ON CONDITIONAL PERMUTABILITY AND
SATURATED FORMATIONS

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Abstract   Two subgroups A and B of a group G are said to be totally completely conditionally permutable (tcc-permutable) in G if X permutes with Y^g for some $g \in \langle X, Y \rangle$, for all $X \leqslant A$ and $Y \leqslant B$. We study the belonging of a finite product of tcc-permutable subgroups to a saturated formation of soluble groups containing all finite supersoluble groups.

Keywords: finite groups; products of subgroups; saturated formations; conditional permutability

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1. Introduction and results

All groups considered in the paper are finite.

This paper is a contribution to the study of products of groups whose factors are linked by certain permutability properties. The origin of this research can be traced to Asaad and Shaalan [**3**], who provide criteria for the product of supersoluble groups to be supersoluble. Based on their paper, Maier [**22**] defines a group $G = HK$ to be the product of the *totally permutable* subgroups H and K if every subgroup of H is permutable with every subgroup of K. We recall that a *formation* is a class $\mathcal{F}$ of groups closed under homomorphic images, such that $G/M \cap N \in \mathcal{F}$ whenever G is a group and M, N are normal subgroups of G with $G/M \in \mathcal{F}$ and $G/N \in \mathcal{F}$. In this case the $\mathcal{F}$-residual $G^{\mathcal{F}}$ of G is the smallest normal subgroup of G such that $G/G^{\mathcal{F}} \in \mathcal{F}$. The formation $\mathcal{F}$ is *saturated* if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$, where $\Phi(G)$ denotes the Frattini subgroup of G. The following result was proved in [**4**, **5**, **7**].

Theorem 1.1. *Let $\mathcal{F}$ be a formation which contains the class $\mathcal{U}$ of all finite super-soluble groups. Let the group $G = G_1 G_2 \cdots G_r$ be a product of pairwise totally permutable subgroups $G_1, G_2, \ldots, G_r$. Then the following hold.*

(i) *If $G_1, G_2, \ldots, G_r \in \mathcal{F}$, then $G \in \mathcal{F}$.*

(ii) *Assume in addition that $\mathcal{F}$ is either saturated or a formation of soluble groups. If $G \in \mathcal{F}$, then $G_1, G_2, \ldots, G_r \in \mathcal{F}$.*

$\square$

Part (i) in this result was first proved for a saturated formation $\mathcal{F}$ such that $\mathcal{U} \subseteq \mathcal{F}$ in [**22**] (for $r = 2$) and [**9**] (for r arbitrary), generalizing a result in [**3**] for $\mathcal{F} = \mathcal{U}$, the saturated formation of all finite supersoluble groups.

Furthermore, products of totally permutable subgroups have been studied in depth both in the frameworks of formation theory [**4**–**10**, **22**] and in the theory of Fitting classes [**16**–**18**]. The structure of products of totally permutable subgroups is nowadays quite well understood. For instance, products $G = HK$ of totally permutable subgroups H and K are close to central products in the sense that the nilpotent residual $H^{\mathcal{N}}$ of H centralizes K, and vice versa [**8**, Theorem 1]; $\mathcal{N}$ denotes the class of all finite nilpotent groups. Also $H \cap K \leqslant F(G)$, the Fitting subgroup of the group G [**22**, Lemma 2], and G modulo its $\mathcal{U}$-hypercentre $Z_{\mathcal{U}}(G)$ is a direct product of images of H and K [**13**, p. 859, Remarks (3)]. (The $\mathcal{U}$-hypercentre $Z_{\mathcal{U}}(G)$ of G is the largest normal subgroup of G such that every chief factor X/Y of G with $Y < X \leqslant Z_{\mathcal{U}}(G)$ is cyclic of prime order.)

More recently this development has been taken further by Guo *et al.* in [**15**], by considering a weaker condition of subgroup permutability, namely *conditional permutability*. We refer to [**14**, **15**] and discuss the following concepts.

Definition 1.2. *Let G be a group. Two subgroups X and Y of G are called condition-ally permutable (c-permutable, for brevity) in G if X permutes with Y^g for some element $g \in G$.*

The subgroups X and Y are called completely conditionally permutable (cc-permutable) in G if X permutes with Y^g for some element $g \in \langle X, Y \rangle$, the subgroup generated by X and Y.

Two subgroups A and B of G are said to be totally completely conditionally permutable (tcc-permutable) in G if X and Y are cc-permutable in G for all $X \leqslant A$ and $Y \leqslant B$.

