac{\eta!}{\eta_1!\eta_2!\eta_3!} \sum_{j=N+1}^{\infty} \tau^{|\eta_1+\eta_2|} D_y^{\eta_1} \varphi_j \cdot D_y^{\eta_2} p_j \cdot D_y^{\eta_3} f(y). \end{aligned}$$

Hence, we can reformulate the integral in $a)$ as

$$\int\Big(\int \frac{1}{G^s(\xi)}\sum_{\eta\in\mathbb{N}_0^d} b_\eta \sum_{\eta_1+\eta_2+\eta_3=\eta}\frac{\eta!}{\eta_1!\eta_2!\eta_3!}\tau^{|\eta_1+\eta_2|}\times \\ \times\int e^{i(x-y)\cdot\xi}\sum_{j=N+1}^{\infty} D_y^{\eta_1}\varphi_j\cdot D_y^{\eta_2}p_j\cdot D_y^{\eta_3}f(y)dyd\xi\Big)g(x)dx. \tag{3.9}$$

When $\varphi_j\neq 0$, and $j_n\leq j<j_{n+1}$, we have $\log\big(\frac{\langle((1-\tau)x+\tau y,\xi)\rangle}{2R}\big)\geq \frac{n}{j}\varphi^*\big(\frac{j}{n}\big)$ (see (3.2)). By Corollary 3.4, for each $n\in\mathbb{N}$, the following estimate holds (as in the hypotheses of this lemma, we denote $D_n'=C'_{nL^{\tilde{p}+1}}>0$)

$$|D_y^{\eta_1}\varphi_j((1-\tau)x+\tau y,\xi)|\leq D_n' e^{nL^{\tilde{p}+1}\varphi^*\big(\frac{|\eta_1|}{nL^{\tilde{p}+1}}\big)}.$$

Moreover, for that $n\in\mathbb{N}$ (as in the hypotheses of this lemma, we denote $D_n=C_{2nL^{\tilde{p}+1}}>0$), by (2.4), we have

$$\begin{aligned}
&|D_y^{\eta_2}p_j((1-\tau)x+\tau y,\xi)|\\
&\quad\leq D_n e^{2nL^{\tilde{p}+1}\rho\varphi^*\big(\frac{|\eta_2|+j}{2nL^{\tilde{p}+1}}\big)}\langle((1-\tau)x+\tau y,\xi)\rangle^{-\rho(|\eta_2|+j)}e^{m\omega((1-\tau)x+\tau y,\xi)}\\
&\quad\leq D_n e^{nL^{\tilde{p}+1}\varphi^*\big(\frac{|\eta_2|}{nL^{\tilde{p}+1}}\big)}e^{nL^{\tilde{p}+1}\rho\varphi^*\big(\frac{j}{nL^{\tilde{p}+1}}\big)}\langle((1-\tau)x+\tau y,\xi)\rangle^{-\rho j}e^{m\omega((1-\tau)x+\tau y,\xi)}\\
&\quad\leq D_n e^{nL^{\tilde{p}+1}\varphi^*\big(\frac{|\eta_2|}{nL^{\tilde{p}+1}}\big)}(2R)^{-\rho j}e^{m\omega((1-\tau)x+\tau y,\xi)}.
\end{aligned}$$

Property (γ) of Definition 2.1 yields that there exists $C>0$ such that $\langle x\rangle\leq Ce^{\omega(\langle x\rangle)}$, $x\in\mathbb{R}^d$. Then, using (2.2),

$$\begin{aligned}
e^{m\omega((1-\tau)x+\tau y,\xi)}&\leq e^{(m+3)\omega(\langle((1-\tau)x+\tau y,\xi)\rangle)}e^{-3\omega(\langle((1-\tau)x+\tau y,\xi)\rangle)}\\
&\leq e^{(m+3)L\omega((1-\tau)x+\tau y,\xi)}e^{(m+3)L}C^3\langle((1-\tau)x+\tau y,\xi)\rangle^{-3}\\
&\leq C^3 e^{(m+3)L\omega((1-\tau)x+\tau y,\xi)}e^{(m+3)L}e^{-3\frac{n}{j}\varphi^*(\frac{j}{n})}.
\end{aligned}$$

By (3.5) (k being as in (3.3)), we obtain

$$\begin{aligned}
e^{(m+3)L\omega((1-\tau)x+\tau y,\xi)}&\leq e^{(m+3)L^{k+1}\omega(x,y,\xi)}e^{(m+3)L^{k+1}+\dots+(m+3)L^2}\\
&\leq e^{(m+3)L^{k+2}(\omega(x)+\omega(y)+\omega(\xi))}e^{(m+3)L^{k+2}+\dots+(m+3)L^2}.
\end{aligned}$$

Take $0<\ell<n$. Later, an additional condition will be imposed on ℓ. Since $f,g\in\mathcal{S}_\omega(\mathbb{R}^d)$, there are $C''_\ell>0$, which depends on ℓ,m, and on τ, and $D>0$ that depends on m and on τ such that

$$\begin{aligned}
|D_y^{\eta_3}f(y)|&\leq C''_\ell e^{\ell L^{\tilde{p}+1}\varphi^*\big(\frac{|\eta_3|}{\ell L^{\tilde{p}+1}}\big)}e^{-((m+3)L^{k+2}+1)\omega(y)};\\
|g(x)|&\leq De^{-((m+3)L^{k+2}+1)\omega(x)}.
\end{aligned}$$

Lemma 2.4, the fact that $\sum\frac{\eta!}{\eta_1!\eta_2!\eta_3!}=3^{|\eta|}$ and the choice of $\tilde{p}\in\mathbb{N}$ provide

$$\begin{aligned}
&\sum_{\eta_1+\eta_2+\eta_3=\eta}\frac{\eta!}{\eta_1!\eta_2!\eta_3!}|\tau|^{|\eta_1+\eta_2|}e^{nL^{\tilde{p}+1}\varphi^*\big(\frac{|\eta_1|}{nL^{\tilde{p}+1}}\big)}e^{nL^{\tilde{p}+1}\varphi^*\big(\frac{|\eta_2|}{nL^{\tilde{p}+1}}\big)}e^{\ell L^{\tilde{p}+1}\varphi^*\big(\frac{|\eta_3|}{\ell L^{\tilde{p}+1}}\big)}\\
&\quad\leq\langle\tau\rangle^{|\eta|}e^{\ell L^{\tilde{p}+1}\varphi^*\big(\frac{|\eta|}{\ell L^{\tilde{p}+1}}\big)}\sum_{\eta_1+\eta_2+\eta_3=\eta}\frac{\eta!}{\eta_1!\eta_2!\eta_3!}\\
&\quad\leq e^{\ell L\varphi^*\big(\frac{|\eta|}{\ell L}\big)}e^{\ell L\sum_{r=1}^{\tilde{p}}L^r}.
\end{aligned}$$

Thus, from (3.8), we estimate (3.9) by

$$\int\Big(\int C_3^s e^{-sC_2\omega(\xi)}\sum_{\eta\in\mathbb{N}_0^d} e^{sC_1}e^{-sC_1\varphi^*\left(\frac{|\eta|}{sC_1}\right)}\Big(\int\sum_{j=N+1}^{\infty} D_nD_n' e^{\ell L\varphi^*\left(\frac{|\eta|}{\ell L}\right)}e^{\ell L\sum_{r=1}^{\tilde{p}}L^r}\times$$
$$\times (2R)^{-\rho j}C^3 e^{(m+3)L+(m+3)L^2+\cdots+(m+3)L^{k+2}}e^{(m+3)L^{k+2}(\omega(x)+\omega(y)+\omega(\xi))}\times$$
$$\times e^{-3\frac{n}{j}\varphi^*(\frac{j}{n})}C_\ell'' e^{-((m+3)L^{k+2}+1)\omega(y)}dy\Big)d\xi\Big)De^{-((m+3)L^{k+2}+1)\omega(x)}dx.$$

Take $s\in\mathbb{N}_0$ such that $sC_2\geq(m+3)L^{k+2}+1$. Choosing $\ell\geq sC_1$ we obtain that the series depending on $\eta\in\mathbb{N}_0^d$

$$\sum_{\eta\in\mathbb{N}_0^d} e^{-sC_1\varphi^*\left(\frac{|\eta|}{sC_1}\right)}e^{\ell L\varphi^*\left(\frac{|\eta|}{\ell L}\right)}$$

converges (see [1, (3.6)]). The constant depending on n is D_nD_n'. We get for $j_l\leq N+1<j_{l+1}$, the following estimate for the integral in a):

$$E_\ell\Big(\int e^{-\omega(x)}dx\Big)\Big(\int e^{-\omega(y)}dy\Big)\Big(\int e^{-\omega(\xi)}d\xi\Big)\Big(\sum_{n=l}^{\infty}\sum_{j=j_n}^{j_{n+1}-1}\frac{D_nD_n'}{(2R)^{\rho j}e^{3\frac{n}{j}\varphi^*(\frac{j}{n})}}\Big),$$

where $E_\ell>0$ is a constant depending on ℓ. The last 3 integrals converge by property (γ) of the weight function. By assumption, we have $3\frac{n}{j}\varphi^*\big(\frac{j}{n}\big)\geq\log D_n+\log D_n'+n$. This finally proves that the integral in a) converges to zero as N tends to infinity.

For the limit in b), we can proceed as in [1, Proposition 4.8] with the above techniques. □

The next example recovers [1, Example 4.9] for $\tau=0$. The proof is straightforward and is left to the reader.

Example 3.7. Let $a(x,y,\xi)$ be an amplitude in $\mathrm{GA}_\rho^{m,\omega}$ and let

$$p_j(x,\xi):=\sum_{|\beta+\gamma|=j}\frac{1}{\beta!\gamma!}\tau^{|\beta|}(1-\tau)^{|\gamma|}\partial_\xi^{\beta+\gamma}(-D_x)^\beta D_y^\gamma\, a(x,y,\xi)\big|_{y=x}.$$

Then, the series $\sum_{j=0}^{\infty}p_j(x,\xi)$ is a formal sum in $\mathrm{FGS}_\rho^{\max\{m,mL\},\omega}$ for all $\tau\in\mathbb{R}$.

The following lemma is taken from [16, Lemma 3.11].

Lemma 3.8. *Let $m\geq n$ and $\frac{1}{e}e^{\frac{m}{j}\varphi^*(\frac{j}{m})}\leq t\leq e^{\frac{n}{j}\varphi^*(\frac{j}{n})}$ for $t>0$. Then*

$$e^{n\varphi^*(\frac{j}{n})}\geq e^{(n-1)\omega(t)}e^{2n\varphi^*(\frac{j}{2n})},$$

for j large enough.

These two lemmas are easy to prove.

Lemma 3.9. *Let $\tau\in\mathbb{R}$ and let $k\in\mathbb{N}_0$ as in (3.3). Then we have*

$$\omega(x,y)\leq L^2\omega((1-\tau)x+\tau y)+L^{k+2}\omega(y-x)+\sum_{j=1}^{k+2}L^j,\qquad x,y\in\mathbb{R}^d.$$

Lemma 3.10. *For all $\tau \in \mathbb{R}$, the inequality*

$$|v|^2 \le C(|v+t\tau w|^2 + |v-t(1-\tau)w|^2)$$

holds for all $v, w \in \mathbb{R}^d$, $0 \le t \le 1$, where $C = 2\max\{(1-\tau)^2, \tau^2\}$.

The following result shows that any pseudodifferential operator can be written as a quantization modulo an ω-regularizing operator and is needed to understand the composition of two different quantizations in the next section. For the proof, it is fundamental the fact that the kernel K of a pseudodifferential operator behaves like a function in $\mathcal{S}_\omega(\mathbb{R}^d)$ in the complement of a strip $\Delta_r = \{(x,y) \in \mathbb{R}^{2d} : |x-y| < r\}$ around the diagonal of $\mathbb{R}^{2d}$, for some $r > 0$. In other words, if χ is as in [1, Lemma 5.1], then $\chi K \in \mathcal{S}_\omega(\mathbb{R}^{2d})$ [1, Theorem 5.2].

