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Weakly contractive multivalued maps and *w*-distances on complete quasi-metric spaces

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Abstract

We obtain versions of the Boyd and Wong fixed point theorem and of the Matkowski fixed point theorem for multivalued maps and *w*-distances on complete quasi-metric spaces. Our results generalize, in several directions, some well-known fixed point theorems.

Keywords: Fixed point, multivalued map, *w*-distance, quasi-metric space

Introduction and preliminaries

Throughout this article, the letters $\mathbb{N}$ and ω will denote the set of positive integer numbers and the set of non-negative integer numbers, respectively.

Following the terminology of [1], by a T_0 quasi-pseudo-metric on a set X, we mean a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:

(i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$;
(ii) $d(x, z) \le d(x, y) + d(y, z)$.

A T_0 quasi-pseudo-metric d on X that satisfies the stronger condition

(i') $d(x, y) = 0 \Leftrightarrow x = y$,

is called a quasi-metric on X.

Our basic references for quasi-metric spaces and related structures are [2] and [3].

We remark that in the last years several authors used the term "quasi-metric" to refer to a T_0 quasi-pseudo-metric and the term "T_1 quasi-metric" to refer to a quasi-metric in the above sense. It is also interesting to recall (see, for instance, [3]) that T_0 quasi-pseudo-metric spaces play a crucial role in some fields of theoretical computer science, asymmetric functional analysis and approximation theory.

Hereafter, we shall simply write T_0 qpm instead of T_0 quasi-pseudo-metric if no confusion arises.

A T_0 qpm space is a pair (X, d) such that X is a set and d is a T_0 qpm on X. If d is a quasi-metric on X, the pair (X, d) is then called a quasi-metric space.

Each T_0 qpm d on a set X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x, r) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Note that if d is a quasi-metric, then τ_d is a T_1 topology on X.

Given a T_0 qpm d on X, the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$, is also a T_0 qpm on X, called the conjugate of d, and the function d^s defined by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a metric on X.

It is well known (see, for instance, [3,4]) that there exist many different notions of completeness for T_0 qpm spaces. In our context, we shall use the following very general notion:

A T_0 qpm space (X, d) is said to be complete if every Cauchy sequence in the metric space (X, d^s) is $\tau_{d^{-1}}$-convergent. In this case, we say that d is a complete T_0 qpm on X. (Note that this notion corresponds with the notion of a d^{-1}-sequentially complete quasi-pseudo-metric space as defined in [4].)

Matthews introduced in [5] the notion of a weightable T_0 qpm space (under the name of a "weightable quasi-metric space"), and its equivalent partial metric space, as a part of the study of denotational semantics of dataflow networks. In fact, partial metric spaces constitute an efficient tool in raising and solving problems in theoretical computer science, domain theory, and denotational semantics for complexity analysis, among others (see [6-17], etc.).

A T_0 qpm space (X, d) is called weightable if there exists a function $w : X \to [0, \infty)$ such that for all $x, y \in X$, $d(x, y) + w(x) = d(y, x) + w(y)$. In this case, we say that d is a weightable T_0 qpm on X. The function w is said to be a weighting function for (X, d).

A partial metric on a set X is a function $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:

(i) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$; (ii) $p(x, x) \leq p(x, y)$; (iii) $p(x, y) = p(y, x)$; (iv) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is a pair (X, p) such that X is a set and p is a partial metric on X.

Each partial metric p on X induces a T_0 topology τ_p on X which has as a base the family of open balls $\{Bp(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < \varepsilon + p(x, x)\}$ for all $x \in X$ and $\varepsilon > 0$.

The precise relationship between partial metric spaces and weightable T_0 qpm spaces is provided in the following result.

Theorem 1.1 (Matthews [5]). (a) *Let (X, d) be a weightable T_0 qpm space with weighting function w. Then the function $p_d : X \times X \to [0, \infty)$ defined by $p_d(x, y) = d(x, y) + w(x)$ for all $x, y \in X$, is a partial metric on X. Furthermore $\tau_d = \tau_{pd}$.*

(b) *Conversely, let (X, p) be a partial metric space. Then, the function $d_p : X \times X \to [0, \infty)$ defined by $d_p(x, y) = p(x, y) - p(x, x)$ for all $x, y \in X$ is a weightable T_0 qpm on X with weighting function w given by $w(x) = p(x, x)$ for all $x \in X$. Furthermore $\tau_p = \tau_{d_p}$.*