Such a type of permutability conditions has been considered by other authors in extending classical results about the influence of permutability properties of certain families of subgroups on the structure of groups (see, for example, [**2**, **14**, **20**, **23**]).

This paper continues a study on products of subgroups, c-permutability and super-solubility carried out in [**1**]. Previous related results involving cc-permutability appear in [**15**, **21**]. In particular, the following result is obtained in [**1**, Theorem 2].

Theorem 1.3. *Let the group $G = AB$ be the product of subgroups A and B such that every subgroup of A is c-permutable in G with every subgroup of B. Then $G^{\mathcal{U}} = A^{\mathcal{U}} B^{\mathcal{U}}$. In particular, if A and B are supersoluble, then G is supersoluble.*

As mentioned in [**1**], in contrast to the previous research on products of totally permutable subgroups, this result does not extend to (saturated) formations containing $\mathcal{U}$, even in the universe of finite soluble groups (see Example 3.3). Nevertheless, we prove in this paper that positive results are possible if c-permutability is strengthened to cc-permutability. It is remarkable that c-permutability fails to satisfy the property of persistence in intermediate subgroups (see Example 3.4). Then cc-permutability appears when requiring c-permutability to satisfy this persistence property and becomes a stronger condition. However, significant structural properties of products of totally permutable subgroups, such as those mentioned previously, are missed when considering cc-permutability instead of permutability (see Examples 3.5 and 3.6).

The main aim of this paper is to prove the following result.

Theorem 1.4. *Let $\mathcal{F}$ be a saturated formation of soluble groups containing $\mathcal{U}$. Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geqslant 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then*

(i) *if $G_i \in \mathcal{F}$ for all $i = 1, \ldots, r$, then $G \in \mathcal{F}$,*

(ii) *if $G \in \mathcal{F}$, then $G_i \in \mathcal{F}$ for all $i = 1, \ldots, r$.*

As a consequence, a stronger version of this theorem is obtained.

Corollary 1.5. *Let $\mathcal{F}$ be a saturated formation of soluble groups containing $\mathcal{U}$. Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geqslant 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then*

(i) *$G_i^{\mathcal{F}} \trianglelefteq G$ for all $i = 1, \ldots, r$,*

(ii) *$G^{\mathcal{F}} = G_1^{\mathcal{F}} \cdots G_r^{\mathcal{F}}$.*

Analogous results to Corollary 1.5 for products of totally permutable subgroups and formations, either saturated or of soluble groups, containing $\mathcal{U}$ were obtained in [**4**,**7**] (for $r = 2$) and [**5**] (for r arbitrary). These results generalize a classical Doerk and Hawkes result which states that for a formation $\mathcal{F}$ of soluble groups the $\mathcal{F}$-residual respects the operation of forming direct products [**12**, §IV, Theorem 1.18].

Section 2 is devoted to prove our main results. Some results and remarks about the behaviour of $\mathcal{F}$-projectors and $\mathcal{F}$-normalizers in such products of (soluble) groups, as well as the above-mentioned examples are collected in §3.

For notation and results on classes of groups we refer to [**12**]. In particular, $\sigma(G)$ denotes the set of all primes dividing the order of the group G. For subgroups X, A of a group G, we define $\langle X^A \rangle = \langle x^a : x \in X, \ a \in A \rangle$; in particular, $\langle X^G \rangle$ is the normal closure of X in G. For a prime p the set of Sylow p-subgroups of G is denoted by $\mathrm{Syl}_p(G)$.

2. Proof of the main results

In order to prove Theorem 1.4 we need some previous results.

Lemma 2.1. *Let the group $G = AB$ be the product of tcc-permutable subgroups A and B. Then*

(i) *X and Y^g are tcc-permutable subgroups of G for any $X \leqslant A$, $Y \leqslant B$ and $g \in G$,*

(ii) *for each $Y \leqslant B$, A permutes with Y^b for some $b \in B$.*

Proof. (i) We take $g = ba \in G = AB = BA$ with $a \in A$ and $b \in B$. Then $Y^g = (Y^b)^a$ with $Y^b \leqslant B$.

We consider $X^{a^{-1}} \leqslant A$ and $Y^b = (Y^g)^{a^{-1}} \leqslant B$, which are tcc-permutable.

