UNIVERSITAT POLITÈCNICA DE VALÈNCIA

DEPARTAMENT DE MATEMÀTICA APLICADA

Operators on weighted spaces of holomorphic functions

TESI DOCTORAL REALITZADA PER:

María José Beltrán Meneu

DIRIGIDA PER:

José Bonet Solves

Carmen Fernández Rosell

VALÈNCIA, FEBRER 2014

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Don **José Bonet Solves**, Catedrático de Universidad de la Universitat Politècnica de València y Doña **Carmen Fernández Rosell**, Profesora Titular del Departamento de Análisis Matemático de la Universitat de València

CERTIFICAN:

que la presente memoria "Operators on weighted spaces of holomorphic functions" ha sido realizada bajo nuestra dirección por María José Beltrán Meneu y constituye su tesis para optar al grado de Doctor en Ciencias Matemáticas, con mención "Doctor Internacional".

Y para que así conste en cumplimiento de la legislación vigente presentamos y apadrinamos ante la Escuela de Doctorado de la Universitat Politècnica de València la referida tesis firmando el presente certificado.

Valencia, Febrero de 2014

Los directores:

José Bonet Solves y Carmen Fernández Rosell

a la memòria de mon tio Juan Alfonso
a la meua família

Agraïments

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Betxí, a 6 de Gener de 2014
María José Beltrán Meneu

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Resumen

La presente memoria "Operadores en espacios ponderados de funciones holomorfas" trata diferentes áreas del análisis funcional tales como espacios de funciones holomorfas, holomorfía en dimensión infinita y dinámica de operadores.

Después de un primer capítulo introductorio en el que incluimos la notación, las definiciones y los resultados básicos que usaremos a lo largo de la tesis, el texto queda dividido en dos partes. Una primera, correspondiente a los Capítulos 1 y 2, centrada en el estudio de espacios (LB)-ponderados de funciones enteras sobre espacios de Banach, y una segunda, correspondiente a los Capítulos 3 y 4, en la que estudiamos el comportamiento dinámico de los operadores de diferenciación e integración actuando sobre diferentes clases de espacios ponderados de funciones enteras. A continuación, presentamos una breve descripción de los capítulos:

En el Capítulo 1, dada una familia decreciente de pesos radiales y continuos sobre un espacio de Banach X, consideramos los límites inductivos ponderados de funciones enteras $VH(X)$ y $VH_0(X)$. Los espacios ponderados de funciones holomorfas aparecen de forma natural en el estudio de condiciones de crecimiento de funciones holomorfas y han sido investigados por varios autores desde los trabajos de Williams en 1967, Rubel y Shields en 1970, y Shields y Williams en 1971. Primero determinamos condiciones suficientes sobre la familia de pesos para que el correspondiente espacio ponderado sea un álgebra o tenga una descomposición de Schauder polinómica. Estudiamos álgebras de Hörmander de funciones enteras definidas sobre un espacio de Banach y damos una descripción de ellas como espacios de sucesiones. Estudiamos homomorfismos de álgebras entre estos espacios y obtenemos un teorema de tipo Banach-Stone para familias de pesos de tipo exponencial. Finalmente, analizamos el espectro de estas álgebras ponderadas, dotándolo de una estructura analítica, y demostramos que cada función $f \in VH(X)$ puede extenderse de forma natural a una función analítica definida sobre el espectro. Dado un homomorfismo de álgebras, también investigamos cómo la aplicación inducida entre los espectros actúa sobre las correspondientes estructuras analíticas. Vemos que en nuestro contexto los operadores de composición tienen un comportamiento diferente del que tienen cuando consideramos funciones holomorfas de

tipo acotado. Este trabajo ha sido motivado por un artículo reciente de Carando, García, Maestre y Sevilla-Peris. Los resultados obtenidos han sido publicados por Beltrán en [14].

En el Capítulo 2 obtenemos una linearización del espacio $VH(X)$ a través de un predual. Éste lo obtenemos aplicando el teorema de completación de Mujica para espacios (LB), que además nos permite probar que el espacio $VH(X)$ es regular y completo. También estudiamos condiciones para asegurar la igualdad $VH_0(X)'' = VH(X)$, observando algunas diferencias entre los casos finito e infinito dimensional. Finalmente damos condiciones suficientes para extender funciones definidas en un subconjunto A de X, con valores en otro espacio de Banach E, y admitiendo algunas extensiones débiles en un espacio de funciones holomorfas, a funciones en el correspondiente espacio de funciones holomorfas sobre X con valores vectoriales. La mayoría de los resultados incluidos en este capítulo han sido publicados por Beltrán en [13].

En el resto de la tesis se analiza el comportamiento dinámico de los siguientes operadores sobre espacios ponderados de funciones enteras: el operador de diferenciación $Df(z) = f'(z)$, el operador de integración $Jf(z) = \int_0^z f(\zeta)d\zeta$ y el operador de Hardy $Hf(z) = \frac{1}{z}\int_0^z f(\zeta)d\zeta$, $z \in \mathbb{C}$.

En el Capítulo 3 estudiamos la dinámica de estos operadores sobre espacios de Banach ponderados de funciones enteras definidos por normas de tipo integral o supremo: los espacios ponderados de funciones enteras $B_{p,q}(v)$, $1 \leq p \leq \infty$, y $1 \leq q \leq \infty$. Cuando $q = \infty$ se conocen como espacios generalizados de Bergman ponderados de funciones enteras, y si además $p = \infty$, se denotan por $H_v(\mathbb{C})$ y $H_v^0(\mathbb{C})$. Analizamos cuándo los operadores son hipercíclicos, caóticos, de potencias acotadas o (uniformemente) ergódicos en media, complementando así el trabajo de Bonet y Ricker sobre operadores de multiplicación ergódicos en media. Además, para pesos verificando ciertas condiciones, estimamos la norma de los operadores y estudiamos su espectro. Se presta especial atención a pesos de tipo exponencial. El contenido de este capítulo se ha publicado en [17] y en [15].

En el caso de operadores diferenciales $\phi(D) : B_{p,q}(v) \to B_{p,q}(v)$, con $D : B_{p,q}(v) \to B_{p,q}(v)$ continuo y ϕ una función entera, estudiamos la hiperciclicidad y el caos. El capítulo finaliza con un ejemplo de un operador hipercíclico y uniformemente ergódico en media. Éste ha sido proporcionado por A. Peris, al que agradecemos haber permitido su inclusión en esta tesis. Creemos que es el primer ejemplo de un operador verificando estas dos propiedades al mismo tiempo.

El último capítulo se ha dedicado al estudio de la dinámica de los operadores de diferenciación e integración actuando sobre límites inductivos y proyectivos ponderados de funciones enteras. Damos condiciones suficientes para que D y J sean continuous sobre los espacios y caracterizamos cuándo el operador de diferenciación es hipercíclico, topológicamente mezclante o caótico en el caso de límites proyectivos. Finalmente investigamos la dinámica de estos operadores sobre las álgebras de Hörmander $A_p(\mathbb{C})$ y $A_p^0(\mathbb{C})$. Los resultados obtenidos en este capítulo han sido incluidos por Bonet, Fernández y la autora en [16].

Resum

La tesi “Operadors en espais ponderats de funcions holomorfes” tracta diferents àrees de l’anàlisi funcional tals com espais de funcions holomorfes, holomorfia de dimensió infinita i dinàmica d’operadors. Després d’un primer capítol introductori en el que incloem la notació, les definicions i els resultats bàsics que emprarem a la tesi, el text queda dividit en dues parts. Una primera, corresponent als Capítols 1 i 2, centrada a l’estudi d’espais (LB)-ponderats de funcions enteres sobre espais de Banach, i una segona, corresponent als Capítols 3 i 4, on estudiem el comportament dinàmic d’operadors de diferenciació i integració actuant sobre diferents classes d’espais ponderats de funcions enteres. A continuació presentem una breu descripció dels diferents capítols:

Al Capítol 1, donada una família decreixent de pesos radials i continus sobre un espai de Banach X, considerem els límits inductius ponderats de funcions enteres $VH(X)$ i $VH_0(X)$. Els espais ponderats de funcions holomorfes apareixen de forma natural a l’estudi de condicions de creixement de funcions holomorfes i han sigut investigats per diversos autors desde els treballs de Williams en 1967, Rubel i Shields en 1970 i Shields i Williams en 1971. Primer determinem condicions sobre la família de pesos per assegurar que el corresponent espai ponderat siga un àlgebra o tinga una descomposició de Schauder polinòmica. Estudiem àlgebres de Hörmander de funcions enteres definides sobre un espai de Banach i donem una descripció d’elles com espais de successions. També estudiem homomorfismes d’àlgebres entre aquests espais i obtenim un teorema de tipus Banach-Stone per una família particular de pesos. Finalment, estudiem l’espectre d’aquestes àlgebres ponderades, dotant-lo d’una estructura analítica, i demostrem que cada funció $f \in VH(X)$ pot extendre’s de forma natural a una funció analítica definida sobre l’espectre. Donat un homomorfisme d’àlgebres, també investiguem com l’aplicació induïda entre els espectres actua sobre les corresponents estructures analítiques. Veiem que en el nostre contexte els operadors de composició tenen un comportament diferent del que tenen quan actuen sobre funcions holomorfes de tipus fitat. Aquest treball ha sigut motivat per un article recent de Carando, García, Maestre i Sevilla-Peris. Els resultats obtinguts han sigut publicats per Beltrán en [14].

Al Capítol 2 obtenim una linearització de l'espai $VH(X)$ a través d'un predual. Aquest l'obtenim aplicant el teorema de completació de Mujica per espais (LB), que a més ens permet provar que l'espai $VH(X)$ és regular i complet. També estudiem condicions per assegurar la igualtat $VH_0(X)'' = VH(X)$, observant algunes diferències entre els casos finit i infinit dimensional. Finalment donem condiciones suficients per extendre funcions definides en un subconjunt A de X, amb valors en un altre espai de Banach E, i admetent algunes extensions dèbils en un espai de funcions holomorfes, a funcions en el corresponent espai de funcions holomorfes sobre X amb valors vectorials. La majoria dels resultats inclosos en aquest capítol han sigut publicats per Beltrán en [13].

A la resta de la tesi s'analitza el comportament dinàmic dels següents operadors sobre espais ponderats de funcions enteres: l'operador de diferenciació $Df(z) = f'(z)$, l'operador d'integració $Jf(z) = \int_0^z f(\zeta)d\zeta$ i l'operador de Hardy $Hf(z) = \frac{1}{z}\int_0^z f(\zeta)d\zeta$, $z \in \mathbb{C}$.

Al Capítol 3 estudiem la dinàmica d'aquests operadors sobre espais de Banach ponderats de funcions enteres definits per normes de tipus integral o suprem: els espais ponderats de funcions enteres $B_{p,q}(v)$, $1 \leq p \leq \infty$, i $1 \leq q \leq \infty$. Quan $q = \infty$ es coneixen com espais generalitzats de Bergman ponderats de funcions enteres, i si a més $p = \infty$, es denoten per $H_v(\mathbb{C})$ i $H_v^0(\mathbb{C})$. Analitzem quan els operadors són hipercíclics, caòtics, de potències fitades o (uniformement) ergòdics en mitja, complementant així el traball de Bonet i Ricker sobre operadors de multiplicació ergòdics en mitja. A més, per a pesos verificant certes condicions, estimem la norma dels operadors i estudiem el seu espectre. Es dóna especial atenció a pesos de tipus exponencial. El contingut d'aquest capítol s'ha publicat en [17] i en [15].

En el cas d'operadores diferencials $\phi(D) : B_{p,q}(v) \to B_{p,q}(v)$, amb $D : B_{p,q}(v) \to B_{p,q}(v)$ continu i ϕ una funció entera, estudiem la hiperciclicitat i el caos. El capítol finalitza amb un exemple d'un operador hipercíclic i uniformement ergòdic en mitja. Aquest ha sigut proporcionat per A. Peris, a qui agraïm haver permés la seua inclusió en aquesta tesi. Creiem que és el primer exemple d'un operador verificant aquestes dues propietats al mateix temps.

L'últim capítol s'ha dedicat a l'estudi de la dinàmica dels operadors de diferenciació i integració actuant sobre límits inductius i projectius ponderats de funciones enteres. Donem condicions suficients perquè D i J siguen continus sobre els espais i caracteritzem quan l'operador de diferenciació és hipercíclic, topològicament mesclant o caòtic en el cas de límits projectius. Finalment investiguem la dinàmica d'aquests operadors sobre les àlgebres de Hörmander $A_p(\mathbb{C})$ i $A_p^0(\mathbb{C})$. Els resultats obtesos en aquest capítol han sigut inclosos per Bonet, Fernández i l'autora en [16].

Summary

The Ph.D. Thesis "Operators on weighted spaces of holomorphic functions" presented here treats different areas of functional analysis such as spaces of holomorphic functions, infinite dimensional holomorphy and dynamics of operators.

After a first chapter that introduces the notation, definitions and the basic results we will use throughout the thesis, the text is divided into two parts. A first one, consisting of Chapters 1 and 2, focused on a study of weighted (LB)-spaces of entire functions on Banach spaces, and a second one, corresponding to Chapters 3 and 4, where we consider differentiation and integration operators acting on different classes of weighted spaces of entire functions to study its dynamical behaviour. In what follows, we give a brief description of the different chapters:

In Chapter 1, given a decreasing sequence of continuous radial weights on a Banach space X, we consider the weighted inductive limits of spaces of entire functions $VH(X)$ and $VH_0(X)$. Weighted spaces of holomorphic functions appear naturally in the study of growth conditions of holomorphic functions and have been investigated by many authors since the work of Williams in 1967, Rubel and Shields in 1970 and Shields and Williams in 1971. We determine conditions on the family of weights to ensure that the corresponding weighted space is an algebra or has polynomial Schauder decompositions. We study Hörmander algebras of entire functions defined on a Banach space and we give a description of them in terms of sequence spaces. We also focus on algebra homomorphisms between these spaces and obtain a Banach-Stone type theorem for a particular decreasing family of weights. Finally, we study the spectra of these weighted algebras, endowing them with an analytic structure, and we prove that each function $f \in VH(X)$ extends naturally to an analytic function defined on the spectrum. Given an algebra homomorphism, we also investigate how the mapping induced between the spectra acts on the corresponding analytic structures and we show how in this setting composition operators have a different behavior from that for holomorphic functions of bounded type. This research is related to recent work by Carando, García, Maestre and Sevilla-Peris. The results included in this chapter are published by Beltrán in [14].

Chapter 2 is devoted to study the predual of $VH(X)$ in order to linearize this space of entire functions. We apply Mujica's completeness theorem for (LB)-spaces to find a predual and to prove that $VH(X)$ is regular and complete. We also study conditions to ensure that the equality $VH_0(X)'' = VH(X)$ holds. At this point, we will see some differences between the finite and the infinite dimensional cases. Finally, we give conditions which ensure that a function f defined in a subset A of X, with values in another Banach space E, and admitting certain weak extensions in a space of holomorphic functions can be holomorphically extended in the corresponding space of vector-valued functions. Most of the results obtained have been published by the author in [13].

The rest of the thesis is devoted to study the dynamical behaviour of the following three operators on weighted spaces of entire functions: the differentiation operator $Df(z) = f'(z)$, the integration operator $Jf(z) = \int_0^z f(\zeta)d\zeta$ and the Hardy operator $Hf(z) = \frac{1}{z}\int_0^z f(\zeta)d\zeta$, $z \in \mathbb{C}$.

In Chapter 3 we focus on the dynamics of these operators on a wide class of weighted Banach spaces of entire functions defined by means of integrals and supremum norms: the weighted spaces of entire functions $B_{p,q}(v)$, $1 \leq p \leq \infty$, and $1 \leq q \leq \infty$. For $q = \infty$ they are known as generalized weighted Bergman spaces of entire functions, denoted by $H_v(\mathbb{C})$ and $H_v^0(\mathbb{C})$ if, in addition, $p = \infty$. We analyze when they are hypercyclic, chaotic, power bounded, mean ergodic or uniformly mean ergodic; thus complementing also work by Bonet and Ricker about mean ergodic multiplication operators. Moreover, for weights satisfying some conditions, we estimate the norm of the operators and study their spectrum. Special emphasis is made on exponential weights. The content of this chapter is published in [17] and [15].

For differential operators $\phi(D) : B_{p,q}(v) \to B_{p,q}(v)$, whenever $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous and ϕ is an entire function, we study hypercyclicity and chaos. The chapter ends with an example provided by A. Peris of a hypercyclic and uniformly mean ergodic operator. To our knowledge, this is the first example of an operator with these two properties. We thank him for giving us permission to include it in our thesis.

The last chapter is devoted to the study of the dynamics of the differentiation and the integration operators on weighted inductive and projective limits of spaces of entire functions. We give sufficient conditions so that D and J are continuous on these spaces and we characterize when the differentiation operator is hypercyclic, topologically mixing or chaotic on projective limits. Finally, the dynamics of these operators is investigated in the Hörmander algebras $A_p(\mathbb{C})$ and $A_p^0(\mathbb{C})$. The results concerning this topic are included by Bonet, Fernández and the author in [16].

Contents

Introduction

Weighted spaces of entire functions defined on a finite or infinite dimensional complex Banach space X constitute the core of this Ph.D. Thesis. We treat different aspects about them. Given a *weight* v we define the *weighted Banach spaces of entire functions* $H_v(X)$ and $H_v^0(X)$. If instead of an only weight v we consider a decreasing sequence of weights $V := \{v_n\}_n$, then we are able to define the *weighted (LB)-spaces of entire functions*

$$VH(X) := ind_n H_{v_n}(X) \quad \text{and} \quad VH_0(X) := ind_n H_{v_n}^0(X),$$

endowed with the inductive limit topology. Analogously, given an increasing sequence of weights $W := \{w_n\}_n$, we define the *weighted Fréchet spaces of entire functions*

$$HW(X) := proj_n H_{w_n}(X) \quad \text{and} \quad HW_0(X) := proj_n H_{w_n}^0(X),$$

endowed with the projective limit topology, that is, the Fréchet topology defined by the norms $\| \, \|_{w_n}, \ n \in \mathbb{N}$.

Weighted spaces of holomorphic functions appear naturally in the study of growth conditions of holomorphic functions and have been investigated by many authors since the work of Williams in 1967, Rubel and Shields in 1970, and Shields and Williams in 1971. They naturally arise in great profusion throughout such fields as linear partial differential equations and convolution equations, distribution theory and representation of distributions as boundary values of holomorphic functions, complex and Fourier analysis, and spectral theory and the holomorphic functional calculus (see [31]). The one variable case has been extensively studied by Rubel and Shields [112], Williams [123], Bierstedt and Summers [32] and Bierstedt and Bonet [25]. Lusky [95] completed the isomorphic classification of $H_v(G)$ when G is a balanced domain in the complex plane and the weight v is radial. Bierstedt, Meise and Summers [31] studied a projective description of weighted inductive limits of holomorphic functions on open subsets of $\mathbb{C}^d$ (see also [28], [40], [44] and [45]), and Bierstedt, Bonet and Galbis [27] achieved significant advances in the knowledge of these spaces on balanced domains G in $\mathbb{C}^d$. Later, García, Maestre and Rueda [71] studied weighted Fréchet spaces of holomorphic functions defined on Banach spaces. We carry on this work studying weighted inductive limits of holomorphic functions defined on Banach spaces.

The starting point of the first chapter has been the article [56] by Carando and Sevilla-Peris about the spectra of weighted Fréchet algebras $HW(X)$, and its generalization to $HW(U)$, U an unbounded open subset of X, given by Carando, García, Maestre and Sevilla-Peris in [55]. We show that a similar situation holds in our context of weighted (LB)-spaces $VH(X)$.

In Section 1.1 we characterize when $VH(X)$ is an algebra in terms of a condition on the family of weights V, we give a sufficient condition on the family in order to ensure the existence of a polynomial Schauder decomposition, and we present some examples. This section collects some facts that are analogue to those for $HW(X)$ in [56] and [71]. Weighted algebras of continuous functions have been studied by Oubbi in [108] and [109].

In Section 1.2 we study the special case of weighted (LB)-algebras $A_p(X)$ determined by a *growth condition* p : the Hörmander algebras of entire functions defined on a Banach space X, and we give a description of them in terms of sequence spaces. They were introduced for the first time by Hörmander in [83] and were intensively studied by Berenstein and Taylor in the context of interpolation of entire functions (see e.g. [19]). The study of the locally convex structure of the algebras $A_p(\mathbb{C})$ was initiated by Meise in [99].

We treat algebra homomorphisms between spaces $VH(X)$ and $VH(Y)$, X, Y Banach spaces, when V is a family of exponential weights, i.e., we consider the algebras of holomorphic functions of exponential type $Exp(X)$ and $Exp(Y)$. This class of functions has been widely studied in function theory in one or several variables since the 1930s ([36], [37]) and, even nowadays, its interest also arises in areas such as harmonic and Fourier analysis, operator theory and partial differential equations in complex domains. The results on algebra homomorphisms allow us to formulate a Banach-Stone type theorem: if $Exp(X) \cong Exp(Y)$ as topological algebras, then $X' \cong Y'$. Some recent articles on this kind of problems are [53], [56] and [119]. A survey on different types of Banach-Stone theorems can be found in [72].

In the last section of Chapter 1, whenever $VH(X)$ is an algebra and X a symmetrically regular Banach space, we endow the spectrum of $VH(X)$, i.e., the space of non-zero continuous multiplicative functionals $V\mathfrak{M}(X)$, with a topology that makes it an analytic variety over X''. In [7, Corollary 2.4], Aron, Galindo, García and Maestre gave the spectrum $\mathfrak{M}_b(U)$ of the algebra of holomorphic functions of bounded type $H_b(U)$, the structure of a Riemann analytic manifold modelled on X'', for U an open subset of X. For the case $U = X$, $\mathfrak{M}_b(X)$ can be viewed as the disjoint union of analytic copies of X'', these copies being the connected components of $\mathfrak{M}_b(X)$. The same situation holds for $V\mathfrak{M}(X)$. In [56], Carando and Sevilla-Peris studied the spectrum of the weighted Fréchet algebras of holomorphic functions $HW(X)$, where W is an increasing family of weights satisfying certain conditions. In [55], Carando, García, Maestre and Sevilla-Peris generalize these results for the spectrum of $HW(U)$, where U is an unbounded open subset of X. Here, the conditions on the weights for entire functions are softer than those in [56]. The first steps towards the description of the spectrum of $H_b(X)$ were taken by Aron, Cole and Gamelin in [6]. A survey with the most relevant recent developments on the research of the spectra of algebras of analytic functions can be found in [54]. We also show, as in the case of $H_b(X)$ and $HW(X)$, that any function $f \in VH(X)$ extends naturally to an analytic function defined on the spectrum. Moreover, under certain conditions on the family of weights, this extension can be seen to belong, in some sense, to $VH(V\mathfrak{M}(X))$.

Finally, given an algebra homomorphism, we investigate how the mapping induced between the spectra acts on the corresponding analytic structures and we also show how in this setting composition operators have a different behavior from that for holomorphic functions of bounded type.

Most of the results included in this chapter are published by Beltrán in [14].

Chapter 2 is devoted to find a linearization of the space $VH(X)$ by means of a predual. The existence for a given locally convex space E of a second locally convex space F such that $F' = E$ and having some additional properties has been shown useful to obtain results in infinite dimensional holomorphy. Recall that the proof of the completeness of the space of the germs of holomorphic functions $H(K) = ind_n H^\infty(V_n)$, where K is a compact subset of a Fréchet space and V_n is a sequence of open neighbourhoods of K such that $K = \cap_n V_n$, was substantially simplified by Mujica [102] using the predual of $H(K)$. The existence of predual is also the key to prove that $H(K)$ is an inductive limit where each canonical injection $J_{n,n+1}$ is weakly compact when K is a compact subset of the Tsirelson space [105]. This is a partial answer to the problem posed by Bierstedt and Meise [30] of characterizing those Fréchet spaces for which the inductive limit above has weakly compact canonical injections. The compact case was solved by these authors in 1977.

As the closed unit ball B_{v_n} of $H_{v_n}(X)$, $n \in \mathbb{N}$, is compact with respect to the compact open topology τ_{co}, we apply Mujica's completeness theorem for (LB)-spaces [102, Theorem 1], which was inspired by a theorem of Banach-Dixmier-Waelbroeck-Ng on dual Banach spaces (cf. Waelbroeck [122, Proposition 1] and Ng [107]), in order to find a predual of $VH(X)$ and to prove the completeness of the space. As a corollary of [25, Corollary 2], we also get that the inductive limit $VH(X)$ is regular. Moreover, for a regularly decreasing sequence of weights, we obtain that the predual F is a quasinormable Fréchet space, and thus, distinguished, that is, $VH(X)$ is topologically isomorphic to the strong dual F'_b. For these sequences of weights we also study, using [25, Theorem 7], some conditions to ensure that the equality $VH_0(X)'' = VH(X)$ holds. Similar questions are treated in [70], [71] and [115] for weighted Fréchet algebras.

Biduality on weighted Banach spaces of entire functions was treated for the first time in the work of Williams [123] and Rubel and Shields [112], and their results were improved by Bierstedt and Summers in [32] for weighted Banach spaces of holomorphic functions defined on an open set G of $\mathbb{C}^d$. In fact, they proved that the spaces $H_v^0(G)''$ and $H_v(G)$ are isometrically isomorphic if and only if the closed unit ball of $H_v^0(G)$ is dense in the closed unit ball of $H_v(G)$ with respect to the compact open topology. This result was generalized to the weighted spaces $VH(G)$ and $HW(G)$ by Bierstedt and Bonet in [25]. At this point, we see some differences between the finite and the infinite dimensional cases. Although several of our results are based on the ones proved there, when moving to infinite dimensions we loose local compactness and, consequently, the fact that the elements of the topological dual of $VH_0(X)$ have an integral representation ([32, Theorem 1.1b]). So, whereas condition (ii) in [25, Theorem 7] is always satisfied in the finite dimensional case by the Hahn-Banach and Riesz representation theorems, in the infinite dimensional case, X must be at least reflexive.