Kada et al. introduced in [18] the notion of w-distance on a metric space and then extended the Caristi-Kirk fixed point theorem [19], the Ekeland variational principle [20] and the nonconvex minimization theorem [21], for w-distances. In [22], Park extended the notion of w-distance to quasi-metric spaces and obtained, among other results, generalized forms of Ekeland's priniciple which improve and unify corresponding results in [18,23,24]. Recently, Al-Homidan et al. [25] introduced the concept of Q-function on a quasi-metric space as a generalization of w-distances, and then obtained a Caristi-Kirk-type fixed point theorem, a Takahashi minimization theorem, and versions of Ekeland's principle and of Nadler's fixed point theorem for a Q-function on a complete quasi-metric space, generalizing in this way, among others, the main results of [22]. This approach has been continued by Hussain et al. [26], Latif and Al-Mezel [27], and Marín et al. [1]. In particular, the authors of [27] and [1] have obtained a Rakotch-type and a Bianchini-Grandolfi-type fixed point theorems, respectively, for multivalued maps and Q-functions on complete quasi-metric spaces and complete T_0 qpm spaces.

In this article, we prove a T_0 qpm version of the celebrated Boyd-Wong fixed point theorem in terms of Q-functions, which generalizes and improves, in several senses, some well-known fixed point theorems. We also discuss the extension of our result to the case of multivalued maps. Although we only obtain a partial result, it is sufficient to be able to deduce a multivalued version of Boyd-Wong's theorem for partial metrics induced by complete weightable T_0 qpm spaces. Finally, we shall show that a multivalued extension for Q-functions on complete T_0 qpm spaces of the famous Matkowski fixed point theorem can be obtained.

We conclude this section by highlighting some pertinent concepts and facts on w-distances and Q-functions on T_0 qpm spaces.

Definition 1.2 ([22]). A w-distance on a T_0 qpm space (X, d) is a function $q : X \times X \to [0, \infty)$ satisfying the following conditions:

(W1) $q(x, z) \leq q(x, y) + q(y, z)$ for all $x, y, z \in X$;

(W2) $q(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous on $(X, \tau_{d^{-1}})$ for all $x \in X$;

(W3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $q(x, y) \leq \delta$ and $q(x, z) \leq \delta$ imply $d(y, z) \leq \varepsilon$.

If in Definition 1.2 above condition (W2) is replaced by

(Q2) if $x \in X$, $M > 0$, and $(y_n)_{n\in\mathbb{N}}$ is a sequence in X that $\tau_{d^{-1}}$-converges to a point $y \in X$ and satisfies $q(x, y_n) \leq M$ for all $n \in \mathbb{N}$, then $q(x, y) \leq M$,

then q is called a Q-function on (X, d) (cf. [25]).

Clearly, every w-distance is a Q-function. Moreover, if (X, d) is a metric space, then d is a w-distance on (X, d). However, Example 3.2 of [25] shows that there exists a T_0 qpm space (X, d) such that d does not satisfy condition (W3), and hence it is not a Q-function on (X, d).

Remark 1.3 ([1]). *Let q be a Q-function on a T_0 qpm space (X, d). Then, for each $\varepsilon > 0$ there exists $\delta > 0$, such that $q(x, y) \leq \delta$ and$q(x, z) \leq \delta$ imply $d^s(y, z) \leq \varepsilon$.*

Remark 1.4 ([1]). *Let (X, d) be a weightable T_0 qpm space. Then, the induced partial metric p_d is a Q-function on (X,d). Actually, it is a w-distance on (X,d).*

The results

Let (X, d) be a T_0 qpm space. A selfmap T on X is called BW -contractive if there exists a function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) < t$ and $\lim_{r\to t^+} \sup \varphi(r) < t$ for all $t > 0$, and such that for each $x, y \in X$,

$$d(Tx, Ty) \leq \varphi(d(x, y)).$$

If $\phi(t) = rt$, with $r \in [0, 1)$ being constant, then T is called contractive.

In their celebrated article [28], Boyd and Wong essentially proved the following general fixed point theorem: *Let (X,d) be complete metric space. Then every BW-contractive selfmap on X has a unique fixed point.*

The following easy example shows that unfortunately Boyd-Wong's theorem cannot be generalized to complete quasi-metric spaces, even for T contractive.

Example 2.1. Let $X = \{1/n : n \in \mathbb{N}\}$ and let d be the quasi-metric on X given by $d(1/n, 1/n) = 0$, and $d(1/n, 1/m) = 1/m$ for all $n, m \in \mathbb{N}$. Clearly, (X, d) is complete (in fact, it is complete in the stronger sense of [1,22,25,27]). Define $T : X \to X$ by $T1/n = 1/2n$. Then, T is contractive but it has not fixed point.