Let $X_0 \leqslant X$ and $Y_0 \leqslant Y^g$. Then $X_0^{a^{-1}} \leqslant X^{a^{-1}}$ and $Y_0^{a^{-1}} \leqslant Y^b$. Hence, there exists $t \in \langle X_0^{a^{-1}}, Y_0^{a^{-1}} \rangle$ such that $X_0^{a^{-1}}(Y_0^{a^{-1}})^t = (Y_0^{a^{-1}})^t X_0^{a^{-1}}$. Then $t^a \in \langle X_0, Y_0 \rangle$ and

$$(X_0 Y_0^{t^a})^{a^{-1}} = X_0^{a^{-1}}(Y_0^{t^a})^{a^{-1}} = (Y_0^{t^a})^{a^{-1}} X_0^{a^{-1}} = (Y_0^{t^a} X_0)^{a^{-1}}.$$

Therefore, $X_0 Y_0^{t^a} = Y_0^{t^a} X_0$ with $t^a \in \langle X_0, Y_0 \rangle$.

(ii) By hypothesis, A permutes with Y^g for some $g = ba \in \langle Y, A \rangle \leqslant BA$, $b \in B$, $a \in A$. Then $A = A^{a^{-1}}$ permutes with $Y^b = (Y^g)^{a^{-1}}$ and we are done. $\qquad\square$

Lemma 2.2 (Çunihin [11]). *Let the group $G = HK$ be the product of subgroups H and K. If $L \trianglelefteq H$ and $L \leqslant K$, then $\langle L^K \rangle \trianglelefteq G$; in particular, $L \leqslant \mathrm{Core}_G(K)$.*

Proof. This follows easily since $L \leqslant \langle L^G \rangle = \langle L^{HK} \rangle = \langle L^K \rangle \leqslant \mathrm{Core}_G(K)$. $\qquad\square$

The following lemma is a key fact in our work. A corresponding result for products of totally permutable subgroups was proved in [**22**, Lemma 2(a)].

Lemma 2.3. *Let the group $1 \neq G = AB$ be the product of tcc-permutable subgroups A and B. Let p be the largest prime divisor of $|G|$. Without loss of generality let $a \in A$ be a p-element of maximal order in $A \cup B$ and let $X_0 \leqslant \langle a \rangle$ with $|X_0| = p$. Then*

(i) *B^g normalizes X_0 for some $g \in G$,*

(ii) *$1 \neq \langle X_0^A \rangle \trianglelefteq G$; in particular, $1 \neq X_0 \leqslant \mathrm{Core}_G(A)$.*

Proof. (i) Set $X = \langle a \rangle$. By hypothesis, $XB^g = B^g X$ for some $g \in G$. Arguing by induction on $|G| + |A| + |B|$ and using Lemma 2.1 (i), we may assume that $G = XB^g$ is the product of the tcc-permutable subgroups X and B^g. Without loss of generality we may also set $B = B^g$.

Assume that q is a prime divisor of $|B|$ such that $q \neq p$. By Lemma 2.1 (ii) and Sylow's theorems, X permutes with B_q for some $B_q \in \mathrm{Syl}_q(B)$. Hence, XB_q is a product of the tcc-permutable supersoluble subgroups X and B_q. It follows that XB_q is supersoluble by Theorem 1.3 and, in particular, B_q normalizes X. Hence, B_q normalizes X_0. On the other hand, there exists $B_p \in \mathrm{Syl}_p(B)$ such that $XB_p = B_pX$ (if $p \notin \sigma(B)$, then $B_p = 1$).

If $XB_p < G$, we deduce by induction that B_p^u normalizes X_0 for some $u \in XB_p$ which implies that X_0 is normal in XB_p. Consequently, $B = \langle B_p, B_q \mid q \neq p, \ q \in \sigma(B) \rangle$ normalizes X_0 and we are done.

Then, we may assume that $G = XB_p$ and $B = B_p \neq 1$.

We may assume that $X < G$ and consider a maximal subgroup M of G containing X. Then $M = X(B \cap M)$ and X and $B \cap M$ are tcc-permutable subgroups of M. By induction we deduce that $(B \cap M)^m$ for some $m \in M$ normalizes X_0 and then M normalizes X_0. Consequently, $M \leqslant N_G(X_0) = XN_B(X_0)$.

Assume that $N_B(X_0) < B$ and consider $b \in B \setminus N_B(X_0)$.