Once we have a predual F, the following natural step is to use it to obtain a linearization of holomorphic mappings. Linearization might be useful because it permits to write spaces of "complicated" maps, for instance, holomorphic maps, as spaces of linear maps defined on another space, which can be easier to handle. Several authors have obtained linearization theorems for various classes of holomorphic mappings. It seems that the first general result of this kind is due to Mazet [98] who obtained a linearization theorem for holomorphic mappings on locally convex spaces, thus improving various results of Schottenloher [118], Aron-Schottenloher [8] and Ryan [116]. By specializing to smaller classes of mappings, in the setting of Banach spaces, Mujica [104] obtained a linearization theorem for bounded holomorphic mappings, whereas Galindo, García and Maestre [69] obtained a linearization theorem for holomorphic mappings of bounded type. In [57], Carando and Zalduendo generalize all these special cases linearizing functions with values in locally convex spaces. We will show that in our case there exists a holomorphic function $\Delta : X \to F$ with the following universal property: for each Banach space E and each function $f \in VH(X, E)$, there is a unique linear continuous operator $T_f : F \to E$ such that $T_f \circ \Delta = f$. The correspondence $f \to T_f$ is an isomorphism between the space $VH(X, E)$ and the (LB)-space of operators $\mathcal{L}_i(F, E)$ induced with the inductive limit topology. Moreover, we obtain a more general linearization result which includes the one given for $VH(X, E)$.

We finish the chapter considering the following general question: given two Banach spaces X and E, consider $A \subseteq X$, $H \subseteq E'$, and $f : A \to E$ such that for every $u \in H$ the function $u \circ f : A \to \mathbb{C}$ has an extension in $VH(X)$ $(HW(X))$. When does this imply that there is an extension F of f in the weighted space of vector-valued holomorphic functions $VH(X,E)$ $(HW(X,E))$? This problem is motivated by the fact that a continuous function $f : X \to E$ belongs to $VH(X,E)$ $(HW(X,E))$ if and only if $u \circ f : X \to \mathbb{C}$ belongs to $VH(X)$ $(HW(X))$ for every $u \in E'$. Despite the weak and the formal definition do not coincide in general for $VH_0(X,E)(HW_0(X,E))$, we also give conditions in order to obtain extensions results using weak extensions on these spaces.

Given a locally convex space E, the problem of deciding when a function $f : \Omega \subseteq \mathbb{C} \to E$ is holomorphic whenever $u \circ f \in \mathcal{H}(\Omega)$ for each $u \in E'$ goes back to Dunford [64], who proved that this happens when E is a Banach space. Grothendieck [81] extended the result for a quasicomplete locally convex space E. Bogdanowicz [38] gives extension results through weak extension proving among other results that if $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}$ are two domains (open and connected subsets), E is a complex, sequentially complete, locally convex Hausdorff space and $f : \Omega_1 \to E$ satisfies that $u \circ f$ admits holomorphic extension for each $u \in E'$ then f admits a holomorphic extension to Ω_2. More recently, Grosse-Erdmann, Arendt and Nikolski, Bonet, Frerick, Wengenroth and Jordá have given results in this way smoothing the conditions on Ω_1 and also requiring extensions of $u \circ f$ only for a proper subset $H \subseteq E'$ (see [4], [47], [67], [79]). Also Laitila and Tylli have recently discussed the difference between strong and weak definitions for important spaces of vector valued functions [89, Section 6]. In [85], Jordá analyzes a weak criterion for holomorphy and uses linearization in order to give extension results for Banach spaces of holomorphic functions defined on a non-void open subset U of a Banach space X.

Most of our results given in Chapter 2 have been published by the author in [13].

The rest of the thesis is devoted to study the dynamical behaviour of the following three operators on weighted spaces of entire functions: the differentiation operator $Df(z) = f'(z)$, the integration operator $Jf(z) = \int_0^z f(\zeta)d\zeta$ and the Hardy operator $Hf(z) = \frac{1}{z}\int_0^z f(\zeta)d\zeta$, $z \in \mathbb{C}$. Although there is a huge literature on the Hardy operator on different function spaces (see e.g. [9]), it seems that it has not yet been studied in this context.

In Chapter 3 we focus the dynamics of the operators on a wide class of weighted Banach spaces of entire functions defined by means of integrals and supremum norms: the weighted spaces of entire functions $B_{p,q}(v)$, $1 \leq p \leq \infty$, and $1 \leq q \leq \infty$, also determined by a weight v. For $q = \infty$ they are known as *generalized weighted Bergman spaces of entire functions*, which are just the spaces $H_v(\mathbb{C})$ and $H_v^0(\mathbb{C})$

if, in addition, $p = \infty$. Similar spaces of holomorphic functions on the disc have been considered by Blasco in [33] and by Blasco and de Souza in [34].

The continuity of the differentiation and the integration operators on the space $H_v(\mathbb{C})$ has been studied by Harutyunyan and Lusky in [82]. They prove that the continuity is determined by the growth or decline of $v(r)e^{\alpha r}$ for some $\alpha > 0$ in an interval $[r_0, \infty[$. In [41], Bonet investigated when the operator of differentiation is hypercyclic or chaotic on $H_v^0(\mathbb{C})$, and in [42], Bonet and Bonilla extend these results to the generalized weighted Bergman spaces showing conditions to ensure that the differentiation operator is chaotic, hypercyclic or frequently hypercyclic. The surjectivity and the spectrum of the differentiation operator on the spaces $B_{p,q}(v)$, $p = q$, were studied by Atzmon and Brive in [10].

In this chapter we continue this work by analyzing the operators of differentiation and of integration and the Hardy operator on the spaces $B_{p,q}(v)$. We study when they are hypercyclic or chaotic, but also other properties like being power bounded, mean ergodic or uniformly mean ergodic; thus complementing also work by Bonet and Ricker [50] about mean ergodic multiplication operators. Moreover, we estimate the norm of the operators and study their spectrum. Special emphasis is made on exponential weights. The results obtained on the weighted Banach spaces of entire functions $H_v(\mathbb{C})$ and $H_v^0(\mathbb{C})$ are published by Bonet, Fernández and the author in [17], and their generalization to weighted Banach spaces of entire functions defined by means of integral norms are published by Beltrán in [15].

For differential operators $\phi(D) : B_{p,q}(v) \to B_{p,q}(v)$, whenever $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous and ϕ is an entire function, we study hypercyclicity and chaos. Godefroy and Shapiro proved that if $T : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$, $T \neq \lambda I$, commutes with D, that is, $TD = DT$, it can be expressed as a differential operator $\phi(D)$ for an entire function ϕ of exponential type [73]. Moreover, they proved that T is chaotic. MacLane also considered the question about what are the possible rates of growth of D-hypercyclic functions. He showed that there exists a D-hypercyclic entire function f of exponential type 1, that is, for all $\varepsilon > 0$ there is $M > 0$ with $|f(z)| \leq Me^{(1+\varepsilon)|z|}$. Bernal and Bonilla [21] have attacked the same problem for general T following the idea of Chan and Shapiro in 1991 of replace $\mathcal{H}(\mathbb{C})$ by a space of entire functions of restricted growth. We continue their work focusing this problem on the weighted spaces of entire functions $B_{p,q}(v)$, $1 \leq p \leq \infty$, $q = 0$ or $1 \leq q < \infty$.

In the last section of the chapter we include an example of a hypercyclic and uniformly mean ergodic operator given by Peris. We thank him for giving us permission to include his example in our Ph.D. Thesis. It consists on the *backward shift operator* B acting on the weighted sequence space $\ell_p(v)$. Examples of operators being mean ergodic and hypercyclic at the same time seem to be unknown until now.

We conclude the thesis with a chapter devoted to the study of the dynamics of the differentiation and the integration operators on weighted inductive and projective limits of spaces of entire functions. We give sufficient conditions to ensure that D and J are continuous on these spaces similar to those for $H_v(\mathbb{C})$ in [82] and [42, Proposition 2.1], and we characterize when the differentiation operator is hypercyclic, topologically mixing or chaotic on $HW(\mathbb{C})$. For the Hörmander algebras $A_p(\mathbb{C})$ and $A_p^0(\mathbb{C})$ we study when the operators are hypercyclic, chaotic, power bounded and (uniformly) mean ergodic in terms of the order of growth of the growth condition p, thus continuing the research by Bonet in [39]. According to [2, Proposition 2.4]), since $A_p(\mathbb{C})$ and $A_p^0(\mathbb{C})$ are complete and Montel, each power bounded operator is uniformly mean ergodic.

Most of our results concerning this topic are included by Bonet, Fernández and the author in [16].

Chapter 0

Preliminaries

The first chapter is devoted to introduce the notation, definitions and the basic results we will use throughout the thesis.

0.1 Notation and basic definitions

We denote the natural numbers by $\mathbb{N} := \{1, 2, 3, ...\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the real numbers by $\mathbb{R}$, the positive real numbers by $\mathbb{R}_+ := (0, \infty)$ and the complex numbers by $\mathbb{C}$. By $D(z_0, \varepsilon) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ we denote the open disc centred at $z_0 \in \mathbb{C}$ of radius $\varepsilon > 0$, and by $\mathbb{D} \subseteq \mathbb{C}$ the open unit disc. For some $z_0 \in \mathbb{C}$ and $R > 0$, we denote by $C(0, R) := \{z \in \mathbb{C}, |z| = R\}$ the circumference centred at z_0 of radius R.

If E is a normed space, we denote by B_E the closed unit ball centred at zero, and by $B(x_0, \varepsilon)$ the closed ball centred at $x_0 \in E$ of radius $\varepsilon > 0$. We say that a subset $U \subseteq E$ is *balanced* if $\lambda U \subseteq U$ for all $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$.

We recall Landau's notation of capital O-growth and little o-growth: given f and g two functions defined on some unbounded subset of the real numbers, one writes $f(x) = O(g(x))$ as $x \to \infty$ if there is a positive constant M and $x_0 > 0$ such that $|f(x)| \leq M|g(x)|$ for all $x > x_0$, and $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ whenever $g(x)$ is non-zero, or at least becomes non-zero beyond a certain point. The expression $f(x) \sim g(x)$ means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and $f(x) \lesssim g(x)$ means that $f(x) = O(g(x))$. When $f(x) \lesssim g(x) \lesssim f(x)$, we simply write $f(x) \approx g(x)$.

Our notation for functional analysis and operator theory is standard. We refer the reader e.g. to [65], [101], [110] and [113].

Given a locally convex space E, we denote by E^* the algebraic dual of E, that is, the space of all linear forms $T : E \to \mathbb{C}$, and by E' its topological dual, i.e., the space of all continuous linear forms on E. The *weak topology* of E, denoted by $\sigma(E, E')$, is defined as follows: a net $\{x_\alpha\}_\alpha$ converges to x_0 in $(E, \sigma(E, E'))$ if and only if for all x' in E', $\{x'(x_\alpha)\}_\alpha$ converges to $x'(x_0)$ in $\mathbb{C}$. A net $\{x'_\alpha\}_\alpha$ converges to x'_0 in the *weak-star topology* on E', usually known as weak* or w^* topology, and denoted by $\sigma(E', E)$, if and only if for all x in E, $\{x'_\alpha(x)\}_\alpha$ converges to $x'_0(x)$ in $\mathbb{C}$. For a locally convex space E, $cs(E)$ denotes a system of continuous seminorms determining the topology of E, and for two locally convex spaces E and F, the set of all continuous linear maps from E to F is denoted by $\mathcal{L}(E, F)$. Each element $T \in \mathcal{L}(E, F)$ is called an operator, and it defines another operator $T' : F' \to E'$, $T'(\lambda)(x) = \lambda(T(x))$, $\lambda \in F'$, $x \in E$, called its *transpose*. The expression $E \cong F$ means that these spaces are topologically isomorphic.

The *strong operator topology* τ_s in $\mathcal{L}(E, F)$ is determined by the family of seminorms

$$q_x(S) := q(Sx),\ S \in \mathcal{L}(E, F),$$

for each $x \in E$ and $q \in cs(F)$. In this case we denote the space by $\mathcal{L}_s(E, F)$. The *uniform operator topology* τ_b of uniform convergence on bounded sets is defined in $\mathcal{L}(E, F)$ via the seminorms

$$q_B(S) := \sup_{x \in B} q(Sx),\ S \in \mathcal{L}(E, F),$$

for each bounded set B of E and each $q \in cs(F)$. In this case the space is denoted by $\mathcal{L}_b(E, F)$. When $F = E$, we simply write $\mathcal{L}(E)$, $\mathcal{L}_s(E)$ and $\mathcal{L}_b(E)$, respectively. For a Banach space E, observe that τ_b is the operator norm topology in $\mathcal{L}(E)$. We denote by $\sigma(T)$ the *spectrum* of $T \in \mathcal{L}(E)$, that is, the set of complex numbers $\lambda \in \mathbb{C}$ such that the operator $T - \lambda I$ has no inverse. As usual, I denotes the identity on E.

A set M is *absorbing* if $\cup_n nM = E$. We say that a Hausdorff locally convex space E is *barrelled* if every *barrel*, that is, if every closed, absolutely convex (i.e., convex and balanced) and absorbing set in the space is a zero-neighbourhood. Banach-Steinhaus theorem still holds on barrelled spaces.

In what follows we give an introduction to countable inductive and projective limits of locally convex spaces. For the definitions, the proofs and more background, see e.g. [24], [101] or [110].

Let E be a linear space, $\{E_n : n = 1, 2, \dots\}$ an increasing sequence of subspaces of E and $J_n : E_n \to E$, $J_{n,n+1} : E_n \to E_{n+1}$, the canonical injections. Suppose that each E_n is endowed with a Hausdorff locally convex topology τ_n such that each $J_{n,n+1} : (E_n, \tau_n) \to (E_{n+1}, \tau_{n+1})$ is continuous. Then $\mathcal{E} := \{(E_n, \tau_n) : n = 1, 2, \dots\}$ is called an *inductive sequence* with respect to the mappings $\{J_n : n = 1, 2, \dots\}$. An inductive sequence is *strict* if each $J_{n,n+1}$ is an isomorphism onto its image and *hyperstrict* if it is strict and each E_n is closed in (E_{n+1}, τ_{n+1}). Each (E_n, τ_n) is called a *step* of $\mathcal{E}$.

Let $\mathcal{E}$ be an inductive sequence and let τ be the finest locally convex topology on E such that each $J_n : (E_n, \tau_n) \to (E, \tau)$ is continuous. Then (E, τ) is called the *inductive limit* of the defining sequence $\mathcal{E}$ and we write $(E, \tau) = ind\ \mathcal{E} = ind\{(E_n, \tau_n) : n = 1, 2, \dots\}$. If $\mathcal{E}$ is strict (resp., hyperstrict), (E, τ) is said to be the strict (resp., hyperstrict) inductive limit of $\mathcal{E}$. If each (E_n, τ_n) of an inductive sequence is a Banach (resp., Fréchet) space, then (E, τ) is said to be an *(LB)-space* (resp., *(LF)-space*).

Proposition 0.1.1 ([110, 0.3.2]) *If $(E, \tau) = ind\{(E_n, \tau_n) : n = 1, 2, \dots\}$ and if*

(i) $\{n(k) : k = 1, 2, \dots\}$ is a strictly increasing sequence of positive integers, then $\mathcal{F} := \{(E_{n(k)}, \tau_{n(k)}) : k = 1, 2, \dots\}$ is also a defining sequence for (E, τ).

(ii) $T : (E, \tau) \to F$, F being a Hausdorff locally convex space, is a linear mapping, then T is continuous if and only if each $T \circ J_n : (E_n, \tau_n) \to F$ is continuous.

(iii) U is an absolutely convex subset of E, then U is a 0-neighbourhood in (E, τ) if and only if each $U \cap E_n$ is a 0-neighbourhood in (E_n, τ_n). Thus a basis of 0-neighbourhoods in (E, τ) can be given by the sets $\Gamma(\bigcup_{n \in \mathbb{N}} U_n)$, where each U_n is a 0-neighborhood in (E_n, τ_n) and Γ denotes the absolutely convex hull.

Theorem 0.1.2 (Grothendieck's Factorization Theorem [110, 1.2.20]) *Let F be a Baire space, $E = ind_n E_n$ a countable inductive limit of Fréchet spaces and $T : F \to E$ a linear mapping with closed graph in $F \times E$. Then, there exists a positive integer k such that $T(F)$ is contained in E_k and $T : F \to (E_k, \tau_k)$ is continuous.*

We say that an inductive limit $E = ind_n E_n$ is *regular* if, for every bounded set $B \subseteq E$, there exists $n \in \mathbb{N}$ such that B is a bounded subset of E_n. E is *boundedly retractive* if, for every bounded subset B of E, there exists $n \in \mathbb{N}$ such that B is bounded in E_n and the topologies of E and E_n coincide on B, and E is said *strongly boundedly retractive* if E is regular and, for every $n \in \mathbb{N}$, there exists some $m \in \mathbb{N}$, $m \geq n$ such that the topology τ of the inductive limit E and the topology of E_m coincide on each bounded subset B of E_n. It is clear that a strongly boundedly retractive inductive limit is boundedly retractive.

Proposition 0.1.3 ([24, page 80]) *If $E = ind_n E_n$ is a separated countable inductive limit of normed spaces, let B_n denote the closed unit ball of E_n for $n = 1, 2, \ldots$. If each B_n is even closed in the inductive limit, then $E = ind_n E_n$ is regular.*

Given E, F and G three topological vector spaces, a bilinear map $P : E \times F \to G$ is said to be *hypocontinuous* if for each bounded set $B \subseteq E$ and for each zero neighbourhood $V \subseteq G$, there exists a zero neighbourhood $U \subseteq F$ such that $P(B \times U) \subseteq V$, and for each bounded set $B' \subseteq F$ and for each zero neighbourhood $V' \subseteq G$, there exists a zero neighbourhood $U' \subseteq E$ such that $P(U' \times B') \subseteq V'$. All continuous bilinear maps are hypocontinuous ([110, see 11.3]).

Proposition 0.1.4 *Let $E = ind_n(E_n, \| \|_n)$ be a regular inductive limit of Banach spaces and $T : E \times E \to E$ a symmetric bilinear map. Then, T is hypocontinuous if and only if for every $m \in \mathbb{N}$ and every bounded set $B \subseteq E$ there exists $n \in \mathbb{N}$, $n \geq m$ and $C > 0$ such that $\|T(x, y)\|_n \leq C\|y\|_m$ for every $x \in B$ and every $y \in E_m$.*

Proof. By definition, since T is symmetric, $T : E \times E \to E$ is hypocontinuous if and only if for every bounded set $B \subseteq E$, the set $\{T_x,\ x \in B\} \subseteq \mathcal{L}(E)$ is equicontinuous, where $T_x : E \to E$ is the linear map defined by $T_x(y) = T(x, y)$, $x, y \in E$. Since (LB)-spaces are barrelled, this is equivalent to the fact of $\{T_x,\ x \in B\}$ being bounded on bounded sets. Since the inductive limit E is regular, this is satisfied if and only if for every $m \in \mathbb{N}$, the set $\{T_x(y) :\ x \in B,\ y \in B_m\}$ is bounded in E, and this is equivalent to the existence of $n \in \mathbb{N}$ and $C > 0$ such that $\|T_x(y)\|_n \leq C$ for every $x \in B$ and every $y \in B_m$. This yields the conclusion. □

Certain properties of locally convex spaces are preserved by the operation of taking inductive limits. In fact, an inductive limit of barrelled spaces is barrelled. However, even though we suppose that each of the locally convex spaces E_n in an inductive limit has a Hausdorff topology, it is possible that the inductive limit topology τ of $E = ind_n E_n$ is not Hausdorff. Regular inductive limits always carry a Hausdorff topology.

Symmetric to the process of constructing inductive limits, is the construction of projective limits of locally convex spaces.

Let $\{F_n : n = 1, 2, \ldots\}$ be a sequence of locally convex spaces. For all m, n with $m \geq n$ let $\pi_{n,m} : F_m \to F_n$ be a continuous linear mapping such that $\pi_{n,n}$ is the identity and $\pi_{n,m} \circ \pi_{m,s} = \pi_{n,s}$ $(n \leq m \leq s)$. The pair $\{(F_n, \{\pi_{n,m}\}_{m \geq n})\}_n$ is called a *projective sequence* and the space

$$F := \left\{ \{x(n)\}_n \in \prod_{n\in\mathbb{N}} F_n : \pi_{n,m}(x(m)) = x(n) \text{ for all } m \geq n \right\},$$

endowed with the induced topology of $\prod_{n\in\mathbb{N}} F_n$ is called its *projective limit*, denoted by $F = proj_n F_n$. If each F_n is Hausdorff, then it is even a closed subspace of $\prod_{n\in\mathbb{N}} F_n$. The canonical projections $F \to F_n$, $\{x(m)\}_m \to x(n)$ will be denoted by π_n. In fact, the projective limit topology τ of $F = proj_n F_n$ is the weakest locally convex topology, but also the weakest vector space topology or even the weakest topology which makes all the canonical mappings $\pi_n : F \to F_n$ continuous. If $\mathcal{V}_n$ is a base of absolutely convex neighbourhoods in F_n, the finite intersections of the sets $\pi_n^{-1}(V_n)$, $V_n \in \mathcal{V}_n$, $n \in \mathbb{N}$, form a basis $\mathcal{V}$ of absolutely convex neighborhoods for F with this topology.

Proposition 0.1.5 *Let E be a convex space and T a linear mapping of E into the projective limit $F = proj_n F_n$ with mappings π_n. Then T is continuous if and only if for each $n \in \mathbb{N}$, $\pi_n \circ T$ is a continuous mapping of E into F_n.*

In a projective limit, we have that a subset A of F is bounded, or precompact, if and only if each $\pi_n(A)$ has the same property for every $n \in \mathbb{N}$. Moreover, contrary to what happens to inductive limits, a projective limit of Hausdorff spaces is Hausdorff.

A projective limit $F = proj_n F_n$ is called *reduced* if the maps $\pi_k : proj_n F_n \to F_k$ have dense range in F_k for every $k \in \mathbb{N}$. There is no real restriction in considering reduced projective limits only, as every projective limit is linearly homeomorphic to a reduced one. When F is reduced, all the transpose mappings $\pi_n' : F_n' \to F'$ are injective, and the inductive limit $ind_n F_n'$ is equal to $F' = (proj_n F_n)'$ algebraically.

Proposition 0.1.6 ([24, page 57]) *For any inductive sequence $\{(E_n, \tau_n) : n = 1, 2, \dots\}$ of locally convex spaces, $\{(E_n', J_{n,n+1}')\}$ is a projective sequence, and we have $(ind_n E_n)' = proj_n E_n'$ algebraically. If $E = ind_n E_n$ is a regular inductive limit, then*

$$(ind_n E_n)_b' = proj_n (E_n)_b'$$

holds algebraically and topologically. Analogously, we get that for a Banach space F,

$$\mathcal{L}_b(ind_n E_n, F) = proj_n(\mathcal{L}_b(E_n, F))$$

if $ind_n E_n$ is regular.

Definition 0.1.7 ([24, page 78]) A locally convex space E is said to be a *(DF)-space* if:

(i) it has a fundamental sequence of bounded sets,

(ii) it is σ-quasibarrelled in the sense that, for each sequence $\{U_n\}_n$ of closed absolutely convex 0-neighborhoods in E such that $U := \cap_n U_n$ absorbs every bounded set, this intersection U must again be a 0-neighborhood in E.

The strong dual of any metrizable locally convex space is a (DF)-space, each normed space also enjoys the (DF)-property, the strong dual of every (DF)-space is Fréchet, and a countable inductive limit $E = ind_n E_n$ of (DF)-spaces E_n is again a (DF)-space.

Definition 0.1.8 A locally convex space E is *quasinormable* if for every absolutely convex 0-neighbourhood U in E there exists an absolutely convex 0-neighbourhood V in E, contained in U, such that for every $a > 0$ there is a bounded subset B in E with $V \subseteq B + aU$.

Definition 0.1.9 Given $F = proj_n F_n$, where $\{(F_n, \{\pi_{n,m}\}_{m\geq n})\}_n$ is a projective sequence of Banach spaces, and a locally convex space E, we consider the inductive limit $ind_n(\mathcal{L}(F_n, E), I_{n,m})$, with $I_{n,m} : \mathcal{L}_b(F_n, E) \to \mathcal{L}_b(F_m, E)$, $I_{n,m}(T) := T \circ \pi_{n,m}$ as canonical injections. Since $\mathcal{L}(F, E) = ind_n(\mathcal{L}(F_n, E), \{I_{n,m}\}_{m\geq n})$ algebraically, we define $\mathcal{L}_i(F, E)$ to be the vector space of all continuous linear operators from F into E, endowed with the inductive limit topology (see [46]). By [110, Proposition 8.3.45], any quasinormable Fréchet space F is *distinguised*, that is $F'_i = F'_b$.

0.2 Weighted Banach spaces of holomorphic functions

The weighted Banach spaces of holomorphic functions $H_v(G)$ and $H_v^0(G)$, where G is an open balanced set in $\mathbb{C}^d$, appear naturally in the study of growth conditions of holomorphic functions and have been investigated in many papers since the works of Shields and Williams in 1978 and 1982. See e.g. [27, 28, 35, 68, 94, 95] and the references therein. In this section we give an introduction to these spaces.

We denote by $\mathcal{H}(G)$ the space of all analytic functions on G, which is usually endowed with the compact-open topology τ_{co}. This is the topology of uniform convergence on the compact subsets of G, defined by the seminorms

$$\{q_K : \mathcal{H}(G) \to [0, \infty),\ K \text{ compact subset of } G\},$$

where $q_K(f) = \|f\|_K := \max_{z\in K}|f(z)|$. Furthermore, we consider sometimes on $\mathcal{H}(G)$ the topology τ_p of pointwise convergence, defined by the seminorms

$$\{q_F : \mathcal{H}(G) \to [0,\infty),\ F \subseteq \mathbb{C} \text{ finite}\}$$

such that $q_F(f) := \sup_{z\in F}|f(z)|$. By $\mathcal{H}^\infty(G)$ we denote the space of all bounded analytic functions on G, by $\mathcal{P}_n(\mathbb{C})$ the polynomials of degree less than or equal to n on $\mathbb{C}$ and by $\mathcal{P}(\mathbb{C})$ the set of all polynomials on $\mathbb{C}$.