Next, we show that it is, however, possible to obtain a nice quasi-metric version of Boyd-Wong's theorem using Q-functions.

Let (X, d) be a T_0 qpm space. A selfmap T on X is called BW-weakly contractive if there exist a Q-function q on (X, d) and a function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(0) = 0$, $\phi(t) < t$ and $\lim_{r\to t^+} \sup \varphi(r) < t$ for all $t > 0$, and such that for each $x, y \in X$,

$$q(Tx, Ty) \leq \varphi(q(x, y)).$$

If $\phi(t) = rt$, with $r \in [0, 1)$ being constant, then T is called weakly contractive.

Theorem 2.2. *Let (X, d) be a complete T_0 qpm space. Then, each BW-weakly contractive selfmap on X has a unique fixed point $z \in X$. Moreover, $q(z, z) = 0$.*

Proof. Let $T : X \to X$ be BW-weakly contractive. Then, there exist a Q-function q on (X, d) and a function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(0) = 0$, $\phi(t) < t$ and $\lim_{r\to t^+} \sup \varphi(r) < t$ for all $t > 0$, such that for each $x, y \in X$,

$$q(Tx, Ty) \leq \varphi(q(x, y)).$$

Fix $x_0 \in X$ and let $x_n = T^n x_0$ for all $n \in \omega$.

We show that $q(x_n, x_{n+1}) \to 0$.

Indeed, if $q(x_k, x_{k+1}) = 0$ for some $k \in \omega$, then $\phi(q(x_k, x_{k+1})) = 0$ and thus $q(x_n, x_{n+1}) = 0$ for all $n \geq k$. Otherwise, $(q(x_n, x_{n+1}))_{n\in\omega}$ is a strictly decreasing sequence in $(0, \infty)$ which converges to 0, as in the classical proof of Boyd-Wong's theorem.

Similarly, we have that $q(x_{n+1}, x_n) \to 0$.

Now, we show that for each $\varepsilon \in (0, 1)$ there exists $n_\varepsilon \in \mathbb{N}$ such that $q(x_n, x_m) < \varepsilon$ whenever $m > n > n_\varepsilon$.

Assume the contrary. Then, there exists $\varepsilon_0 \in (0, 1)$ such that, for each $k \in \mathbb{N}$, there exist $n(k), j(k) \in \mathbb{N}$ with $j(k) > n(k) > k$ and $q(x_{n(k)}, x_{j(k)}) \geq \varepsilon_0$.

Since $q(x_n, x_{n+1}) \to 0$, there exists $n_{\varepsilon_0} \in \mathbb{N}$ such that $q(x_n, x_{n+1}) < \varepsilon_0$ for all $n > n_{\varepsilon_0}$.

For each $k > n_{\varepsilon_0}$, we denote by $m(k)$ the least $j(k) \in \mathbb{N}$ satisfying the following three conditions:

$$\begin{aligned} &j(k) > n(k), \\ &q(x_{n(k)}, x_{j(k)}) \geq \varepsilon_0, \quad \text{and} \\ &q(x_{n(k)}, x_{j(k)-1}) < \varepsilon_0. \end{aligned}$$

Note that there exists such a $m(k)$ because $q(x_{n(k)}, x_{n(k)+1}) < \varepsilon_0$. Then, for each $k > n_{\varepsilon_0}$, we obtain

$$\begin{aligned} \varepsilon_0 \leq q(x_{n(k)}, x_{m(k)}) &\leq q(x_{n(k)}, x_{m(k)-1}) + q(x_{m(k)-1}, x_{m(k)}) \\ &< \varepsilon_0 + q(x_{m(k)-1}, x_{m(k)}). \end{aligned}$$