Theorem 3.11. *Let $a(x,y,\xi)$ be an amplitude in $\mathrm{GA}_\rho^{m,\omega}$ with associated pseudodifferential operator A. Then, for any $\tau \in \mathbb{R}$, we can write A uniquely as*

$$A = P_\tau + R,$$

where R is an ω-regularizing operator and P_τ is the pseudodifferential operator given by

$$P_\tau u(x) = \iint e^{i(x-y)\cdot\xi} p_\tau((1-\tau)x+\tau y, \xi)u(y)dyd\xi, \qquad u \in \mathcal{S}_\omega(\mathbb{R}^d),$$

being $p_\tau \in \mathrm{GS}_\rho^{\max\{m,mL\},\omega}$. Moreover, we have

$$p_\tau(x,\xi) \sim \sum_{j=0}^\infty p_j(x,\xi) = \sum_{j=0}^\infty \sum_{|\beta+\gamma|=j} \frac{1}{\beta!\gamma!}\tau^{|\beta|}(1-\tau)^{|\gamma|}\partial_\xi^{\beta+\gamma}(-D_x)^\beta D_y^\gamma\, a(x,y,\xi)\big|_{y=x}.$$

The symbol $p_\tau(x,\xi)$ is called *τ-symbol of the pseudodifferential operator A.* When $\tau = 0, 1, 1/2$, these symbols are called the *left, right, and Weyl symbols of A.*

Proof. We consider the sequence $(j_n)_n$ as in the statement of Lemma 3.6, with $\frac{n}{j}\varphi^*\left(\frac{j}{n}\right) \ge \max\{n, \log(C_{4nL^{\tilde{p}+3}}), \log(D_{4nL^{\tilde{p}+3}})\}$, where $(C_n)_n$ and $(D_n)_n$ denote the sequences of constants that come from Definition 2.8 and Corollary 3.4, and $\tilde{p} \in \mathbb{N}_0$ is so that $\max\{|1-\tau|, 2|\tau|\} \le e^{\tilde{p}}$. Put

$$p_j(x,\xi) := \sum_{|\beta+\gamma|=j} \frac{1}{\beta!\gamma!}\tau^{|\beta|}(1-\tau)^{|\gamma|}\partial_\xi^{\beta+\gamma}(-D_x)^\beta D_y^\gamma\, a(x,y,\xi)\big|_{y=x}.$$

By Example 3.7, $\sum_j p_j \in \mathrm{FGS}_\rho^{\max\{m,mL\},\omega}$. Now, we write

$$p_\tau(x,\xi) := \sum_{j=0}^\infty \varphi_j(x,\xi)p_j(x,\xi),$$

where $(\varphi_j)_j$ is the sequence described in (3.2). By [1, Theorem 4.6] we obtain that $p_\tau(x,\xi) \in \mathrm{GS}_\rho^{\max\{m,mL\},\omega}$ and $p_\tau \sim \sum p_j$. We set, for $u \in \mathcal{S}_\omega(\mathbb{R}^d)$,

$$P_\tau u(x) := \iint e^{i(x-y)\cdot\xi} p_\tau((1-\tau)x+\tau y, \xi)u(y)dyd\xi.$$

By Lemma 3.6, P_τ is the limit of $S_{N,\tau}$ in $L(\mathcal{S}_\omega(\mathbb{R}^d), \mathcal{S}'_\omega(\mathbb{R}^d))$, where $S_{N,\tau}$ is the pseudodifferential operator with amplitude $\sum_{j=0}^N(\varphi_j - \varphi_{j+1})((1-\tau)x+\tau y,\xi)(\sum_{l=0}^j p_l((1-\tau)x+\tau y,\xi))$ in

$\mathrm{GA}_\rho^{\max\{0,m'L\},\omega}$, m' as in (3.4). On the other hand, from Lemma 3.5, $A = \sum_{N=0}^\infty A_N$, where A_N is the pseudodifferential operator with amplitude $a(x,y,\xi)(\varphi_N - \varphi_{N+1})((1-\tau)x + \tau y, \xi)$ in $\mathrm{GA}_\rho^{m,\omega} \subseteq \mathrm{GA}_\rho^{\max\{0,m'L\},\omega}$. Thus, for $u \in \mathcal{S}_\omega(\mathbb{R}^d)$,

$$Au(x) = \sum_{N=0}^\infty \iint e^{i(x-y)\cdot\xi}(\varphi_N - \varphi_{N+1})((1-\tau)x + \tau y, \xi) a(x,y,\xi) u(y) dy d\xi$$

and

$$P_\tau u(x) = \lim_{N\to\infty} \iint e^{i(x-y)\cdot\xi} \big[\sum_{j=0}^N (\varphi_j - \varphi_{j+1})((1-\tau)x+\tau y,\xi) \big(\sum_{l=0}^j p_l((1-\tau)x+\tau y,\xi) \big) \big] u(y) dy d\xi.$$

Hence, we can write $A - P_\tau$ as the series $\sum_{N=0}^\infty P_{N,\tau}$, where each $P_{N,\tau}$ corresponds to the pseudodifferential operator associated to the amplitude in $\mathrm{GA}_\rho^{\max\{0,m'L\},\omega}$:

$$(\varphi_N - \varphi_{N+1})((1-\tau)x + \tau y, \xi)\Big(a(x,y,\xi) - \sum_{j=0}^N p_j((1-\tau)x + \tau y, \xi) \Big).$$

Our purpose is to show that the kernel K of $A - P_\tau$ belongs to $\mathcal{S}_\omega(\mathbb{R}^{2d})$. To that purpose, we write

$$\begin{aligned} K(x,y) &= \sum_{N=0}^\infty K_N(x,y) \\ &= \sum_{N=0}^\infty \int e^{i(x-y)\cdot\xi}(\varphi_N - \varphi_{N+1})((1-\tau)x + \tau y, \xi)\Big(a(x,y,\xi) - \sum_{j=0}^N p_j((1-\tau)x + \tau y, \xi)\Big) d\xi. \end{aligned}$$

Now, we take $r > 0$ and $\chi \in \mathcal{E}_\omega(\mathbb{R}^{2d})$ such that $\chi \equiv 1$ in $\mathbb{R}^{2d} \setminus \Delta_{2r}$, and $\chi \equiv 0$ in $\overline{\Delta_r}$ (see [1, Lemma 5.1]). Then we can write

$$K = \chi K + (1-\chi) \lim_{N\to\infty} \sum_{j=0}^N K_j.$$

We follow the lines of [25, Theorem 23.2], and also the scheme of the proof of [16, Theorem 3.13], as well as [1, Theorem 5.4]. We make the following change of variables:

$$v = (1-\tau)x + \tau y; \qquad w = x - y.$$

Similarly as in [16, Theorem 3.13], we write the partial sums of K as

$$\sum_{j=0}^N K_j = K_0 + \sum_{j=1}^N I_j + \sum_{j=1}^N Q_j - W_N,$$

where

$$
\begin{aligned}
I_j(x,y) &:= \sum_{|\beta+\gamma|=j} \sum_{0\neq\alpha\leq\beta+\gamma} \frac{(\beta+\gamma)!}{\beta!\gamma!} \frac{1}{\alpha!(\beta+\gamma-\alpha)!} \times \\
&\quad \times \int e^{i(x-y)\cdot\xi} \tau^{|\beta|}(1-\tau)^{|\gamma|}(-1)^{|\gamma|} D_\xi^\alpha \varphi_j(v,\xi) \big(\partial_x^\beta \partial_y^\gamma D_\xi^{\beta+\gamma-\alpha} a\big)(v,v,\xi) d\xi; \\
Q_j(x,y) &:= \sum_{|\beta+\gamma|=j+1} \sum_{\alpha\leq\beta+\gamma} \frac{(\beta+\gamma)!}{\beta!\gamma!} \frac{1}{\alpha!(\beta+\gamma-\alpha)!} \tau^{|\beta|}(1-\tau)^{|\gamma|}(-1)^{|\gamma|} \times \\
&\quad \times \int e^{i(x-y)\cdot\xi} D_\xi^\alpha (\varphi_j - \varphi_{j+1})(v,\xi) D_\xi^{\beta+\gamma-\alpha} \omega_{\beta\gamma}(x,y,\xi) d\xi; \\
\omega_{\beta\gamma}(x,y,\xi) &:= (j+1) \int_0^1 \big(\partial_x^\beta \partial_y^\gamma a\big)(v+t\tau w, v-(1-\tau)tw, \xi)(1-t)^j dt; \qquad (3.10) \\
W_N(x,y) &:= \sum_{|\beta+\gamma|=1}^{N} \sum_{0\neq\alpha\leq\beta+\gamma} \frac{(\beta+\gamma)!}{\beta!\gamma!} \frac{1}{\alpha!(\beta+\gamma-\alpha)!} \times \\
&\quad \times \int e^{i(x-y)\cdot\xi} \tau^{|\beta|}(1-\tau)^{|\gamma|}(-1)^{|\gamma|} D_\xi^\alpha \varphi_{N+1}(v,\xi) \big(\partial_x^\beta \partial_y^\gamma D_\xi^{\beta+\gamma-\alpha} a\big)(v,v,\xi) d\xi.
\end{aligned}
$$

We have $\chi K \in \mathcal{S}_\omega(\mathbb{R}^{2d})$ [1, Lemma 5.1, Theorem 5.2]. Moreover, it is easy to see that $K_0 \in \mathcal{S}_\omega(\mathbb{R}^{2d})$. Indeed, we have

$$
K_0(x,y) = \int e^{i(x-y)\cdot\xi}(1-\varphi_1)((1-\tau)x+\tau y,\xi)(a(x,y,\xi)-a(x,x,\xi))d\xi.
$$

Since $1-\varphi_1 \in \mathcal{S}_\omega(\mathbb{R}^{2d})$, following [1, Lemma 3.5(a)] one obtains the desired property for K_0 by Lemma 3.9.

First step. First of all, we compute $D_x^\theta D_y^\epsilon I_j(x,y)$ for $\theta,\epsilon \in \mathbb{N}_0^d$. We use integration by parts with the formula

$$
e^{i(x-y)\cdot\xi} = \frac{1}{G(y-x)} G(-D_\xi) e^{i(x-y)\cdot\xi}, \qquad (3.11)
$$

for a suitable power $G^s(D)$ of $G(D)$, being $G(\xi)$ the entire function considered in Proposition 2.11. We obtain, as in [1, Theorem 5.4],

$$
\begin{aligned}
&D_x^\theta D_y^\epsilon I_j(x,y) \\
&\quad = \sum_{|\beta+\gamma|=j} \sum_{0\neq\alpha\leq\beta+\gamma} \frac{(\beta+\gamma)!}{\beta!\gamma!} \frac{1}{\alpha!(\beta+\gamma-\alpha)!} \frac{1}{G^s(y-x)} \sum_{\eta\in\mathbb{N}_0^d} b_\eta \sum_{\substack{\eta_1+\eta_2+\eta_3=\eta \\ \theta_1+\theta_2+\theta_3=\theta \\ \epsilon_1+\epsilon_2+\epsilon_3=\epsilon}} (-1)^{|\gamma+\epsilon_1|} \times \\
&\quad \times \frac{\eta!}{\eta_1!\eta_2!\eta_3!} \frac{\theta!}{\theta_1!\theta_2!\theta_3!} \frac{\epsilon!}{\epsilon_1!\epsilon_2!\epsilon_3!} \frac{(\theta_1+\epsilon_1)!}{(\theta_1+\epsilon_1-\eta_1)!} \tau^{|\beta+\epsilon_2|}(1-\tau)^{|\gamma+\theta_2|} \times \\
&\quad \times \int e^{i(x-y)\cdot\xi} \xi^{\theta_1+\epsilon_1-\eta_1} D_x^{\theta_2} D_y^{\epsilon_2} D_\xi^{\alpha+\eta_2} \varphi_j(v,\xi) D_x^{\theta_3} D_y^{\epsilon_3} \big(\partial_x^\beta \partial_y^\gamma D_\xi^{\beta+\gamma-\alpha+\eta_3} a\big)(v,v,\xi) d\xi.
\end{aligned}
$$

Fix $\lambda > 0$ and set $n \geq \lambda$ large enough that may depend on τ, m, ρ, L, and R. According to Lemma 3.9, it is enough to take $s \in \mathbb{N}$ such that $sC_2 \geq \lambda L^{k+2}$, where $C_2 > 0$ comes from (3.8) and $k \in \mathbb{N}_0$ as in (3.3). For the convergence of the series depending on $\eta \in \mathbb{N}_0^d$, let n satisfy in addition that $n \geq sC_1$, where $C_1 > 0$ comes from (3.8). Now, proceeding as in [16, Theorem 3.13] (and using Proposition 2.3 and Lemma 3.8) we can show that $\sum_{j=1}^\infty I_j \in \mathcal{S}_\omega(\mathbb{R}^{2d})$.