Definition 0.2.1 A *weight* is a continuous bounded strictly positive function $v : G \to \mathbb{R}_+$. If $G \subseteq \mathbb{C}$ and $v(z) = v(|z|)$ for all $z \in G$, then v is called *radial*. Furthermore, if $G = \mathbb{D}$, we call a weight *typical* if it is radial, non-increasing with respect to $|z|$ and satisfies $\lim_{|z|\to 1^-} v(z) = 0$. Hence, a typical weight v on $\mathbb{D}$ has a continuous extension to the boundary of $\mathbb{D}$ by zero. We say that a weight v on $\mathbb{C}$ is *rapidly decreasing at infinity* if it is radial and $\lim_{r\to\infty} r^k v(r) = 0$ for all $k \in \mathbb{N}$. Throughout the thesis we consider all the weights on $\mathbb{D}$ typical and on $\mathbb{C}$ rapidly decreasing.

We say that a function $g : G \to \mathbb{R}$ *vanishes at infinity on* G if for each $\varepsilon > 0$ there exists a compact subset $K \subseteq G$ such that $|g(z)| < \varepsilon$ for all $z \in G \setminus K$.

Definition 0.2.2 For an arbitrary weight v on G we define the *weighted spaces of holomorphic functions* (defined on G) with O- and o-growth conditions as

$$H_v(G) := \{f \in \mathcal{H}(G) : \|f\|_v := \sup_{z\in G} v(z)|f(z)| < \infty\}$$

and

$$H_v^0(G) := \{f \in \mathcal{H}(G) : v|f| \text{ vanishes at infinity on } G\ \}.$$

By definition, $\lim_{|z|\to 1\_} v(z)|f(z)| = 0$ for all $f \in H_v^0(\mathbb{D})$. If we let $v \equiv 1$ on $\mathbb{D}$, we get the Hardy space $H_v(\mathbb{D}) = H^\infty(\mathbb{D})$, and the Maximum Modulus Theorem [114, Theorem 10.24] shows that $H_v^0(\mathbb{D}) = \{0\}$. Both spaces $(H_v(G), \|\ \|_v)$ and $(H_v^0(G), \|\ \|_v)$ are Banach spaces, and $(H_v^0(G), \|\ \|_v) \hookrightarrow (H_v(G), \|\ \|_v) \hookrightarrow (\mathcal{H}(G), \tau_{co})$ with continuous inclusions. Since we assume all weights v on $\mathbb{C}$ rapidly decreasing, $H_v^0(\mathbb{C})$ and $H_v(\mathbb{C})$ contain the polynomials.

We denote by B_v and B_v^0 the closed unit balls of $H_v(G)$ and $H_v^0(G)$, respectively. Since

$$B_v = \{f \in \mathcal{H}(G) :\ |f(z)| \leq \frac{1}{v(z)} \text{ for all } z \in G\},$$

the closed unit ball is uniformly bounded on the compact subsets of G, and since the evaluation map $\delta_z : \mathcal{H}(G) \to \mathbb{C}$, $f \mapsto f(z)$ is τ_{co}-continuous on $\mathcal{H}(G)$ and

$$B_v = \bigcap_{z\in G} \delta_z^{-1}(\overline{D(0, 1/v(z))}),$$

it is τ_{co}-closed and τ_{co}-compact in the Montel space $\mathcal{H}(G)$. Thus, the topology induced by τ_{co} and the topology induced by τ_p coincide in B_v.

Given a weight v on G, its *associated weight* $\widetilde{v}$ is defined as

$$\widetilde{v}(x) := \frac{1}{\sup\{|f(x)| : f \in H_v(G), \|f\|_v \leq 1\}}.$$

It is known (see [28, Proposition 1.2]) that $v \leq \widetilde{v}$ and $H_v(G) = H_{\widetilde{v}}(G)$ isometrically. $H^0_{\widetilde{v}}(G)$ is a closed subspace of $H^0_v(G)$, but these spaces do not coincide in general. Two weights v and w on G are called *equivalent* if there exist constants $C, C' > 0$ such that $Cv(z) \leq w(z) \leq C'v(z)$ for all $z \in G$. Moreover, a weight v on G is said to be *essential* if there exists a constant $C > 0$ such that $v(z) \leq \widetilde{v}(z) \leq Cv(z)$ for all $z \in G$. As mentioned by Bierstedt, Bonet and Taskinen in [28], many results on weighted spaces of analytic functions and on composition operators defined on them have to be formulated in terms of the associated weights and not directly on the given weights, since they satisfy nice additional properties.

Example 0.2.3 ([28, 1.7 and 1.9]) The following weights verify that $\widetilde{v} = v$ for C and α positive constants and n a fixed natural number. Then, they all are essential:

(i) $G = \mathbb{C},\ v(z) = \exp(-C|z|^n)$,

(ii) $G = \mathbb{D},\ v(z) = \exp\left(-\frac{C}{(1-|z|)^\alpha}\right)$,

(iii) $G = \mathbb{D},\ v(z) = (1 - |z|)^\alpha$,

(iv) $G = \mathbb{D},\ v(z) = \max(1, -C\log(1 - |z|))$,

(v) $G = \mathbb{C},\ v(z) = (1 + |z|)^{-n} \exp(-n|Imz|)$,

For examples of essential weights such that the constant C is not equal to one, we refer to [28, Section 3.A and 3.B] and we remark that Example 1.7 yields the existence of non-essential weights.

In [32, Theorem 1.1 and Corollary 1.2], Bierstedt and Summers obtain that the expression $H^0_v(G)'' = H^\infty_v(G)$ holds isometrically if and only if the closed unit ball B^0_v is τ_{co}-dense in B_v. For $G = \mathbb{D}$, it is enough that the weight is typical, and for $G = \mathbb{C}$, that the weight is rapidly decreasing.

Given G a balanced subset of $\mathbb{C}$, each $f \in \mathcal{H}(G)$ has a Taylor series representation around zero $f(z) = \sum_{k=0}^{\infty} a_k z^k,\ z \in G$, with $a_k \in \mathbb{C}, k \in \mathbb{N}_0$. The *Cesàro means* of the partial sums of the Taylor series of f around zero are denoted by $C_n f$, $n = 0, 1, \ldots$, and defined as

$$(C_n f)(z) := \frac{1}{n+1} \sum_{i=0}^{n} \left(\sum_{k=0}^{i} a_k z^k \right),\ z \in G.$$

Observe that for each $n \in \mathbb{N}_0$ and $f \in \mathcal{H}(G)$, the function $C_n f$ is a polynomial of degree less or equal to n and $C_n f \to f$ uniformly on every compact subset of G. Moreover, by the Cauchy inequalities, the coefficients of the Taylor polynomials, and hence the polynomials $C_n f$, depend continuously on $f \in \mathcal{H}(G)$ with respect to τ_{co} on $\mathcal{H}(G)$. Moreover, since for $f \in \mathcal{H}(G)$ and $z \in G$ we have

$$|(C_n f)(z)| \leq \max_{|\lambda|=1} |f(\lambda z)|,\ n = 0, 1, ...,$$

we get that given a weight v on G such that $H_v^0(G)$ contains all the polynomials, $C_n f$ converge to f in $\| \ \|_v$ for every $f \in H_v^0(G)$. Hence, the polynomials are dense in $H_v^0(G)$. Moreover, for each $n \in \mathbb{N}_0$, the operator $C_n : f \to C_n f,\ f \in H_v^\infty(G)$ is a continuous linear operator of finite rank from $H_v(G)$ into $H_v^0(G)$ satisfying $\|C_n f\|_v \leq \|f\|_v$ for every $f \in H_v(G)$. See [27] for the details.

Weighted Banach spaces of entire functions are treated in Chapter 3, Section 3.6, where we study dynamical properties of the differentiation, the integration and the Hardy operator acting on them. See [27] for more details and background about weighted spaces of holomorphic functions defined on a balanced set G of $\mathbb{C}^d$.

0.3 Weighted spaces of entire functions on Banach spaces

In this section we deal with weighted spaces of entire functions defined on a Banach space X. First we introduce some basic results concerning infinite dimensional holomorphy, that is, the study of holomorphic functions on Banach spaces. We refer the reader to [63] or [103] for background information.

A mapping $P : X \to \mathbb{C}$ is said to be a *k-homogeneous polynomial* if there exists a continuous k-linear mapping $A : X \times \overset{k)}{\cdots} \times X \to \mathbb{C}$ such that $P(x) = A(x, ..., x)$ for every $x \in X$. Given an homogeneous polynomial P, there exists a unique continuous symmetric k-linear mapping A, that is, a k-linear mapping satisfying

$$A(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = A(x_1, \ldots, x_n)$$

for all $x_1, \dots, x_n \in X$ and any permutation σ of the first n natural numbers, such that $P(x) = A(x, \dots, x)$. It can be obtained through the *Polarization formula*:

$$A(x_1, \dots, x_n) = \frac{1}{2^n n!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \dots \varepsilon_n P(\sum_{i=1}^{n} \varepsilon_i x_i).$$

Given an homogeneous polynomial P, we denote by $\check{P}$ its associated continuous symmetric k-linear mapping, and given a continuous k-linear mapping A, we denote by $\hat{A}$ its corresponding k-homogeneous polynomial $\hat{A}(x) = A(x, \dots, x)$, $x \in X$. We shall denote by $\mathcal{P}(^k X)$ the vector space of all k-homogeneous polynomials. It is a Banach space under the supremum norm given by $\|P\| = \sup_{\|x\| \leq 1} |P(x)|$, $P \in \mathcal{P}(^k X)$. A mapping $P : X \to \mathbb{C}$ is said to be a *polynomial of degree at most* n if it can be represented as a sum $P = P_0 + P_1 + \dots + P_n$, where $P_k \in \mathcal{P}(^k X)$ for $k = 0, \dots, n$. We denote by $\mathcal{P}(X)$ the vector space of all polynomials, and by $\mathcal{L}(^k X)$ ($\mathcal{L}_s(^k X)$) the vector space of all continuous (symmetric) k-linear mappings.

A mapping $f : X \to \mathbb{C}$ is said to be *holomorphic* on X or *entire* if it has a complex Fréchet derivative at each point of X. Equivalently, f is holomorphic if for each $x_0 \in X$ there exists a ball $B(x_0, r)$, centred at x_0 with radius $r > 0$, and a sequence of polynomials $\{P_k\}_k$, $P_k \in \mathcal{P}(^k X)$, such that $f(x) = \sum_{k=0}^{\infty} P_k(x - x_0)$ uniformly for $x \in B(x_0, r)$. We shall denote by $\mathcal{H}(X)$ the vector space of all holomorphic mappings from X into $\mathbb{C}$. The sequence $\{P_k\}_k$ is uniquely determined by f and x_0. We shall write $P_k f(x_0) = P_k$, $P_k f = P_k f(0)$ and we will denote by $A_k f(x_0)$ and $A_k f$ its respectively associated symmetric k-linear mappings. The series $\sum_{k=0}^{\infty} P_k f(x_0)(x - x_0)$ is called the *Taylor series* of f at x_0. Its radius of uniform convergence is the supremum of all $r \geq 0$ such that the series converges uniformly on the closed ball centred at x_0 and radius r. This radius R is given by the Cauchy-Hadamard Formula $1/R = \limsup_{k \to \infty} \|P_k\|^{1/k}$.

If X is an infinite dimensional Banach space, it is no longer true that the Taylor series of each function $f \in \mathcal{H}(X)$ at each $x_0 \in X$ converges uniformly on any ball around x_0. In fact, there exist holomorphic mappings such that the Taylor series at each $x_0 \in X$ does not converge uniformly on some ball around x_0. This is because a holomorphic function in this case is not necessarily bounded on bounded sets. Therefore, we consider the space $H_b(X)$ of *holomorphic functions of bounded type* on X, i.e., holomorphic functions bounded on bounded sets. This space is endowed with the locally convex topology τ_b of uniform convergence on the bounded subsets of X.

We say that a map $v : X \to]0, \infty[$ is a *weight* if there exists a continuous decreasing function $\eta : [0, \infty[\to]0, \infty[$ *rapidly decreasing*, that is, $\lim_{r \to \infty} \eta(r) r^k = 0$ for all $k \in \mathbb{N}$, such that $v(x) = \eta(\|x\|)$ for all $x \in X$. Given a weight v, we define the *weighted Banach spaces of entire functions*

$$H_v(X) = \{f : X \to \mathbb{C} \text{ holomorphic: } \|f\|_v := \sup_{x\in X} v(x)|f(x)| < \infty\}$$

and

$$H_v^0(X) = \{f \in H_v(X) : v|f| \text{ vanishes at infinity outside bounded sets}\}.$$

We recall that a function f on X *vanishes at infinity outside bounded sets (o.b.s)* if for all $\varepsilon > 0$ there exists a bounded set $A \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \in X \setminus A$. We also denote by B_v and B_v^0 their closed unit balls, respectively. Since

$$\sup_{x\in X} v(x)|p_k(x)| = \sup_{x\in X} v(x)\|x\|^k \left|p_k\left(\frac{x}{\|x\|}\right)\right| \leq \sup_{x\in X} v(x)\|x\|^k\|p_k\|$$

for every $p_k \in \mathcal{P}(^kX)$ and the weights are rapidly decreasing, $\mathcal{P}(X) \subseteq H_v^0(X)$ with continuous inclusion.

As in Section 0.2, given a weight v on X, its *associated weight* $\widetilde{v}$ is defined as

$$\widetilde{v}(x) := \frac{1}{\sup\{|f(x)| : f \in H_v(X), \|f\|_v \leq 1\}}.$$

As in the case $X = \mathbb{C}$, from the definition, $v \leq \widetilde{v}$ and $H_v(X) = H_{\widetilde{v}}(X)$ isometrically. If there exists a constant $C > 0$ for which $\widetilde{v} \leq Cv$, the weight is also called *essential*.

In [71], García, Maestre and Rueda defined and studied weighted spaces of holomorphic functions defined on Banach spaces. In the paper, given an increasing sequence of weights $W := \{w_n\}_n$, they define the *weighted Fréchet spaces of entire functions on Banach spaces*

$$HW(X) := \{f \in \mathcal{H}(X) : \ \|f\|_n := \sup_{x\in X} w_n(x)|f(x)| < \infty \text{ for all } n \in \mathbb{N}\}$$

and

$$HW_0(X) := \{f \in \mathcal{H}(X) : \ w_n|f| \text{ vanishes at } \infty \text{ o.b.s for all } n \in \mathbb{N}\},$$

endowed with the projective limit topologies. That is,

$$HW(X) = proj_n H_{w_n}(X) \quad \text{and} \quad HW_0(X) = proj_n H_{w_n}^0(X)$$

are Fréchet spaces with norms $\{\| \ \|_n\}_n$.

On the other hand, weighted spaces of holomorphic functions defined by a decreasing sequence of weights $V := \{v_n\}_n$, i.e., $v_n \geq v_{n+1}$ for all $n \in \mathbb{N}$, were studied in depth by Bierstedt, Meise and Summers [31] for open subsets of $\mathbb{C}^d$ (see also [28], [40], [44] and [45]). For X a Banach space, we denote the *weighted inductive limits of spaces of holomorphic functions* by

$$VH(X) := \{f \in \mathcal{H}(X) : \ \exists n \in \mathbb{N} \text{ such that } \sup_{x\in X} v_n(x)|f(x)| < \infty\}$$

and

$$VH_0(X) := \{f \in \mathcal{H}(X) : \ \exists n \in \mathbb{N} \text{ such that } \ v_n|f| \text{ vanishes at } \infty \text{ o.b.s } \},$$

endowed with the inductive limit topologies τ and τ', respectively, that is,

$$VH(X) := ind_n H_{v_n}(X) \quad \text{and} \quad VH_0(X) := ind_n H^0_{v_n}(X).$$

Observe that $VH_0(X)$ is continuously included in $VH(X)$, but we cannot assure a priori that it is a topological subspace. We denote by B_n and B_n^0 the closed unit balls of $H_{v_n}(X)$ ($H_{w_n}(X)$) and $H^0_{v_n}(X)$ ($H^0_{w_n}(X)$), respectively.

Countable locally convex inductive limits of weighted spaces of holomorphic functions naturally arise in great profusion throughout such fields as linear partial differential equations and convolution equations, distribution theory and representation of distributions as boundary values of holomorphic functions, complex analysis in one and several variables, and spectral theory and the holomorphic functional calculus (see [31]).

As in the finite dimensional case, given a weight v on X and a function $f \in H^0_v(X)$, the Cesàro means

$$C_N f = \frac{1}{N+1}\sum_{l=0}^{N}\left(\sum_{k=0}^{l} P_k f\right), \ N \in \mathbb{N},$$

converge to f in $H^0_v(X)$ (see [71, Proposition 4]). Hence, given a decreasing or an increasing family of weights V or W, respectively, for each $n \in \mathbb{N}$, $\mathcal{P}(X)$ is dense in $H^0_{w_n}(X)$ and in $H^0_{v_n}(X)$, and thus, in $HW_0(X)$ and $VH_0(X)$. Since our weights are defined through continuous and decreasing functions, $HW(X) \hookrightarrow H_b(X)$ and $VH(X) \hookrightarrow H_b(X)$ with continuous inclusions, and thus, $HW(X)$, $HW_0(X)$, $VH(X)$ and $VH_0(X)$ are Hausdorff spaces. The linear map $\delta_x : H_b(X) \to \mathbb{C}, f \mapsto f(x)$ is τ_{co}-continuous for every $x \in X$, so, given a weight v, the closed unit ball

$$B_v = \bigcap_{x\in X} \delta_x^{-1}(\overline{D(0, 1/v(x))})$$

is τ_{co}-closed. Moreover, if we fix a compact set $K \subseteq X$, then

$$\sup_{f\in B_v}\sup_{x\in K}|f(x)| \le \max_{x\in K}\frac{1}{v(x)} < \infty,$$

so, B_v is uniformly bounded on the compact subsets of X. Therefore, by [103, Proposition 9.15], B_v is equicontinuous, and since $(H_b(X), \tau_{co})$ is semi-Montel (see [25, p.130]), B_v is τ_{co}-compact. Thus, by Proposition 0.1.3, $VH(X)$ is a regular inductive limit.

In many concrete situations it is important to know an explicit formulation of the continuous seminorms of the inductive limit topology. In order to obtain the seminorms describing the topology of $VH(X)$, Bierstedt, Meise and Summers studied in [31] the problem of projective description for weighted inductive limits of spaces of holomorphic functions.

Definition 0.3.1 Given a decreasing sequence of weights $V = \{v_n\}_n$ we define the *maximal Nachbin family of weights* associated to V by

$$\overline{V} := \{\overline{v} : X \to \mathbb{R} \ : \ \overline{v} \geq 0, \ \overline{v} \ \text{ u.s. and } \sup_{x \in X} \frac{\overline{v}(x)}{v_n(x)} < \infty \text{ for all } n \in \mathbb{N}\},$$

where $\overline{v}$ u.s. denotes $\overline{v}$ upper semicontinuous. Clearly, for every sequence $\{\alpha_n\}_n$ of strictly positive numbers, $\inf_n \alpha_n v_n$ is upper semicontinuous and belongs to $\overline{V}$.

Definition 0.3.2 The *projective hulls* of the weighted inductive limits are defined as follows:

$$H\overline{V}(X) := \{f \in \mathcal{H}(X) : \ p_{\overline{v}}(f) := \sup_{x \in X} \overline{v}(x)|f(x)| < \infty \text{ for every } \overline{v} \in \overline{V}\}$$

and

$$H\overline{V}_0(X) := \{f \in \mathcal{H}(X) : \ \overline{v}f \text{ vanishes at infinity o.b.s. for all } \overline{v} \in \overline{V}\},$$

both endowed with the locally convex topology defined by the seminorms $\{p_{\overline{v}}, \ \overline{v} \in \overline{V}\}$.

$H\overline{V}_0(X)$ is a closed subspace of $H\overline{V}(X)$, and $H\overline{V}(X)$ is Hausdorff (the canonical injection $H\overline{V}(X) \hookrightarrow (C(X), \tau_p)$ is continuous). We always have the continuous injections $VH(X) \hookrightarrow H\overline{V}(X)$ and $VH_0(X) \hookrightarrow H\overline{V}_0(X)$, and the spaces $VH(X)$ and $H\overline{V}(X)$ coincide algebraically.

We define now the spaces for the vector-valued case:

Given a Hausdorff locally convex space G, we denote by $\mathcal{H}(X, G)$ the space of *vector-valued holomorphic functions*. If G is a Banach space, a function $f \in \mathcal{H}(X, G)$ is holomorphic if and only if $u \circ f \in \mathcal{H}(X)$ for every $u \in G'$. See [103, Chapter 2] for the definition and more background about vector-valued holomorphic functions.

Given a weight $v : X \to]0, \infty[$, we define the *weighted spaces of vector-valued holomorphic functions* by

$$H_v(X,G) = \{f \in \mathcal{H}(X,G) : \ r_{v,q}(f) := \sup_{x \in X} v(x) q(f(x)) < \infty \text{ for all } q \in cs(G)\}, \tag{3.1}$$

$$H_v^0(X,G) = \{f \in \mathcal{H}(X,G) : \ v(q \circ f) \text{ vanishes at } \infty \text{ o.b.s. } \forall q \in cs(G)\},$$

which are Banach spaces in the case G is a Banach space.

Given a decreasing (increasing) sequence of weights V (W), the associated *weighted spaces of vector-valued holomorphic functions* are defined by

$$VH(X,G) := \{f \in \mathcal{H}(X,G) : \ \exists n \in \mathbb{N} : \sup_{x \in X} v_n(x) q(f(x)) < \infty \text{ for all } q \in cs(G)\},$$

endowed with the inductive limit topology, i.e., $VH(X,G) := ind_n H_{v_n}(X,G)$, and by

$$HW(X,G) := \{f \in \mathcal{H}(X,G) : \ \sup_{x \in X} v_n(x) q(f(x)) < \infty \ \forall q \in cs(G), \forall n \in \mathbb{N}\},$$

endowed with the projective limit topology, i.e., $HW(X,G) = proj_n H_{w_n}(X,G)$. It is a locally convex space with seminorms $\{r_{n,q}\}_{n,q}$, $r_{n,q}(f) := \sup_{x \in X} v_n(x) q(f(x))$.

Weighted (LB)-spaces of entire functions on Banach spaces are the spaces we work with in Chapters 1 and 2. In Chapter 3 we deal with the weighted Banach spaces of entire functions $H_v(\mathbb{C})$ and $H_v^0(\mathbb{C})$, together with other weighted spaces of entire functions defined by integral norms. The spaces $HW(\mathbb{C})$, $HW_0(\mathbb{C})$, $VH(\mathbb{C})$ and $VH_0(\mathbb{C})$ appear in Chapter 4. The vector-valued case is considered in Sections 2.2 and 2.4.

0.4 Arens and Aron-Berner extensions

In this section we consider the problem of extending holomorphic functions of bounded type from a Banach space X to its bidual X'' as a holomorphic function. For the details and more background, see, e.g., [5] and [63, Section 6.2].

Various approaches to extensions have been proposed and developed in recent years and, whether explicit or implicit, each successful effort employs one or both of the canonical mappings $J_X : X \to X''$, $J_{X'} : X' \to X'''$. We concentrate here on the *Aron-Berner extension* operator $AB_k : \mathcal{P}(^k X) \to \mathcal{P}(^k X'')$ defined in [5].

By *Goldstine's theorem*, B_X is weak*-dense in $B_{X''}$. If $L \in \mathcal{L}(^kX)$, then $\mathcal{E}_k(L)$ or $\widetilde{L}$ denotes the k-linear mapping on $(X'')^k$ defined by

$$\widetilde{L}(x_1'', \dots, x_k'') = \lim_{\alpha_1} \cdots \lim_{\alpha_k} L(x_{\alpha_1}, \dots, x_{\alpha_k}), \tag{4.2}$$

where $\{x_{\alpha_j}\}_{\alpha_j}$ is a net in X such that $J_X x_{\alpha_j} \to x_j''$ in the $\sigma(X'', X')$ topology for all $j \in \{1, \dots, k\}$. $\mathcal{E}_k$ is well defined, since for each $j \in \{1, \dots, k-1\}$, $\{x_i\}_{i=1}^j \in X^j$ and $\{x_i''\}_{i=j+2}^k \in (X'')^{k-j-1}$, the mapping

$$x'' \in X'' \to \widetilde{L}(J_X x_1, \dots, J_X x_j, x'', x_{j+2}'', \dots, x_k'')$$

is $\sigma(X'', X')$ continuous. It is important to note the order, right to left, in which the limits are taken in (4.2). A different ordering may give rise to a different extension, because the extension $\mathcal{E}_k(L)$ is not necessarily separately weak-star continuous in each variable. From the proof of [63, Proposition 1.53] we have $\widetilde{L} \in \mathcal{L}(^kX'')$ and $\|L\| = \|\widetilde{L}\|$. If $\hat{L} = P \in \mathcal{P}(^kX)$, then its Aron-Berner extension is defined by $AB_k(P)(z) := \widetilde{L}(z, \dots, z)$ for all $z \in X''$, and is also norm-preserving, i.e., $\|\widetilde{P}\| = \|P\|$ for all $P \in \mathcal{P}(^kX)$. In fact, Davie and Gamelin showed in [60] that if $z \in X''$, there is $\{x_\alpha\}_\alpha \subseteq X$ such that $\|x_\alpha\| \le \|z\|$ for all α, and $P(x_\alpha) \underset{\alpha}{\to} \widetilde{P}(z)$ for all polynomial P on X. For convenience of notation we sometimes write $\widetilde{P}$ instead of $AB_k(P)$.

Now we consider the question of permuting the order of taking limits in (4.2). If $L \in \mathcal{L}(^kX)$ and the order can be interchanged, then we say that L is *Arens regular*, and if every $L \in \mathcal{L}(^kX)$ is Arens regular, we say that X is Arens regular. In this case, $\mathcal{E}_k(\mathcal{L}^s(^kX)) \subseteq \mathcal{L}^s(^kX'')$, but this is not satisfied in general.