Since $q(x_{m(k)-1}, x_{m(k)}) \to 0$, it follows from the preceding inequalities that $r_k \to \varepsilon_0^+$ where $r_k = q(x_{n(k)}, x_{m(k)})$. Hence,

$$\limsup_{r_k \to \varepsilon_0^+} \varphi(r_k) < \varepsilon_0.$$

Choose δ >0 with $\limsup_{r_k \to \varepsilon_0^+} \varphi(r_k) < \delta < \varepsilon_0$. Let $k_0 > n_{\varepsilon_0}$ such that $q(x_{n(k)}, x_{n(k)+1}) <$ $(\varepsilon_0 - \delta)/2$, and $q(x_{m(k)+1}, x_{m(k)}) < (\varepsilon_0 - \delta)/2$,
for all $k > k_0$.
Then,

$$\begin{aligned} q(x_{n(k)}, x_{m(k)}) &\le q(x_{n(k)}, x_{n(k)+1}) + q(x_{n(k)+1}, x_{m(k)+1}) + q(x_{m(k)+1}, x_{m(k)}) \\ &< \frac{\varepsilon_0 - \delta}{2} + \varphi(q(x_{n(k)}, x_{m(k)})) + \frac{\varepsilon_0 - \delta}{2} < \varepsilon_0, \end{aligned}$$

for some $k > k_0$, which contradicts that $\varepsilon_0 \le q(x_{n(k)}, x_{m(k)})$ for all $k > n_{\varepsilon_0}$. We conclude that for each $\varepsilon \in (0, 1)$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$q(x_n, x_m) < \varepsilon \quad \text{whenever } m > n > n_\varepsilon. \qquad (*)$$

Next, we show that $(x_n)_{n \in \omega}$ is a Cauchy sequence in the metric space (X, d^s). Indeed, let ε >0, and let $\delta = \delta(\varepsilon)$ >0 as given in Definition 1.2 (W3). Then, for $n, m > n_\delta$ we obtain $q(x_{n_\delta}, x_n) < \delta$, and $q(x_{n_\delta}, x_m) < \delta$, and hence from Remark 1.3, $d^s(x_n, x_m) \le \varepsilon$. Consequently, $(x_n)_{n \in \omega}$ is a Cauchy sequence in (X, d^s).

Now, let $z \in X$ such that $d(x_n, z) \to 0$. Then $q(x_n, z) \to 0$ by (Q2) and condition (*) above. Hence, $q(Tx_n, Tz) \to 0$. From Remark 1.3, we conclude that $d^s(z, Tz) = 0$, i.e., $z = Tz$.

Next, we show the uniqueness of the fixed point. Let $y = Ty$. If $q(y, z)$ >0, $q(Ty, Tz) = q(y, z) \le \phi(q(y, z)) < q(y, z)$, a contradiction. Hence, $q(y, z) = 0$. Interchanging y and z, we also have $q(z, y) = 0$. Therefore, $y = z$ from Remark 1.3.

Finally, $q(z, z) = 0$ since otherwise we obtain $q(z, z) = q(Tz, Tz) \le \phi(q(z, z)) < q(z, z)$, a contradiction. □

The following is an example of a non-BW-contractive selfmap T on a complete T_0 qpm space (X, d) for which Theorem 2.2 applies.

Example 2.3. Let $X = [0, 1)$ and d be the weightable T_0 qpm on X given by $d(x, y) = \max\{y - x, 0\}$ for all $x, y \in X$. Clearly (X, d) is complete because $d(x, 0) = 0$ for all $x \in X$, and thus every sequence in X converges to 0 with respect to $\tau_{d^{-1}}$.

Now, define $T : X \to X$ by $Tx = x^2$ for all $x \in X$. Then, T is not BW-contractive because $d(Tx, Ty) = y^2 - x^2 > y - x = d(x, y)$, whenever $0 < x < y < 1 < x + y$. However, T is BW-weakly contractive for the partial metric p_d induced by d (recall that, from Remark 1.4, pd is a Q-function on (X, d)), and the function $\phi : [0, \infty) \to [0, \infty)$ defined by $\phi(t) = t^2$ for $0 \le t < 1$ and $\varphi(t) = \sqrt{t/2}$ for $t \ge 1$. Indeed, for each $x, y \in X$ we have,

$$p_d(Tx, Ty) = \max\{x^2, y^2\} = \varphi(\max\{x, y\}) = \varphi(p_d(x, y)).$$

Hence, we can apply Theorem 2.2, so that T has a unique fixed point: in fact, 0 is the only fixed point of T, and $p_d(0, 0) = 0$. (Note that in this example, there exists not $r \in [0, 1)$ such that $p_d(Tx, Ty) \le r p_d(x, y)$ for all $x, y \in X$.)

In the light of the applications of w-distances and Q-functions to the fixed point theory for multivalued maps on metric and quasi-metric spaces, it seems interesting to investigate the extension of our version of Boyd-Wong's theorem to the case of multivalued maps. In Theorem 2.6 below, we shall prove a positive result for the case of symmetry Q-functions, which are defined as follows:

Definition 2.4. A symmetric Q-function on a T_0 qpm space (X, d) is a Q-function q on (X, d) such that

$$(SY)\ q(x,y) = q(y,x) \quad \text{for all } x, y \in X.$$

If q is a w-distance satisfying (SY), we then say that it is a symmetric w-distance on (X, d).