By hypothesis $X\langle b\rangle^r = \langle b\rangle^r X$ for some $r \in \langle X, \langle b\rangle\rangle$. Since $|X| \geqslant |\langle b\rangle|$, we can deduce from [**19**, §VI, Satz 10.1] that X_0 is normal in $\langle b\rangle^r X$. We are now assuming that G is a p-group and so $M \trianglelefteq G$. Then $r = b^i s \in G = M\langle b\rangle$ for some $s \in M = N_G(X_0)$ and some integer i. Hence, $X_0^{b^s} = X_0^{b^{i}s} = X_0^{b^r} = X_0$ and so $X_0^b = X_0^{s^{-1}b} = X_0^{s^{-1}} = X_0$, which contradicts the choice of b and concludes the proof of part (i).

(ii) From (i) we consider B^g, with $g \in G$, which normalizes X_0. Then $G = AB = A(X_0 B^g)$ and the result follows from Lemma 2.2. $\qquad\square$

Remark 2.4. Lemma 2.3 states that for a group $1 \neq G = AB$ which is the product of tcc-permutable subgroups A and B, there exists $1 \neq N \trianglelefteq G$ such that either $N \leqslant A$ or $N \leqslant B$.

Lemma 2.5. *Let the group $1 \neq G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geqslant 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then there exists $1 \neq N \trianglelefteq G$ such that $N \leqslant G_i$ for some $i \in \{1, \ldots, r\}$.*

Proof. Let p be the largest prime divisor of $|G|$ and let $i \in \{1, \ldots, r\}$ such that the factor G_i contains a p-element of $G_1 \cup \cdots \cup G_r$ of maximal order. Then Lemma 2.3 implies that G_i contains a non-trivial normal subgroup of G. $\qquad\square$

Lemma 2.6. *Let the group $G = LN$ be the product of subgroups L and N with $N \trianglelefteq G$, N abelian and $N \cap L = 1$. If L is c-permutable in G with a subgroup X of N, then L normalizes X. In particular, if L is tcc-permutable with a subgroup N_0 of N, then L normalizes every subgroup of N_0.*

Proof. Assume that L is c-permutable in G with a subgroup X of N. Then L permutes with X^n for some $n \in N$ as $G = LN$. Since N is abelian, L permutes with $X = X^n$ and then $X = (N \cap L)X = N \cap LX \trianglelefteq LX$ and L normalizes X. The rest is easily deduced. $\qquad\square$

Lemma 2.7. *Let the group $G = AB$ be the product of tcc-permutable subgroups A and B. Then B normalizes $A^{\mathcal{U}}$, and vice versa.*

Proof. Let p be a prime divisor of $|B|$. By Sylow's theorems and Lemma 2.1 we deduce that A permutes with B_p for some $B_p \in \mathrm{Syl}_p(B)$, and, moreover, A and B_p

are tcc-permutable subgroups. In particular, $A^{\mathcal{U}} = (AB_p)^{\mathcal{U}} \trianglelefteq AB_p$ by Theorem 1.3. It follows now that B normalizes $A^{\mathcal{U}}$. $\square$

Lemma 2.8. *Let the group $1 \neq G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geqslant 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Assume in addition that G is a soluble primitive group; let N be the unique minimal normal subgroup of G and let p be the prime divisor of $|N|$. Then either G is supersoluble or the following conditions are satisfied:*

(i) *without loss of generality $N \leqslant G_1$;*

(ii) *$G_2 \cdots G_r$ is a cyclic group whose order divides $p - 1$;*

(iii) *there exists a maximal subgroup M of G with $\mathrm{Core}_G(M) = 1$ such that $M = (M \cap G_1)(G_2 \cdots G_r)$ and $M \cap G_1$ centralizes $G_2 \cdots G_r$.*

Proof. Assume that G is not supersoluble. Let M be a maximal subgroup of G with $\mathrm{Core}_G(M) = 1$. In particular, $G = NM$, $N \cap M = 1$ and N is an elementary abelian p-group. We prove the following.

Step 1 ($N \not\leqslant \cap_{i=1}^r G_i$). Otherwise $G_i = N(M \cap G_i)$ for each $i = 1, \ldots, r$. It follows from Lemma 2.6 that $M \cap G_i$ normalizes every subgroup of N for all $i = 1, \ldots, r$. Hence, N has order p and G is supersoluble: a contradiction.

By Lemma 2.5 and without loss of generality assume that $N \leqslant \cap_{i=1}^s G_i$ with $1 \leqslant s < r$ and $N \not\leqslant G_j$ for all $j = s + 1, \ldots, r$. In particular, we note that $G_i = N(M \cap G_i)$ and $N \cap (M \cap G_i) = 1$ for all $i = 1, \ldots, s$.