See the following alternative construction of the Aron-Berner extension. We only require the $k = 2$ case but the procedure can be carried out for all k. If $A \in \mathcal{L}(^2X)$ we define $T_A \in \mathcal{L}(X, X')$ by the formula $[T_A(x)](y) = A(x, y)$ for all $x, y \in X$. The double transpose of T_A, T_A'', maps X'' into X''' and $\mathcal{E}_2(A)(x'', y'') = \widetilde{A}(x'', y'') = [T_A''(x'')](y'')$.

Given a Banach space X, we say that $T \in \mathcal{L}(X, X')$ is *weakly compact* if T maps the unit ball of X into a weakly relative compact subset of X'. A complex Banach space X is said to be *(symmetrically) regular* if every continuous (symmetric) linear mapping $T : X \to X'$ is weakly compact. Recall that T is symmetric if $Tx_1(x_2) = Tx_2(x_1)$ for all $x_1, x_2 \in X$.

Proposition 0.4.1 ([63, Proposition 6.13]) *If X is a symmetrically regular Banach space, then for all $k \ge 1$, each continuous symmetric k-linear form on X extends to a separately weak-star continuous symmetric k-linear form on X''.*

For regular spaces we have the next result:

Proposition 0.4.2 ([63, Proposition 6.14]) *The following are equivalent for a Banach space X :*

(i) *each $L \in \mathcal{L}(^k X)$ is Arens regular,*

(ii) *if $L \in \mathcal{L}(^k X,)$ then $\mathcal{E}_k(L)$ is separately weak-star continuous in each variable,*

(iii) *each continuous linear mapping from X to X' is weakly compact, i.e., X is regular.*

The Aron-Berner extension of a bounded holomorphic function on a Banach space X into its bidual X'' is defined through its Taylor series:

Proposition 0.4.3 ([63, Proposition 6.16]) *If X is a Banach space, then there exists a multiplicative, continuous linear mapping*

$$AB : H_b(X) \to H_b(X^{''}),\ AB(f) := \sum_{k \geq 0} AB_k(P_k f)$$

such that $AB(f)|_X = f$. We say that $AB(f)$ is the Aron-Berner extension of f to X'', and in order to simplify, we denote it by $\widetilde{f}$.

0.5 An introduction to linear dynamics

In this section we introduce some basic definitions and results about linear dynamical systems. For motivation, more examples and background about linear dynamics we refer the reader to the books by Bayart and Matheron [12] and by Grosse-Erdmann and Peris [80], the article by Godefroy and Shapiro [73] and the surveys [76] and [78] by Grosse-Erdmann.

Let X be a topological vector space and $T : X \to X$ an operator, that is, a linear and continuous mapping. We call the pair (X, T) a *linear dynamical system.* For $x \in X$, we denote by $Orb(x, T) := \{x, Tx, T^2x, ...\}$ its *orbit under* T, and we say that a point $x \in X$ is *periodic* if there is some $n \in \mathbb{N}$ such that $T^n x = x$. The smallest $n \in \mathbb{N}$ which verifies this condition is called the *period of* x.

An operator $T : X \to X$ is called *topologically transitive* if, for any pair U, V of non-empty open subsets of X, there exists some $n \in \mathbb{N}_0$ such that $T^n(U) \cap V \neq \emptyset$, and T is called *topologically mixing* if, for any pair U, V of non-empty open subsets of X, there exists some $N \in \mathbb{N}_0$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \in \mathbb{N}$, $n \geq N$.

An operator $T : X \to X$ is said to be *hypercyclic* (Beauzamy, 1986) if it has a dense orbit, that is, if there is some $x \in X$ such that its orbit is dense in X. Any such vector is called a *hypercyclic vector* and the set of hypercyclic vectors is denoted by $HC(T)$.

For X a complete metric vector space, an operator $T : X \to X$ is called *chaotic* (Godefroy, Shapiro, 1991) if it is hypercyclic and it has a dense set of periodic points.

By the *Birkhoff's transitivity criterion* (see [80, Theorem 1.16]), if X is a separable complete metric vector space, then T is hypercyclic if and only if it is topologically transitive.

The first simple criterion to ensure that an operator T on a separable complete metrizable topological vector space is hypercyclic (even topologically mixing), was presented by Kitai in her Thesis (1982) (see [80, Theorem 3.4]). It was discovered independently by Gethner and Shapiro (1987) and was improved by several authors. A weakening of the Kitai-Gethner-Shapiro criterion is the famous Hypercyclicity Criterion due to Bès and Peris in 1999 (see [20] and [22]). The condition in this weaker criterion do not imply that the operator is topologically mixing.

Theorem 0.5.1 (Hypercyclicity Criterion.) *Let $T : X \to X$ be an operator on a separable complete metrizable topological vector space X. Suppose that there are dense subsets $Y_0, Y_1 \subseteq X$, an increasing sequence $\{n_k\}_k$ of positive integers, and maps $S_{n_k} : Y_1 \to X$, $k \geq 1$, not necessarily linear nor continuous, such that:*

(i) $T^{n_k} x \to 0$ *for each* $x \in Y_0$,

(ii) $S_{n_k} y \to 0$ *for each* $y \in Y_1$, *and*

(iii) $T^{n_k} S_{n_k} y \to y$ *for each* $y \in Y_1$,

then T is hypercyclic.

If the Hypercyclicity Criterion is satisfied for the sequence of all positive integers, then the proof shows that the operator T is even topologically mixing. Bès and Peris proved that an operator T satisfies the assumptions of the Hypercyclicity Criterion if and only if $T \oplus T$ is hypercyclic on $X \oplus X$. Only very recently, De La Rosa and Read ([61]) were able to exhibit hypercyclic operators which do not satisfy the hypercyclicity criterion, thus solving a long standing problem. Their example was improved later by Bayart and Matheron ([12]), who presented examples defined on classical Banach sequence spaces. The next result, first observed by Kitai, gives a sufficient condition for an operator not to be hypercyclic.

Lemma 0.5.2 (A "non-hypercyclicity criterion" [80, Lemma 2.53]) *Suppose T is an operator on a metrizable topological vector space X, and that $T' : X' \to X'$ has an eigenvalue. Then T is not hypercyclic.*

A vector $x \in X$ is called *frequently hypercyclic* for T if, for every non-empty open subset U of X,

$$\underline{\text{dens}}\ \{n \in \mathbb{N} : \ T^n x \in U\} > 0,$$

where

$$\underline{\text{dens}}(A) = \liminf_{N \to \infty} \frac{|\{n \in A : \ n \leq N\}|}{N}$$

denotes the lower density of a subset A of $\mathbb{N}$ and $|S|$ denotes the cardinality of a set S. The operator T is called *frequently hypercyclic* if it possesses a frequently hypercylic vector.

The orbit of a frequently hypercyclic vector is therefore, in the specified sense, frequently recurrent. Obviously, frequent hypercyclicity is a stronger notion than hypercyclicity.

According to Bayart and Grivaux [11], a bounded operator T on a Banach space X is said to have a *perfectly spanning set of eigenvectors associated to unimodular eigenvalues* if there exists a continuous probability measure σ on the unit circle $\mathbb{T}$ such that for every σ-measurable subset A of $\mathbb{T}$ which is of σ-measure 1, $\text{span}(\cup\{\text{Ker}(T - \lambda I) : \ \lambda \in A\})$ is dense in X.

Theorem 0.5.3 ([74, Theorem 1.4]) *If a bounded operator T on a Banach space X has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues, then T is frequently hypercyclic on X.*

See [80, Section 9] for more background and details about frequently hypercyclic operators.

0.5.1 Linear dynamics on locally convex spaces

Since much of modern analysis also occurs in locally convex Hausdorff spaces which are not metrizable, there is some interest in extending the definitions and properties about dynamical systems to this more general setting. See [80, Chapter 12] and [88] for more background and details.

Since no topological vector space E has isolated points, every hypercyclic operator T on E is topologically transitive, but the converse is not true in general (see [80, Example 12.9]). In this context, the hypothesis of the Hypercyclicity Criterion in 0.5.1 does not imply hypercyclicity, but topologically transitivity (see [80, Theorem 12.33]). Following [80, Definition 12.11], since hypercyclicity and topological transitivity no longer coincide, we adopt Devaney's original definition of chaos, that is, $T : E \to E$ is chaotic if it is topologically transitive and has a dense set of periodic points.

In what follows, consider E is a barrelled locally convex space, and $T : E \to E$ an operator. T is said to be *power bounded* if the sequence $\{T^N\}_N$ is an equicontinuous set of $\mathcal{L}(E)$. By the uniform boundedness principle, this is equivalent to the fact that the set $\{T^N x\}_N$ is bounded for every $x \in E$, and if E is a Banach space, to $\sup_N \|T^N\| < \infty$.

Given $T \in \mathcal{L}(E)$, let

$$T_{[N]} := \frac{1}{N}\sum_{j=1}^{N} T^j, \ N \in \mathbb{N}, \tag{5.3}$$

denote de Cesàro means of the iterates of T. The operator T is said to be *Cesàro power bounded* if the sequence $\{T_{[N]}\}_N$ is equicontinuous, and *mean ergodic* if the limits $Px := \lim_{N\to\infty} T_{[N]}x, \ x \in X$, exist in E. If $\{T_{[N]}\}_N$ is convergent in the uniform operator topology τ_b, then T is called *uniformly mean ergodic.*

A power bounded operator T is mean ergodic precisely when

$$E = \mathrm{Ker}(I - T) \oplus \overline{\mathrm{Im}(I - T)}, \tag{5.4}$$

where $\mathrm{Im}(I - T)$ denotes the range of $I - T$ and the bar denotes the closure in E. In general, $\overline{\mathrm{Im}(I - T)}$ is the set of all $x \in E$ for which the sequence $\{T_{[N]}x\}_N$ converges to 0 in E. The space E itself is called mean ergodic (resp. uniformly mean ergodic) if every power bounded operator on E is mean ergodic (resp. uniformly mean ergodic). F. Riesz showed in 1938 that all spaces L^p $(1 < p < \infty)$ are mean ergodic. In 1939, E.R. Lorch proved that all reflexive Banach spaces are mean ergodic.

Mean ergodic operators in Fréchet spaces and barrelled locally convex spaces have been considered by Albanese, Bonet and Ricker in [1] and [2].

Proposition 0.5.4 ([1, Proposition 2.8]) *Let E be any (DF) or (LF)-space which is Montel. Then X is uniformly mean ergodic.*

Clearly, since

$$\frac{T^N}{N} = \frac{1}{N}\sum_{j=1}^{N} T^j - \frac{N-1}{N}\frac{1}{N-1}\sum_{j=1}^{N-1} T^j,$$

if T is mean ergodic (uniformly mean ergodic), then T^N/N converges to zero in the strong operator topology (uniform operator topology).

The next two results will be used in Chapter 3 in order to show the uniform mean ergodicity of some operators on Banach spaces.

Theorem 0.5.5 (Lin [91]) *Let X be a Banach space and $T \in \mathcal{L}(X)$ such that $\|T^N/N\| \to 0$. Then, T is uniformly mean ergodic if and only if* $\mathrm{Im}(I-T)$ *is closed.*

A Banach space X is called a *Grothendieck space* (see [65, Exercise 3.42]) if every weak*-convergent sequence in X' is weak-convergent. X has the *Dunford-Pettis property* (see [65, Definition 13.41]) if $x'_n(x_n) \to 0$ whenever $x_n \in X$ and $x'_n \in X'$, $n \in \mathbb{N}$, satisfy $x_n \to 0$ in $(X, \sigma(X, X'))$ and $x'_n \to 0$ in $(X', \sigma(X', X))$. ℓ_∞ and the Hardy space H^∞ are examples of Grothendieck Banach spaces with de Dunford-Pettis property.

Theorem 0.5.6 (Lotz [92]) *Let X be a Grothendieck Banach space satisfying the Dunford-Pettis property. If $T \in \mathcal{L}(X)$ is such that $\|T^N/N\| \to 0$, then T is mean ergodic if and only if T is uniformly mean ergodic.*

The spaces we are going to consider are either (LB)-spaces or Fréchet spaces which can be expressed as intersection of a decreasing sequence of Banach spaces. In both cases, the spaces are barrelled. Therefore, an operator $T \in \mathcal{L}(E)$ is power bounded if and only if for each bounded set B of E, the set $\{T^k(x) : \ k \in \mathbb{N},\ x \in B\}$ is bounded in E. In particular, we get the next equivalence:

Proposition 0.5.7 *(i) Let $E = ind_n E_n$ be a regular inductive limit of Banach spaces. An operator $T \in \mathcal{L}(E)$ is power bounded if for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ and some constant $C > 0$ such that $\|T^k x\|_n \le C\|x\|_m$ for every $k \in \mathbb{N}$ and $x \in E_m$.*

(ii) Given $F = proj_n F_n$ a projective limit of Banach spaces, the operator $T : F \to F$ is power bounded if for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ and some constant $D > 0$ such that $\|T^k x\|_n \le D\|x\|_m$ for every $k \in \mathbb{N}$ and every $x \in F$.

Proof. (i) Since the inductive limit E is regular, T is power bounded if and only if for every $m \in \mathbb{N}$, the set $\{T^k(x): \ k \in \mathbb{N}, \ x \in B_m\}$ is bounded in E. And this happens if and only if there exists $n \in \mathbb{N}$ and $C > 0$ such that $\|T^k(x)\|_n \leq C$ for every $k \in \mathbb{N}$ and every $x \in B_m$, which yields (i). (ii) follows using the same arguments we use for the characterization of equicontinuous operators on Fréchet spaces. □

Lemma 0.5.8 *Let $E = ind_n E_n$ be an inductive limit of Banach spaces. If $T : E \to E$ is a linear map such that $T_| : E_n \to E_n$ is continuous and mean ergodic (power bounded) for every $n \geq n_0$, $n_0 \in \mathbb{N}$, then the operator $T : E \to E$ is continuous and mean ergodic (power bounded). If E is regular and $T_| : E_n \to E_n$ is uniformly mean ergodic for every $n \in \mathbb{N}$, then $T : E \to E$ is uniformly mean ergodic.*

Proof. Only uniform mean ergodicity needs a proof, since the other implications are direct. Since every $T_| : E_n \to E_n$ is mean ergodic, $T : E \to E$ is mean ergodic. Let $P(x) := \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} T^j x$, $x \in E$. Take a bounded set B in E. Since E is regular, there exists some $n \in \mathbb{N}$ such that $B \subseteq E_n$ and it is bounded in E_n. As $T_| : E_n \to E_n$ is uniformly mean ergodic, $\frac{1}{N}\sum_{j=1}^{N} T^j$ converges uniformly on B with respect to $\| \ \|_n$, and thus, it converges to P uniformly on B with respect to the inductive limit topology. □

Analogously,

Lemma 0.5.9 *Let $E = proj_n E_n$ be a projective limit of Banach spaces. If $T : E_1 \to E_1$ is an operator such that $T_| : E_n \to E_n$ is continuous and (uniformly) mean ergodic for every $n \geq n_0$, $n_0 \in \mathbb{N}$, then the operator $T : E \to E$ is continuous and (uniformly) mean ergodic. If $T_| : E_n \to E_n$ is power bounded for every $n \in \mathbb{N}$, then T is power bounded on E.*

Lemma 0.5.10 ([39, Lemma 3]) *Let T be a continuous linear operator on a locally convex space E. Let F be a locally convex space which is continuously and densely contained in E. If $T|_F : F \to F$ is well-defined, continuous and hypercyclic (resp. chaotic), then T is also hypercyclic (resp. chaotic) on E.*

Lemma 0.5.11 *Let E be a barrelled locally convex space and $T \in \mathcal{L}(E)$.*

(i) If T is topologically transitive, then $\{(T')^k(v)\}_k$ is unbounded in E'_b for every $v \in E'$, $v \neq 0$.

(ii) If T is topologically mixing, then for each $v \in E'$, $v \neq 0$, no infinite subset of $\{(T')^k(v)\}_k$ is bounded in E'_b. In particular, if E is a Banach space, T topologically mixing implies $\lim_{k\to\infty} \|(T')^k(v)\| = \infty$ for each $v \in E'$, $v \neq 0$.

Proof. (i) Assume that for some $0 \neq v \in E'$, the set $U := \{(T')^k(v)\}_k$ is bounded in E'_b. Since U is bounded and E barrelled, the polar U° is a 0-neighbourhood, and thus, has non-empty interior (see [110, Proposition 3.1.1]). Therefore, if $V := \{x \in E : |\langle x, v\rangle| > 2\}$, we have $T^k(U^\circ) \cap V \neq \emptyset$ for some k, which is a contradiction. (ii) Analogously, assume that for some $0 \neq v \in E'$ there exists some sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that the set $U := \{(T')^{n_k}(v)\}_k$ is bounded in E'_b. The polar U° has non-empty interior, therefore, if $V := \{x \in E : |\langle x, v\rangle| > 2\}$, we have $T^k(U^\circ) \cap V \neq \emptyset$ for every $k \in \mathbb{N}$, $k \geq n_0$, for some $n_0 \in \mathbb{N}$, which is a contradiction. □

Chapter 1

Spectra of weighted (LB)-algebras of entire functions on Banach spaces

In this chapter, given a decreasing family of weights on a Banach space X, we consider the weighted inductive limits of spaces of entire functions $VH(X)$ and $VH_0(X)$ that have been introduced in section 0.3. Motivated by recent research by Carando and Sevilla-Peris on weighted Fréchet algebras of entire functions on Banach spaces [56], we determine conditions on the family of weights to ensure that the corresponding weighted space is an algebra or has polynomial Schauder decompositions. We study Hörmander algebras of entire functions defined on a Banach space and we give a description of them in terms of sequence spaces. We also focus on algebra homomorphisms between these spaces and obtain a Banach-Stone type theorem for a particular decreasing family of weights. Finally, we study the spectra of these weighted algebras, endowing them with an analytic structure. Most of the results included in this chapter are published by Beltrán in [14].

By an *algebra* we understand a locally convex algebra, that is, an algebra $\mathcal{A}$ with a locally convex structure so that multiplication is jointly continuous. The *spectrum* of $\mathcal{A}$ is the space of non-zero continuous multiplicative functionals. We denote by $\mathfrak{M}_b(X)$ the spectrum of the space of entire functions of bounded type $H_b(X)$, and in the case $VH(X)$ is an algebra, we denote its spectrum by $V\mathfrak{M}(X)$.

1.1 Weighted algebras of holomorphic functions

In this section we characterize when $VH(X)$ is an algebra in terms of a condition on the family of weights V. By the next lemma, it es enough to show that multiplication is well defined:

Lemma 1.1.1 ([43, Remark 2.2]) *In every (LB)-space, pointwise multiplication is well defined if and only if it is jointly continuous.*

We have seen in section 0.3 that given a weight v on X, the unit ball B_v is compact with respect to the compact open topology τ_{co}. Hence, for every $x_0 \in X$, since the evaluation map $\delta_{x_0} : H_b(X) \to \mathbb{C}$ is τ_{co}-continuous, the supremum in the definition of associated weight (see section 0.3) is even a maximum. Thus, there exists some $f \in H_v(X)$ with $\|f\|_v \leq 1$ such that $\widetilde{v}(x_0) = \frac{1}{|f(x_0)|}$. Using this fact and following the same ideas of [43, Proposition 2.1] and [56, Proposition 1], we prove the following theorem:

Theorem 1.1.2 *$VH(X)$ is an algebra if and only if for each $m \in \mathbb{N}$ there exist $n \in \mathbb{N}$, $n \geq m$, and $C > 0$ such that*

$$v_n(x) \leq C\widetilde{v}_m^2(x) \text{ for all } x \in X. \tag{1.1}$$

Proof. Let us begin by assuming that $VH(X)$ is an algebra. Multiplication is a continuous bilinear map, in particular hypocontinuous. Then, by Proposition 0.1.4, given a bounded set $B \subseteq VH(X)$ and $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ and $C > 0$ such that $\|fg\|_n \leq C\|g\|_m$ for every $g \in H_{v_m}(X)$ and every $f \in B$. For a fixed $x_0 \in X$ and $m \in \mathbb{N}$, there exists some $f \in H_{v_m}(X)$ with $\|f\|_{v_m} \leq 1$ such that $\widetilde{v}_m(x_0) = \frac{1}{|f(x_0)|}$. So, if we consider $B := B_m$, there exists $n \geq m$ and $C > 0$ such that

$$v_n(x_0) = |f^2(x_0)|v_n(x_0)\frac{1}{|f^2(x_0)|} \leq \|f^2\|_{v_n}\frac{1}{|f^2(x_0)|} \leq C\widetilde{v}_m^2(x_0).$$

The converse is clear from (1.1) and the fact that $\|f\|_{\widetilde{v_j}} = \|f\|_{v_j}$ for every $f \in H_{v_j}(X)$ and every $j \in \mathbb{N}$. By Lemma 1.1.1, multiplication is jointly continuous. □

Corollary 1.1.3 *Given a weight v, the weighted Banach space $H_v(X)$ is never an algebra.*

Proof. Given a weight v, by [28, Proposition 1.2], $H_v(X) = H_{\widetilde{v}}(X)$. If we assume it is an algebra, applying Theorem 1.1.2 to $H_{\widetilde{v}}(X)$ we get that $\widetilde{v}$ is bounded below, since $\widetilde{\widetilde{v}} = \widetilde{v}$ also by [28, Proposition 1.2]. So, $H_v(X)$ is the space of bounded entire functions on X, that is, the space of constant functions, which does not contain the homogeneous polynomials. A contradiction, since we are considering rapidly decreasing weights. $\square$

Corollary 1.1.4 *If the family V consists of essential weights, $VH(X)$ is an algebra if and only if for each $m \in \mathbb{N}$ there exist $n \in \mathbb{N}$, $n \geq m$, and $C > 0$ such that $v_n \leq C v_m^2$.*

Example 1.1.5 Given a weight v on X, the inductive limit $VH(X)$, where $V = \{v^n\}_n$, is always an algebra.

In Section 0.3, we have seen that given a function $f \in H_v^0(X)$, the Cesàro means $C_n f$ converge to f in $H_v^0(X)$. Moreover, if for each function $f \in VH(X)$ (resp. $VH_0(X)$) the Taylor series expansion of f at zero converges to f for the inductive limit topology τ, then $\{\mathcal{P}(^kX)\}_{k\geq 0}$ is said to be a Schauder decomposition of $VH(X)$ ($VH_0(X)$). We are going to see that this condition, which is not satisfied in general, holds when we introduce an additional condition on the weights.

Definition 1.1.6 A sequence $\{E_n\}_n$ of subspaces of a locally convex space E is a *Schauder decomposition* of E if:

(i) for all $x \in E$ there exists a unique sequence $\{x_n\}_n$, $x_n \in E_n$, such that $x = \sum_{n=1}^{\infty} x_n := \lim_{m\to\infty} \sum_{n=1}^{m} x_n$.

(ii) the projections $\{u_m\}_m$, $u_m(\sum_{n=1}^{\infty} x_n) := \sum_{n=1}^{m} x_n$ are continuous.

Definition 1.1.7 Let S denote the set of all scalar sequences $\alpha := \{\alpha_n\}_{n=1}^{\infty}$ such that $\limsup_{n\to\infty} |\alpha_n|^{1/n} \leq 1$. A Schauder decomposition $\{E_n\}_n$ of a locally convex space E is an *S-absolute decomposition* if for all $\alpha \in S$ and $x = \sum_{n=1}^{\infty} x_n \in E$, $x_n \in E_n$,

(i) $\alpha \cdot x := \sum_{n=1}^{\infty} \alpha_n x_n \in E$,

(ii) for each continuous seminorm p and $\alpha \in S$, $p_\alpha(\sum_{n=1}^{\infty} x_n) := \sum_{n=1}^{\infty} |\alpha_n| p(x_n)$ defines a continuous seminorm on E.

When the last condition holds for the unit constant sequence $\mathbb{1}$, i.e., $p_{\mathbb{1}}(\sum_{n=1}^{\infty} x_n) := \sum_{n=1}^{\infty} p(x_n)$ is a continuous seminorm for all continuous seminorm p on E, we say that $\{E_n\}_n$ is an *absolute decomposition* of E.

Definition 1.1.8 An absolute Schauder decomposition $\{E_n\}_n$ of a locally convex space E is a *γ-complete decomposition* if given $\{x_n\}_{n=1}^{\infty}$, $x_n \in E_n$, such that $\sum_{n=1}^{\infty} p(x_n) < \infty$ for all continuous seminorm p, the series $\sum_{n=1}^{\infty} x_n$ converges in E.

Lemma 1.1.9 ([71, Proposition 3]) *If V is a decreasing family of weights, the map $P_k : VH(X) \to VH(X),\ f \mapsto P_k f$, is continuous for each $k \in \mathbb{N}$. Moreover, $\|P_k f\|_{v_n} \le \|f\|_{v_n}$ for all $f \in H_{v_n}(X),\ n \in \mathbb{N}$.*

Definition 1.1.10 Let $V = \{v_n\}_n$ be a decreasing sequence of weights. We say that V satisfies *Condition (A)* if for each $m \in \mathbb{N}$ there exist $R > 1, D > 0$ and $n \in \mathbb{N}, n \ge m$, such that

$$\|P_k f\|_{v_n} \le D \frac{1}{R^k} \|f\|_{v_m} \ \forall f \in H_{v_m}(X),\ k = 0, 1, 2, ...$$

Observe that this condition is analogous to Condition II in [71, Definition 7]. As in [71, Proposition 8], the next proposition gives a condition that implies Condition (A) and that is easier to check.

Proposition 1.1.11 *Let $V = \{v_n\}_n$ be a decreasing sequence of weights. If for each $m \in \mathbb{N}$ there exist $R > 1, D > 0$ and $n \in \mathbb{N}, n \ge m$ such that*

$$v_n(x) \le D v_m(Rx)\ \forall x \in X \quad \textit{(Condition (A'))},$$

then the family V satisfies Condition (A).