Second step. Since $1-\chi$ is supported in Δ_{2r}, we estimate $|D_x^\theta D_y^\epsilon Q_j(x,y)|$ for $\theta,\epsilon\in\mathbb{N}_0^d$, $(x,y)\in\Delta_{2r}$. By the formula of integration by parts given in (3.11) for a suitable power of $G(D)$, $G^s(D)$, we have

$$\begin{aligned}
&D_x^\theta D_y^\epsilon Q_j(x,y)\\
&\quad=\sum_{|\beta+\gamma|=j+1}\sum_{\alpha\le\beta+\gamma}\frac{(\beta+\gamma)!}{\beta!\gamma!}\frac{1}{\alpha!(\beta+\gamma-\alpha)!}\frac{1}{G^s(y-x)}\sum_{\eta\in\mathbb{N}_0^d}b_\eta\sum_{\substack{\theta_1+\theta_2+\theta_3=\theta\\ \epsilon_1+\epsilon_2+\epsilon_3=\epsilon\\ \eta_1+\eta_2+\eta_3=\eta}}(-1)^{|\epsilon_1+\gamma|}\times\\
&\qquad\times\frac{\theta!}{\theta_1!\theta_2!\theta_3!}\frac{\epsilon!}{\epsilon_1!\epsilon_2!\epsilon_3!}\frac{\eta!}{\eta_1!\eta_2!\eta_3!}\frac{(\theta_1+\epsilon_1)!}{(\theta_1+\epsilon_1-\eta_1)!}\tau^{|\beta|}(1-\tau)^{|\gamma|}(1-\tau)^{|\theta_2|}\tau^{|\epsilon_2|}\times\\
&\qquad\times\int e^{i(x-y)\cdot\xi}\xi^{\theta_1+\epsilon_1-\eta_1}D_x^{\theta_2}D_y^{\epsilon_2}D_\xi^{\alpha+\eta_2}(\varphi_j-\varphi_{j+1})(v,\xi)D_x^{\theta_3}D_y^{\epsilon_3}(D_\xi^{\beta+\gamma-\alpha+\eta_3}\omega_{\beta\gamma})d\xi,
\end{aligned}$$

where $\omega_{\beta\gamma}=\omega_{\beta\gamma}(x,y,\xi)$ is defined in (3.10). Fix $\lambda>0$ and take $n\ge\lambda$ to be determined later. We consider in this step $\widetilde{p}\in\mathbb{N}$ such that

$$\max\{2(1+|\tau|),(1+2r)^\rho\}\le e^{\rho\widetilde{p}}.$$

We put $\widetilde{n}\in\mathbb{N}_0$, $\widetilde{n}\ge n$, such that (where $q\in\mathbb{N}_0$ satisfies $2^q\ge 3R$)

$$\widetilde{n}\ge\frac{L^{q+1}}{\rho}(\lambda L^{\widetilde{p}+2}+mL^3+1)+1.$$

By Lemma 3.10 and the properties of φ^*, proceeding as in the proof of the second step of [1, Theorem 5.4] we obtain, for some $C_{\widetilde{n}}>0$,

$$\begin{aligned}
&|D_x^{\theta_3}D_y^{\epsilon_3}(D_\xi^{\beta+\gamma-\alpha+\eta_3}\omega_{\beta\gamma})(x,y,\xi)|\\
&\quad\le C_{\widetilde{n}}e^{16\widetilde{n}L^{\widetilde{p}+3}\rho\sum_{p=1}^{3\widetilde{p}+1}L^p}e^{mL^{k+3}+\cdots+mL}(j+1)\langle(v,\xi)\rangle^{-\rho|2\beta+2\gamma-\alpha|}\times\\
&\qquad\times e^{16\widetilde{n}L^{\widetilde{p}+3}\rho\varphi^*\left(\frac{|2\beta+2\gamma-\alpha+\theta_3+\epsilon_3+\eta_3|}{16\widetilde{n}L^{\widetilde{p}+3}}\right)}e^{mL^3\omega(v)}e^{mL^{k+3}\omega(w)}e^{mL\omega(\xi)}\int_0^1|1-t|^j dt.
\end{aligned}$$

For the estimate of the derivatives of $Q_j(x,\xi)$ we can proceed similarly as in the first step to show finally that $(1-\chi)\sum_{j=1}^\infty Q_j\in\mathcal{S}_\omega(\mathbb{R}^{2d})$.

Third step. Let $T_N:\mathcal{S}_\omega(\mathbb{R}^d)\to\mathcal{S}_\omega(\mathbb{R}^d)$ be the operator with kernel $(1-\chi)W_N$. As in the proof of [1, Theorem 5.4], it follows that (T_N) converges to an operator $T:\mathcal{S}_\omega(\mathbb{R}^d)\to\mathcal{S}_\omega(\mathbb{R}^d)$ in $L(\mathcal{S}_\omega(\mathbb{R}^d),\mathcal{S}'_\omega(\mathbb{R}^d))$. We show that $T=0$. To this aim, fix $N\in\mathbb{N}$, $j_n\le N+1<j_{n+1}$ and set $a_N:=Re^{\frac{n}{N+1}\varphi^*\left(\frac{N+1}{n}\right)}$. For the support of the derivatives of φ_{N+1}, we may assume that

$$2a_N\le\langle((1-\tau)x+\tau y,\xi)\rangle\le 3a_N.$$

For $f,g\in\mathcal{S}_\omega(\mathbb{R}^d)$, we have

$$\langle T_Nf,g\rangle=\int T_Nf(x)g(x)dx=\int\Big(\int(1-\chi)(x,y)W_N(x,y)f(y)dy\Big)g(x)dx.$$

Fixed $N\in\mathbb{N}$, we can use Fubini's theorem (since $f,g\in\mathcal{S}_\omega(\mathbb{R}^d)$ and $|\xi|\le 3a_N$) and we obtain

$$\begin{aligned}
\langle T_Nf,g\rangle=\int\Big(\int\sum_{|\beta+\gamma|=1}^N\sum_{0\ne\alpha\le\beta+\gamma}\frac{(\beta+\gamma)!}{\beta!\gamma!}\frac{1}{\alpha!(\beta+\gamma-\alpha)!}\Big\{\int e^{i(x-y)\cdot\xi}\tau^{|\beta|}(1-\tau)^{|\gamma|}(-1)^{|\gamma|}\times\\
\times D_\xi^\alpha\varphi_{N+1}(v,\xi)\big(\partial_x^\beta\partial_y^\gamma D_\xi^{\beta+\gamma-\alpha}a\big)(v,v,\xi)d\xi\Big\}f(y)(1-\chi)(x,y)dy\Big)g(x)dx.
\end{aligned}$$

An integration by parts with (3.11) for a suitable power $s \in \mathbb{N}$, to be determined, gives

$$
\begin{aligned}
&e^{i(x-y)\cdot\xi}\frac{1}{G^s(\xi)}G^s(D_y)\big\{D_\xi^\alpha\varphi_{N+1}(v,\xi)\big(\partial_x^\beta\partial_y^\gamma D_\xi^{\beta+\gamma-\alpha}a\big)(v,v,\xi)f(y)(1-\chi)(x,y)g(x)\big\}\\
&\quad= e^{i(x-y)\cdot\xi}\frac{1}{G^s(\xi)}\sum_{\eta\in\mathbb{N}_0^d}b_\eta\sum_{\eta_1+\eta_2+\eta_3+\eta_4=\eta}\frac{\eta!}{\eta_1!\eta_2!\eta_3!\eta_4!}\tau^{|\eta_1|}D_y^{\eta_1}D_\xi^\alpha\varphi_{N+1}(v,\xi)\times\\
&\qquad\times D_y^{\eta_2}\big(\partial_x^\beta\partial_y^\gamma D_\xi^{\beta+\gamma-\alpha}a\big)(v,v,\xi)D_y^{\eta_3}f(y)D_y^{\eta_4}(1-\chi)(x,y)g(x).
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
\langle T_N f,g\rangle = &\sum_{|\beta+\gamma|=1}^{N}\sum_{0\neq\alpha\leq\beta+\gamma}\frac{(\beta+\gamma)!}{\beta!\gamma!}\frac{1}{\alpha!(\beta+\gamma-\alpha)!}\sum_{\eta\in\mathbb{N}_0^d}b_\eta\sum_{\eta_1+\eta_2+\eta_3+\eta_4=\eta}\frac{\eta!}{\eta_1!\eta_2!\eta_3!\eta_4!}\times\\
&\times\tau^{|\eta_1+\beta|}(1-\tau)^{|\gamma|}(-1)^{|\gamma|}\int\!\!\int e^{i(x-y)\cdot\xi}\frac{1}{G^s(\xi)}\int D_y^{\eta_1}D_\xi^\alpha\varphi_{N+1}(v,\xi)\times\\
&\times D_y^{\eta_2}\big(\partial_x^\beta\partial_y^\gamma D_\xi^{\beta+\gamma-\alpha}a\big)(v,v,\xi)D_y^{\eta_3}f(y)D_y^{\eta_4}(1-\chi)(x,y)g(x)dyd\xi dx.
\end{aligned}
$$

To estimate $|\langle T_N f,g\rangle|$, let $\widetilde{p}\in\mathbb{N}_0$ be as at the beginning of the proof ($\max\{|1-\tau|,2|\tau|\}\leq e^{\widetilde{p}}$). By Definition 2.8 and Corollary 3.4, for all $n\in\mathbb{N}$ there exist $C_n = C_{4nL^{\widetilde{p}+3}}>0$ and $D_n = D_{4nL^{\widetilde{p}+3}}>0$ such that, by the chain rule,

$$
\begin{aligned}
&|D_y^{\eta_2}(\partial_x^\beta\partial_y^\gamma D_\xi^{\beta+\gamma-\alpha}a)(v,v,\xi)|\\
&\qquad\leq 2^{|\eta_2|}|\tau|^{|\eta_2|}C_n\langle(v,\xi)\rangle^{-\rho|2\beta+2\gamma+\eta_2-\alpha|}e^{4nL^{\widetilde{p}+3}\rho\varphi^*\left(\frac{|2\beta+2\gamma+\eta_2-\alpha|}{4nL^{\widetilde{p}+3}}\right)}e^{m\omega(v,v,\xi)},\\
&|D_y^{\eta_1}D_\xi^\alpha\varphi_{N+1}(v,\xi)|\leq D_n\langle(v,\xi)\rangle^{-\rho|\eta_1+\alpha|}e^{4nL^{\widetilde{p}+3}\rho\varphi^*\left(\frac{|\eta_1+\alpha|}{4nL^{\widetilde{p}+3}}\right)}.
\end{aligned}
$$

By the choice of $\widetilde{p}\in\mathbb{N}_0$,

$$
|\tau|^{|\eta_1+\beta|}(2|\tau|)^{|\eta_2|}|1-\tau|^{|\gamma|}\leq e^{\widetilde{p}|\eta_1+\eta_2+\beta+\gamma|}.
$$

Since $2a_N\leq\langle(v,\xi)\rangle$ and $1\leq|\beta+\gamma|\leq N<N+1$, we use that $\varphi^*(x)/x$ is increasing to get

$$
\begin{aligned}
\langle(v,\xi)\rangle^{-\rho|\eta_1+\alpha|}\langle(v,\xi)\rangle^{-\rho|2\beta+2\gamma+\eta_2-\alpha|}&\leq\langle(v,\xi)\rangle^{-\rho|2\beta+2\gamma|}\\
&\leq(2R)^{-2\rho|\beta+\gamma|}e^{-2n\rho\varphi^*\left(\frac{|\beta+\gamma|}{n}\right)}.
\end{aligned}
$$

Put $\ell<n$. Since $f,g\in\mathcal{S}_\omega(\mathbb{R}^d)$ and $1-\chi\in\mathcal{E}_\omega(\mathbb{R}^{2d})$, there exist $E_\ell,E'_\ell,E>0$ such that (where k is as in (3.3))

$$
\begin{aligned}
|D_y^{\eta_3}f(y)|&\leq E_\ell e^{\ell L^3\varphi^*\left(\frac{|\eta_3|}{\ell L^3}\right)}e^{-((mL+L)L^{k+1}+1)\omega(y)};\\
|D_y^{\eta_4}(1-\chi)(x,y)|&\leq E'_\ell e^{\ell L^3\varphi^*\left(\frac{|\eta_4|}{\ell L^3}\right)};\\
|g(x)|&\leq E'e^{-((mL+L)L^{k+1}+1)\omega(x)}.
\end{aligned}
$$

We use (3.8). Since $\frac{(\beta+\gamma)!}{\beta!\gamma!}\leq 2^{|\beta+\gamma|}\leq e^{|\beta+\gamma|}$, we have by the properties of φ^* that $|\langle T_N f,g\rangle|$ is less than or equal to

$$
\begin{aligned}
&\sum_{|\beta+\gamma|=1}^{N}\sum_{0\neq\alpha\leq\beta+\gamma}\Big(\frac{e^{\widetilde{p}+1}}{(2R)^{2\rho}}\Big)^{|\beta+\gamma|}\frac{1}{\alpha!(\beta+\gamma-\alpha)!}e^{sC_1}\Big(\sum_{\eta\in\mathbb{N}_0^d}e^{\ell L\varphi^*\left(\frac{|\eta|}{\ell L}\right)}e^{-sC_1\varphi^*\left(\frac{|\eta|}{sC_1}\right)}\Big)e^{\ell L\sum_{t=1}^{\widetilde{p}+2}L^t}\times\\
&\times\int\Big(\int C_3^s e^{-sC_2\omega(\xi)}\Big(\int C_nD_nE_\ell E'_\ell E'e^{m\omega(v,v,\xi)}e^{-((mL+L)L^{k+1}+1)(\omega(y)+\omega(x))}dy\Big)d\xi\Big)dx.
\end{aligned}
$$

Set $s \in \mathbb{N}$ such that $sC_2 \geq (mL+L)L^{k+1}+1$, and take $\ell \geq sC_1$ to get that the series is convergent. It is easy to see for such $s \in \mathbb{N}$ that there exists $C_k > 0$ such that

$$\begin{aligned} &e^{m\omega(v,v,\xi)}e^{-((mL+L)L^{k+1}+1)\omega(y)}e^{-((mL+L)L^{k+1}+1)\omega(x)}e^{-sC_2\omega(\xi)} \\ &\quad \leq C_k e^{-\omega(\langle(v,\xi)\rangle)}e^{-\omega(x)}e^{-\omega(y)}e^{-\omega(\xi)}. \end{aligned}$$

So, we have

$$\begin{aligned} &\iiint_{2a_N \leq \langle(v,\xi)\rangle \leq 3a_N} e^{-\omega(\langle(v,\xi)\rangle)}e^{-\omega(x)-\omega(y)-\omega(\xi)}dyd\xi dx \\ &\quad \leq e^{-\omega(2a_N)} \iiint_{\mathbb{R}^{3d}} e^{-\omega(x)-\omega(y)-\omega(\xi)}dyd\xi dx, \end{aligned}$$

By property (γ) of Definition 2.1, there exists $C > 0$ such that $3\log(t) \leq \omega(t) + C$, $t \geq 0$. Thus,

$$e^{-\omega(2a_N)} \leq (2a_N)^{-3}e^C.$$

We recall that C_nD_n is the only constant that depends on n. By the choice of the sequence $(j_n)_n$, we have

$$e^n C_n D_n \leq a_N^3.$$

Hence, there exists $C' > 0$ such that

$$\begin{aligned} |\langle T_N f, g\rangle| &\leq C' \sum_{|\beta+\gamma|=1}^{N} \sum_{0\neq\alpha\leq\beta+\gamma} \Big(\frac{e^{\widetilde{p}+1}}{(2R)^{2\rho}}\Big)^{|\beta+\gamma|} \frac{1}{\alpha!(\beta+\gamma-\alpha)!}\frac{C_nD_n}{a_N^3} \\ &\leq \frac{C'}{e^n}\sum_{l=1}^{N}\frac{1}{l!}\Big(\frac{de^{\widetilde{p}+1}}{(2R)^{2\rho}}\Big)^l. \end{aligned}$$

Since the series converges for $R \geq 1$ large enough (which may depend on τ), and since $n \to \infty$ when $N \to \infty$, we show that $|\langle T_N f, g\rangle|$ tends to zero when $N \to \infty$.