Step 2 ($N \cap G_j = 1$ for all $j = s+1, \ldots, r$). Assume that $N \cap G_j \neq 1$ for some $j \in \{s+1, \ldots, r\}$. By Lemma 2.6 we deduce that $N \cap G_j$ is normalized by G_i for all $i = 1, \ldots, s$. Then $N \cap G_j \trianglelefteq G_1 \cdots G_s G_j$ and Lemma 2.2 implies that $N \leqslant \mathrm{Core}_G(G_{s+1} \cdots G_r) \neq 1$. On the other hand, by Lemma 2.5 one of the factors $G_{s+1}, \ldots, G_r$ contains a minimal normal subgroup R of $G_{s+1} \cdots G_r$, $R \neq N$. If $R \leqslant N$, it follows as above from Lemma 2.6 that R is normal in G and $R = N$: a contradiction. Hence, $[R, N] \leqslant R \cap N = 1$ and $R \leqslant C_G(N) = N$: again a contradiction, which proves that $N \cap G_j = 1$.

Step 3 (G_j normalizes every subgroup of N, and therefore G_j is a cyclic group whose order divides $p - 1$, for all $j = s+1, \ldots, r$). Let $j \in \{s+1, \ldots, r\}$. It follows from Step 2 and Lemma 2.6 that G_j normalizes every subgroup of N. Then for each $g \in G_j$, $n^g = n^t$ for some t, $1 \leqslant t \leqslant p - 1$, for every $n \in N$. Since $C_G(N) = N$ and $N \cap G_j = 1$, we deduce that G_j is a cyclic group whose order divides $p - 1$.

Step 4 ($s = 1$). If $s \geqslant 2$ we deduce from Lemma 2.6 that G_i normalizes every subgroup of N for all $i = 1, \ldots, s$. Consequently, N has order p, by Step 3, and G is supersoluble: a contradiction.

Step 5 ($G_2 \cdots G_r$ is a cyclic group whose order divides $p - 1$ and $[G_2 \cdots G_r, G] \leqslant N$). By Steps 3 and 4, $G_2 \cdots G_r$ normalizes every subgroup of N and $G_2 \cdots G_r \cap N = 1$. The result is now deduced by using arguments as in Step 3; in particular we notice that $[G_2 \cdots G_r, G] \leqslant C_G(N) = N$.

Step 6 (without loss of generality we may assume that $G_2 \cdots G_r \leqslant M$ and $M = (M \cap G_1)(G_2 \cdots G_r)$; moreover, $M \cap G_1$ centralizes $G_2 \cdots G_r$). By Step 5 and Hall's theorems, $G_2 \cdots G_r$ is contained in a Hall p'-subgroup of G and then $G_2 \cdots G_r \leqslant M^g$ for some $g \in G$. Without loss of generality we may set $M = M^g$. Since $G = G_1(G_2 \cdots G_r)$, we have that $M = (M \cap G_1)(G_2 \cdots G_r)$. Moreover, Step 5 implies also that $[G_2 \cdots G_r, M \cap G_1] \leqslant M \cap N = 1$. □

We recall the following basic result on formations.

Lemma 2.9. *Let $\mathcal{H}$ be a formation and let the group $G = HK$ be the product of subgroups H and K such that H centralizes K. If $H, K \in \mathcal{H}$, then $G \in \mathcal{H}$.*

Proof. Since H centralizes K, G is a factor group of the external direct product $H \times K$, and the result is easily deduced. □

Proof of Theorem 1.4. (i) Assume that the result is not true and let the group $G = G_1 \cdots G_r$ be a counterexample of minimal order. We notice that, for any normal subgroup N of G, the factor group $G/N = (G_1 N/N) \cdots (G_r N/N)$ satisfies the hypotheses of the theorem. Since $\mathcal{F}$ is a saturated formation, G is a primitive group with a unique minimal normal subgroup, say N, and $G/N \in \mathcal{F}$; in particular, G/N is a soluble group. By Lemma 2.5 we may assume without loss of generality that $N \leqslant G_1 \in \mathcal{F}$. Hence, N is soluble; consequently, G is a soluble primitive group and Lemma 2.8 can be applied. We consider $M = (M \cap G_1)(G_2 \cdots G_r)$ a maximal subgroup of G as in Lemma 2.8. Let F denote the canonical local definition of $\mathcal{F}$. Since $\mathcal{U} \subseteq \mathcal{F}$, we have that $G_2 \cdots G_r \in \mathsf{F}(p)$ for p the prime divisor of $|N|$. Moreover, the fact that $G_1 \in \mathcal{F}$ implies $M \cap G_1 \in \mathsf{F}(p)$. Therefore, $M = (M \cap G_1)(G_2 \cdots G_r) \in \mathsf{F}(p)$ by Lemma 2.9 as $\mathsf{F}(p)$ is a formation. Since $G/C_G(N) = G/N \cong M \in \mathsf{F}(p)$ and $G/N \in \mathcal{F}$, it follows that $G \in \mathcal{F}$: a contradiction, which proves (i).