Proof. Given $m \in \mathbb{N}$ and $f \in H_{v_m}(X)$, by Lemma 1.1.9, $\|P_k f\|_{v_m} \le \|f\|_{v_m}$, $k = 0, 1, ...$ Hence, by hypothesis there exist $R > 1,\ D > 0$ and $n \in \mathbb{N},\ n \ge m$, such that

$$R^k \|P_k f\|_{v_n} = \sup_{x \in X} v_n(x) |P_k f(Rx)| \le D \sup_{x \in X} v_m(Rx) |P_k f(Rx)| \le D \|f\|_{v_m}.$$

□

Lemma 1.1.12 *Given a decreasing sequence of weights $V = \{v_n\}_n$, the spaces $H_{v_1}(X),\ VH(X)$ and $H_b(X)$ induce the same topology on $\mathcal{P}(^k X)$ for all $k \in \mathbb{N}$. It is equivalent to the supremum norm topology on $\mathcal{P}(^k X)$.*

Proof. Since the weights are rapidly decreasing,

$$\mathcal{P}(^k X) \hookrightarrow H_{v_1}(X) \hookrightarrow VH(X) \hookrightarrow H_b(X)$$

with continuous inclusions. Hence, since the supremum norm on $\mathcal{P}(^k X)$ coincides with the uniform convergence topology τ_b, we have the equivalence of all this topologies on $\mathcal{P}(^k X)$. □

The next proposition is analogous to [71, Corollary 12], but its proof is essentially different.

Proposition 1.1.13 *If V is a family of weights satisfying Condition (A), then the spaces $VH(X)$ and $VH_0(X)$ coincide algebraically and topologically.*

Proof. Given $m \in \mathbb{N}$ and a function $f \in H_{v_m}(X)$, by Condition (A) there exists $n \geq m,\ n \in \mathbb{N}$, such that $\sum_{k\in\mathbb{N}} P_k f$ converges absolutely to some g in $H^0_{v_n}(X)$. By the uniqueness of the Taylor series, $f = g$, and therefore, the identity map $i : H_{v_m}(X) \hookrightarrow H^0_{v_n}(X)$ is continuous. □

Now, we can proceed as in the proof of [71, Theorem 11] to obtain the next theorem:

Theorem 1.1.14 *Let V be a family of weights on X satisfying Condition (A). Then $\{\mathcal{P}(^kX)\}_{k\geq 0}$ is an S-absolute, γ-complete Schauder decomposition of $VH(X)$.*

Proof. By Lemma 1.1.9 and the proof of Proposition 1.1.13, $\{\mathcal{P}(^kX)\}_{k\geq 0}$ is a Schauder decomposition of $VH(X)$. Let us see that it is S-absolute: consider

$$S := \{\{a_j\}_j \in \mathbb{C}^{\mathbb{N}} :\ \limsup_{j\to\infty} |a_j|^{1/j} \leq 1\}.$$

Given $f \in VH(X)$, there exists $m \in \mathbb{N}$ such that $f \in H_{v_m}(X)$. Thus, by Condition (A), there exists $R > 1,\ D > 0$ and $n \in \mathbb{N},\ n \geq m$ such that $\|P_k f\|_{v_n} \leq D\frac{1}{R^k}\|f\|_{v_m}$ for every $k \in \mathbb{N}$. Given $\{a_j\}_j \in S$ we can find $j_0 \in \mathbb{N}$ such that $|a_j|^{1/j} < (1+R)/2$ for each $j \geq j_0$. For $c > 0$ large enough we have $|a_j| \leq c\left(\frac{1+R}{2}\right)^j$ for all $j \in \mathbb{N}_0$, and then,

$$\sup_{x\in X} v_n(x)|a_j P_j f(x)| \leq D\frac{|a_j|}{R^j}\|f\|_{v_m} \leq cD\left(\frac{1+R}{2R}\right)^j \|f\|_{v_m}.$$

Since $0 < \frac{1+R}{2R} < 1$, we have that $\sum_{j=0}^{\infty} a_j P_j f$ converges in $H^0_{v_n}(X)$, and thus, on $VH(X)$. Moreover, given a continuous seminorm p on $VH(X)$ and $\{a_j\}_j \in S$, the map $q : H_{v_m}(X) \to [0,\infty[,\ q(f) := \sum_{j=0}^{\infty} |a_j| p(P_j f)$ is continuous. In fact, since $H_{v_n}(X) \hookrightarrow VH(X)$ is continuous, there exists $C_n > 0$ such that $p(f) \leq C_n\|f\|_{v_n}$ for all $f \in H_{v_n}(X)$. Thus, for each $j \in \mathbb{N}$,

$$|a_j|p(P_j f)/C_n \leq |a_j|\|P_j f\|_{v_n} \leq D\frac{|a_j|}{R^j}\|f\|_{v_m} \leq cD\left(\frac{1+R}{2R}\right)^j \|f\|_{v_m}.$$

Hence, since $q : H_{v_m}(X) \to [0,\infty[$ is continuous for every $m \in \mathbb{N}$, q defines a continuous seminorm on $VH(X)$.

Finally, let us show that it is a γ-complete decomposition. As $VH(X)$ is regular, given a sequence $\{Q_j\}_j,\ Q_j \in \mathcal{P}(^jX)$, such that $\{\sum_{j=0}^{k} Q_j\}_k$ is bounded, there exists $m \in \mathbb{N}$ such that $\{\sum_{j=0}^{k} Q_j\}_k$ is included and bounded in $H_{v_m}(X)$. Hence, by Condition (A), there exist $R > 1,\ D,\ C > 0$ and $n \in \mathbb{N},\ n \geq m$, such that for each $k \in \mathbb{N}$,

$$\sum_{j=0}^{k} \|Q_j\|_{v_n} = \sum_{j=0}^{k} \left\| P_j \left(\sum_{i=0}^{k} Q_i \right) \right\|_{v_n} \leq D \frac{R}{R-1} \left\| \sum_{j=0}^{k} Q_j \right\|_{v_m} < C.$$

Therefore, $\{\sum_{j=0}^{k} Q_j\}_k$ converges in $H_{v_n}(X)$, and thus, on $VH(X)$. □

Given a rapidly decreasing continuous function $\eta : [0, \infty[\to]0, 1]$, we are going to consider the sequences of weights $V = \{v_n\}_n$, $v_n(x) = \eta(\|x\|)^n$, and $W = \{w_n\}_n$, $w_n(x) = \eta(n\|x\|)$, $n \in \mathbb{N}$. The real function η can be radially extended to a weight on $\mathbb{C}$ by $\eta(z) = \eta(|z|)$ for $z \in \mathbb{C}$, and its associated weight is given by

$$\widetilde{\eta}(z) = \frac{1}{\sup\{|g(z)| : g \in \mathcal{H}(\mathbb{C}), |g| \leq 1/\eta \text{ on } \mathbb{C}\}}.$$

Proposition 1.1.15 ([56, Proposition 2 and Remark 1]) *Let X be a Banach space and v a weight defined by $v(x) = \eta(\|x\|)$ for $x \in X$. Then, $\widetilde{v}(x) = \widetilde{\eta}(\|x\|)$ for all $x \in X$, and $\widetilde{w}_n(x) = \widetilde{\eta}(\|x\|n)$ for each $n \in \mathbb{N}$, $x \in X$.*

A consequence of this result is that v is essential if and only if η is so. As we have seen in Example 1.1.5, $VH(X)$ is always an algebra. We proceed as in the proof of [56, Proposition 4] in order to characterize when the space $WH(X)$ is an algebra.

Proposition 1.1.16 *$WH(X)$ is an algebra if and only if there exist $k > 1$ and $C > 0$ such that, for all $t \geq 0$,*

$$\eta(kt) \leq C\widetilde{\eta}(t)^2. \tag{1.2}$$

If, furthermore, η is essential, then $WH(X)$ is an algebra if and only if there exist $k > 1$ and $C > 0$ so that, for all $t \geq 0$,

$$\eta(kt) \leq C\eta(t)^2. \tag{1.3}$$

In this case, $VH(X) \hookrightarrow WH(X)$ continuously and there exist positive constants a, b and α such that $\eta(t) \leq ae^{-bt^{\alpha}}$ for all $t \geq 0$.

Proof. Assume $WH(X)$ is an algebra. Therefore, by Theorem 1.1.2, given $m = 1$ there exist $k \in \mathbb{N},\ k > 1$ and $C > 0$ such that

$$\eta(k\|x\|) = w_k(x) \leq C\widetilde{w}_1^2(x) = C\widetilde{\eta}(\|x\|)^2$$

for all $x \in X$. On the other hand, if (1.2) is satisfied, given $m \in \mathbb{N}$, $x \in X$, we have

$$w_{km}(x) = \eta(km\|x\|) \leq C\widetilde{\eta}(m\|x\|)^2 = C\widetilde{w}_m^2(x).$$

By Theorem 1.1.2, $WH(X)$ is an algebra.

If η is essential, condition (1.2) is equivalent to (1.3). In this case, $\eta(t) \leq C^{2^n-1}\eta(t/k^n)^{2^n}$ for all $t > 0$ and $n \in \mathbb{N}$. Hence, given $n \in \mathbb{N}$ take $m \geq k^n$ in order to get

$$w_m(x) = \eta(m\|x\|) \leq C^{2^n-1}\eta(\|x\|)^{2^n} \leq C^{2^n-1}\eta(\|x\|)^n = C^{2^n-1}v_n(x),$$

which yields the continuity of $VH(X) \hookrightarrow WH(X)$.

The last statement is proved in [56, Proposition 4]. We include here the details for the sake of completeness. Since $\eta(t)$ tends to zero as t tends to infinity, we can choose $r > 0$ such that $C\eta(r) < 1$. We have

$$\eta(k^n r) \leq C^{2^n-1}\eta(r)^{2^n} \leq (C\eta(r))^{2^n}$$

for all $n \in \mathbb{N}$ since we can assume $C > 1$. Now, for any $t > r$, let n be such that $k^n r \leq t < k^{n+1}r$. As $2^n = k^{nlog_k 2} \geq \frac{1}{2}\left(\frac{t}{r}\right)^{log_k 2}$, then

$$\eta(t) \leq \eta(k^n r) \leq (C\eta(r))^{2^n} \leq (C\eta(r))^{1/2(t/r)^{log_k 2}},$$

which is bounded by ae^{-bt^α} for all $t > 0$ and some positive constants a, b and α, since there exists $L \geq 0$ such that $e^{-L} = C\eta(r) < 1$. □

Proposition 1.1.17 *The family W satisfies Condition (A') and $\{\mathcal{P}(^k X)\}_{k\geq 0}$ is an S-absolute γ-complete Schauder decomposition of $WH(X)$.*

As in [56, Proposition 6], we characterize when the family V satisfies Condition (A') in terms of the function η. This condition also imposes a relationship between $VH(X)$ and $WH(X)$.

Proposition 1.1.18 *The family V satisfies Condition (A') if and only if there exist $R > 1$ and $\alpha,\ C > 0$ so that, for all $t > 0$,*

$$\eta(t)^\alpha \leq C\eta(Rt). \tag{1.4}$$

In this case, $WH(X) \hookrightarrow VH(X)$ continuously.

Proof. If V satisfies Condition (A'), given $m = 1$, there exist $n \geq 1,\ D > 0$ and $R > 1$ such that $\eta(t)^n \leq D\eta(Rt)$, and (1.4) is satisfied. On the other hand, if (1.4) is satisfied, for any $m \in \mathbb{N}$, choose $n \geq \max(\alpha m, m)$. Hence, $\eta(t)^n \leq \eta(t)^{\alpha m} \leq C^m\eta(Rt)^m$. In this case, we have that for all $n \in \mathbb{N}$, $H_{w_n}(X) \hookrightarrow VH(X)$ is continuous. In fact, if we take $k \in \mathbb{N}$ such that $R^k > n$, from (1.4) we get

$$\eta(t)^{\alpha^k} \leq C^{\frac{\alpha^k-1}{\alpha-1}}\eta(R^k t) \leq C^{\frac{\alpha^k-1}{\alpha-1}}\eta(nt).$$

Hence, take $m > \alpha^k, \ m \in \mathbb{N}$, in order to have $H_{w_n}(X) \hookrightarrow H_{v_m}(X)$ continuously. □

Corollary 1.1.19 *If the function η is essential, $WH(X)$ is an algebra and V satisfies Condition (A'), then $WH(X) = VH(X)$ topologically.*

1.2 Algebra homomorphisms between weighted algebras

The aim of this section is to find conditions to ensure that two weighted algebras defined on two different Banach spaces by the same family of weights are homeomorphic as locally convex algebras. This aim is achieved under certain conditions for a large class of algebras known as Hörmander algebras.

Definition 1.2.1 A *growth condition* is an increasing continuous function $p : [0, \infty[\to [0, \infty[$ with the following properties:

(α) $\varphi : r \to p(e^r)$ is convex,

(β) $\log(1 + r^2) = o(p(r))$ as r tends to ∞,

(γ) there exists $\lambda \geq 0$ with $p(2r) \leq \lambda(p(r) + 1)$ for all $r \geq 0$.

Example 1.2.2 The following functions are easily seen to be growth conditions:

(i) $p(r) = r^d, \ d > 0$,

(ii) $p(r) = \begin{cases} (\log r)^\alpha & \text{if } r \geq 1 \\ 0 & \text{if } r \leq 1 \end{cases}, \quad \alpha > 1.$

The *Young conjugate* $\varphi^* : [0, \infty[\to \mathbb{R}$ of φ is given by

$$\varphi^*(s) := \sup\{sr - \varphi(r), r \geq 0\}.$$

There is no loss of generality to assume that p vanishes on $[0, 1]$. Thus, φ^* has only non-negative values, it is convex and $\varphi^*(r)/r$ is increasing and tends to ∞ as $r \to \infty$. In fact, $\varphi^*(t) \geq nt - \varphi(n)$ for all $n \in \mathbb{N}$. Hence, $\frac{\varphi^*(t)}{t} \geq n - \frac{\varphi(n)}{t}$ for all $n \in \mathbb{N}, t > 0$, and thus, for each $n \in \mathbb{N}$ there exists some $t_n > 0$ such that, for all $t \geq t_n$, $\frac{\varphi^*(t)}{t} \geq n - 1$. Therefore, $\lim_{t\to\infty} \frac{t}{\varphi^*(t)} = 0$. As an immediate consequence, since $n\varphi^*(j/n) > j$ for j big enough, the series $\sum_{j\in\mathbb{N}} \exp(-n\varphi^*(j/n))$ converges for all $n \in \mathbb{N}$.

For a growth condition p, we set $V = \{v_n\}_n$, $v_n(x) = \eta(\|x\|)^n$, $x \in X$, where $\eta(t) = e^{-p(t)}$, and we consider the *Hörmander algebra*

$$A_p = A_p(X) := VH(X).$$

The algebras $A_p(\mathbb{C})$ were introduced for the first time by Hörmander in [83]. They were intensively studied by Berenstein and Taylor in the context of interpolation of entire functions (see e.g. [19]). The study of the locally convex structure of the algebras $A_p(\mathbb{C})$ was initiated by Meise in [99]. We refer the reader to Chapter 2 in [18] for a detailed exposition of the role of the algebras $A_p(\mathbb{C})$ in interpolation.

By Condition (γ) in Definition 1.2.1, the family V satisfies Condition (A'). Indeed, by hypothesis, there exists $\lambda > 0$ such that $-\lambda p(t) \leq \lambda - p(2t)$ for each $t \geq 0$. Hence, there exists $C > 0$ such that $\eta(t)^\lambda \leq C\eta(2t)$ and (1.4) is satisfied. By Condition (β), the weights are rapidly decreasing. In fact, given $n \in \mathbb{N}$, there exists $r_n > 0$ such that $\log(1+r^2)^n \leq p(r)$ for all $r \geq r_n$, and thus, $r^n e^{-p(r)} \leq \left(\frac{r}{1+r^2}\right)^n$, which tends to zero as r tends to infinity.

Corollary 1.2.3 *The sequence $\{\mathcal{P}(^k X)\}_k$ is an S-absolute γ-complete Schauder decomposition of the algebra A_p for every growth condition p.*

The fact that the polynomials on two different spaces of holomorphic functions are isomorphic does not in general imply that the spaces are isomorphic, even having a decomposition like that in Corollary 1.2.3. The aim now is to show that, under certain conditions, for Hörmander algebras this does hold. To see this we give a representation of $A_p(X)$ as a sequence space (Theorem 1.2.7). First, the next lemma is needed. It is a generalization of [100, Proposition 1.10] (see also [99]), where the case $X = \mathbb{C}$ is considered.

Lemma 1.2.4 *Given a growth condition p and $n \in \mathbb{N}$, the following holds:*

(i) If $f \in H_b(X)$ satisfies $\sup_{x\in X} |f(x)| \exp(-np(\|x\|)) = A$, then $\sup_j \|P_j f\| \exp(n\varphi^*(j/n)) \leq A$.

(ii) If $\sup_j \|P_j\| \exp(n\varphi^*(j/n)) = A$, where $P_j \in \mathcal{P}(^j X)$, then $\sum_{j\geq 0} P_j$ converges to some $f \in H_b(X)$ with $\sup_{x\in X} |f(x)| \exp(-np(2\|x\|) \leq 2A$.

Proof. (i) By [103, 7.3], for all $j \in \mathbb{N}$ and $R > 0$,

$$\|P_j f\| \leq \frac{1}{R^j} \sup_{\|x\|\leq R} |f(x)| \leq \frac{\|f\|_{v_n}}{R^j} \sup_{\|x\|\leq R} \exp(np(\|x\|)) \leq \frac{\|f\|_{v_n}}{R^j} \exp(np(R)).$$

Thus, since

$$
\begin{aligned}
\sup_{R>0}\{R^j \exp(-np(R))\} &= \exp(n \sup_{R>0}\{\frac{j}{n}\log R - p(R)\}) \\
&= \exp(n \sup_{R\geq 1}\{\frac{j}{n}\log R - p(R)\}) \\
&= \exp(n \sup_{r\geq 0}\{\frac{j}{n} r - p(e^r)\}) \\
&= \exp(n\varphi^*(j/n)), \qquad (2.5)
\end{aligned}
$$

we get

$$
\sup_j \|P_j f\| \exp(n\varphi^*(j/n)) \leq \|f\|_{v_n}. \qquad (2.6)
$$

(ii) As $\|P_j\| \leq A\exp(-n\varphi^*(j/n))$ for every $j \in \mathbb{N}$ and $\varphi^*(r)/r$ tends to ∞ as $r \to \infty$, by the Cauchy-Hadamard formula, the series $\sum_{j\geq 0} P_j$ converges in $(H_b(X), \tau_b)$ to some f, and thus, $P_j = P_j f$ for every $j \in \mathbb{N}$. Since by (2.5)

$$
\begin{aligned}
|f(x)| \exp(-np(2\|x\|) &\leq \sum_{j\geq 0} |P_j f(x)| \exp(-np(2\|x\|) \\
&\leq \sum_{j\geq 0} \frac{1}{2^j}\|P_j f\| \|2x\|^j \exp(-np(\|2x\|)) \\
&\leq 2\sup_{j\geq 0}\|P_j f\| \exp(n\varphi^*(j/n))
\end{aligned}
$$

for every $x \in X$, the conclusion follows. □

Definition 1.2.5 Given a Banach space X and a growth condition p, for each $n \in \mathbb{N}$ consider the Banach space

$$
\ell_{\infty,n}(X,p) := \left\{ \{P_j\}_j \in \prod_{j\in\mathbb{N}} \mathcal{P}(^j X) : |||\{P_j\}_j|||_n := \sup_j \|P_j\| \exp(n\varphi^*(\frac{j}{n})) < \infty \right\}.
$$

The inclusions $\ell_{\infty,n}(X,p) \hookrightarrow \ell_{\infty,n+1}(X,p)$ are continuous, therefore, we denote by $\kappa_\infty(X,p)$ the locally convex space $ind_n(\ell_{\infty,n}(X,p), |||\cdot|||_n)$ endowed with the inductive limit topology.

Proposition 1.2.6 *Given a Banach space X and a growth condition p, the space $\kappa_\infty(X,p)$ is an algebra with multiplication* $\{P_j\}_j \cdot \{Q_k\}_k := \{\sum_{j+k=h} P_j Q_k\}_h$.

Proof. If we take $\{P_j\}_j$, $\{Q_k\}_k \in \kappa_\infty(X,p)$, then there exist $n,m \in \mathbb{N}$ such that $\{P_j\}_j \in \ell_{\infty,n}(X,p)$ and $\{Q_k\}_k \in \ell_{\infty,m}(X,p)$. By (2.5) and Condition (γ) in Definition 1.2.1, there exists $\lambda \in \mathbb{N}$ such that, for all $h \in \mathbb{N}$,

$$\begin{aligned}
\|\sum_{j+k=h} P_j Q_k\| &\leq \sum_{j+k=h} \|P_j\|\|Q_k\| \\
&\leq CD \sum_{j+k=h} \exp(-n\varphi^*(j/n)) \exp(-m\varphi^*(k/m)) \\
&= CD \sum_{j+k=h} \inf_{r\geq 0} (2r)^{-j} \exp(np(2r)) \inf_{r\geq 0} r^{-k} \exp(mp(r)) \\
&\leq CD \inf_{r\geq 0} r^{-h} \exp(np(2r) + mp(r)) \sum_{0\leq j\leq h} \frac{1}{2^j} \\
&\leq 2CDe^{n\lambda} \inf_{r\geq 0} r^{-h} \exp((m+n\lambda)p(r)) \\
&= 2CDe^{n\lambda} \exp(-(m+n\lambda)\varphi^*(h/(m+n\lambda))),
\end{aligned}$$

where $C = |||\{P_j\}_j|||_n$ and $D = |||\{Q_k\}_k|||_m$. Therefore, $\{\sum_{j+k=h} P_j Q_k\}_h \in \ell_{\infty,m+n\lambda}(X,p)$ and the product is well defined and separately continuous. By Lemma 1.1.1, multiplication is continuous. □

Theorem 1.2.7 *For a growth condition p, the map $\phi : A_p \to \kappa_\infty(X,p)$, $f = \sum_j P_j f \mapsto \{P_j f\}_j$, where $P_j f \in \mathcal{P}({}^j X)$, is an algebra topological isomorphism.*

Proof. By Lemma 1.2.4(i), $|||\{P_j f\}|||_n \leq \|f\|_{v_n}$ for each $f \in H_{v_n}(X)$, $n \in \mathbb{N}$. Then, ϕ is well defined and continuous. On the other hand, by Lemma 1.2.4(ii), if there exists $n \in \mathbb{N}$ such that $|||\{P_j\}_j|||_n < \infty$, then $\sum_{j\geq 0} P_j$ converges to some $f \in H_b(X)$ and

$$\sup_{x\in X} |f(x)| \exp(-np(2\|x\|) \leq 2|||\{P_j\}_j|||_n.$$

By Condition (γ) in Definition 1.2.1, there exists some $\lambda \in \mathbb{N}$ such that $\|f\|_{n\lambda} \leq 2e^{n\lambda}|||\{P_j\}_j|||_n$. Therefore, ϕ is a topological isomorphism. By Proposition 1.2.6, ϕ is also an algebra homomorphism. □

Definition 1.2.8 We say that a growth condition p satisfies the *BMM Condition* if there exists $H > 1$ such that $2p(r) \leq p(Hr) + H$ for all $r \geq 0$. This condition for the weights was introduced by Bonet, Meise and Melikhov in [49].

The next proposition is inspired by [70, Theorem 9], although the proof uses different techniques.

Lemma 1.2.9 *Let X and Y be Banach spaces. Suppose that for each $j \in \mathbb{N}$ there exist an isomorphism $\phi_j : \mathcal{P}(^jX) \to \mathcal{P}(^jY)$ and constants $a, A, b, B > 0$ such that*

$$\|\phi_j(P_j)\| \leq aA^j\|P_j\| \text{ and } \|P_j\| \leq bB^j\|\phi_j(P_j)\| \text{ for all } P_j \in \mathcal{P}(^jX). \tag{2.7}$$

Then, the spaces $\kappa_\infty(X,p)$ and $\kappa_\infty(Y,p)$ are topologically isomorphic for every growth condition p. Moreover, if

$$\phi_k(P_jQ_r) = \phi_j(P_j)\phi_r(Q_r) \text{ for all } P_j \in \mathcal{P}(^jX),\ Q_r \in \mathcal{P}(^rX),\ j+r=k, \tag{2.8}$$

is satisfied, then they are also algebra isomorphic.

On the other hand, if p satisfies the BMM Condition, then we have the converse result: if $\phi : \kappa_\infty(X,p) \to \kappa_\infty(Y,p)$ is a topological isomorphism such that $\phi = (\phi_j)_j$, where $\phi_j : \mathcal{P}(^jX) \to \mathcal{P}(^jY)$, then ϕ_j is a topological isomorphism satisfying (2.7).

Proof. Consider $\phi : \kappa_\infty(X,p) \to \kappa_\infty(Y,p)$, $\phi(\{P_j\}_j) = \{\phi_j(P_j)\}_j$. As $p(2r) = O(p(r))$, given $A > 0$ as in the hypothesis, we can find $k \in \mathbb{N}$, $\lambda \in \mathbb{N}$ and $\mu > 0$ such that $p(Ar) \leq \lambda^k p(r) + \mu$ for every $r \geq 0$. Fix $n \in \mathbb{N}$ and consider $\{P_j\}_j$ such that $\sup_j \|P_j\| \exp(n\varphi^*(j/n)) < \infty$. We get

$$\begin{aligned} |||\{\phi_j(P_j)\}_j|||_{n\lambda^k} &\leq a \sup_j A^j\|P_j\| \sup_{r\geq 0} r^j \exp(-n\lambda^k p(r)) \\ &\leq a \sup_j \|P_j\| \sup_{r\geq 0} (Ar)^j \exp(-np(Ar) + n\mu) \\ &= ae^{n\mu}|||\{P_j\}_j|||_n. \end{aligned}$$

Hence, we have that ϕ is well defined and continuous. From the fact that ϕ_j is a topological isomorphism and from the second inequality in (2.7), ϕ^{-1} is also well defined and continuous. If (2.8) is satisfied, it is easy to check that ϕ is an algebra topological isomorphism.