It only remains to prove the uniqueness of the pseudodifferential operator modulo an ω-regularizing operator. We notice that every global amplitude as in Definition 2.8 defines an ω-ultradistribution. Then, as in [22, 25], the identities in $\mathcal{S}'_\omega(\mathbb{R}^{2d})$ for the Fourier transform

$$K_\tau(x,y) = (2\pi)^d \mathcal{F}^{-1}_{\xi\mapsto x-y}\big(a_\tau((1-\tau)x+\tau y,\xi)\big)$$

and

$$a_\tau(v,\xi) = (2\pi)^{-d}\mathcal{F}_{w\mapsto\xi}\big(K_\tau(v+\tau w, v-(1-\tau)w)\big)$$

yield the uniqueness of the τ-symbol since the kernel K_τ is also unique. □

As a consequence of Theorem 3.11, we can describe the precise relation between different quantizations for a given global symbol in terms of equivalence of formal sums as the following result shows.

Theorem 3.12. *If $a_{\tau_1}(x,\xi)$ and $a_{\tau_2}(x,\xi)$ are the τ_1 and τ_2-symbol of the same pseudodifferential operator A, then*

$$a_{\tau_2}(x,\xi) \sim \sum_{j=0}^{\infty}\sum_{|\alpha|=j}\frac{1}{\alpha!}(\tau_1-\tau_2)^{|\alpha|}\partial_\xi^\alpha D_x^\alpha a_{\tau_1}(x,\xi).$$

Proof. By Theorem 3.11, the pseudodifferential operator A is determined via the τ_1-symbol $a_{\tau_1}((1-\tau_1)x+\tau_1 y,\xi)$ modulo an ω-regularizing operator. Again by Theorem 3.11, its τ_2-symbol has the following asymptotic expansion

$$\begin{aligned}
a_{\tau_2}(x,\xi) &\sim \sum_{j=0}^{\infty}\sum_{|\beta+\gamma|=j}\frac{(-1)^{|\beta|}}{\beta!\gamma!}\tau_2^{|\beta|}(1-\tau_2)^{|\gamma|}\partial_\xi^{\beta+\gamma}D_x^\beta D_y^\gamma(a_{\tau_1}((1-\tau_1)x+\tau_1 y,\xi)\big|_{y=x})\\
&=\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\Big(\sum_{\beta+\gamma=\alpha}\frac{1}{\beta!\gamma!}((1-\tau_2)\tau_1)^{|\gamma|}(-\tau_2(1-\tau_1))^{|\beta|}\Big)\partial_\xi^\alpha D_x^\alpha a_{\tau_1}(x,\xi)\\
&=\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\frac{1}{\alpha!}((1-\tau_2)\tau_1-\tau_2(1-\tau_1))^{|\alpha|}\partial_\xi^\alpha D_x^\alpha a_{\tau_1}(x,\xi)\\
&=\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\frac{1}{\alpha!}(\tau_1-\tau_2)^{|\alpha|}\partial_\xi^\alpha D_x^\alpha a_{\tau_1}(x,\xi).
\end{aligned}$$

□

4 Transposition and composition of operators

By [1, Proposition 3.10], we deduce that if A has as amplitude $a((1-\tau)x+\tau y,\xi)$, then its transpose ${}^tA$ has the amplitude $a((1-\tau)y+\tau x,-\xi)$. Hence, if $a_\tau(x,\xi)$ is the τ-symbol of A, then ${}^ta_{1-\tau}(x,\xi)$ is the $(1-\tau)$-symbol of ${}^tA$ given by

$$ {}^ta_{1-\tau}((1-\tau)x+\tau y,\xi) := a_\tau((1-\tau)y+\tau x,-\xi). \tag{4.1}$$

In particular we have ${}^ta_\tau(x,\xi)=a_{1-\tau}(x,-\xi)$. On the other hand, for $\tau=0$, ${}^ta_1(y,-\xi)$ coincides with $a_0(x,\xi)$. Now, we show the corresponding generalization of [1, Proposition 5.5].

Theorem 4.1. *Let A be the pseudodifferential operator with τ-symbol $a_\tau(x,\xi)$. Then its transpose restricted to $\mathcal{S}_\omega(\mathbb{R}^d)$ can be decomposed as ${}^tA=Q+R$, where R is an ω-regularizing operator and Q is the pseudodifferential operator associated to the τ-symbol given by*

$$q(x,\xi)\sim\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\frac{1}{\alpha!}(1-2\tau)^{|\alpha|}\partial_\xi^\alpha D_x^\alpha a_\tau(x,-\xi).$$

Proof. By assumption we deduce that ${}^tA$ has the $(1-\tau)$-symbol ${}^ta_{1-\tau}(x,\xi)$ given by formula (4.1) restricted to $y=x$. Moreover, from Theorem 3.12, the τ-symbol of ${}^tA$ satisfies

$${}^ta_\tau(x,\xi)\sim\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\frac{1}{\alpha!}(1-2\tau)^{|\alpha|}\partial_\xi^\alpha D_x^{\alpha t}a_{1-\tau}(x,\xi)=\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\frac{1}{\alpha!}(1-2\tau)^{|\alpha|}\partial_\xi^\alpha D_x^\alpha a_\tau(x,-\xi).$$

□

Let us deal with the composition of two pseudodifferential operators given by their corresponding quantizations of symbols.

Theorem 4.2. *Let $a_{\tau_1}(x,\xi)\in \mathrm{GS}_\rho^{m_1,\omega}$ be the τ_1-symbol of A_1 and $b_{\tau_2}(x,\xi)\in\mathrm{GS}_\rho^{m_2,\omega}$ be the τ_2-symbol of A_2, being A_1 and A_2 their corresponding pseudodifferential operators. The τ-symbol*

$c_\tau(x,\xi) \in \mathrm{GS}_\rho^{m_1+m_2,\omega}$ *of* $A_1 \circ A_2$ *has the asymptotic expansion*

$$\sum_{j=0}^{\infty} \sum_{\substack{|\alpha+\beta-\alpha_1-\alpha_2|=j \\ \alpha+\beta=\gamma+\delta}} c_{\alpha\beta\gamma\delta\alpha_1\alpha_2} \partial_\xi^\gamma D_x^\alpha a_{\tau_1}(x,\xi) \cdot \partial_\xi^\delta D_x^\beta b_{\tau_2}(x,\xi), \tag{4.2}$$

where the coefficients $c_{\alpha\beta\gamma\delta\alpha_1\alpha_2}$ *are*

$$\frac{(2\pi)^d}{\gamma!\delta!} \sum_{k,l=0}^{\infty} \sum_{\substack{|\alpha_1|=k \\ |\alpha_2|=l}} (-1)^{|\alpha-\alpha_1+\alpha_2|} \binom{\alpha+\beta-\alpha_1-\alpha_2}{\alpha-\alpha_1} \binom{\gamma}{\alpha_1} \binom{\delta}{\alpha_2} \tau^{|\alpha-\alpha_1|}(1-\tau)^{|\beta-\alpha_2|} \tau_1^{|\alpha_1|}(1-\tau_2)^{|\alpha_2|}.$$

Proof. We first assume $\tau_1 = 0$ and $\tau_2 = 1$. In this case, $a_{\tau_1}((1-\tau_1)x+\tau_1 y,\xi)$ and $b_{\tau_2}((1-\tau_2)x+\tau_2 y,\xi)$ coincide with $a_0(x,\xi)$ and $b_1(y,\xi)$. Then

$$(A_1 \circ A_2)u(x) = \int e^{ix\cdot\xi} a_0(x,\xi)\widehat{A_2 u}(\xi) d\xi, \qquad x \in \mathbb{R}^d.$$

It is not difficult to see that $A_2 u(x) = \widehat{I}(-x)$, where $I(\xi) = \int e^{-iy\cdot\xi} b_1(y,\xi)u(y)dy$. Hence $\widehat{A_2 u}(\xi) = (2\pi)^d I(\xi)$ and

$$(A_1 \circ A_2)u(x) = \iint e^{i(x-y)\cdot\xi} c(x,y,\xi)u(y)dyd\xi, \qquad x \in \mathbb{R}^d,$$

where $c(x,y,\xi) = (2\pi)^d a_0(x,\xi)b_1(y,\xi)$ is an amplitude in $\mathrm{GA}_\rho^{m_1+m_2,\omega}$. So, by Theorem 3.11, the τ-symbol $c_\tau(x,\xi)$ has the asymptotic expansion:

$$c_\tau(x,\xi) \sim (2\pi)^d \sum_{j=0}^{\infty} \sum_{|\beta+\gamma|=j} \frac{(-1)^{|\beta|}}{\beta!\gamma!} \tau^{|\beta|}(1-\tau)^{|\gamma|} \partial_\xi^{\beta+\gamma} D_x^\beta D_y^\gamma \big(a_0(x,\xi)b_1(y,\xi)\big)\big|_{y=x} \tag{4.3}$$

$$= (2\pi)^d \sum_{j=0}^{\infty} \sum_{\substack{|\beta+\gamma|=j \\ \delta+\epsilon=\beta+\gamma}} \frac{(-1)^{|\beta|}(\beta+\gamma)!}{\delta!\epsilon!\beta!\gamma!} \tau^{|\beta|}(1-\tau)^{|\gamma|} \partial_\xi^\delta D_x^\beta a_0(x,\xi) \cdot \partial_\xi^\epsilon D_x^\gamma b_1(x,\xi). \tag{4.4}$$

For the general case, by Theorem 3.12, we have

$$a_0(x,\xi) \sim \sum_{j_1=0}^{\infty} \sum_{|\alpha_1|=j_1} \frac{1}{\alpha_1!} \tau_1^{|\alpha_1|} \partial_\xi^{\alpha_1} D_x^{\alpha_1} a_{\tau_1}(x,\xi);$$

$$b_1(x,\xi) \sim \sum_{j_2=0}^{\infty} \sum_{|\alpha_2|=j_2} \frac{(-1)^{|\alpha_2|}}{\alpha_2!} (1-\tau_2)^{|\alpha_2|} \partial_\xi^{\alpha_2} D_x^{\alpha_2} b_{\tau_2}(x,\xi).$$

Thus, from (4.4), we get

$$\begin{aligned} c_\tau(x,\xi) \sim (2\pi)^d \sum_{j=0}^{\infty} \sum_{\substack{|\beta+\gamma|=j \\ \delta+\epsilon=\beta+\gamma}} \frac{(-1)^{|\beta|}(\beta+\gamma)!}{\delta!\epsilon!\beta!\gamma!} \tau^{|\beta|}(1-\tau)^{|\gamma|} \times \\ \times \partial_\xi^\delta D_x^\beta \Big(\sum_{j_1=0}^{\infty} \sum_{|\alpha_1|=j_1} \frac{1}{\alpha_1!} \tau_1^{|\alpha_1|} \partial_\xi^{\alpha_1} D_x^{\alpha_1} a_{\tau_1}(x,\xi) \Big) \times \\ \times \partial_\xi^\epsilon D_x^\gamma \Big(\sum_{j_2=0}^{\infty} \sum_{|\alpha_2|=j_2} \frac{(-1)^{|\alpha_2|}}{\alpha_2!} (1-\tau_2)^{|\alpha_2|} \partial_\xi^{\alpha_2} D_x^{\alpha_2} b_{\tau_2}(x,\xi) \Big). \end{aligned}$$

We make the change of variables $\gamma' = \alpha_1 + \delta$, $\alpha' = \alpha_1 + \beta$, $\delta' = \alpha_2 + \epsilon$, $\beta' = \alpha_2 + \gamma$. Then

$$
\begin{aligned}
c_\tau(x,\xi) \sim (2\pi)^d \sum_{j=0}^{\infty} \sum_{\substack{|\alpha'+\beta'-\alpha_1-\alpha_2|=j \\ \alpha'+\beta'=\delta'+\gamma'}} & \frac{1}{\gamma'!\delta'!} \partial_\xi^{\gamma'} D_x^{\alpha'} a_{\tau_1}(x,\xi) \partial_\xi^{\delta'} D_x^{\beta'} b_{\tau_2}(x,\xi) \times \\
& \times \sum_{k,l=0}^{\infty} \sum_{\substack{|\alpha_1|=k \\ |\alpha_2|=l}} (-1)^{|\alpha'-\alpha_1+\alpha_2|} \frac{(\alpha'+\beta'-\alpha_1-\alpha_2)!}{(\alpha'-\alpha_1)!(\beta'-\alpha_2)!} \frac{\gamma'!}{\alpha_1!(\gamma'-\alpha_1)!} \frac{\delta'!}{\alpha_2!(\delta'-\alpha_2)!} \times \\
& \times \tau^{|\alpha'-\alpha_1|}(1-\tau)^{|\beta'-\alpha_2|} \tau_1^{|\alpha_1|}(1-\tau_2)^{|\alpha_2|}.
\end{aligned}
$$

The proof follows since

$$
\frac{(\alpha'+\beta'-\alpha_1-\alpha_2)!}{(\alpha'-\alpha_1)!(\beta'-\alpha_2)!} \frac{\gamma'!}{\alpha_1!(\gamma'-\alpha_1)!} \frac{\delta'!}{\alpha_2!(\delta'-\alpha_2)!} = \binom{\alpha'+\beta'-\alpha_1-\alpha_2}{\alpha'-\alpha_1} \binom{\gamma'}{\alpha_1} \binom{\delta'}{\alpha_2}.
$$

□

The coefficients appearing in formula (4.2) are sometimes simplified for some particular $\tau \in \mathbb{R}$. For example, if $\tau = 0$, by formula (4.3), we obtain

$$
c(x,\xi) = c_0(x,\xi) \sim (2\pi)^d \sum_{j=0}^{\infty} \sum_{|\gamma|=j} \frac{1}{\gamma!} \partial_\xi^\gamma D_y^\gamma \big(a_0(x,\xi) b_1(y,\xi)\big)\big|_{y=x}.
$$

On the other hand, from formula (4.1), $b_1(x,\xi) = {}^t b_0(x,-\xi)$. Hence, by [1, Lemma 5.6], we have

$$
c_0(x,\xi) \sim (2\pi)^d \big(a_0(x,\xi) \circ b_0(x,\xi)\big) = (2\pi)^d \sum_{j=0}^{\infty} \sum_{|\gamma|=j} \frac{1}{\gamma!} \partial_\xi^\gamma a_0(x,\xi) D_x^\gamma b_0(x,\xi),
$$

which in particular gives [1, Theorem 5.7] (cf. [25, Theorem 23.7]).