Example 2.5. Of course, if (X, d) is a metric space, then d is a symmetric w-distance on (X, d). Moreover, it follows from Remark 1.4, that for every weightable T_0 qpm space (X, d) its induced partial metric p_d is a symmetric w-distance on (X, d). Note also that the w-distance constructed in Lemma 2 of [29] is also a symmetric w-distance.

Given a T_0 qpm space (X, d), we denote by 2^X and by $Cl_{d^s}(X)$ the collection of all nonempty subsets of X and the collection of all nonempty τ_{d^s}-closed subsets of X, respectively.

Generalizing the notions of a q-contractive multivalued map [[25], Definition 6.1] and of a generalized q-contractive multivalued map [27], we say that a multivalued map T from a T_0 qpm space (X, d) to 2^X, is BW-weakly contractive if there exists a Q-function q on (X, d) and a function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(0) = 0$, $\phi(t) < t$ and $\lim_{r \to t^+} \sup \varphi(r) < t$ for all $t > 0$, and such that, for each $x, y \in X$ and each $u \in Tx$ there exists $v \in Ty$ with $q(u, v) \leq \phi(q(x, y))$.

Theorem 2.6. *Let (X, d) be a complete T_0 qpm space and $T : X \to Cl_{d^s}(X)$ be BW-weakly contractive for a symmetric Q-function q on (X,d). Then, there is $z \in X$ such that $z \in Tz$ and $q(z, z) = 0$.*

Proof. By hypothesis, there is a function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(0) = 0$, $\phi(t) < t$ and $\lim_{r \to t^+} \sup \varphi(r) < t$ for all $t > 0$, and such that for each $x, y \in X$ and $u \in Tx$ there is $v \in Ty$ with

$$q(u,v) \leq \varphi(q(x,y)).$$

Fix $x_0 \in X$ and let $x_1 \in Tx_0$. Then, there exists $x_2 \in Tx_1$ such that $q(x_1, x_2) \leq \phi(q(x_0, x_1)$. Following this process, we obtain a sequence $(x_n)_{n\in\omega}$ with $x_n \in Tx_{n-1}$ and $q(x_n, x_{n+1}) \leq \phi(q(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$.

As in Theorem 2.2, $q(x_n, x_{n+1}) \to 0$.

Now, we show that for each $\varepsilon \in (0, 1)$, there exists $n_\varepsilon \in \mathbb{N}$ such that $q(x_n, x_m) < \varepsilon$ whenever $m > n > n_\varepsilon$.

Assume the contrary. Then, there exists $\varepsilon_0 \in (0, 1)$ such that for each $k \in \mathbb{N}$, there exist $n(k), j(k) \in \mathbb{N}$ with $j(k) > n(k) > k$ and $q(x_{n(k)}, x_{j(k)}) \geq \varepsilon_0$.

Again, by repeating the proof of Theorem 2.2, and using symmetry of q, we derive that

$$\begin{aligned} q(x_{n(k)}, x_{m(k)}) &\leq q(x_{n(k)}, x_{n(k)+1}) + q(x_{n(k)+1}, x_{m(k)+1}) + q(x_{m(k)+1}, x_{m(k)}) \\ &< \frac{\varepsilon_0 - \delta}{2} + \varphi(q(x_{n(k)}, x_{m(k)})) + q(x_{m(k)}, x_{m(k)+1}) \\ &< \frac{\varepsilon_0 - \delta}{2} + \delta + \frac{\varepsilon_0 - \delta}{2} = \varepsilon_0, \end{aligned}$$

a contradiction.

From Remark 1.3, it follows that $(x_n)_{n\in\omega}$ is a Cauchy sequence in (X, d^s) (compare the proof of Theorem 2.2), and so there exists $z \in X$ such that $d(x_n, z) \to 0$, and thus $q(x_n, z) \to 0$.

Therefore, for each $n \in \omega$ there exists $v_{n+1} \in Tz$ with

$$q(x_{n+1}, v_{n+1}) \le \varphi(q(x_n, z)).$$

Since $q(x_n, z) \to 0$ we have $q(x_{n+1}, v_{n+1}) \to 0$, and so $d^s(z, v_{n+1}) \to 0$ from Remark 1.3. Consequently, $z \in Tz$ because Tz is closed in (X, d^s).