Let us see now the other direction. Obviously, for any Banach space Z and any growth condition p, we can define $s_j : \mathcal{P}(^jZ) \to \kappa_\infty(Z,p)$ by $s_j(P_j) = \{P_k\}_k$, with $P_k = 0$ if $k \neq j$. Consider ϕ_j for some $j \in \mathbb{N}$ and assume $\phi_j(P_j) = \phi_j(Q_j)$. Since $\phi(s_j(P_j)) = \phi(s_j(Q_j))$ and ϕ is injective, we get $P_j = Q_j$. Observe that it is also onto. Take $Q_j \in \mathcal{P}(^jY)$. As ϕ is onto, there exists $\{P_k\}_k \in \kappa_\infty(X,p)$ such that $\phi(\{P_k\}_k) = s_j(Q_j)$. Thus, there exists $P_j \in \mathcal{P}(^jX)$ such that $\phi_j(P_j) = Q_j$. Observe that the BMM Condition in Definition 1.2.8 is equivalent to the next one: for each $n \in \mathbb{N}$ there exist some $c_n, \delta_n > 0$ such that $p(s) \leq \frac{1}{n}p(c_n s) + \delta_n$ for all $s \geq 0$. In this case, for each $n \in \mathbb{N}$ there are $d_n, \delta_n > 0$ such that

$$\varphi^*(t) \leq n\varphi^*(\frac{t}{n}) + d_n t + \delta_n \ \forall t \geq 0. \tag{2.9}$$

In fact, if we take $d_n > 0$ such that $c_n = e^{d_n}$, since

$$\frac{1}{n}\varphi(s) = \frac{1}{n}p(e^s) \geq p(\frac{e^s}{c_n}) - \delta_n = \varphi(s - d_n) - \delta_n,$$

we have

$$\begin{aligned}\varphi^*(t) &= n\sup_{s\geq 0}\{\frac{t}{n}s - \frac{1}{n}\varphi(s)\} \\ &\leq n\sup_{s\geq d_n}\{\frac{t}{n}(s-d_n) - \varphi(s-d_n)\} + d_n t + \delta_n \\ &\leq n\sup_{s\geq 0}\{\frac{t}{n}s - \varphi(s)\} + d_n t + \delta_n = n\varphi^*(\frac{t}{n}) + d_n t + \delta_n.\end{aligned}$$

On the other hand, since ϕ is continuous, there exist $n \geq 1$ and $C > 0$ such that $|||\phi(s_j(P_j))|||_n \leq C|||s_j(P_j)|||_1$ for all $P_j \in \mathcal{P}(^jX)$, $j \in \mathbb{N}$, i.e.,

$$\|\phi_j(P_j)\| \leq C\|P_j\|\exp(\varphi^*(j) - n\varphi^*(j/n)).$$

Therefore, by (2.9), there exist d_n and $\delta_n > 0$ such that $\|\phi_j(P_j)\| \leq Ce^{\delta_n}(e^{d_n})^j\|P_j\|$ for every $j \in \mathbb{N}$. Moreover, as ϕ^{-1} is continuous, we have that there exist $m \geq 1$ and $D > 0$ such that $|||s_j(P_j)|||_m \leq D|||\phi(s_j(P_j))|||_1$ for all $P_j \in \mathcal{P}(^jX)$, $j \in \mathbb{N}$, i.e.,

$$\|P_j\| \leq D\|\phi_j(P_j)\|\exp(\varphi^*(j) - m\varphi^*(j/m)).$$

Again, (2.9) yields the isomorphism and the requested bound. □

As an immediate consequence of Lemma 1.2.9 and Theorem 1.2.7 we get:

Proposition 1.2.10 *Let X and Y be Banach spaces such that for each $j \in \mathbb{N}$ there exists an isomorphism $\phi_j : \mathcal{P}(^jX) \to \mathcal{P}(^jY)$ satisfying (2.7). Then, $A_p(X)$ and $A_p(Y)$ are topologically isomorphic for every growth condition p. If (2.8) is satisfied, then they are isomorphic also as algebras.*

The BMM Condition is satisfied, for instance, for $p(r) = r^d$ for each $d > 0$, but it is not satisfied for an arbitrary growth condition. If $p(r) = \max(0, (\log r)^2)$, $r \geq 0$, we get that for $r \geq 1$ and $H \geq 1$, the expression $2p(r) - p(Hr) = \log^2 r - 2\log H \log r - \log^2 H$ cannot be bounded.

Now, we are going to see that, under certain conditions on the spaces and with the particularly well behaved growth condition $p(r) = r,\ r \geq 0$, it is enough that the 1-homogeneous polynomials coincide to obtain that the weighted algebras A_p are isomorphic, thus obtaining a Banach-Stone type theorem. Observe that in this case, the Hörmander algebras coincide with the space of holomorphic functions of exponential type $Exp(X)$.

We recall that a (continuous) mapping T between two locally convex spaces X and Y is *affine* if there exist some (continuous) $L \in \mathcal{L}(X, Y)$ and $y_0 = T(0) \in Y$ such that $T(x) = L(x) + y_0$ for all $x \in X$. We say that it is *weakly affine* if for each $y' \in Y'$, $y' \circ T : X \to \mathbb{C}$ is affine.

Remark 1.2.11 If $f : Y \to \mathbb{C}$ is a holomorphic function such that there exist $A, B > 0$ with $|f(y)| \leq A\|y\| + B$ for all $y \in Y$, then it is affine. In fact, as $f \in H(Y)$, there exists a sequence $\{P_j f\}_j$, $P_j f \in \mathcal{P}(^j Y)$, such that $f(y) = \sum_{j \in \mathbb{N}} P_j f(y)$ for all $y \in Y$. By the Cauchy inequalities (see [103, 7.4]),

$$t^j |P_j f(y)| = |P_j f(ty)| \leq \sup_{|\theta|=1} |f(\theta t y)| \leq At\|y\| + B.$$

Hence, $t^j \|P_j f\| \leq At + B$ for all $t \in \mathbb{C}$. For $j \geq 2$, we get $\|P_j f\| = 0$, as otherwise the polynomial $t^j \|P_j f\| - At$ is not bounded on $\mathbb{C}$. Therefore, there exist $P_1 f \in Y'$, $P_0 f = f(0) \in \mathbb{C}$ such that $f(y) = P_1 f(y) + P_0 f$ for all $y \in Y$.

Remark 1.2.12 If a continuous mapping $T : X \to Y$ between two locally convex spaces X and Y is weakly affine, then it is also affine: by hypothesis, for each $y' \in Y'$, there exists $x' \in X'$ and $c(y') \in \mathbb{C}$ such that, for all $x \in X$, $(y' \circ T)(x) = y'(T(x)) = x'(x) + c(y')$, where $c(y') = y'(T(0))$ for all $y' \in Y'$. Consider now the continuous map $S : X \to Y$, $S(x) := T(x) - T(0)$. Observe that it is lineal: given x_1, $x_2 \in X$, then,

$$\begin{aligned} &y'(T(x_1 + x_2) - T(0) - T(x_1) + T(0) - T(x_2) + T(0)) \\ &\quad = y'(T(x_1 + x_2)) - y'(T(x_1)) - y'(T(x_2)) + y'(T(0)) \\ &\quad = x'(x_1) + x'(x_2) + c(y') - x'(x_1) - c(y') - x'(x_2) - c(y') + c(y') = 0. \end{aligned}$$

As this happens for all $y' \in Y'$ and $\langle X, X' \rangle$ is a dual pair, we have that $S(x_1 + x_2) = S(x_1) + S(x_2)$. Take now $\lambda \in \mathbb{C}$. We have

$$\begin{aligned} &y'(T(\lambda x) - T(0) - \lambda T(x) + \lambda T(0)) \\ &\quad = y'(T(\lambda x)) - y'(T(0)) - \lambda y'(T(x)) + \lambda y'(T(0)) \\ &\quad = \lambda x'(x) + c(y') - y'(T(0)) - \lambda x'(x) - \lambda c(y') + \lambda y'(T(0)) = 0. \end{aligned}$$

Therefore, $S(\lambda x) = \lambda S(x)$, and the map S is linear and continuous.

The following lemma is similar to [56, Lemma 3].

Lemma 1.2.13 *Let $A : Exp(X) \to Exp(Y)$ be a continuous algebra homomorphism. Then Ax' is a degree 1 polynomial for all $x' \in X'$ (i.e., A maps linear forms on X to affine forms on Y).*

Proof. A is continuous, then given $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C > 0$ such that, for every $f \in H_{v_n}(X)$,

$$\sup_{y\in Y} |Af(y)|e^{-m\|y\|} \leq C \sup_{x\in X} |f(x)|e^{-n\|x\|}.$$

Take $x' \in X'$ and define $f_M(x) := \sum_{j=0}^{M} \frac{x'(x)^j n^j}{\|x'\|^j j!} \in \mathcal{P}(^M X) \subseteq H_{v_n}(X)$, $M \in \mathbb{N}$. Since A is an algebra homomorphism,

$$\begin{aligned}\sup_{y\in Y} \left| \sum_{j=0}^{M} \frac{(Ax')(y)^j n^j}{\|x'\|^j j!} \right| e^{-m\|y\|} &\leq C \sup_{x\in X} \left| \sum_{j=0}^{M} \frac{x'(x)^j n^j}{\|x'\|^j j!} \right| e^{-n\|x\|} \\ &\leq C \sup_{x\in X} \sum_{j=0}^{M} \frac{\|x\|^j n^j}{j!} e^{-n\|x\|} \\ &\leq C \sup_{x\in X} e^{n\|x\|} e^{-n\|x\|} = C\end{aligned}$$

for every $M \in \mathbb{N}$, hence $\sup_{y\in Y} \left| e^{\frac{nAx'(y)}{\|x'\|}} \right| e^{-m\|y\|} \leq C$. Then, if we take $n = 1$, there exists $K_1 > 0$ such that $\Re(\frac{Ax'}{\|x'\|}(y)) \leq K_1\|y\| + K_2$ for all $y \in Y$, where $\Re$ stands for the real part of a complex number. Also, if $|\lambda| = 1$ we have $\Re(\lambda\frac{Ax'}{\|x'\|}(y)) = \Re(A\frac{\lambda x'}{\|x'\|}(y)) \leq K_1\|y\| + K_2$. This gives $\left|A\frac{x'}{\|x'\|}(y)\right| \leq K_1\|y\| + K_2$ for all $y \in Y$. By Remark 1.2.11, $A\frac{x'}{\|x'\|}$ is affine on y, and so is Ax'. □

By Lemma 1.2.13 and Remark 1.2.12, we obtain the next corollary:

Corollary 1.2.14 *If $\phi : Y \to X$ is a holomorphic mapping and the composition operator $C_\phi : Exp(X) \to Exp(Y)$ given by $C_\phi(f) = f \circ \phi$ is continuous, then ϕ is affine.*

The proof of the next theorem follows the same steps as the proof of [56, Theorem 2].

Theorem 1.2.15 *If $Exp(X) \cong Exp(Y)$ as topological algebras, then $X' \cong Y'$. Moreover, if X and Y are symmetrically regular, or X is regular, then $Exp(X) \cong Exp(Y)$ if and only if $X' \cong Y'$.*

Proof. Let $A : Exp(X) \to Exp(Y)$ be an isomorphism between these spaces. By Lemma 1.2.13, Ax' is affine for every $x' \in X'$. Define $S : X' \to Y'$ by $Sx' = Ax' - Ax'(0_Y)$. S is linear and continuous. Consider also $\tilde{S} : Y' \to X'$ given by $\tilde{S}y' = A^{-1}y' - A^{-1}y'(0_X)$. Since $Ax'(0_Y)$ and $A^{-1}y'(0_X)$ are constants, and so, invariant for both A and A^{-1}, it is easily seen that S and $\tilde{S}$ are inverse one to each other. So, X' and Y' are isomorphic.

Conversely, if X and Y are symmetrically regular and $S : X' \to Y'$ is an isomorphism, by [90, Theorem 4] the mapping $\hat{S}_j : \mathcal{P}(^j X) \to \mathcal{P}(^j Y)$ given by $\hat{S}_j(P) = \widetilde{P} \circ S' \circ J_Y$, where J_Y denotes the canonical embedding of Y into Y'' and $\tilde{P}$ the Aron-Berner extension of P to X'' (see section 0.4), is an isomorphism. Moreover, since the Aron-Berner extension is norm preserving for polynomials, for every $P \in \mathcal{P}(^j X),\ j \in \mathbb{N}$,

$$\|\hat{S}_j(P)\| = \sup_{\|y\| \le 1} \left| \widetilde{P}\left(\frac{S'(y)}{\|S'(y)\|} \right) \right| \|S'(y)\|^j \le \|S\|^j \|P\|,$$

and analogously for S^{-1}. The fact that $\mathcal{P}(^j X)$ and $\mathcal{P}(^j Y)$ are a Schauder decomposition of $Exp(X)$ and $Exp(Y)$ with isomorphisms $\hat{S}_j$, $j \in \mathbb{N}$, satisfying equation (2.7), together with Proposition 1.2.10, gives the topological isomorphism. By the multiplicative nature of the Aron-Berner extension, i.e., $\widetilde{fg} = \tilde{f}\tilde{g}$ for f, g holomorphic functions, also equation (2.8) in Lemma 1.2.9 is satisfied, which gives the conclusion. The other case is analogous, since X or Y regular and $X' \cong Y'$ implies by [52, Proposition 1] that both of them are regular. □

Remark 1.2.16 Observe that given two dual-isomorphic symmetrically regular Banach spaces X and Y (or X regular), we can proceed as in the proof of Theorem 1.2.15 in order to obtain $A_p(X) \cong A_p(Y)$ as topological algebras for a general growth condition p.

An example of two dual-isomorphic regular Banach spaces can be found in [52, Section 2]. They consist on the spaces $X = C[0,1]$ and $Y = c_0(J, C[0,1])$, which satisfy

$$X' = \ell_1\left(J, \ell_1(\mathbb{N}) \oplus_1 L^1[0,1]\right) = \ell_1\left(J \times J, \ell_1(\mathbb{N}) \oplus_1 L^1[0,1]\right) = \ell_1(J, X') = Y',$$

where J is a set having the power of the continuum. By Remark 1.2.16, for every growth condition p, we get $A_p(X) \cong A_p(Y)$ as topological algebras.

1.3 The spectrum

In this section we study the analytic structure of the spectrum of $VH(X)$, denoted by $V\mathfrak{M}(X)$, when X is a symmetrically regular Banach space and $VH(X)$ is an algebra. We show, in the spirit of the results given in [6], [7], [55] and [56], that the spectrum can be viewed as the disjoint union of analytic copies of X'', these copies being the connected components of $V\mathfrak{M}(X)$. The copies of the bidual are constructed laying a copy of X'' around every element ϕ in the spectrum. We consider, for each $z \in X''$, the homomorphism that on $f \in VH(X)$ takes the value $\phi(x \in X \rightsquigarrow \widetilde{f}(J_X x + z))$, where J_X denotes the canonical embedding of X into X'' and $\widetilde{f}$ the Aron-Berner extension of f to X''. See Section 0.4, [5], [63, Chapter 6.2] and [66, Section 6] for more background about Aron-Berner extensions. If we move z in X'', we obtain a subset of the spectrum homeomorphic to X''. So, we have to show that the function $x \in X \rightsquigarrow \tilde{f}(J_X x + z)$ belongs to $VH(X)$.

Definition 1.3.1 Following [55, Section 1] we say that a sequence of weights $V = \{v_n\}_n$ has *good local control* if for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$, $n \geq m$, such that, for each $s > 0$ there exists $C_s > 0$ with $v_n(x) \leq C_s v_m(x+y)$ for all $x, y \in X$ with $\|y\| \leq s$.

Proposition 1.3.2 *If a sequence of weights $V = \{v_n\}_n$ satisfies Condition (A'), then it has good local control.*

Proof. By Condition (A'), given $m \in \mathbb{N}$ there exist $R > 1$, $D > 0$ and $n \in \mathbb{N}$, $n \geq m$, such that $v_n(x) \leq D v_m(Rx)$ for each $x \in X$. Fix $s > 0$ and consider $y \in X$ such that $\|y\| \leq s$. Therefore, if $\|x\| > \frac{s}{R-1} \geq \frac{\|y\|}{R-1}$, then $\|x+y\| \leq R\|x\|$ and $v_n(x) \leq D v_m(Rx) \leq D v_m(x+y)$. On the other hand, if $\|x\| \leq \frac{s}{R-1}$, then $\|x+y\| \leq \frac{R}{R-1}s$. Hence, there exists $C_s > 0$ such that $\sup_{x \in X} \frac{v_n(x)}{v_m(x+y)} \leq C_s$ for each $y \in X, \|y\| \leq s$. □

Lemma 1.3.3 *If the sequence of weights $V = \{v_n\}_n$ has good local control, then the mapping $VH(X) \to VH(X)$ given by $f \mapsto f(\cdot + y)$ is well defined and continuous for every fixed $y \in X$. Moreover, given $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$, $n \geq m$, such that for all $s > 0$ there exists a constant $C_s > 0$ such that $\|f(\cdot + y)\|_{v_n} \leq C_s \|f\|_{v_m}$ for each $y \in X$, $\|y\| \leq s$.*

Proof. By Definition 1.3.1, given $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$, $n \geq m$, such that, for each $s > 0$ there exists $C_s > 0$ with $v_n(x) \leq C_s v_m(x+y)$ for all $x, y \in X$ with $\|y\| \leq s$. Therefore, for all $y \in X$ with $\|y\| \leq s$,

$$\begin{aligned}\|f(\cdot \ + y)\|_{v_n} &= \sup_{x \in X} \frac{v_n(x)}{v_m(x+y)} |f(x+y)| v_m(x+y) \\ &\leq \sup_{x \in X} \frac{v_n(x)}{v_m(x+y)} \|f\|_{v_m} \leq C\|f\|_{v_m},\end{aligned}$$

and the map $H_{v_m}(X) \to VH(X)$, $f \mapsto f(\cdot + y)$ is well defined and continuous. □

In what follows, we *consider a family of weights V satisfying Condition (A') and such that $VH(X)$ is an algebra.* Then the polynomials form a Schauder decomposition. In order to study the spectrum $V\mathfrak{M}(X)$, we follow the notation of [63, Section 6.3] for $\mathfrak{M}_b(X)$.

As linear functionals belong to $VH(X)$ because the weights are rapidly decreasing, we define

$$\pi : V\mathfrak{M}(X) \to X'' \ , \ \pi(\phi) = \phi|_{X'}.$$

Since the Aron-Berner extension $AB : H_b(X) \to H_b(X'')$, $f \to \widetilde{f}$ is continuous, we can also define

$$\delta : X'' \to V\mathfrak{M}(X), \ \delta(z)(f) = \tilde{f}(z).$$

As $\pi(\delta(z))(x') = \delta(z)_{|X'}(x') = z(x')$ for every $x' \in X'$, π is an onto map. Each $f \in VH(X)$ defines a mapping $\hat{f} : V\mathfrak{M}(X) \to \mathbb{C}$ by $\hat{f}(\phi) = \phi(f)$. It is called the *Gel'fand transform* of f.

Proposition 1.3.4 *Given a weight v, the Aron-Berner extension is an isometry from $(\mathcal{P}(^kX), \| \ \|_v)$ into $(\mathcal{P}(^kX''), \| \ \|_v)$ for all $k \in \mathbb{N}$. Therefore, given a sequence of weights V satisfying Condition (A), the Aron-Berner extension $AB : VH(X) \to VH(X'')$, $f = \sum_k P_k f \mapsto \sum_k \widetilde{P_k f}$ is continuous, linear and multiplicative.*

Proof. If $P \in \mathcal{P}(^kX)$, clearly $\|P\|_{v_n} \leq \|\widetilde{P}\|_{v_n}$. Davie and Gamelin showed in [60] that given $z \in X''$ we can choose $\{x_\alpha\}_\alpha \subseteq X$ in such a way that $\|x_\alpha\| \leq \|z\|$ and

$$v_n(z)\widetilde{P}(z) = \lim_\alpha v_n(z)|P(x_\alpha)| \leq \sup_\alpha v_n(x_\alpha)|P(x_\alpha)| \leq \|P\|_{v_n}.$$

Therefore, $\|P\|_{v_n} = \|\widetilde{P}\|_{v_n}$. This implies that for each $m \in \mathbb{N}$, the map $AB : H_{v_m}(X) \to VH(X'')$, $f \mapsto \widetilde{f}$ is continuous. In fact, given $f \in H_{v_m}(X)$, by Condition (A), there exist $n \in \mathbb{N}, n > m$ and $R > 1$ such that $\sum_{k=0}^\infty P_k f$ converges to f in $H_{v_n}(X)$. Therefore, we obtain

$$\|\widetilde{f}\|_{v_n} \leq \sum_{k=0}^\infty \|\widetilde{P_k(f)}\|_{v_n} = \sum_{k=0}^\infty \|P_k(f)\|_{v_n} \leq \frac{R}{R-1}\|f\|_{v_m}$$

and we get the continuity. By Proposition 0.4.3, it is also multiplicative. □

For a fixed $z \in X''$, we consider $\tau_z(x) = J_X x + z$ for $x \in X$. Since there is no risk of confusion we also denote by $\tau_z : VH(X) \to VH(X)$ the mapping given by $(\tau_z f)(x) = \tilde{f}(J_X x + z) = (\tilde{f} \circ \tau_z)(x)$. This is well defined, multiplicative and continuous because the Aron-Berner extension $AB : VH(X) \to VH(X'')$ is so by Proposition 1.3.4 and we can apply Lemma 1.3.3 to X'' in order to get $\widetilde{f} \circ \tau_z \in VH(X'')$. Thus, $(\widetilde{f} \circ \tau_z)_{|X}$ belongs to $VH(X)$. As a consequence, we get $\phi \circ \tau_z \in V\mathfrak{M}(X)$ for every $\phi \in V\mathfrak{M}(X)$ and $z \in X''$. Since X is symmetrically regular and $VH(X) \hookrightarrow H_b(X)$ continuously, we can apply [63, Lemma 6.28] in order to get $\tau_{z+w}f = (\tau_z \circ \tau_w)f$ for all $f \in VH(X)$ and for all $z, w \in X''$. If $z \in X''$ and $x' \in X'$, then $\tau_z(x') = x' + z(x')$, where $z(x')$ is the constant mapping on X, and $\pi(\phi \circ \tau_z) = \pi(\phi) + z$ for all $\phi \in V\mathfrak{M}(X)$.

For each $\phi \in V\mathfrak{M}(X)$ and $\varepsilon > 0$ we consider

$$V_{\phi,\varepsilon} = \{\phi \circ \tau_z : z \in X'',\ \|z\| < \varepsilon\}.$$

As in [63, Section 6.3], we obtain that $V_\phi := \{V_{\phi,\varepsilon}\}_{\varepsilon>0}$ forms a neighborhood basis at ϕ for a Hausdorff topology on $V\mathfrak{M}(X)$. In fact, if $\phi_1 \in V_{\phi,\varepsilon}$ then $\phi_1 = \phi \circ \tau_z$ for some $z \in X''$, $\|z\| < \varepsilon$. Let $\delta = \varepsilon - \|z\|$. If $\omega \in X''$, $\|\omega\| < \delta$, since associativity is easily checked, then

$$\phi_1 \circ \tau_\omega = (\phi \circ \tau_z) \circ \tau_\omega = \phi \circ (\tau_z \circ \tau_\omega) = \phi \circ \tau_{z+\omega} \in V_{\phi,\varepsilon}$$

and hence $V_{\phi_1,\delta} \subseteq V_{\phi,\varepsilon}$. Let us see that the topology is Hausdorff: consider ϕ, $\phi_1 \in V\mathfrak{M}(X)$. Let $2\varepsilon = \|\pi(\phi) - \pi(\phi_1)\|$ if $\pi(\phi) \neq \pi(\phi_1)$ and suppose $\phi_2 \in V_{\phi,\varepsilon} \cap V_{\phi_1,\varepsilon}$. We have $\phi_2 = \phi \circ \tau_z = \phi_1 \circ \tau_{z_1}$ for some $z, z_1 \in X''$, $\|z\| < \varepsilon$ and $\|z_1\| < \varepsilon$. Hence

$$\pi(\phi) + z = \pi(\phi \circ \tau_z) = \pi(\phi_1 \circ \tau_{z_1}) = \pi(\phi_1) + z_1$$

and

$$2\varepsilon = \|\pi(\phi) - \pi(\phi_1)\| = \|z - z_1\| < 2\varepsilon.$$

This shows that $V_{\phi,\varepsilon} \cap V_{\phi_1,\varepsilon} = \emptyset$. Now suppose $\pi(\phi) = \pi(\phi_1)$. If $\phi_2 \in V_{\phi,r} \cap V_{\phi_1,s}$ for r, s positive then $\phi_2 = \phi \circ \tau_z = \phi_1 \circ \tau_{z_1}$ for some $z, z_1 \in X''$ and

$$\pi(\phi) + z = \pi(\phi \circ \tau_z) = \pi(\phi_1 \circ \tau_{z_1}) = \pi(\phi_1) + z_1.$$

Since $\pi(\phi) = \pi(\phi_1)$ this implies $z = z_1$ and

$$\phi = \phi \circ \tau_z \circ \tau_{-z} = \phi_1 \circ \tau_{z_1} \circ \tau_{-z} = \phi_1 \circ \tau_{z_1 - z} = \phi_1.$$

So, the topology is Hausdorff and $\pi(\phi) = \pi(\phi_1)$ implies $\phi = \phi_1$ or $V_{\phi,r} \cap V_{\phi_1,s} = \emptyset$ for all $r, s > 0$. The mapping π restricted to $V_{\phi,\varepsilon}$ has the form $\pi(\phi \circ \tau_z) = \pi(\phi) + z$ and clearly maps $V_{\phi,\varepsilon}$ homeomorphically onto the open ball in X'' centred at $\pi(\phi)$ and radius ε. Hence, π is a local homeomorphism. Thus, $(V\mathfrak{M}(X), \pi)$ is a Riemann domain spread over X''. In particular, $V\mathfrak{M}(X)$ is a Riemann analytic manifold spread over X''.

Clearly, $V_{\phi,\infty} := \cup_{\varepsilon>0} V_{\phi,\varepsilon}$ is mapped homeomorphically onto X'' and it is an open subset of $V\mathfrak{M}(X)$. Proceeding as in [63, page 430], we also get that it is closed: if $\phi \circ \tau_{z_n} \to \phi_1 \in V\mathfrak{M}(X)$ as $n \to \infty$, then $\pi(\phi) + z_n \to \pi(\phi_1)$ as $n \to \infty$ and $\{z_n\}_n$ converges to some point $z \in X''$. Hence $\phi \circ \tau_z = \lim_n \phi \circ \tau_{z_n} = \phi_1$ and $\phi_1 \in V_{\phi,\infty}$.