(ii) We argue as in (i) and consider $G = G_1 \cdots G_r$ to be a counterexample of minimal order. We deduce here that G has a unique minimal normal subgroup, say N, and again assume $N \leqslant G_1$ by Lemma 2.5. Moreover, now $G \in \mathcal{F}$ and so G is soluble. Assume first that G is primitive. From Lemma 2.8 and with the same notation we have that $G_j \in \mathcal{U} \subseteq \mathcal{F}$ for all $j = 2, \ldots, r$. In addition, $M = (M \cap G_1)(G_2 \cdots G_r) \cong G/N \in \mathsf{F}(p)$ because $G \in \mathcal{F}$. Since $G_2 \cdots G_r$ is a normal nilpotent subgroup of M, it follows from [**12**, §IV, Theorem 1.14] that $M \cap G_1 \in \mathsf{F}(p)$, which implies $G_1 = N(M \cap G_1) \in \mathcal{F}$: a contradiction.

Consider now the case $N \leqslant \Phi(G)$, the Frattini subgroup of G. We notice that $G_i N/N \in \mathcal{F}$ for all $i = 1, \ldots, r$. Assume that $N \leqslant G_j$ for some $j \neq 1$. Then for $k = 1, j$, we have that $G_k = N F_k$, with F_k an $\mathcal{F}$-projector of G_k. Since $N \leqslant G_1 \cap G_j$, N and F_k are tcc-permutable subgroups and part (i) implies that $G_k = N F_k \in \mathcal{F}$. On the other hand, if $N \not\leqslant G_j$ for some $j \neq 1$, then $\mathrm{Core}_G(G_j) = 1$ and we can deduce from Lemma 2.7 that $G_j^{\mathcal{U}} = 1$, that is, $G_j \in \mathcal{U} \subseteq \mathcal{F}$. Consequently, it follows that $G_1 \notin \mathcal{F}$ and $G_j \in \mathcal{U}$ for all $j = 2, \ldots, r$. By the hypothesis, we notice that the $\mathcal{F}$-projector F_1 of G_1 permutes with $G_j^{n_j}$ for some $n_j \in N$ for each $j = 2, \ldots, r$. Therefore, F_1 permutes with $\langle G_2^{n_2}, \ldots, G_r^{n_r} \rangle$ and $G = N F_1 \langle G_2^{n_2}, \ldots, G_r^{n_r} \rangle = F_1 \langle G_2^{n_2}, \ldots, G_r^{n_r} \rangle$, since $N \leqslant \Phi(G)$. Moreover, F_1 and

$G_j^{n_j}$ for each $j = 2, \ldots, r$, are tcc-permutable subgroups by Lemma 2.1 (i), which implies by Lemma 2.7 that $F_1^{\mathcal{U}}$ is normalized by $\langle G_2^{n_2}, \ldots, G_r^{n_r} \rangle$ and then $F_1^{\mathcal{U}}$ is normal in G. We notice that $G_1^{\mathcal{F}} \leqslant N$ is an abelian group, which implies by [**12**, §IV, Theorem 5.18] that $G_1^{\mathcal{F}} \cap F_1 = 1$. If $F_1^{\mathcal{U}} \neq 1$, then $G_1^{\mathcal{F}} \leqslant N \leqslant F_1^{\mathcal{U}} \leqslant F_1$ which implies $G_1^{\mathcal{F}} = 1$, that is, $G_1 \in \mathcal{F}$: a contradiction. Therefore, $F_1 \in \mathcal{U}$. Consequently, $G/N = NF_1 G_2 \cdots G_r/N \in \mathcal{U}$ by (i), which implies $G \in \mathcal{U}$ since $N \leqslant \Phi(G)$, and obviously $G_i \in \mathcal{U} \subseteq \mathcal{F}$ for all $i = 1, \ldots, r$: the final contradiction. $\qquad\square$

The obtained results (in particular, Theorem 1.4 and Lemmas 2.1 and 2.3) allow us to adapt the arguments in the proofs of [**7**, Lemmas 5 and 6] (see also the proof of Lemma 2.7) to deduce the following result.