It remains to be shown that $q(z, z) = 0$. Indeed, since $z \in Tz$ we can construct a sequence $(z_n)_{n\in\mathbb{N}}$ in X such that $z_1 \in Tz$, $z_{n+1} \in Tz_n$, $q(z, z_1) \le \phi(q(z, z_n))$ and $q(z, z_{n+1}) \le \phi(q(z, z_n))$ for all $n \in \mathbb{N}$. Hence $(q(z, z_n))_{n\in\mathbb{N}}$ is a nonincreasing sequence in $[0, \infty)$ that converges to 0. From Remark 1.3, the sequence $(z_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d^s). Let $u \in X$ such that $d(z_n, u) \to 0$. It follows from condition (Q2) that $q(z, u) = 0$. Since $q(x_n, z) \to 0$, we deduce by condition (Q1) that $q(x_n, u) \to 0$. Therefore, $d^s(z, u) \le \varepsilon$ for all $\varepsilon >0$, from Remark 1.3. We conclude that $z = u$, and thus $q(z, z) = 0$. □

Although we do not know if symmetric of q can be omitted in Theorem 2.6, it can be applied directly to obtain the following fixed point result for multivalued maps on partial metric spaces, which substantially improves Theorem 5.3 of [5].

Corollary 2.7. *Let (X, p) be a partial metric space such that the induced weightable T_0 qpm d_p is complete and $T : X \to Cl_{d^s}(X)$be BW-weakly contractive for p. Then, there is $z \in X$ such that $z \in Tz$ and $p(z, z) = 0$.*

We conclude this article by showing, nevertheless, that it is possible to prove a multivalued version of the celebrated Matkowski's fixed point theorem [30], which provides a nice generalization of Boyd-Wong's theorem when ϕ is nondecreasing.

Theorem 2.8. *Let (X, d) be a complete T_0 qpm space and let $T : X \to Cl_{d^s}(X)$. If there exist a Q-function q on (X, d) and a nondecreasing function $\phi : (0, \infty) \to (0, \infty)$ satisfying $\phi^n(t) \to 0$ for all $t >0$, such that for each $x, y \in X$ and each $u \in Tx$, there exists $v \in Ty$ with*

$$q(u, v) \le \varphi(q(x, y)),$$

then, there exists $z \in X$ such that $z \in Tz$ and $q(z, z) = 0$.

Proof. Let $\phi(0) = 0$. Fix $x_0 \in X$ and let $x_1 \in Tx_0$. Then, there exists $x_2 \in Tx_1$ such that $q(x_1, x_2) \le \phi(q(x_0, x_1))$. Following this process, we obtain a sequence $(x_n)_{n\in\omega}$ with $x_n \in Tx_{n-1}$ and $q(x_n, x_{n+1}) \le \phi(q(x_{n-1}, x_n))$ for all $n \in \mathbb{N}$. Therefore,

$$q(x_n, x_{n+1}) \le \varphi^n(q(x_0, x_1))$$

for all $n \in \mathbb{N}$. Since $\phi^n(q(x_0, x_1)) \to 0$, it follows that $q(x_n, x_{n+1}) \to 0$.

Now, choose an arbitrary $\varepsilon >0$. Since $\phi^n(\varepsilon) \to 0$, then $\phi(\varepsilon) < \varepsilon$, so there is $n_\varepsilon \in \mathbb{N}$ such that

$$q(x_n, x_{n+1}) < \varepsilon - \varphi(\varepsilon),$$

for all $n \ge n_\varepsilon$. Note that then,

$$\begin{aligned} q(x_n, x_{n+2}) &\le q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(q(x_n, x_{n+1})) \le \varepsilon, \end{aligned}$$

for all $n \geq n_\varepsilon$, and following this process

$$q(x_n, x_{n+k}) < \varepsilon,$$

for all $n \geq n_\varepsilon$ and $k \in \mathbb{N}$. Applying Remark 1.3, we deduce that $(x_n)_{n\in\omega}$ is a Cauchy sequence in (X, d^s). Then, there is $z \in X$ such that $d(x_n, z) \to 0$ and thus $q(x_n, z) \to 0$ by condition (Q2). The rest of the proof follows similarly as the proof of Theorem 2.6. We conclude that $z \in Tz$ and $q(z, z) = 0$. □

Remark 2.9. The above theorem improves, among others, Theorem 3.3 of [1] (compare also Theorem 1 of [31]).

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Authors' contributions
The three authors have equitably contributed in obtaining the new results presented in this article.
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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