Observe that, if $V_{\phi_1,\infty} \cap V_{\phi_2,\infty} \neq \emptyset$, with $\phi_1, \phi_2 \in V\mathfrak{M}(X)$, then these two copies of X'' must coincide. In fact, if there exists $\phi \in V\mathfrak{M}(X)$ such that $\phi = \phi_1 \circ \tau_{z_1} = \phi_2 \circ \tau_{z_2}$, $z_1, z_2 \in X''$, then $\phi_2 = \phi_2 \circ \tau_{z_2} \circ \tau_{-z_2} = \phi_1 \circ \tau_{z_1} \circ \tau_{-z_2} = \phi_1 \circ \tau_{z_1 - z_2} \in V_{\phi_1,\infty}$. Then, $V_{\phi_2,\infty} \subseteq V_{\phi_1,\infty}$. Analogously, we get the other inclusion. Therefore, $V\mathfrak{M}(X)$ can be viewed as the disjoint union of analytic copies of X'', these copies being the connected components of $V\mathfrak{M}(X)$. Hence, we can visualize the spectrum as a collection of sheets laying one over the other in such a way that all the points in a vertical line are projected by π on the same element of X''.

Since the weights are rapidly decreasing, the polynomials belong to $VH(X)$ and the inclusion $VH(X) \hookrightarrow H_b(X)$ has dense range. Hence, we have a one to one identification $\mathfrak{M}_b(X) \hookrightarrow V\mathfrak{M}(X)$, but we do not know if they coincide.

In the case of $H_b(X)$ $(HW(X))$, the Gel'fand transform $\hat{f}$ of each $f \in H_b(X)$ $(f \in HW(X))$ is holomorphic on $\mathfrak{M}_b(X)$ $(\mathfrak{M}W(X))$ and belongs, in some sense, to $H_b(\mathfrak{M}_b(X))$ $(HW(\mathfrak{M}W(X)))$ (see [63, Proposition 6.30] and [56, Theorem 1]). Now, we show, using the techniques given in [55], that the analogous situation holds in this setting, i.e., we can extend each $f \in VH(X)$ to a holomorphic function on $V\mathfrak{M}(X)$. By the Riemann domain structure of $V\mathfrak{M}(X)$, "holomorphic" means that $\hat{f} \circ (\pi|_{V_{\phi,\infty}})^{-1}$ is holomorphic on X'' for all $\phi \in V\mathfrak{M}(X)$. Observe that the restriction of $\hat{f}$ to X'' coincides with the Aron-Berner extension $\tilde{f}$.

In what follows, given $f \in VH(X)$, we denote by P_k and $P_k(x_0)$ the k-homogeneous polynomials of its Taylor series centred at 0 and $x_0 \in X$, respectively, and we denote by A_k and $A_k(x_0)$ the unique symmetric k-linear mappings such that $A_k(x, \ldots, x) = P_k(x)$ and $A_k(x_0)(x, \ldots, x) = P_k(x_0)(x)$, $x \in X$.

Proceeding similarly to [55, page 5, Lemma 2.10 and Lemma 2.11], we obtain the next proposition. The "local Aron-Berner extension" defined there is just the Aron-Berner extension for the entire case.

Proposition 1.3.5 *If V is a family of weights satisfying Condition (A'), then for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$, $n \geq m$, such that, for every $s > 0$ there exists a constant $C_s > 0$ satisfying:*

(i) $\widetilde{P_k(\cdot)}(z) \in H_{v_n}(X)$ *with* $\|\widetilde{P_k(\cdot)}(z)\|_{v_n} \leq C_s \|f\|_{v_m}$ *for every* $k \in \mathbb{N}$, $f \in H_{v_m}(X)$ *and* $z \in X''$, $\|z\| \leq s$.

(ii) $\sum_k \|\widetilde{P_k(\cdot)}(z)\|_{v_n} \leq 2C_s \|f\|_{v_m}$ *for every* $f \in H_{v_m}(X)$ *and every* $z \in X''$, $\|z\| \leq s/2$. *Therefore,* $\sum_k \widetilde{P_k(\cdot)}(z)$ *converges absolutely and uniformly in* $H_{v_n}(X)$ *on the bounded subsets of* X''.

(iii) $\widetilde{A_k(\cdot)}(z_1, \ldots, z_k) \in H_{v_n}(X)$ *with*

$$\|\widetilde{A_k(\cdot)}(z_1, \ldots, z_k)\|_{v_n} \leq C_s \frac{k^k}{k! s^k} \|f\|_{v_m} \|z_1\| \ldots \|z_k\|$$

for each $z_1, \ldots, z_k \in X''$, $k \in \mathbb{N}$, *and* $f \in H_{v_m}(X)$.

Proof. Given $m \in \mathbb{N}$, by Proposition 1.3.2, there exists $n \in \mathbb{N}$, $n \geq m$, such that, for each $s > 0$, there exists $C_s > 0$ with $v_n(x) \leq C_s v_m(J_X x + z)$ for all $x \in X$, $z \in X''$, $\|z\| \leq s$.

(i) By [7, page 552], $\widetilde{P_k(\cdot)}(z) \in H_b(X)$ for all $z \in X''$ and $\widetilde{f}(z) = \sum_k \widetilde{P_k(x)}(z - x)$ for every $f \in H_{v_m}(X)$ and $x \in X$. By the Cauchy integral formula, we get

$$v_n(x)|\widetilde{P_k(x)}(z)| \leq \frac{1}{2\pi} \int_{|\lambda|=1} v_n(x)|\widetilde{f}(x + \lambda z)| d|\lambda|.$$

Fix now $x \in X$ and $z \in X''$ with $\|z\| \leq s$. By [60, Lemma page 355], there exists some $\{x_\alpha\}_\alpha \subseteq X$ with $\|x_\alpha\| \leq \|z\| < s$ for all α such that $|\widetilde{f}(x + \lambda z)| = \lim_\alpha |f(x + \lambda x_\alpha)|$. Then, we have

$$\begin{aligned} v_n(x)|\widetilde{f}(x + \lambda z)| &= \lim_\alpha v_n(x)|f(x + \lambda x_\alpha)| \leq \sup_\alpha v_n(x)|f(x + \lambda x_\alpha)| \\ &\leq \sup_\alpha C_s v_m(x + \lambda x_\alpha)|f(x + \lambda x_\alpha)| \leq C_s \|f\|_{v_m}. \end{aligned}$$

This yields $v_n(x)|\widetilde{P_k(x)}(z)| \leq C_s \|f\|_{v_m}$ for all $x \in X$ and all $z \in X''$, $\|z\| \leq s$.

(ii) By (i),

$$\begin{aligned} \|\widetilde{P_k(\cdot)}(z)\|_{v_n} &= \sup_{x \in X} v_n(x) \left| \widetilde{P_k(x)} \left(\frac{s}{\|z\|} z \right) \left(\frac{\|z\|}{s} \right)^k \right| \\ &= \left\| \widetilde{P_k(\cdot)} \left(\frac{s}{\|z\|} z \right) \right\|_{v_n} \left(\frac{\|z\|}{s} \right)^k \leq \frac{C_s}{2^k} \|f\|_{v_m} \end{aligned}$$

for every $z \in X''$ with $\|z\| \leq s/2$. Therefore, the mapping $x \rightsquigarrow \sum_k \widetilde{P_k(x)}(z)$ is absolutely and uniformly convergent in $H_{v_n}(X)$ on each bounded subset of X''.

(iii) By the Polarization formula, given $f \in H_{v_m}(X)$,

$$\begin{aligned} \widetilde{A_k(x)}(z_1, \ldots, z_k) &= \frac{1}{2^k k!} \sum_{\varepsilon_l = \pm 1} \varepsilon_1 \ldots \varepsilon_k \widetilde{P_k(x)}(\varepsilon_1 z_1 + \cdots + \varepsilon_k z_k) \\ &= \frac{k^k}{2^k k!} \sum_{\varepsilon_l = \pm 1} \varepsilon_1 \ldots \varepsilon_k \widetilde{P_k(x)} \left(\frac{\varepsilon_1 z_1 + \cdots + \varepsilon_k z_k}{k} \right), \end{aligned}$$

and so, $\widetilde{A_k(\cdot)}(z_1,\dots,z_k) \in H_{v_n}(X)$. Moreover, by (i),

$$v_n(x)|\widetilde{A_k(x)}(z_1,\dots,z_k)| \leq C_s \frac{k^k}{2^k k!} \sum_{\varepsilon_l=\pm 1} \|f\|_{v_m}$$

for all $\|z_1\|,\dots,\|z_k\| \leq s$, $k \in \mathbb{N}$, since $\left\|\frac{\varepsilon_1 z_1 + \dots \varepsilon_k z_k}{k}\right\| < s$. For general $z_1,\dots,z_k \in X''$, the conclusion follows easily. □

Lemma 1.3.6 *Let V be a family of weights satisfying Condition (A'). Then, given $f \in VH(X)$ and $\phi \in V\mathfrak{M}(X)$, the mapping $z \rightsquigarrow \phi(\widetilde{P_k(\cdot)}(z))$ belongs to $\mathcal{P}(^kX'')$ for every $k \in \mathbb{N}$.*

Proof. Given $f \in VH(X)$ there exists $m \in \mathbb{N}$ such that $f \in H_{v_m}(X)$. By Proposition 1.3.5, the mapping $z \rightsquigarrow \phi(\widetilde{P_k(\cdot)}(z))$ is well defined since there exists some $n \in \mathbb{N}$ with $\widetilde{P_k(\cdot)}(z) \in H_{v_n}(X)$ for every $z \in Z''$. Let $\widetilde{A_k(x)} : X'' \times \dots \times X'' \to \mathbb{C}$ be the symmetric k-linear mapping associated to $\widetilde{P_k(x)}$. Thus, since by Proposition 1.3.5, $x \rightsquigarrow \widetilde{A_k(x)}(z_1,\dots,z_k)$ belongs to $H_{v_n}(X)$ for all $z_1,\dots,z_k \in X''$ and $\phi(\widetilde{A_k(\cdot)}(z,\dots,z)) = \phi(\widetilde{P_k(\cdot)(z)})$ for every $z \in X''$, it is enough to show that the mapping $(z_1,\dots,z_k) \rightsquigarrow \phi(\widetilde{A_k(\cdot)}(z_1,\dots,z_k))$ belongs to $\mathcal{L}(^kX'')$. k-linearity is clear. Moreover, using Proposition 1.3.5 and the fact that $\phi : H_{v_n}(X) \to \mathbb{C}$ is continuous, there exists $M > 0$ such that, for every $s > 0$ there exists $C_s > 0$ with

$$\begin{aligned}\sup_{\|z_i\|\leq 1, 1\leq i\leq k} |\phi(\widetilde{A_k(\cdot)}(z_1,\dots,z_k))| &\leq M \sup_{\|z_i\|\leq 1, 1\leq i\leq k} \|\widetilde{A_k(\cdot)}(z_1,\dots,z_k)\|_{v_n} \\ &\leq MC_s \frac{k^k}{k! s^k} \|f\|_{v_m}.\end{aligned}$$

Then, the mapping $z \rightsquigarrow \phi(\widetilde{P_k(\cdot)}(z))$ belongs to $\mathcal{P}(^kX'')$ for every $k \in \mathbb{N}$. □

Theorem 1.3.7 *Let X be a symmetrically regular Banach space and V a family of weights satisfying Condition (A') and such that $VH(X)$ is an algebra. Then, for every $f \in VH(X)$, the Gel'fand transform $\hat{f} : V\mathfrak{M}(X) \to \mathbb{C}$ is a holomorphic function of bounded type.*

Proof. Given $f \in VH(X)$ there exists $m \in \mathbb{N}$ such that $f \in H_{v_m}(X)$. For any $\phi \in V\mathfrak{M}(X)$ and $z \in X''$ we have

$$(\hat{f} \circ (\pi|_{V_{\phi,\infty}})^{-1})(\pi(\phi) + z) = \hat{f}(\phi \circ \tau_z) = (\phi \circ \tau_z)(f) = \phi(\tau_z f).$$

We shall prove that the mapping $z \in X'' \rightsquigarrow \phi(\tau_z f) = \phi(x \mapsto \widetilde{f}(J_X x + z))$ is holomorphic. Let us consider the Taylor series at zero of $f = \sum_k P_k$, where

$P_k \in \mathcal{P}(^k X)$ for all $k \in \mathbb{N}$. As in [7, page 552], we get a pointwise representation

$$(\tau_z f)(x) = \widetilde{f}(J_X x + z) = \sum_{j=0}^{\infty} \widetilde{P}_j(J_X x + z) = \sum_{k=0}^{\infty} \widetilde{P_k(x)}(z).$$

By Proposition 1.3.5, there exists $n \in \mathbb{N}$, $n \geq m$, such that the last series is absolutely convergent in $H_{v_n}(X)$. Then, $\phi(\tau_z f) = \sum_{k=0}^{\infty} \phi(\widetilde{P_k(\cdot)}(z))$ is well defined and, by Lemma 1.3.6, $\phi(\widetilde{P_k(\cdot)}(z))$ belongs to $\mathcal{P}(^k X'')$. Moreover, for each $s > 0$ and $z \in X''$ with $\|z\| \leq s/2$,

$$\sum_{k=0}^{\infty} |\phi(\widetilde{P_k(\cdot)}(z))| \leq M \sum_{k=0}^{\infty} \|\widetilde{P_k(\cdot)}(z)\|_{v_n} \leq 2MC_s \|f\|_{v_m}$$

for some $M > 0$ and C_s the constant in Proposition 1.3.5. Therefore, the series converges uniformly on the bounded subsets of X'', and then, $z \in X'' \rightsquigarrow \phi(\tau_z f)$ is a holomorphic function of bounded type. □

Remark 1.3.8 We have shown that $\hat{f} \in H_b(V\mathfrak{M}(X))$ for every $f \in VH(X)$. If we assume that for each $m, n \in \mathbb{N}$, $n \geq m$, there exist $q \in \mathbb{N}$ and $\delta > 0$ such that $\sup_{x \in X} \frac{v_n(x)}{v_m(J_X x + z)} \leq \frac{\delta}{v_q(z)}$ for every $z \in X''$, then we can even get that in some sense it belongs to $VH(V\mathfrak{M}(X))$. To be more precise, let $\phi \in V\mathfrak{M}(X)$ and $f \in H_{v_m}(X)$. By the proof of Theorem 1.3.7, for every $s > 0$ and every $z \in X''$ with $\|z\| \leq s/2$,

$$|\hat{f}(\phi \circ \tau_z)| = |\phi(\tau_z f)| \leq 2MC_s \|f\|_{v_m} < \infty,$$

where M and $C_s > 0$ are the constants in the proof. As $C_s > 0$ in Lemma 1.3.3 can be improved by $\sup_{x \in X} \frac{v_n(x)}{v_m(J_X x + z)}$ for every $z \in X''$, by hypothesis we get

$$v_q(z)|\hat{f}(\phi \circ \tau_z)| \leq 2M v_q(z) \sup_{x \in X} \frac{v_n(x)}{v_m(J_X x + z)} \|f\|_{v_m} \leq 2M\delta \|f\|_{v_m}$$

for each $z \in X''$. Therefore, $\hat{f}$ belongs to VH of each copy of X'' in the spectrum. This assumption is satisfied, for instance, when $V = \{v^n\}$, $v(x) = \eta(\|x\|)$, and η is a function such that there is $\alpha > 0$ with $\eta(s)\eta(t) \leq \alpha\eta(s+t)$. In fact,

$$\frac{\eta(\|x\|)^n}{\eta(\|J_X x + z\|)^m} \leq \alpha^m \frac{\eta(\|x\|)^n}{\eta(\|x\|)^m \eta(\|z\|)^m} \leq \frac{\alpha^m}{\eta(\|z\|)^m}.$$

A simple example of such a function is $\eta(t) = e^{-t},\ t \geq 0$.

Every algebra homomorphism $A : Exp(X) \to Exp(Y)$ induces a mapping $\theta_A : V\mathfrak{M}(Y) \to V\mathfrak{M}(X)$ defined by $\theta_A(\phi) = \phi \circ A$. By the analytic structure of $V\mathfrak{M}(Y)$,

θ_A is continuous if and only if θ_A maps sheets into sheets continuously. So, proceeding as in [56, Theorem 3], we characterize the continuity of θ_A. We include the proof for the sake of completeness.

Theorem 1.3.9 *Let X and Y be symmetrically regular Banach spaces and let $A : Exp(X) \to Exp(Y)$ be an algebra homomorphism. The following are equivalent.*

(i) There exist $\phi \in V\mathfrak{M}(X)$ and $T : Y'' \to X''$ affine and w^*-w^*-continuous so that $Af(y) = \phi(\tilde{f}(\cdot + Ty))$ for all $y \in Y$.

(ii) θ_A maps sheets into sheets.

(iii) θ_A maps Y'' into a sheet.

In particular, θ_A is continuous if and only if it is continuous on Y''.

Proof. Let us note first that $T : Y'' \to X''$ is affine and w^*-w^*-continuous if and only if there exist $R : X' \to Y'$ linear and continuous and $x_0'' \in X''$ so that $T(y'') = R'(y'') + x_0''$.

We begin by assuming that (i) holds. If A has such a representation, let us see that then the Aron-Berner extension of Af is of the form

$$\widetilde{Af}(y'') = \phi(\widetilde{f}(\cdot + Ty'')). \tag{3.10}$$

Indeed, let $h(z) = \phi(f(\cdot + z)) = \phi(x \to f(x + z))$ for $z \in X$. By [6, Theorem 6.12], its Aron-Berner extension is given by $\widetilde{h}(x'') = \phi(\widetilde{f}(\cdot + x'')) = \phi(x \to \widetilde{f}(x + x''))$. We define now $\overline{h}(y'') = \phi(\widetilde{f}(\cdot + Ty''))$. Then

$$\overline{h}(y'') = (\widetilde{h} \circ T)(y'') = \widetilde{h}(R'(y'') + x_0'') = \left(\tau_{x_0''}(\widetilde{h}) \circ R'\right)(y'').$$

Since $\widetilde{h}$ is the Aron-Berner extension of a function, $\tau_{x_0''}(\widetilde{h})$ is the Aron-Berner extension of some other function ([6, Theorem 6.12]). On the other hand, by [6, Lemma 9.1], the composition of an Aron-Berner extension with the transpose of a linear mapping is again the Aron-Berner extension of some function. Hence, $\overline{h} = \tau_{x_0''}(\widetilde{h}) \circ R'$ is the Aron-Berner extension of a function; but $\overline{h}$ coincides with Af on Y, therefore $\overline{h} = \widetilde{Af}$ and (3.10) holds.

Now, to see that θ_A maps sheets into sheets it is enough to find $S : Y'' \to X''$ such that $\theta_A(\psi \circ \tau_{y''}) = (\theta_A\psi) \circ \tau_{Sy''}$. We define $Sy'' = Ty'' - x_0''$. First we have

$$\begin{aligned}
\theta_A(\psi \circ \tau_{y''})(f) &= (\psi \circ \tau_{y''})(Af) = \psi[y \to \widetilde{Af}(y + y'')] \\
&= \psi[y \to \phi[x \to \widetilde{f}(x + T(y + y''))]] \\
&= \psi[y \to \phi[x \to \widetilde{f}(x + Ty + Sy'')]].
\end{aligned}$$

Let us call $g(x) = \widetilde{f}(x + Sy'')$. As above, we can check that its Aron-Berner extension is $\widetilde{g}(x'') = \widetilde{f}(x'' + Sy'')$. With this we obtain

$$\begin{aligned}(\theta_A\psi \circ \tau_{Sy''})(f) &= \theta_A\psi[x \to \widetilde{f}(x + Sy'')] = \psi[y \to Ag(y)] \\ &= \psi[y \to \phi[x \to \widetilde{g}(x + Ty)]] = \psi[y \to \phi[x \to \widetilde{f}(x + Ty + Sy'')]]\end{aligned}$$

and (ii) holds. Clearly, (ii) implies (iii).

Let us suppose that θ_A maps Y'' into a single sheet. Hence, $\theta_A(\delta_{y''}) = \theta_A(\delta_0) \circ \tau_{Sy''} = \phi \circ \tau_{Sy''}$ for some Sy'' in X''. This means that $\delta_{y''}(Af) = (\phi \circ \tau_{Sy''})(f)$ for all $f \in VH(X)$ and from this $\widetilde{Af}(y'') = \phi(\widetilde{f}(\cdot + Sy''))$. Let us see that S is affine. Let $x' \in X'$, then Ax' is a degree one polynomial and so is $\widetilde{Ax'}$. Also,

$$\begin{aligned}\widetilde{Ax'}(y'') &= \phi[x \to AB(x')(x + Sy'')] \\ &= \phi[x \to x'(x) + Sy''(x')] = \phi(x') + S(y'')(x'). \qquad (3.11)\end{aligned}$$

This shows that S is w^* affine; hence, in a similar way than in Remark 1.2.12, S is affine.

Let us finish by proving that S is w^*-w^*-continuous. Indeed, let $\{y''_\alpha\}_\alpha$ be a net w^*-converging to y''. By Lemma 1.2.13 we have, for every $x' \in X'$, $Ax' = y'_{x'} + \lambda_{x'}$. Then $\widetilde{Ax'}(y''_\alpha) = y''_\alpha(y'_{x'}) + \lambda_{x'}$ for some $y'_{x'} \in Y'$ and $\lambda_{x'} \in \mathbb{C}$, and this converges to $y''(y'_{x'}) + \lambda_{x'} = \widetilde{Ax'}(y'')$. Finally, $\lim_\alpha S(y''_\alpha)(x') = \lim_\alpha \widetilde{Ax'}(y''_\alpha) - \phi(x') = \widetilde{Ax'}(y'') - \phi(x') = S(y'')(x')$, and this completes the proof. □

In the case θ_A maps Y'' into X'', ϕ in the last theorem must coincide with $\delta_{T_1(0)}$ for some T_1. Then

$$\widetilde{Af}(y'') = \delta_{T_1(0)}[x \mapsto \tilde{f}(x + Ty'')] = \tilde{f}(T_1(0) + Ty'') = (\tilde{f} \circ T_2)(y'').$$

We say that $A : Exp(X) \to Exp(Y)$ is an *AB-composition homomorphism* if there exists an affine mapping $g : Y'' \to X''$ such that $\widetilde{Af}(y'') = \tilde{f}(g(y''))$ for all $f \in Exp(X)$ and all $y'' \in Y''$ (see [53]). Similar to [56, Corollary 2], we get the following:

Corollary 1.3.10 *Let X and Y be symmetrically regular Banach spaces and $A : Exp(X) \to Exp(Y)$ an algebra homomorphism. Then $\theta_A(Y'') \subseteq X''$ if and only if A is the AB-composition homomorphism associated to an affine mapping.*

Proof. We shall see the converse implication, since the direct one is proved above. By hypothesis, there exists $g : Y'' \to X''$ such that $\widetilde{A(f)}(y'') = \widetilde{f}(g(y''))$. Thus, last theorem is satisfied with $T = g$ and $\phi = \delta_0 \in V\mathfrak{M}(X)$. Hence, proceeding as

in the proof of Theorem 4.3.5, we get $\theta_A(\delta_0 \circ \tau_{y''}) = (\theta_A(\delta_0)) \circ \tau_{g(y'')} = \delta_0 \circ \tau_{g(y'')}$.
□

As in [56], there are some differences between the weighted algebras $Exp(X)$ and $H_b(X)$. By Theorem 4.3.5 and the comments above, each AB-composition homomorphism A induces a continuous θ_A. In [53], we find examples of composition homomorphisms inducing discontinuous θ_A. Analogously to [56, page 901], Corollary 1.3.10 implies that, if the spectrum of $Exp(X)$ does not coincides with X'', there are homomorphisms on $Exp(X)$ that are not AB-composition ones (see [56]).

Chapter 2

Linearization of weighted (LB)-spaces of entire functions

This chapter is devoted to study the predual of $VH(X)$ in order to linearize this space of entire functions. We also show that $VH(X)$ is complete and we study some conditions to ensure that the equality $VH_0(X)'' = VH(X)$ holds. At this point, we will see some differences between the finite and the infinite dimensional cases. Finally, we give conditions which ensure that a function f defined in a subset A of X, with values in another Banach space E, and admitting certain weak extensions in a space of holomorphic functions can be holomorphically extended in the corresponding space of vector-valued functions. Most of our results concerning this topic have been published by the author in [13].

2.1 Preduals and biduality of weighted (LB)-spaces of entire functions

2.1.1 Predual of $VH(X)$

In Section 0.3 it is shown that given a weight v on X, the closed unit ball B_v is compact with respect to the compact open topology τ_{co}. This fact allows as to obtain, as a corollary of Mujica's completeness theorem for (LB)-spaces [102, Theorem 1], which was inspired by a theorem of Banach-Dixmier-Waelbroeck-Ng on dual Banach spaces (cf. Waelbroeck [122, Proposition 1] and Ng [107]), the predual of the space $VH(X)$. As a corollary of [25, Corollary 2], we also get that the inductive limit $VH(X)$ is complete.

Theorem 2.1.1 (Mujica) *Let $(E, \tau) = ind_n E_n$ be an (LB)-space, and suppose that*
(∗) there exists a locally convex Hausdorff topology $\tilde{\tau} \leq \tau$ on E such that the closed unit ball B_n of each E_n is $\tilde{\tau}$-compact, $n = 1, 2, \ldots$

a) Then

$$F := \{u \in E^* : \ u_{|B_n} \text{ is } \tilde{\tau}\text{-continuous for each } n \in \mathbb{N}\},$$

endowed with the topology of uniform convergence on the sets B_n, is a Fréchet space (in fact, a closed subspace of E'_b) such that the evaluation mapping $J : E \to F'$ given by $[J(x)](u) := u(x)$ for all $x \in E$ and $u \in F$, yields a topological isomorphism of E onto the inductive dual F'_i, and hence E must be complete.

b) If, in addition,
(CNC) τ has a 0-neighbourhood base of convex, balanced $\tilde{\tau}$-closed sets, then F is distinguished, i.e., $F'_i = F'_b$, and E is topologically isomorphic to the strong dual F'_b.