Lemma 2.10. *Let $\mathcal{F}$ be a saturated formation of soluble groups containing $\mathcal{U}$. Let the group $G = AB$ be the product of tcc-permutable subgroups A and B. Then*

(i) *if $B \in \mathcal{F}$, then $G^{\mathcal{F}} = A^{\mathcal{F}}$,*

(ii) *$A^{\mathcal{F}}$ and $B^{\mathcal{F}}$ are normal subgroups of G.*

Proof of Corollary 1.5. From Lemma 2.10 we deduce that $G_i^{\mathcal{F}}$ for all $i = 1, \ldots, r$, and $K := G_1^{\mathcal{F}} \cdots G_r^{\mathcal{F}}$ are normal subgroups of G. We notice that

$$G/G^{\mathcal{F}} = (G_1 G^{\mathcal{F}}/G^{\mathcal{F}}) \cdots (G_r G^{\mathcal{F}}/G^{\mathcal{F}})$$

satisfies the hypotheses of the result and then Theorem 1.4 (ii) implies that $K \leqslant G^{\mathcal{F}}$. By now considering $G/K = (G_1 K/K) \cdots (G_r K/K)$, it follows that $G^{\mathcal{F}} \leqslant K$ from Theorem 1.4 (i). Consequently, $G^{\mathcal{F}} = K$. $\qquad\square$

3. Some consequences and examples

We deduce next some results about the behaviour of $\mathcal{F}$-projectors and $\mathcal{F}$-normalizers in products of tcc-permutable (soluble) subgroups. We follow here previous related results for products of totally permutable subgroups from [**4**, **5**].

Corollary 3.1. *Let $\mathcal{F}$ be a saturated formation of soluble groups containing $\mathcal{U}$. Let the group $G = AB$ be the product of tcc-permutable subgroups A and B.*

(i) *There exist $\mathcal{F}$-projectors X of A and Y of B such that X is permutable with Y. In this case XY is an $\mathcal{F}$-projector of G.*

(ii) *Assume in addition that G is soluble. There exist $\mathcal{F}$-normalizers U of A and V of B such that U is permutable with V and UV is an $\mathcal{F}$-normalizer of G.*

Proof. (i) Let X and Y be $\mathcal{F}$-projectors of A and B, respectively. Since A and B are tcc-permutable, there exists $g = ba \in G = BA$ with $a \in A$ and $b \in B$ such that X permutes with $Y^g = Y^{ba}$. Then $X^{a^{-1}}$ permutes with $(Y^g)^{a^{-1}} = (Y^{ba})^{a^{-1}} = Y^b$ and $X^{a^{-1}}$ is an $\mathcal{F}$-projector of A and Y^b is an $\mathcal{F}$-projector of B.

Assume now that X and Y are $\mathcal{F}$-projectors of A and B respectively, such that X permutes with Y. By using the results in this paper, in particular, Lemmas 2.1 and 2.3 and Theorem 1.4, the proof of [**4**, Theorem B] can be adapted to deduce that XY is an $\mathcal{F}$-projector of G.

(ii) We argue by induction on $|G| + |A| + |B|$. By using Theorem 1.4, Lemma 2.10 and the arguments in the proof of [**4**, Theorem C], we may assume that $A \notin \mathcal{F}$ and deduce the following facts:

- there exists a normal subgroup T of A such that $T \leqslant A^{\mathcal{F}} \leqslant G^{\mathcal{F}}$, $T \leqslant F(A) \cap F(G)$;

- there exists a maximal subgroup M of A such that $A = TM = A^{\mathcal{F}}M = F(A)M$, whence M is $\mathcal{F}$-critical in A.

By Lemma 2.1 (ii) there exists $a \in A$ such that M^a permutes with B. Without loss of generality we may assume that M permutes with B. If MB is a proper subgroup of G, we notice that $G = T(MB) = G^{\mathcal{F}}(MB) = F(G)(MB)$ and MB is a $\mathcal{F}$-critical maximal subgroup of G. In both cases, whether $G = MB$ or $MB < G$, the result follows, as in the proof of [**4**, Theorem C], by induction and [**12**, §V, Lemma 3.7], which states that $\mathcal{F}$-normalizers of $\mathcal{F}$-critical maximal subgroups of a group are $\mathcal{F}$-normalizers of the whole group. $\qquad\square$

Remark 3.2. Let $\mathcal{F}$ be a saturated formation of soluble groups containing $\mathcal{U}$. Let the group $G = G_1 \cdots G_r$ be the product of the pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geqslant 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. It is an open question whether or not there exist $\mathcal{F}$-projectors X_i of G_i for each $i = 1, \ldots, r$, such that $X_1, \ldots, X_r$ are pairwise permutable.