Another interesting case is when dealing with $\tau = 1/2$. We will obtain it as a consequence of a more general result (cf. [25, Problem 23.2]). First, we need a lemma, taken from [3, Theorem 5.5]:

Lemma 4.3. *The formula*

$$
\frac{(\beta+\gamma)!}{(\beta+\gamma-\epsilon)!\epsilon!} \frac{1}{\beta!\gamma!} = \sum_{\substack{0 \le \delta \le \beta \\ \beta-\epsilon \le \delta \le \beta-\epsilon+\gamma}} \frac{1}{(\beta-\delta)!(\beta-\epsilon+\gamma-\delta)!\delta!(\delta-\beta+\epsilon)!},
$$

holds for all $\beta, \gamma, \epsilon \in \mathbb{N}_0^d$ with $\epsilon \le \beta + \gamma$.

Example 4.4. Given two pseudodifferential operators A and B, the τ-symbol of the composition operator $C = A \circ B$ is given by

$$
c_\tau(x,\xi) \sim (2\pi)^d \sum_{j=0}^{\infty} \sum_{|\beta+\gamma|=j} \frac{(-1)^{|\beta|}}{\beta!\gamma!} \tau^{|\beta|}(1-\tau)^{|\gamma|} (\partial_\xi^\gamma D_x^\beta a_\tau(x,\xi))(\partial_\xi^\beta D_x^\gamma b_\tau(x,\xi)).
$$

Proof. Formula (4.4) states that $c_\tau(x,\xi)$ is equivalent to (since $\delta = \beta + \gamma - \epsilon$)

$$
(2\pi)^d \sum_{j=0}^{\infty} \sum_{|\beta+\gamma|=j} (-1)^{|\beta|} \tau^{|\beta|}(1-\tau)^{|\gamma|} \sum_{\epsilon \le \beta+\gamma} \frac{(\beta+\gamma)!}{(\beta+\gamma-\epsilon)!\epsilon!} \frac{1}{\beta!\gamma!} \partial_\xi^{\beta+\gamma-\epsilon} D_x^\beta a_0(x,\xi) \cdot \partial_\xi^\epsilon D_x^\gamma b_1(x,\xi).
$$

Moreover, by Lemma 4.3, it is equal to

$$(2\pi)^d \sum_{j=0}^{\infty} \sum_{|\beta+\gamma|=j} (-1)^{|\beta|}\tau^{|\beta|}(1-\tau)^{|\gamma|} \times$$
$$\times \sum_{\epsilon\leq\beta+\gamma} \sum_{\substack{0\leq\delta\leq\beta \\ \beta-\epsilon\leq\delta\leq\beta-\epsilon+\gamma}} \frac{1}{(\beta-\delta)!(\beta-\epsilon+\gamma-\delta)!\delta!(\delta-\beta+\epsilon)!} \partial_\xi^{\beta+\gamma-\epsilon} D_x^\beta a_0(x,\xi) \cdot \partial_\xi^\epsilon D_x^\gamma b_1(x,\xi).$$

We put $\mu = \beta-\delta$, $\nu = \beta-\epsilon+\gamma-\delta$, and $\theta = \delta-\beta+\epsilon$. Therefore,

$$c_\tau(x,\xi) \sim (2\pi)^d \sum_{j=0}^{\infty} \sum_{|\nu+\theta+\mu+\delta|=j} \frac{(-1)^{|\mu+\delta|}}{\mu!\nu!\delta!\theta!} \tau^{|\mu+\delta|}(1-\tau)^{|\nu+\theta|} \times$$
$$\times\, \partial_\xi^{\nu+\delta} D_x^{\mu+\delta} a_0(x,\xi) \cdot \partial_\xi^{\theta+\mu} D_x^{\nu+\theta} b_1(x,\xi),$$

and taking $j = j_1 + j_2 + j_3$, $j_1, j_2, j_3 \in \mathbb{N}_0$, we have

$$c_\tau(x,\xi) \sim (2\pi)^d \sum_{j_1=0}^{\infty} \sum_{|\nu+\mu|=j_1} \frac{(-1)^{|\mu|}}{\mu!\nu!} \tau^{|\mu|}(1-\tau)^{|\nu|} \partial_\xi^\nu D_x^\mu \Big(\sum_{j_2=0}^{\infty} \sum_{|\delta|=j_2} \frac{(-1)^{|\delta|}}{\delta!} \tau^{|\delta|} \partial_\xi^\delta D_x^\delta a_0(x,\xi) \Big) \times$$
$$\times\, \partial_\xi^\mu D_x^\nu \Big(\sum_{j_3=0}^{\infty} \sum_{|\theta|=j_3} \frac{1}{\theta!} (1-\tau)^{|\theta|} \partial_\xi^\theta D_x^\theta b_1(x,\xi) \Big).$$

We get the result since Theorem 3.12 gives

$$a_\tau(x,\xi) \sim \sum_{k=0}^{\infty} \sum_{|\delta|=k} \frac{(-1)^{|\delta|}}{\delta!} \tau^{|\delta|} \partial_\xi^\delta D_x^\delta a_0(x,\xi), \qquad b_\tau(x,\xi) \sim \sum_{k=0}^{\infty} \sum_{|\theta|=k} \frac{1}{\theta!} (1-\tau)^{|\theta|} \partial_\xi^\theta D_x^\theta b_1(x,\xi).$$

□

Corollary 4.5. *Given two pseudodifferential operators A and B, the Weyl symbol of the composition operator $C = A \circ B$ is given by*

$$c_w(x,\xi) \sim (2\pi)^d \sum_{j=0}^{\infty} \sum_{|\beta+\gamma|=j} \frac{(-1)^{|\beta|}}{\beta!\gamma!} 2^{-|\beta+\gamma|} (\partial_\xi^\gamma D_x^\beta a_w(x,\xi))(\partial_\xi^\beta D_x^\gamma b_w(x,\xi)).$$

5 Parametrices and ω-regularity

In this section we give a sufficient condition for *ω-regularity* of a global pseudodifferential operator. We say that a pseudodifferential operator $P : \mathcal{S}'_\omega(\mathbb{R}^d) \to \mathcal{S}'_\omega(\mathbb{R}^d)$ is *ω-regular* if given $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ such that $Pu \in \mathcal{S}_\omega(\mathbb{R}^d)$, we have $u \in \mathcal{S}_\omega(\mathbb{R}^d)$. See [4] for a study of ω-regularity of linear partial differential operators with polynomial coefficients using quadratic transformations (cf. [21] for the non-isotropic case).

We use the well-known method of the construction of a parametrix for the symbol of the operator, using symbolic calculus. We follow the lines of [15, 26]. From [23], we know that a weight function σ is equivalent to a subadditive weight function if and only if it satisfies

(α_0) $\exists C > 0,\ \exists t_0 > 0\ \forall \lambda \geq 1 :\ \sigma(\lambda t) \leq \lambda C \sigma(t), \qquad t \geq t_0.$

We refer to [14, 23] for applications and characterizations of property (α_0) on the weight function. The following result is taken from [15, Lemma 3.3].

Lemma 5.1. *Let ω be a subadditive weight function. For all $\lambda > 0$ and $j, k \in \mathbb{N}$, we have*

$$\frac{e^{\lambda\varphi^*_\omega(\frac{j}{\lambda})}}{j!}\frac{e^{\lambda\varphi^*_\omega(\frac{k}{\lambda})}}{k!} \leq \frac{e^{\lambda\varphi^*_\omega(\frac{j+k}{\lambda})}}{(j+k)!}.$$

The following lemma states Vandermonde's identity.

Lemma 5.2. *For any $m, n, r \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}.$$

Lemma 5.3. *If $\sum_j a_j \in \mathrm{FGS}^{m_1,\omega}_\rho$ and $b(x,\xi) \in \mathrm{GS}^{m_2,\omega}_\rho$, then $\sum_j a_j(x,\xi)b(x,\xi) \in \mathrm{FGS}^{m_1+m_2,\omega}_\rho$.*

The following result is in the spirit of Zanghirati [26] and Fernández, Galbis, and Jornet [15] (see also Cappiello, Pilipović, and Prangoski [12]).

Theorem 5.4. *Let ω be a weight function and let σ be a subadditive weight function with $\omega(t^{1/\rho}) = o(\sigma(t))$ as $t \to \infty$. Let $p(x,\xi) \in \mathrm{GS}^{|m|,\omega}_\rho$ be such that, for some $R \geq 1$:*

(i) $|p(x,\xi)| \geq \dfrac{1}{R} e^{-|m|\omega(x,\xi)}$ *for* $\langle (x,\xi)\rangle \geq R$;

(ii) There exist $C > 0$ and $n \in \mathbb{N}$ such that

$$|D^\alpha_x D^\beta_\xi p(x,\xi)| \leq C^{|\alpha+\beta|}\langle (x,\xi)\rangle^{-\rho|\alpha+\beta|} e^{\frac{1}{n}\varphi^*_\sigma(n|\alpha|)} e^{\frac{1}{n}\varphi^*_\sigma(n|\beta|)}|p(x,\xi)|,$$

for $\alpha, \beta \in \mathbb{N}^d_0$, $\langle (x,\xi)\rangle \geq R$.

Then there exists $q(x,\xi) \in \mathrm{GS}^{|m|,\omega}_\rho$ such that $q \circ p \sim 1$ in $\mathrm{FGS}^{|m|,\omega}_\rho$.

Proof. We set

$$q_0(x,\xi) = \frac{1}{p(x,\xi)}, \qquad \langle (x,\xi)\rangle \geq R.$$

We show by induction on $|\alpha + \beta|$ that there exists $C_1 > 0$ such that

$$|D^\alpha_x D^\beta_\xi q_0(x,\xi)| \leq C_1^{|\alpha+\beta|}\langle (x,\xi)\rangle^{-\rho|\alpha+\beta|} e^{\frac{1}{n}\varphi^*_\sigma(n|\alpha|)} e^{\frac{1}{n}\varphi^*_\sigma(n|\beta|)}|q_0(x,\xi)| \tag{5.1}$$

for all $\alpha, \beta \in \mathbb{N}^d_0$, $\langle (x,\xi)\rangle \geq R$. Indeed, the inequality is true for $\alpha = \beta = 0$. Now, differentiating the formula $p(x,\xi)q_0(x,\xi) = 1$, we obtain

$$p(x,\xi)D^\alpha_x D^\beta_\xi q_0(x,\xi) = -\sum_{0\neq(\widehat{\alpha},\widehat{\beta})\leq(\alpha,\beta)} \frac{\alpha!}{\widehat{\alpha}!(\alpha-\widehat{\alpha})!}\frac{\beta!}{\widehat{\beta}!(\beta-\widehat{\beta})!} D^{\widehat{\alpha}}_x D^{\widehat{\beta}}_\xi p(x,\xi) D^{\alpha-\widehat{\alpha}}_x D^{\beta-\widehat{\beta}}_\xi q_0(x,\xi).$$