In this case $X_1 \cdots X_r$ would be an $\mathcal{F}$-projector of G. The proof of this would follow as above, by using the results in this paper, in particular, Lemmas 2.1 and 2.5 and Theorem 1.4, and the proof of [**5**, Theorem 5] (see also the proof of [**4**, Theorem B]).

Assume in addition that the group $G = G_1 \cdots G_r$ is soluble; equivalently, by Theorem 1.4 assume that the factors G_i for all $i = 1, \ldots, n$ are soluble. It is also an open question whether or not there exist $\mathcal{F}$-normalizers X_i of G_i for each $i = 1, \ldots, r$, such that $X_1 \cdots X_r$ is an $\mathcal{F}$-normalizer of G.

Next we gather the examples mentioned in §1. They are taken from [**1**] and included here for completeness.

Example 3.3 (Arroyo-Jordá *et al.* [1, Final Remark]). This example shows the failure of Theorem 1.4 when cc-permutability is weakened to c-permutability.

Let $G = \mathrm{Sym}(4) = AY$ be the symmetric group of degree 4, with $A = \mathrm{Alt}(4)$ the alternating subgroup of G and Y a subgroup of G of order 2 generated by a transposition. Then every subgroup of A is c-permutable in G with every subgroup of Y; however, for $\mathcal{N}^2$ the saturated formation of metanilpotent groups, we have that $\mathcal{U} \subseteq \mathcal{N}^2$, $A, Y \in \mathcal{N}^2$ but $G \notin \mathcal{N}^2$.

Example 3.4 (Arroyo-Jordá *et al.* [1, Example 4]). We show now that c-permutability does not satisfy the property of persistence in intermediate subgroups, unlike cc-permutability.

We consider, as above, $G = \mathrm{Sym}(4)$ to be the symmetric group of degree 4 and Y a subgroup of G of order 2 generated by a transposition. Let V be the normal subgroup of G of order 4 and X a subgroup of V of order 2, $X \neq Z(VY)$. Then we observe that X and Y are c-permutable in G but they are not c-permutable in $\langle Y, X \rangle$.

The next examples show the failure of important structural properties of products of totally permutable subgroups when considering tcc-permutability.

Example 3.5 (Arroyo-Jordá *et al*. [1, Example 2]). We consider $G = \mathrm{Sym}(3) = AB$, the symmetric group of degree 3 and the trivial factorization with $A = G$ and B a Sylow 2-subgroup of G. Then A and B are tcc-permutable in G but $B = B \cap A \nleqslant F(G)$, the Fitting subgroup of G.

(We recall that if a group $G = HK$ is the product of totally permutable subgroups H and K, then $H \cap K \leqslant F(G)$ [**22**, Lemma 2].)

Example 3.6 (Arroyo-Jordá *et al*. [1, Example 3]). Let $V = \langle a, b \rangle \cong Z_5 \times Z_5$ and $Z_6 \cong C = \langle \alpha, \beta \rangle \leqslant \mathrm{Aut}(V)$ as given by

$$a^\alpha = a^{-1}, \qquad b^\alpha = b^{-1}; \qquad a^\beta = b, \qquad b^\beta = a^{-1}b^{-1}.$$

Let $G = [V]C$ be the corresponding semidirect product of V with C. Set $A = \langle \alpha \rangle$ and $B = V\langle \beta \rangle$. Then $G = AB$ and A and B are tcc-permutable subgroups of G but they are not totally permutable. In fact, $B^{\mathcal{N}} = B^{\mathcal{U}} = V$ does not centralize A, in contrast to properties of products of totally permutable subgroups. We remark also that $Z_{\mathcal{U}}(G) = 1$, and obviously G modulo $Z_{\mathcal{U}}(G)$ is not a direct product of the images of A and B.

(It is known that if $G = HK$ is the product of totally permutable subgroups H and K, then $X^{\mathcal{N}}$ centralizes Y for $\{H, K\} = \{X, Y\}$ [**8**, Theorem 1]; also G modulo $Z_{\mathcal{U}}(G)$ is a direct product of the images of H and K [**13**, Remarks (3), p. 859].)

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