Now, we assume that the inequality (5.1) is true for $(\widehat{\alpha},\widehat{\beta}) < (\alpha,\beta)$. Using condition (ii), we obtain

$$\begin{aligned}
&|p(x,\xi)D^\alpha_x D^\beta_\xi q_0(x,\xi)| \\
&\quad\leq \sum_{0\neq(\widehat{\alpha},\widehat{\beta})\leq(\alpha,\beta)} \frac{\alpha!}{\widehat{\alpha}!(\alpha-\widehat{\alpha})!}\frac{\beta!}{\widehat{\beta}!(\beta-\widehat{\beta})!} C^{|\widehat{\alpha}+\widehat{\beta}|}\langle (x,\xi)\rangle^{-\rho|\widehat{\alpha}+\widehat{\beta}|} e^{\frac{1}{n}\varphi^*_\sigma(n|\widehat{\alpha}|)} e^{\frac{1}{n}\varphi^*_\sigma(n|\widehat{\beta}|)}|p(x,\xi)| \times \\
&\quad\times C_1^{|\alpha-\widehat{\alpha}+\beta-\widehat{\beta}|}\langle (x,\xi)\rangle^{-\rho|\alpha-\widehat{\alpha}+\beta-\widehat{\beta}|} e^{\frac{1}{n}\varphi^*_\sigma(n|\alpha-\widehat{\alpha}|)} e^{\frac{1}{n}\varphi^*_\sigma(n|\beta-\widehat{\beta}|)}|q_0(x,\xi)|.
\end{aligned}$$

Since $\frac{\alpha!}{\widehat{\alpha}!(\alpha-\widehat{\alpha})!}\frac{\beta!}{\widehat{\beta}!(\beta-\widehat{\beta})!} \leq \frac{|\alpha|!}{|\widehat{\alpha}|!|\alpha-\widehat{\alpha}|!}\frac{|\beta|!}{|\widehat{\beta}|!|\beta-\widehat{\beta}|!}$, we obtain, by Lemma 5.1,

$$|\alpha|!\frac{e^{\frac{1}{n}\varphi^*_\sigma(n|\widehat{\alpha}|)}}{|\widehat{\alpha}|!}\frac{e^{\frac{1}{n}\varphi^*_\sigma(n|\alpha-\widehat{\alpha}|)}}{|\alpha-\widehat{\alpha}|!}|\beta|!\frac{e^{\frac{1}{n}\varphi^*_\sigma(n|\widehat{\beta}|)}}{|\widehat{\beta}|!}\frac{e^{\frac{1}{n}\varphi^*_\sigma(n|\beta-\widehat{\beta}|)}}{|\beta-\widehat{\beta}|!} \leq e^{\frac{1}{n}\varphi^*_\sigma(n|\alpha|)}e^{\frac{1}{n}\varphi^*_\sigma(n|\beta|)}.$$

Thus

$$|D^\alpha_x D^\beta_\xi q_0(x,\xi)| \leq C_1^{|\alpha+\beta|}\langle(x,\xi)\rangle^{-\rho|\alpha+\beta|}e^{\frac{1}{n}\varphi^*_\sigma(n|\alpha|)}e^{\frac{1}{n}\varphi^*_\sigma(n|\beta|)}|q_0(x,\xi)|\sum_{0\neq(\widehat{\alpha},\widehat{\beta})\leq(\alpha,\beta)}\left(\frac{C}{C_1}\right)^{|\widehat{\alpha}+\widehat{\beta}|}.$$

Finally, the fact that

$$\sum_{0\neq(\widehat{\alpha},\widehat{\beta})\leq(\alpha,\beta)}\left(\frac{C}{C_1}\right)^{|\widehat{\alpha}+\widehat{\beta}|} \leq \sum_{k=1}^{|\alpha+\beta|}\sum_{|\eta|=k}\left(\frac{C}{C_1}\right)^k \leq \sum_{k=1}^{|\alpha+\beta|}\left(\frac{dC}{C_1}\right)^k$$

completes the proof of (5.1) if we take $C_1 > 0$ such that

$$\sum_{k=1}^{\infty}\left(\frac{dC}{C_1}\right)^k < 1.$$

For $j \in \mathbb{N}$, we define recursively

$$q_j(x,\xi) := -q_0(x,\xi)\sum_{0<|\epsilon+\gamma|\leq j}\frac{(-1)^{|\epsilon|}}{\epsilon!\gamma!}\tau^{|\epsilon|}(1-\tau)^{|\gamma|}(\partial^\gamma_\xi D^\epsilon_x q_{j-|\epsilon+\gamma|}(x,\xi))(\partial^\epsilon_\xi D^\gamma_x p(x,\xi)).$$

We show that there exist constants $C_2, C_3 > 0$ with $C_1 < C_2 < C_3$ such that

$$|D^\alpha_x D^\beta_\xi q_j(x,\xi)| \leq C_2^{|\alpha+\beta|}C_3^j\langle(x,\xi)\rangle^{-\rho(|\alpha+\beta|+2j)}e^{\frac{1}{n}\varphi^*_\sigma(n(|\alpha+\beta|+2j))}e^{|m|\omega(x,\xi)}, \tag{5.2}$$

for all $\alpha,\beta \in \mathbb{N}_0^d$, $\langle(x,\xi)\rangle \geq R$. We proceed by induction on $j \in \mathbb{N}_0$. First, observe that formula (5.1) implies formula (5.2) for $j=0$, since $|q_0(x,\xi)| \leq Re^{|m|\omega(x,\xi)}$ for $\langle(x,\xi)\rangle \geq R$ (from condition (i)). Now, assume that (5.2) holds for all $0 \leq l < j$ (where $C_3 > C_2 > C_1$, and $C_2, C_3 > 0$ are large enough). Then, by the definition of $q_j(x,\xi)$, we have

$$\begin{aligned}|D^\alpha_x D^\beta_\xi q_j(x,\xi)| \leq &\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\ \beta_1+\beta_2+\beta_3=\beta}}\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\frac{\beta!}{\beta_1!\beta_2!\beta_3!}|D^{\alpha_1}_x D^{\beta_1}_\xi q_0(x,\xi)|\sum_{0<|\epsilon+\gamma|\leq j}\frac{1}{\epsilon!\gamma!}\times\\ &\times|\tau|^{|\epsilon|}|1-\tau|^{|\gamma|}|D^{\alpha_2+\epsilon}_x D^{\beta_2+\gamma}_\xi q_{j-|\epsilon+\gamma|}(x,\xi)||D^{\alpha_3+\gamma}_x D^{\beta_3+\epsilon}_\xi p(x,\xi)|.\end{aligned}$$

We use formula (5.1) for the derivatives of $q_0(x,\xi)$, the inductive hypothesis (5.2) for the ones of

$q_{j-|\mu|}(x,\xi)$, and condition (ii) for the derivatives of $p(x,\xi)$. All this implies

$$
\begin{aligned}
|D_x^\alpha D_\xi^\beta q_j(x,\xi)| \leq & \sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\ \beta_1+\beta_2+\beta_3=\beta}} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\frac{\beta!}{\beta_1!\beta_2!\beta_3!} C_1^{|\alpha_1+\beta_1|}\langle(x,\xi)\rangle^{-\rho|\alpha_1+\beta_1|}e^{\frac{1}{n}\varphi_\sigma^*(n|\alpha_1|)}\times\\
&\times e^{\frac{1}{n}\varphi_\sigma^*(n|\beta_1|)}|q_0(x,\xi)|\sum_{0<|\epsilon+\gamma|\leq j}\frac{1}{\epsilon!\gamma!}|\tau|^{|\epsilon|}|1-\tau|^{|\gamma|}C_2^{|\alpha_2+\epsilon+\beta_2+\gamma|}C_3^{j-|\epsilon+\gamma|}\times\\
&\times\langle(x,\xi)\rangle^{-\rho(|\alpha_2+\epsilon+\beta_2+\gamma|+2(j-|\epsilon+\gamma|))}e^{\frac{1}{n}\varphi_\sigma^*(n(|\alpha_2+\epsilon+\beta_2+\gamma|+2(j-|\epsilon+\gamma|)))}e^{|m|\omega(x,\xi)}\times\\
&\times C^{|\alpha_3+\gamma+\beta_3+\epsilon|}\langle(x,\xi)\rangle^{-\rho|\alpha_3+\gamma+\beta_3+\epsilon|}e^{\frac{1}{n}\varphi_\sigma^*(n|\alpha_3+\gamma|)}e^{\frac{1}{n}\varphi_\sigma^*(n|\beta_3+\epsilon|)}|p(x,\xi)|\\
=\langle(x,\xi)\rangle^{-\rho(|\alpha+\beta|+2j)}e^{|m|\omega(x,\xi)}&\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\ \beta_1+\beta_2+\beta_3=\beta}}\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\frac{\beta!}{\beta_1!\beta_2!\beta_3!}C_1^{|\alpha_1+\beta_1|}\times\\
&\times e^{\frac{1}{n}\varphi_\sigma^*(n|\alpha_1|)}e^{\frac{1}{n}\varphi_\sigma^*(n|\beta_1|)}\sum_{0<|\epsilon+\gamma|\leq j}\frac{1}{\epsilon!\gamma!}|\tau|^{|\epsilon|}|1-\tau|^{|\gamma|}C_2^{|\alpha_2+\epsilon+\beta_2+\gamma|}C_3^{j-|\epsilon+\gamma|}\times\\
&\times e^{\frac{1}{n}\varphi_\sigma^*(n(|\alpha_2+\beta_2|+2j-|\epsilon+\gamma|))}C^{|\alpha_3+\gamma+\beta_3+\epsilon|}e^{\frac{1}{n}\varphi_\sigma^*(n|\alpha_3+\gamma|)}e^{\frac{1}{n}\varphi_\sigma^*(n|\beta_3+\epsilon|)}.
\end{aligned}
\tag{5.3}
$$

To estimate the right-hand side of (5.3) we multiply and divide by

$$(|\alpha_2+\beta_2|+2j-|\epsilon+\gamma|)!|\alpha_3+\gamma|!|\beta_3+\epsilon|!$$

Then, as

$$\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\frac{\beta!}{\beta_1!\beta_2!\beta_3!}\leq\frac{|\alpha|!}{|\alpha_1|!|\alpha_2|!|\alpha_3|!}\frac{|\beta|!}{|\beta_1|!|\beta_2|!|\beta_3|!},$$

we have, by Lemma 5.1,

$$
\begin{aligned}
&\frac{e^{\frac{1}{n}\varphi_\sigma^*(n|\alpha_1|)}}{|\alpha_1|!}\frac{e^{\frac{1}{n}\varphi_\sigma^*(n|\beta_1|)}}{|\beta_1|!}\frac{e^{\frac{1}{n}\varphi_\sigma^*(n(|\alpha_2+\beta_2|+2j-|\epsilon+\gamma|))}}{(|\alpha_2+\beta_2|+2j-|\epsilon+\gamma|)!}\frac{e^{\frac{1}{n}\varphi_\sigma^*(n|\alpha_3+\gamma|)}}{|\alpha_3+\gamma|!}\frac{e^{\frac{1}{n}\varphi_\sigma^*(n|\beta_3+\epsilon|)}}{|\beta_3+\epsilon|!}\\
&\quad\leq\frac{1}{(|\alpha+\beta|+2j)!}e^{\frac{1}{n}\varphi_\sigma^*(n(|\alpha+\beta|+2j))}.
\end{aligned}
$$

Now, we see that

$$\frac{|\alpha|!}{|\alpha_2|!|\alpha_3|!}\frac{|\beta|!}{|\beta_2|!|\beta_3|!}|\alpha_3+\gamma|!|\beta_3+\epsilon|!\frac{(|\alpha_2+\beta_2|+2j-|\epsilon+\gamma|)!}{(|\alpha+\beta|+2j)!}\leq 2^{|\alpha_1+\alpha_3|}2^{|\beta_1+\beta_3|}. \tag{5.4}$$

Indeed, we multiply and divide by $(|\alpha_1+\alpha_3|+|\beta_1+\beta_3|+|\epsilon+\gamma|)!$ to get, by the properties of the multinomial coefficients,

$$
\begin{aligned}
&\frac{|\alpha|!}{|\alpha_2|!|\alpha_3|!}\frac{|\beta|!}{|\beta_2|!|\beta_3|!}\frac{|\alpha_3+\gamma|!|\beta_3+\epsilon|!}{(|\alpha_1+\alpha_3|+|\beta_1+\beta_3|+|\epsilon+\gamma|)!}\frac{1}{\binom{|\alpha+\beta|+2j}{|\alpha_2+\beta_2|+2j-|\epsilon+\gamma|}}\\
&\quad\leq\frac{|\alpha|!}{|\alpha_2|!|\alpha_3|!}\frac{|\beta|!}{|\beta_2|!|\beta_3|!}\frac{1}{|\alpha_1|!|\beta_1|!}\frac{1}{\binom{|\alpha+\beta|+2j}{|\alpha_2+\beta_2|+2j-|\epsilon+\gamma|}}.
\end{aligned}
$$

As we have, for $\alpha=\alpha_1+\alpha_2+\alpha_3$,

$$\frac{|\alpha|!}{|\alpha_1|!|\alpha_2|!|\alpha_3|!}=\frac{|\alpha_1+\alpha_3|!}{|\alpha_1|!|\alpha_3|!}\binom{|\alpha|}{|\alpha_2|}\leq 2^{|\alpha_1+\alpha_3|}\binom{|\alpha|}{|\alpha_2|},$$

(and in the same way for $\beta = \beta_1 + \beta_2 + \beta_3$), we deduce formula (5.4) by Lemma 5.2. We then have, from (5.3),

$$\begin{aligned}|D_x^\alpha D_\xi^\beta q_j(x,\xi)| \leq{}& \langle (x,\xi)\rangle^{-\rho(|\alpha+\beta|+2j)} e^{\frac{1}{n}\varphi_\sigma^*(n(|\alpha+\beta|+2j))} e^{|m|\omega(x,\xi)} \times \\ &\times \sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\ \beta_1+\beta_2+\beta_3=\beta}} 2^{|\alpha_1+\alpha_3|} 2^{|\beta_1+\beta_3|} C_1^{|\alpha_1+\beta_1|} C_2^{|\alpha_2+\beta_2|} C_3^j C^{|\alpha_3+\beta_3|} \times \\ &\times \sum_{0<|\epsilon+\gamma|\leq j} \frac{1}{\epsilon!\gamma!} |\tau|^{|\epsilon|} |1-\tau|^{|\gamma|} C_2^{|\epsilon+\gamma|} C_3^{-|\epsilon+\gamma|} C^{|\epsilon+\gamma|}.\end{aligned}$$

Since

$$\begin{aligned}C_2^{|\alpha+\beta|} C_3^j \sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\ \beta_1+\beta_2+\beta_3=\beta}} \Big(\frac{2C_1}{C_2}\Big)^{|\alpha_1+\beta_1|} \Big(\frac{2C}{C_2}\Big)^{|\alpha_3+\beta_3|} &\leq C_2^{|\alpha+\beta|} C_3^j \sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\ \beta_1+\beta_2+\beta_3=\beta}} \Big(\frac{2CC_1}{C_2}\Big)^{|\alpha_1+\alpha_3+\beta_1+\beta_3|} \\ &\leq C_2^{|\alpha+\beta|} C_3^j \sum_{k=0}^{|\alpha+\beta|} \sum_{|\eta|=k} \Big(\frac{2CC_1}{C_2}\Big)^k,\end{aligned}$$

we take $C_2 > 0$ large enough so that

$$\sum_{k=0}^{\infty} \Big(\frac{2dCC_1}{C_2}\Big)^k < 2.$$

In addition, we put $C_3 > 0$ large enough satisfying

$$\begin{aligned}\sum_{0<|\epsilon|\leq j} \frac{1}{\epsilon!} \Big(\frac{CC_2|\tau|}{C_3}\Big)^{|\epsilon|} \sum_{0<|\gamma|\leq j} \frac{1}{\gamma!} \Big(\frac{CC_2|1-\tau|}{C_3}\Big)^{|\gamma|} &\leq \Big(\sum_{0<k\leq j} \frac{1}{k!} \Big(\frac{d^2CC_2 \max\{|\tau|,|1-\tau|\}}{C_3}\Big)^k\Big)^2 \\ &\leq \Big(\sum_{k=1}^{\infty} \frac{1}{k!} \Big(\frac{d^2CC_2 \max\{|\tau|,|1-\tau|\}}{C_3}\Big)^k\Big)^2 < 1/2.\end{aligned}$$

This proves (5.2). Furthermore, by [1, Lemma 2.9(1)] we have that for all $\ell \in \mathbb{N}$ there exists $C_\ell > 0$ such that, for each j,

$$|D_x^\alpha D_\xi^\beta q_j(x,\xi)| \leq C_\ell C_2^{|\alpha+\beta|} C_3^j \langle (x,\xi)\rangle^{-\rho(|\alpha+\beta|+2j)} e^{\ell\rho\varphi_\omega^*\left(\frac{|\alpha+\beta|+2j}{\ell}\right)} e^{|m|\omega(x,\xi)},$$

for all $\alpha, \beta \in \mathbb{N}_0^d$ and $\langle (x,\xi)\rangle \geq R$ and, in particular, the estimate of Definition 3.1 follows.

Now, we extend $q_j(x,\xi)$ to $C^\infty(\mathbb{R}^{2d})$ for each $j \in \mathbb{N}_0$. To this aim, we take $\phi \in \mathcal{D}_\sigma(\mathbb{R}^{2d})$, supported in $\{(x,\xi) \in \mathbb{R}^{2d} : \langle (x,\xi)\rangle \leq 2R\}$ and equal to 1 when $\langle (x,\xi)\rangle \leq R$. Then, we set $\widetilde{q_j}(x,\xi) := q_j(x,\xi)(1-\phi)(x,\xi)$, which satisfies $\widetilde{q_j} = q_j$ if $\langle (x,\xi)\rangle > 2R$ and vanishes if $\langle (x,\xi)\rangle \leq R$. It is easy to see that $1-\phi \in \mathrm{GS}_\rho^{0,\omega}$. Hence, by Lemma 5.3, $\widetilde{q_j}(x,\xi) \in \mathrm{FGS}_\rho^{|m|,\omega}$.

We identify $\widetilde{q_j} = q_j$ and we show that $\sum q_j \circ p \sim 1$. For $j > 0$, by the definition of $q_j(x,\xi)$ we have

$$\begin{aligned}q_j(x,\xi)p(x,\xi) &= -\sum_{0<|\epsilon+\gamma|\leq j} \frac{(-1)^{|\epsilon|}}{\epsilon!\gamma!} \tau^{|\epsilon|}(1-\tau)^{|\gamma|} (\partial_\xi^\gamma D_x^\epsilon q_{j-|\epsilon+\gamma|}(x,\xi))(\partial_\xi^\epsilon D_x^\gamma p(x,\xi)) \\ &= -r_j(x,\xi) + q_j(x,\xi)p(x,\xi),\end{aligned}$$

where $\sum r_j := \sum q_j \circ p$ (cf. [1, Proposition 4.13]). Thus, $r_j(x,\xi) = 0$ for $j > 0$. Also, by the definition of composition, $r_0(x,\xi) = q_0(x,\xi)p(x,\xi) = 1$ if $\langle (x,\xi)\rangle > 2R$, which shows that $\sum q_j \circ p \sim 1$. Since $\sum q_j$ is a formal sum in $\mathrm{FGS}_\rho^{|m|,\omega}$, by [1, Theorem 4.6] there exists $q(x,\xi) \in \mathrm{GS}_\rho^{|m|,\omega}$ such that $q \sim \sum q_j$. Finally, [1, Proposition 4.14] yields $q \circ p \sim 1$, and the proof is complete. □

Corollary 5.5. *Let ω be a weight function and let σ be a weight function that satisfies (α_0) with $\omega(t^{1/\rho}) = o(\sigma(t))$ as $t \to \infty$. If $p(x,\xi) \in \mathrm{GS}_\rho^{m,\omega}$ satisfies the hypotheses of Theorem 5.4, any quantization of the corresponding pseudodifferential operator P is ω-regular.*

Proof. By Theorem 5.4 there is a pseudodifferential operator Q such that $Q \circ P = I + R$, being I the identity operator and R an ω-regularizing operator (as a direct consequence of Theorems 4.2 and 3.11 for $\tau = 0$). Then, $u = Q(Pu) - Ru \in \mathcal{S}_\omega(\mathbb{R}^d)$ for any $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ with $Pu \in \mathcal{S}_\omega(\mathbb{R}^d)$. The same argument is valid for an arbitrary quantization. □

6 Global ω-hypoellipticity for mixed classes

In what follows, $m, m_0 \in \mathbb{R}$, $m_0 \le m$, $0 < \rho \le 1$, and for any given weight function ω, σ denotes a Gevrey weight function, i.e. $\sigma(t) = t^a$, for some $0 < a < 1$, such that

$$\omega(t^{1/\rho}) = o(\sigma(t)), \qquad t \to \infty. \tag{6.1}$$

Definition 6.1. *Let $a \in \mathrm{GS}_\rho^{m,\omega}$. We say that a is an* ω-hypoelliptic symbol *in the class $\mathrm{HGS}_\rho^{m,m_0;\omega}$, and we write $a \in \mathrm{HGS}_\rho^{m,m_0;\omega}$, if there exist a Gevrey weight function σ satisfying (6.1) and $R \ge 1$ such that*

(i) There exist $C_1, C_2 > 0$ such that

$$C_1 e^{m_0\omega(x,\xi)} \le |a(x,\xi)| \le C_2 e^{m\omega(x,\xi)}, \qquad \langle (x,\xi)\rangle \ge R.$$

(ii) There exist $C > 0$, $n \in \mathbb{N}$ such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C^{|\alpha+\beta|} \langle (x,\xi)\rangle^{-\rho|\alpha+\beta|} e^{\frac{1}{n}\varphi_\sigma^*(n|\alpha|)} e^{\frac{1}{n}\varphi_\sigma^*(n|\beta|)} |a(x,\xi)|,$$

for $\langle (x,\xi)\rangle \ge R$, $\alpha, \beta \in \mathbb{N}_0^d$.

We show in Theorem 6.8 below that Definition 6.1 is independent on the quantization τ for the case $m_0 = m$. Hence, we extend [3, Proposition 8.4], showing that ω-hypoelliptic symbol classes are not perturbed by a change of quantization. We observe that any pseudodifferential operator defined by an ω-hypoelliptic symbol is also ω-regular by Theorem 5.4, but the converse is not true. For instance, the twisted Laplacian in $\mathbb{R}^2$,

$$L = \Big(D_x - \frac{1}{2}y\Big)^2 + \Big(D_y - \frac{1}{2}x\Big)^2$$

is ω-regular for every weight function ω as it is shown in [4, Example 5.4], but its corresponding symbol is not ω-hypoelliptic for any given weight function ω by [4, Remark 5.5].

For technical reasons, the class of global symbols for which Theorem 6.8 holds needs to be smaller than the one introduced in Section 2. Namely, we need to introduce some kind of mixed conditions. The following is the corresponding definition for symbols:

Definition 6.2. *We say that* $a \in \widetilde{\mathrm{GS}}_\rho^{m,\omega}$ *if* $a \in C^\infty(\mathbb{R}^{2d})$ *and there exists a Gevrey weight function* σ *satisfying* (6.1) *such that for all* $\lambda > 0$ *there is* $C_\lambda > 0$ *with*

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_\lambda \langle (x,\xi)\rangle^{-\rho|\alpha+\beta|} e^{\lambda\varphi_\sigma^*\left(\frac{|\alpha+\beta|}{\lambda}\right)} e^{m\omega(x,\xi)}, \qquad \alpha,\beta \in \mathbb{N}_0^d,\ x,\xi \in \mathbb{R}^d.$$

Definitions 6.1 and 6.2 are independent of the weight function σ, since given two Gevrey weight functions σ_1 and σ_2 with (6.1), the Gevrey weight function $\sigma(t) := \min\{\sigma_1(t), \sigma_2(t)\}$, $t > 1$, satisfies (6.1) too.

According to condition (6.1), we have, by [1, Lemma 2.9(1)], that for all $\lambda, \mu > 0$ there exists $C > 0$ such that

$$\lambda\varphi_\sigma^*\Big(\frac{j}{\lambda}\Big) \leq C + \mu\rho\varphi_\omega^*\Big(\frac{j}{\mu}\Big), \qquad j \in \mathbb{N}_0. \tag{6.2}$$

As an immediate consequence we have $\widetilde{\mathrm{GS}}_\rho^{m,\omega} \subseteq \mathrm{GS}_\rho^{m,\omega}$.

Lemma 6.3. *Let* $a \in \widetilde{\mathrm{GS}}_\rho^{m,\omega}$. *Then* $a \in \mathrm{HGS}_\rho^{m,m;\omega}$ *if and only if there exist* $R \geq 1$ *and* $C_1' > 0$ *such that* $|a(x,\xi)| \geq C_1' e^{m\omega(x,\xi)}$ *for* $\langle (x,\xi)\rangle \geq R$.

Proof. The necessity is obvious. For the sufficiency, since $a \in \widetilde{\mathrm{GS}}_\rho^{m,\omega}$, for σ as in (6.1) there exists $C > 0$ with

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C \langle (x,\xi)\rangle^{-\rho|\alpha+\beta|} e^{\varphi_\sigma^*(|\alpha+\beta|)} e^{m\omega(x,\xi)}, \qquad \alpha,\beta \in \mathbb{N}_0^d,\ x,\xi \in \mathbb{R}^d, \tag{6.3}$$

which in particular yields

$$C_1' e^{m\omega(x,\xi)} \leq |a(x,\xi)| \leq C e^{m\omega(x,\xi)}, \qquad \langle (x,\xi)\rangle \geq R. \tag{6.4}$$

This shows Definition 6.1(i). For condition (ii), by (2.4), $e^{\varphi_\sigma^*(|\alpha+\beta|)} \leq e^{\frac12\varphi_\sigma^*(2|\alpha|)} e^{\frac12\varphi_\sigma^*(2|\beta|)}$. Thus, by (6.3) and (6.4), we have (since $C_1' \leq C$)

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq \Big(\frac{C}{C_1'}\Big)^{|\alpha+\beta|} \langle (x,\xi)\rangle^{-\rho|\alpha+\beta|} e^{\frac12\varphi_\sigma^*(2|\alpha|)} e^{\frac12\varphi_\sigma^*(2|\beta|)} |a(x,\xi)|,$$

for $\langle (x,\xi)\rangle \geq R$, $\alpha,\beta \in \mathbb{N}_0^d$. Since $a \in \widetilde{\mathrm{GS}}_\rho^{m,\omega} \subseteq \mathrm{GS}_\rho^{m,\omega}$, the result follows. $\square$

Similar mixed conditions are imposed to amplitudes and formal sums.

Definition 6.4. *An amplitude* $a(x,y,\xi) \in C^\infty(\mathbb{R}^{3d})$ *belongs to* $\widetilde{\mathrm{GA}}_\rho^{m,\omega}$ *if there exists a Gevrey weight function* σ *satisfying* (6.1) *such that for all* $\lambda > 0$ *there is* $C_\lambda > 0$ *with*

$$|D_x^\alpha D_y^\beta D_\xi^\gamma a(x,y,\xi)| \leq C_\lambda \frac{\langle x-y\rangle^{\rho|\alpha+\beta+\gamma|}}{\langle (x,y,\xi)\rangle^{\rho|\alpha+\beta+\gamma|}} e^{\lambda\varphi_\sigma^*\left(\frac{|\alpha+\beta+\gamma|}{\lambda}\right)} e^{m\omega(x,\xi)}, \quad \alpha,\beta,\gamma \in \mathbb{N}_0^d,\ x,y,\xi \in \mathbb{R}^d.$$

Definition 6.5. *A formal sum* $\sum p_j$ *is in* $\widetilde{\mathrm{FGS}}_\rho^{m,\omega}$ *if* $p_j \in C^\infty(\mathbb{R}^{2d})$ *and there exist a Gevrey weight function* σ *satisfying* (6.1) *and* $R \geq 1$ *such that for all* $n \in \mathbb{N}$ *there exists* $C_n > 0$ *such that*

$$|D_x^\alpha D_\xi^\beta p_j(x,\xi)| \leq C_n \langle (x,\xi)\rangle^{-\rho(|\alpha+\beta|+j)} e^{n\varphi_\sigma^*\left(\frac{|\alpha+\beta|+j}{n}\right)} e^{m\omega(x,\xi)},$$

for each $j \in \mathbb{N}_0$, $\alpha,\beta \in \mathbb{N}_0^d$, $\log\big(\frac{\langle (x,\xi)\rangle}{R}\big) \geq \frac{n}{j}\varphi_\omega^*\big(\frac{j}{n}\big)$.

Definition 6.6. *We say that* $\sum a_j \sim \sum b_j$ *in* $\widetilde{\mathrm{FGS}}_\rho^{m,\omega}$ *if there exist a Gevrey weight function* σ *satisfying* (6.1) *and* $R \geq 1$ *such that for all* $n \in \mathbb{N}$ *there exist* $C_n > 0$, $N_n \in \mathbb{N}$ *such that*

$$\big|D_x^\alpha D_\xi^\beta \sum_{j<N}(a_j - b_j)\big| \leq C_n \langle (x,\xi)\rangle^{-\rho(|\alpha+\beta|+N)} e^{n\varphi_\sigma^*\left(\frac{|\alpha+\beta|+N}{n}\right)} e^{m\omega(x,\xi)},$$

for all $N \geq N_n$, $\alpha, \beta \in \mathbb{N}_0^d$, $\log\left(\frac{\langle (x,\xi)\rangle}{R}\right) \geq \frac{n}{N}\varphi_\omega^*\left(\frac{N}{n}\right)$.

Again by (6.2) it is also clear that $\widetilde{\mathrm{GA}}_\rho^{m,\omega} \subseteq \mathrm{GA}_\rho^{m,\omega}$ and $\widetilde{\mathrm{FGS}}_\rho^{m,\omega} \subseteq \mathrm{FGS}_\rho^{m,\omega}$.

The amplitudes introduced in Definition 6.4 do not have exponential growth in the variable y to avoid the increasing in the order $m \in \mathbb{R}$ in some results in Section 3. For instance, if $a \in \widetilde{\mathrm{GA}}_\rho^{m,\omega}$, then, following Example 3.7,

$$p_j(x,\xi) := \sum_{j=0}^\infty \sum_{|\beta+\gamma|=j} \frac{1}{\beta!\gamma!}\tau^{|\beta|}(1-\tau)^{|\gamma|}\partial_\xi^{\beta+\gamma}(-D_x)^\beta D_y^\gamma a(x,y,\xi)\Big|_{y=x} \in \widetilde{\mathrm{FGS}}_\rho^{m,\omega}. \tag{6.5}$$

It is easy to check that φ_j (defined in (3.2)) belongs to $\widetilde{\mathrm{GS}}_\rho^{0,\omega}$. Hence the corresponding symbolic calculus is developed in the same manner as for the global symbol class $\mathrm{GS}_\rho^{m,\omega}$. In particular, by [1, Theorem 4.6], we have, from (6.5),

$$p_\tau(x,\xi) := \sum_{j=0}^\infty \varphi_j(x,\xi)p_j(x,\xi) \in \widetilde{\mathrm{GS}}_\rho^{m,\omega} \tag{6.6}$$

for all $\tau \in \mathbb{R}$. Such symbol is called is the τ*-symbol of the pseudodifferential operator associated to the amplitude* $a(x,y,\xi) \in \widetilde{\mathrm{GA}}_\rho^{m,\omega}$. In addition, as a consequence of Theorem 3.11 we obtain Theorem 3.12 for mixed classes.

Theorem 6.7. *Let* $\tau_1, \tau_2 \in \mathbb{R}$. *If* $a_{\tau_1}(x,\xi), a_{\tau_2}(x,\xi) \in \widetilde{\mathrm{GS}}_\rho^{m,\omega}$ *are the* τ_1*-symbol and the* τ_2*-symbol of the pseudodifferential operator* A, *then*

$$a_{\tau_2}(x,\xi) \sim \sum_{j=0}^\infty \sum_{|\alpha|=j} \frac{1}{\alpha!}(\tau_1-\tau_2)^{|\alpha|}\partial_\xi^\alpha D_x^\alpha a_{\tau_1}(x,\xi)$$

in $\widetilde{\mathrm{FGS}}_\rho^{m,\omega}$.

Now we are ready to prove the main theorem of this section.

Theorem 6.8. *Let* $\tau_1, \tau_2 \in \mathbb{R}$ *and let* $a_{\tau_1} \in \widetilde{\mathrm{GS}}_\rho^{m,\omega}$. *If* $a_{\tau_1} \in \mathrm{HGS}_\rho^{m,m;\omega}$, *then* $a_{\tau_2} \in \mathrm{HGS}_\rho^{m,m;\omega}$.

Proof. By (6.6) we have $a_{\tau_2} \in \widetilde{\mathrm{GS}}_\rho^{m,\omega}$. Therefore, by Lemma 6.3, it is enough to show that there exist $R \geq 1$, $D > 0$ such that

$$|a_{\tau_2}(x,\xi)| \geq De^{m\omega(x,\xi)} \tag{6.7}$$

for $\langle (x,\xi)\rangle \geq R$. In fact, by assumption, by the same result there are $R_1 \geq 1$, $D_1 > 0$ such that

$$|a_{\tau_1}(x,\xi)| \geq D_1 e^{m\omega(x,\xi)} \tag{6.8}$$

for $\langle (x,\xi)\rangle \geq R_1$. By Theorem 6.7 and Definition 6.6, there exist a Gevrey weight function σ_1 satisfying (6.1) and $R_2 \geq 1$ such that there exist $C_1 > 0$, $N_1 \in \mathbb{N}$:

$$\Big|a_{\tau_2}(x,\xi) - \sum_{j<N}\sum_{|\alpha|=j}\frac{1}{\alpha!}(\tau_1-\tau_2)^{|\alpha|}\partial_\xi^\alpha D_x^\alpha a_{\tau_1}(x,\xi)\Big| \leq C_1\langle (x,\xi)\rangle^{-\rho N} e^{\varphi_{\sigma_1}^*(N)} e^{m\omega(x,\xi)}$$

for $N \geq N_1$ and $\log\big(\frac{\langle (x,\xi)\rangle}{R_2}\big) \geq \frac{1}{N}\varphi^*_\omega(N)$. By (6.2), there exists $A_1 > 0$ such that $\varphi^*_{\sigma_1}(N) \leq A_1 + \rho\varphi^*_\omega(N)$ for all $N \in \mathbb{N}$. Then,

$$\Big|a_{\tau_2}(x,\xi) - \sum_{j<N}\sum_{|\alpha|=j}\frac{1}{\alpha!}(\tau_1-\tau_2)^{|\alpha|}\partial^\alpha_\xi D^\alpha_x a_{\tau_1}(x,\xi)\Big| \leq C_1 e^{A_1} R_3^{-\rho N} e^{m\omega(x,\xi)}, \tag{6.9}$$

for all $N \geq N_1$ and $\langle (x,\xi)\rangle \geq R_3 e^{\frac{1}{N}\varphi^*_\omega(N)}$, where $R_3 \geq R_2$ will be determined later.

We fix $N = N_1 \in \mathbb{N}$ and we claim that

$$\Big|\sum_{j=0}^{N-1}\sum_{|\alpha|=j}\frac{1}{\alpha!}(\tau_1-\tau_2)^{|\alpha|}\partial^\alpha_\xi D^\alpha_x a_{\tau_1}(x,\xi)\Big| \geq \frac{D_1}{2}e^{m\omega(x,\xi)}, \tag{6.10}$$

if $\langle (x,\xi)\rangle$ is large enough. The inequality is immediate for $N = 1$ by (6.8) for $\langle (x,\xi)\rangle \geq R_1$, so we shall assume that $N > 1$. First, we estimate

$$\Big|\sum_{j=1}^{N-1}\sum_{|\alpha|=j}\frac{1}{\alpha!}(\tau_1-\tau_2)^{|\alpha|}\partial^\alpha_\xi D^\alpha_x a_{\tau_1}(x,\xi)\Big|.$$

Since $a_{\tau_1}(x,\xi) \in \widetilde{\mathrm{GS}}^{m,\omega}_\rho$, there exists a Gevrey weight function σ_2 satisfying (6.1) such that there is $C_2 > 0$ with

$$|D^\alpha_x D^\alpha_\xi a_{\tau_1}(x,\xi)| \leq C_2\langle (x,\xi)\rangle^{-2\rho} e^{2\varphi^*_{\sigma_2}(N-1)} e^{m\omega(x,\xi)},$$

for all $x,\xi \in \mathbb{R}^d$ and $1 \leq |\alpha| \leq N-1$. Again by (6.2), there exists $A_2 > 0$ such that $\varphi^*_{\sigma_2}(N-1) \leq A_2 + \rho\varphi^*_\omega(N-1)$. Consider $\langle (x,\xi)\rangle$ large enough so that

$$\langle (x,\xi)\rangle \geq R_4 e^{\varphi^*_\omega(N-1)},$$

with $R_4 \geq 1$ to be determined. Then

$$\begin{aligned}|D^\alpha_x D^\alpha_\xi a_{\tau_1}(x,\xi)| &\leq C_2 e^{2A_2}\langle (x,\xi)\rangle^{-2\rho} e^{2\rho\varphi^*_\omega(N-1)} e^{m\omega(x,\xi)}\\ &\leq C_2 e^{2A_2}(R_4)^{-2\rho} e^{m\omega(x,\xi)},\end{aligned}$$

for $\langle (x,\xi)\rangle \geq R_4 e^{\varphi^*_\omega(N-1)}$, $1 \leq |\alpha| \leq N-1$. On the other hand, by formula [22, (0.3.1)], we obtain

$$\sum_{j=1}^{N-1}\sum_{|\alpha|=j}\frac{|\tau_1-\tau_2|^{|\alpha|}}{\alpha!} \leq \sum_{j=1}^{N-1}\frac{(d|\tau_1-\tau_2|)^j}{j!} \leq e^{d|\tau_1-\tau_2|}.$$

So, we deduce

$$\Big|\sum_{j=1}^{N-1}\sum_{|\alpha|=j}\frac{1}{\alpha!}(\tau_1-\tau_2)^{|\alpha|}\partial^\alpha_\xi D^\alpha_x a_{\tau_1}(x,\xi)\Big| \leq C_2 e^{2A_2}(R_4)^{-2\rho} e^{d|\tau_1-\tau_2|} e^{m\omega(x,\xi)}, \tag{6.11}$$

for $\langle (x,\xi)\rangle \geq R_4 e^{\varphi^*_\omega(N-1)}$. Hence, by the triangular inequality, from formulas (6.11) and (6.8) we have

$$\begin{aligned}\Big|\sum_{j=0}^{N-1}\sum_{|\alpha|=j}\frac{1}{\alpha!}(\tau_1-\tau_2)^{|\alpha|}\partial^\alpha_\xi D^\alpha_x a_{\tau_1}(x,\xi)\Big| &\geq D_1 e^{m\omega(x,\xi)} - C_2 e^{2A_2}(R_4)^{-2\rho} e^{d|\tau_1-\tau_2|} e^{m\omega(x,\xi)}\\ &\geq \frac{D_1}{2}e^{m\omega(x,\xi)},\end{aligned}$$

which shows (6.10) provided R_4 be so that

$$(R_4)^{2\rho} \geq \frac{2}{D_1} C_2 e^{2A_2} e^{d|\tau_1 - \tau_2|},$$

and $\langle (x,\xi) \rangle \geq \max\{R_1, R_4 e^{\varphi^*_\omega(N-1)}\}$. Finally we obtain, by (6.10) and (6.9),

$$|a_{\tau_2}(x,\xi)| \geq \frac{D_1}{2} e^{m\omega(x,\xi)} - C_1 e^{A_1} R_3^{-\rho N} e^{m\omega(x,\xi)} \geq \frac{D_1}{4} e^{m\omega(x,\xi)}$$

if $R_3^{\rho N} \geq \frac{4}{D_1} C_1 e^{A_1}$ and $\langle (x,\xi) \rangle \geq R := \max\{R_1, R_4 e^{\varphi^*_\omega(N-1)}, R_3 e^{\frac{1}{N}\varphi^*_\omega(N)}\}$. Then (6.7) is satisfied for $D = \frac{D_1}{4} > 0$ and $R \geq 1$, and the proof is complete. □

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