Corollary 2.1.2 ([25, Corollary 2]) *If one identifies E with F'_i via J, then each bounded subset of E is equicontinuous on F, and F'_i is a regular inductive limit.*

Proposition 2.1.3 *Given a decreasing sequence of weights V, the space*

$$F := \{u \in VH(X)^* : \ u_{|B_n} \text{ is } \tau_{co}\text{-continuous for each } n \in \mathbb{N}\},$$

where B_n denotes the closed unit ball of $H_{v_n}(X)$, endowed with the topology of uniform convergence on B_n, $n \in \mathbb{N}$, is a Fréchet space (in fact, a closed subspace of $VH(X)'_b$). The evaluation mapping $J : VH(X) \to F'$ given by $J(f)(u) = u(f)$ for all $f \in VH(X)$ and $u \in F$ yields a topological isomorphism from $VH(X)$ onto the inductive dual F'_i, and hence $VH(X)$ must be complete. Moreover, each bounded subset of $VH(X)$ is equicontinuous on F, and $VH(X)$ is a regular inductive limit.

Remark 2.1.4 (i) Since $VH(X)$ is regular, $\{nB_n\}_n$ is a fundamental sequence of bounded sets in $VH(X)$. Therefore, equivalently,

$$F = \{u \in VH(X)^* : \ u_{|B} \text{ is } \tau_{co}\text{-continuous } \forall \text{ bounded set } B \subseteq VH(X)\}.$$

(ii) Given a decreasing sequence of weights V, the inclusion

$$(B_n, \tau_{co}) \hookrightarrow (B_n, \sigma(VH(X), F))$$

is continuous since $u_{|B_n}$ is τ_{co}-continuous for any $u \in F$. Moreover, since B_n is τ_{co}-compact and $(B_n, \sigma(VH(X), F))$ is Hausdorff, we obtain that the other inclusion is also continuous, and thus, τ_{co} and $\sigma(VH(X), F)$ induce the same topology on B_n for every $n \in \mathbb{N}$. Therefore, we also get

$$F = \{u \in VH(X)^* : \ u_{|B_n} \text{ is } \sigma(VH(X), F)\text{-continuous for each } n \in \mathbb{N}\}.$$

Definition 2.1.5 ([31, Definition 2.1]) A sequence of weights $V = \{v_n\}_n$ is *regularly decreasing* if, given $n \in \mathbb{N}$, there exists $m \geq n$ so that, for every $\varepsilon > 0$ and every $k \geq m$, it is possible to find $\delta > 0$ such that, if

$$v_m(x) \geq \varepsilon v_n(x), \text{ then } v_k(x) \geq \delta v_n(x).$$

In other words, V is regularly decreasing if, and only if, given $n \in \mathbb{N}$, there exists $m \geq n$ such that, on each subset of X on which the quotient v_m/v_n is bounded away from zero, also all quotients v_k/v_n, $k \geq m$, are bounded away form zero.

Lemma 2.1.6 ([31, Proposition 2.2]) *V is a regularly decreasing family of weights if and only if*

$$\forall n \in \mathbb{N}, \ \exists m \geq n \textit{ such that, } \forall \varepsilon > 0, \exists \, \overline{v} \in \overline{V} : v_m \leq \max(\varepsilon v_n, \overline{v}),$$

where $\overline{V}$ is the maximal Nachbin family of weights associated to V (see Section 0.3).

The next proposition can be found in [31] for X a Hausdorff locally compact topological space and $VC(X)$ a weighted inductive limit of spaces of continuous functions.

Proposition 2.1.7 *If V is a regularly decreasing sequence of weights, then $VH(X)$ is boundedly retractive. As a consequence, the predual F of $VH(X)$ is a quasinormable Fréchet space, and thus, distinguished. Therefore, $VH(X)$ is topologically isomorphic to the strong dual F'_b. In this case, $VH(X)$ has a 0-neighbourhood basis of absolutely convex $\sigma(VH(X), F)$-closed sets.*

Proof. We start by showing that $VH(X)$ is boundedly retractive. As $VH(X)$ is a regular inductive limit, it is enough to show that for every B_n, $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that the topologies of $VH(X)$ and $H_{v_m}(X)$ coincide on B_n, and by [111, Lemma 1 in page 102], it is enough to check that the 0-neighbourhoods coincide. Apply Lemma 2.1.6 in order to find $m \geq n$ such that, for all $\varepsilon > 0$ there is $\overline{v} \in \overline{V}$ with $v_m \leq \max(\varepsilon v_n, \overline{v})$. Hence, as every $f \in B_n$ satisfies $\sup_{x \in X} |f(x)| v_n(x) \varepsilon < \varepsilon$, $B_n \cap B_{\overline{v}}(0, \varepsilon) \subseteq B_n \cap B_{v_m}(0, \varepsilon)$, and thus, $H_{v_m}(X)$ and $H\overline{V}(X)$ induce the same 0-neighbourhoods on B_n. Since the inclusions $H_{v_m}(X) \hookrightarrow VH(X) \hookrightarrow H\overline{V}(X)$ hold continuously, $VH(X)$ is boundedly retractive, which implies by [25, Remark 4] that F is quasinormable. By

[110, Proposition 8.3.45], any quasinormable Fréchet space is distinguished, i.e., $F_i' = F_b'$. Therefore, by Proposition 2.1.3, $F_b' = VH(X)$. Since (CNC) holds for $\tilde{\tau} = \sigma'(E, F)$ when a locally convex space E is the strong dual of some (quasi-) barrelled locally convex space F, the last assertion holds (see [25, page 116]). □

2.1.2 Biduality of $VH_0(X)$

In this section we characterize when the identity $(VH_0(X)_b')_i' = VH(X)$ holds canonically, i.e., under which conditions the restriction map $R : F \to (VH_0(X))_b'$ is a topological isomorphism onto. Moreover, we show that condition (ii) in the next theorem is not always satisfied for X an infinite dimensional Banach space, differing from the finite dimensional case.

Theorem 2.1.8 ([25, Theorem 7]) *The restriction mapping $R : F \to (VH_0(X))_b'$ given by $R(u) = u_{|VH_0(X)}$, $u \in F$, is always well-defined, linear and continuous. Moreover, it is a topological isomorphism onto if and only if the following two conditions are satisfied:*

(i) For each $n \in \mathbb{N}$ there are $m \geq n$ and $M \geq 1$ with $B_n \subseteq \overline{MB_m^0}$ (where the closure on the right hand side is taken in $(VH(X), \sigma(VH(X), F))$.

(ii) The restriction of each $u \in VH_0(X)'$ to any B_n^0 is $\sigma(VH(X), F)$-continuous, $n = 1, 2, \ldots$

Whenever R is a topological isomorphism onto, $VH_0(X)$ is a topological subspace of $VH(X)$ and $VH(X)$ is canonically the bidual $(VH_0(X)_b')_i'$.

Lemma 2.1.9 *If V is a decreasing sequence of weights, then $B_n \subseteq \overline{B_n^0}$ for every $n \in \mathbb{N}$, where the closure is taken in $(VH(X), \sigma(VH(X), F))$.*

Proof. By Remark 2.1.4(ii), it is enough to show that if v is a rapidly decreasing weight, then $B_v \subseteq \overline{B_v^0}^{\tau_{co}}$. Indeed, given $f \in B_v$, it is known that the Cesàro means $\{C_j f\}_j \subseteq \mathcal{P}(X)$ of the partial sums of the Taylor series of f at the origin converge to f in the compact open topology τ_{co} (see [27]). Since by [27, Proposition 1.2] $\|C_j f\|_v \leq \|f\|_v$ for every $j \in \mathbb{N}$, we get the conclusion. □

If $VH(X)$ is reflexive, then each Banach space $(\mathcal{P}(^k X), \|\cdot\|)$, $k \in \mathbb{N}$, must be reflexive, since any closed subspace of a reflexive space is also reflexive. Under the assumption that the family of weights V satisfies Condition (A) (see Definition 1.1.10), proceeding as in [71, Corollary 13], we get the converse:

Proposition 2.1.10 *If V is a sequence of weights satisfying Condition (A), then $VH_0(X) = VH(X)$ is reflexive if and only if $(\mathcal{P}(^k X), \|\cdot\|)$ is a reflexive Banach space for every $k \in \mathbb{N}$.*

Proof. By Proposition 1.1.13, $VH(X)$ and $VH_0(X)$ coincide algebraically and topologically. Since by Theorem 1.1.14 the sequence $\{\mathcal{P}(^kX)\}_k$ is an S-absolute decomposition of $VH(X)$, by [62, Corollary 3.14], it is a shrinking decomposition. Moreover, it is γ-complete, and then, [86, Theorem 3.2] gives the conclusions, since by [117, page 144] a locally convex space E is reflexive if and only if it is semi-reflexive and barrelled. □

Example 2.1.11 Alencar, Aron and Dineen gave in [3] the first example of an infinite dimensional Banach space X for which the space $\mathcal{P}(^kX)$ is reflexive for every $k \in \mathbb{N}$. This space is the Tsirelson space T^*. By the last proposition, given V a sequence of weights satisfying Condition (A), the space $VH(T^*)$ is reflexive.

Lemma 2.1.12 *Let X be a Banach space and denote by $B_{X'}$ the closed unit ball of X'. If $z''_{|B_{X'}}$ is $\sigma(X', X)$-continuous for every $z'' \in X''$, then X must be reflexive.*

Proof. Our assumption yields that $(B_{X'}, \sigma(X', X)) \hookrightarrow (B_{X'}, \sigma(X', X''))$ is continuous. By Alaouglu-Bourbaki, the unit ball $B_{X'}$ is $\sigma(X', X)$-compact. This implies that $B_{X'}$ is also $\sigma(X', X'')$-compact, and thus, X', or equivalently, X, is reflexive. □

Recall that a set of functions $\mathcal{F}$ defined on an open subset Ω of a locally convex space and taking their values in a normed linear space is said to be *locally bounded (l.b.)* if and only if for each $x \in \Omega$ there exists a neighbourhood V_x of x contained in Ω such that $\sup_{f\in\mathcal{F}} \sup_{y\in V_x} \|f(y)\| < \infty$ (see [63, page 24]). From [63, Example 1.24 and Lemma 1.23] we get the next lemma.

Lemma 2.1.13 *If X is a normed space, then the locally bounded subsets of $\mathcal{P}(^nX)$ are bounded with respect to the supremum norm.*

Proposition 2.1.14 ([63, Proposition 1.17]) *If E is a locally convex space then*

$$\mathcal{P}(^nE) \cong (\widehat{\bigotimes_{n,s,\pi}} E)',$$

where $\widehat{\bigotimes}_{n,s,\pi} E$ *denotes the completion of the space* $\bigotimes_{n,s} E := \mathrm{span}\{x\otimes\cdots\otimes x : x \in E\}$ *endowed with the locally convex topology π or* projective topology. *In the case that E is a Banach space,* $\widehat{\bigotimes}_{n,s,\pi} E$ *is also a Banach space (see [63, page 19]).*

Proposition 2.1.15 *If V is a decreasing sequence of weights and the restriction of each $u \in VH_0(X)'$ to any B_n^0 is $\sigma(VH(X), F)$-continuous, then $(\mathcal{P}(^kX), \|\cdot\|)$ is reflexive for every $k \in \mathbb{N}$.*

Proof. Take $p' \in \mathcal{P}(^kX)'$ for a fixed $k \in \mathbb{N}$ and define $\overline{p}' : VH_0(X) \to \mathbb{C}$, $f \mapsto \langle p', P_k f\rangle$. Fix $n \in \mathbb{N}$ and consider the restriction $\overline{p}' : H^0_{v_n}(X) \to \mathbb{C}$. As $H_{v_n}(X)$ induces the supremum norm topology on $\mathcal{P}(^kX)$ for all $k, n \in \mathbb{N}$ (see Lemma 1.1.12), we get that $\overline{p}' \in VH_0(X)'$, since there exists some $C > 0$ such that, for every $f \in H^0_{v_n}(X)$,

$$|\langle p', P_k f\rangle| \leq \|p'\|_{\mathcal{P}(^kX)'}\|P_k f\| \leq C\|p'\|_{\mathcal{P}(^kX)'}\|P_k f\|_{v_n} \leq C\|p'\|_{\mathcal{P}(^kX)'}\|f\|_{v_n}.$$

By hypothesis, $\overline{p}'_{|B^0_n}$ is $\sigma(VH(X), F)$-continuous. As the inclusion $J : \mathcal{P}(^kX) \to H^0_{v_n}(X)$ is continuous, there exists $D_n > 0$ such that $J(\frac{1}{D_n}B_{\mathcal{P}(^kX)}) \subseteq B^0_n(X)$ for all $n \in \mathbb{N}$, where $B_{\mathcal{P}(^kX)}$ denotes the unit ball of $(\mathcal{P}(^kX), \|\cdot\|)$. Then, we get that $p' = \overline{p}' \circ J : \frac{1}{D_n}B_{\mathcal{P}(^kX)} \to \mathbb{C}$ is $\sigma(VH(X), F)$-continuous. Since by Lemma 1.1.12, $VH(X)$ induces the supremum norm topology on $\mathcal{P}(^kX)$, by Lemma 2.1.13, the locally bounded subsets of $\mathcal{P}(^kX)$ are τ-bounded. Therefore, by [63, Proposition 1.28] and Remark 2.1.4, we have $F \subseteq \mathcal{Q}(^kX) \cong \widehat{\bigotimes}_{k,s,\pi} X$, where

$$\mathcal{Q}(^kX) := \{\phi \in \mathcal{P}(^kX)^* : \ \phi \text{ is } \tau_{co}\text{-continuous on the l.b. subsets of } \mathcal{P}(^kX)\}$$

is endowed with the topology of uniform convergence on the locally bounded subsets of $\mathcal{P}(^kX)$. Therefore, $p' : \frac{1}{D_n}B_{\mathcal{P}(^kX)} \to \mathbb{C}$ is $\sigma(\mathcal{P}(^kX), \widehat{\bigotimes}_{n,s,\pi} X)$-continuous. Since this holds for every $p' \in \mathcal{P}(^kX)'$, by Proposition 2.1.14 and Lemma 2.1.12, we conclude that $\mathcal{P}(^kX)$ is reflexive. □

Proposition 2.1.15 shows a difference between the finite dimensional and the infinite dimensional case that must be stressed. Whereas for a finite dimensional Banach space X condition (ii) in Theorem 2.1.8 is always satisfied by the Hahn-Banach and Riesz representation theorems (see [25, Section 3.B]), in the infinite dimensional case, X has to be at least reflexive.

2.2 Linearization of weighted spaces of entire functions

In this section we obtain a representation of the weighted space of vector-valued holomorphic functions $VH(X, E)$, X and E Banach spaces, as a space of operators $\mathcal{L}_i(F, E)$ defined on the predual F of the corresponding weighted space $VH(X)$. Our results here complement [27], [29] and [48].

Remark 2.2.1 In Proposition 2.1.3 we have seen that

$$F = \{u \in VH(X)^* : \ u_{|B} \text{ is } \tau_{co}\text{-continuous for each bounded set } B \subseteq VH(X)\}$$

is a Fréchet space such that the evaluation map $J : VH(X) \to F_i'$, $f \mapsto Jf(u) = u(f)$ is a topological isomorphism. Observe that for the case $V = \{v\}$, a result given by Ng in [107] ensures that the space F is a Banach space and that the mapping J is an isometric isomorphism.

In what follows, we denote by $\{q_n\}_n$ the fundamental sequence of continuous seminorms in F. Recall that they are given by $q_n(u) := \sup_{f \in B_n} |u(f)|$ for all $u \in F$.

Since for every $x \in X$, the evaluation map $\delta_x : VH(X) \to \mathbb{C}$, $\delta_x(f) := f(x)$ for all $f \in VH(X)$, belongs to F, the mapping $\Delta : X \to F$, $\Delta(x) := \delta_x$, $x \in X$, is well defined. Moreover, since B_n is equicontinuous for every $n \in \mathbb{N}$ by Proposition 2.1.3, it is easy to see that Δ is continuous. Observe that each function in $VH(X)$ linearizes through Δ since $f = Jf \circ \Delta$ and $F_i' = VH(X)$ topologically. Therefore, by [57, Corollary 2], the predual F must be isomorphic to the predual $VH_*(X)$ given in [57, Definition 2]. As in [57, Proposition 2], the next lemma shows that the map Δ inherits properties of $f \in VH(X)$.

Lemma 2.2.2 *The map $\Delta : X \to F$, $\Delta(x) := \delta_x$, $x \in X$, is holomorphic. Moreover, it belongs to $VH(X, F)$ if and only if the family V contains only a weight v. In this case, $\|\Delta\|_v \leq 1$.*

Proof. Since F is complete and Δ is continuous, it follows from a classical result of Grothendieck [81] that it is enough to show that Δ is weakly holomorphic, that is, $\phi \circ \Delta : X \to \mathbb{C}$ is holomorphic for each $\phi \in F'$. Since $F' = J(VH(X))$, we check this property for each Jf, $f \in VH(X)$. We have

$$(Jf \circ \Delta)(x) = Jf(\delta_x) = f(x),$$

therefore, $Jf \circ \Delta$ is holomorphic. Moreover, observe that Δ belongs to $VH(X, F)$ if and only if there exists some $v \in V$ such that

$$\sup_{x \in X} v(x) q_n(\delta_x) = \sup_{x \in X} v(x) \sup_{f \in B_n} |f(x)| = \sup_{x \in X} v(x)/\widetilde{v_n}(x) < \infty$$

for every $n \in \mathbb{N}$, i.e., if $H_{v_n}(X) \subseteq H_v(X)$ for every $n \in \mathbb{N}$, since $H_{v_n}(X) = H_{\widetilde{v_n}}(X)$. □

Lemma 2.2.3 *The set $\Delta(X) := \{\delta_x, \ x \in X\}$ is total in F, i.e., the linear span of $\Delta(X)$ is dense in F.*

Proof. The conclusion follows from the Hahn-Banach theorem. Indeed, suppose that $f \in F' = VH(X)$ satisfies that $f(x) = \langle \delta_x, f \rangle = 0$ for each $x \in X$. This implies $f \equiv 0$. □

We consider now the linearization of the vector-valued case. It must be compared to the one given in [48, Theorem 3.3] for the weighted Fréchet space $HW(G,E)$, where G is an open connected domain in $\mathbb{C}^d$ and E is a complete locally convex space.

Theorem 2.2.4 *Let X be a Banach space and let V be a decreasing sequence of weights. For every Banach space E and every $f \in VH(X,E)$, there exists a unique continuous operator $T_f \in \mathcal{L}(F,E)$ such that $T_f \circ \Delta = f$. Moreover, the mapping $f \in VH(X,E) \to T_f \in \mathcal{L}_i(F,E)$ is a topological isomorphism.*

Proof. For $f \in H_{v_n}(X,E) \subseteq VH(X,E)$, we define $T_f : F \to (E')^*$ by $T_f(u) : E' \to \mathbb{C}$, $(T_f(u))(z') := u(z' \circ f)$. Since $z' \circ f : X \to \mathbb{C}$ is holomorphic and

$$\sup_{x\in X} |(z' \circ f)(x)|v_n(x) \leq \|z'\| \sup_{x\in X} \|f(x)\|v_n(x) < \infty,$$

$z' \circ f \in H_{v_n}(X) \subseteq VH(X)$ for every $z' \in E'$, and the map $T_f(u)$ is well-defined and linear for each $u \in F \subseteq VH(X)'$. Since $u \in H_{v_n}(X)'$, by Proposition 2.1.3,

$$|T_f(u)(z')| = |u(z' \circ f)| \leq q_n(u)\|z' \circ f\|_{v_n} \leq q_n(u)\|z'\|\|f\|_{v_n}.$$

Therefore, $T_f(u) \in E''$ with $\|T_f(u)\| \leq q_n(u)\|f\|_{v_n}$ for each $u \in F$. Thus, since by the very definition $T_f : F \to E''$ is linear, $T_f \in \mathcal{L}(F,E'')$.

On the other hand, $T_f(\delta_x) \in E$ for each $x \in X$, since

$$(T_f(\delta_x))(z') = \delta_x(z' \circ f) = z'(f(x)),$$

which yields that $\langle z', T_f(\delta_x)\rangle$ in the pair (E', E'') coincides with $\langle z', f(x)\rangle$ in the pair (E',E) for every $z' \in E'$. Since $\{\delta_x, x \in X\}$ is total in F by Lemma 2.2.3, $T_f \in \mathcal{L}(F,E'')$, and E is complete, we get $T_f(F) \subseteq E$ and $T_f \in \mathcal{L}(F,E)$ with $T_f(\delta_x) = f(x)$ for every $x \in X$. Observe that if we get $T \in \mathcal{L}(F,E)$ such that $(T \circ \Delta)(x) = f(x)$, then $T(\delta_x) = T_f(\delta_x)$ for every $x \in X$. Since $\{\delta_x, x \in X\}$ is total in F, then $T = T_f$ in F.

We claim now that the map $\psi : VH(X,E) \to \mathcal{L}_i(F,E),\ f \to \psi(f) := T_f$, which is clearly linear, is also continuous. Indeed, for each $n \in \mathbb{N}$, $\psi : H_{v_n}(X,E) \to \mathcal{L}(F_n,E)$, where $F = proj_n F_n$, is continuous, since

$$\sup_{q_n(u)\leq 1} \|T_f(u)\| \leq \|f\|_{v_n} \tag{2.1}$$

for each $f \in H_{v_n}(X,E)$.

In order to find the isomorphism, we now define $\chi : \mathcal{L}_i(F,E) \to VH(X,E)$ by $\chi(T) := T \circ \Delta,\ T \in \mathcal{L}_i(F,E)$. By Lemma 2.2.2, $\chi(T) \in \mathcal{H}(X,E)$. Moreover, it belongs to $VH(X,E)$. Indeed, since $T \in \mathcal{L}_i(F,E)$, there exists some $m \in \mathbb{N}$ such that $T \in \mathcal{L}(F_m, E)$. Hence,

$$\begin{aligned} \|\chi(T)(x)\| &= \|T \circ \Delta(x)\| = \|T(\delta_x)\| \\ &\leq \sup_{\substack{u\in F \\ q_m(u)\leq 1}} \|T(u)\| \, q_m(\delta_x) \leq \sup_{\substack{u\in F \\ q_m(u)\leq 1}} \|T(u)\|/v_m(x). \end{aligned}$$

Therefore, $\chi(T) \in H_{v_m}(X,E) \subseteq VH(X,E)$ with

$$\|\chi(T)\|_{v_m} \leq \sup_{\substack{u\in F \\ q_m(u)\leq 1}} \|T(u)\|. \tag{2.2}$$

Then, clearly χ is linear and continuous.

To complete the proof, it remains to show that the two mappings defined above are the inverse of each other. Observe that $\psi \circ \chi$ coincides with the identity on $\mathcal{L}_i(F,E)$. For $T \in \mathcal{L}_i(F,E)$, $(\psi \circ \chi)(T) = \psi(T \circ \Delta) \in \mathcal{L}_i(F,E)$. For each $x \in X$, we have (as we proved above)

$$(\psi(T \circ \Delta))(\delta_x) = (T \circ \Delta)(x) = T(\delta_x).$$

Consequently, $\psi(T \circ \Delta)$ and T are continuous linear maps which coincide on the total subset $\Delta(X)$ of F. Therefore, $\psi(T\circ\Delta) = T$. On the other hand, $\chi\circ\psi$ coincides with the identity on $VH(X,E)$. Indeed, if $f \in VH(X,E)$ and $x \in X$, we have

$$(\chi \circ \psi)(f)(x) = (\chi(\psi(f)))(x) = \psi(f)(\delta_x) = f(x).$$

□

The problem of the topological identity $\mathcal{L}_b(F,E) = \mathcal{L}_i(F,E)$ for a Fréchet space F and a Banach space E is related to the problem of topologies of Grothendieck, and it was investigated thoroughly since its solution by Taskinen in the mid 1980's. We refer the reader to Section 6 in [26] for further information and detailed references.

Remark 2.2.5 In the case $V = \{v\}$, observe that the mapping $f \in VH(X,E) \to T_f \in \mathcal{L}_b(F,E)$ is an isometric isomorphism. Indeed, by Remark 2.2.1, equation (2.1) and doing $T = T_f$ in equation (2.2), we easily get $\|T_f\| = \|f\|_v$ for every $f \in H_v(X,E)$.

We translate now certain properties of a mapping $f \in VH(X,E)$ into properties of the corresponding operator $T_f \in \mathcal{L}_i(F,E)$. If X is a set and Y is a vector space, then a mapping $f : X \to Y$ is said to have *finite rank* if the subspace N of Y generated by $f(X)$ is finite dimensional. In that case, we define the *rank* of f

to be the dimension of N. Proceeding as in [104, Proposition 3.1 and Proposition 3.4], we get the next proposition:

Proposition 2.2.6 *Let X and E be Banach spaces.*

(i) *A mapping $f \in VH(X,E)$ has finite rank if and only if the corresponding operator $T_f \in \mathcal{L}(F,E)$ has finite rank. In that case, $rank f = rank\ T_f$.*

(ii) *A mapping $f \in VH(X,E)$ has a relatively compact range (resp. relatively weakly compact range) if and only if the corresponding operator $T_f \in \mathcal{L}(F,E)$ is compact (resp. weakly compact).*

Remark 2.2.7 There exist several obstructions when trying to extend the linearization Theorem 2.2.4 replacing a Banach space E by a Fréchet space E :

The first one is that in general $\cup_n \mathcal{L}(F_n, E) \neq \mathcal{L}(F,E)$ algebraically if E is a Fréchet space. This follows trivially e.g. taking $E = F$, since $\cup_n \mathcal{L}(F_n, E)$ coincides with the space $\mathcal{LB}(F,E)$ of linear maps that send a neighbourhood of F into a bounded subset of E. The question of the coincidence of $\mathcal{L}(F,E)$ and $\mathcal{LB}(F,E)$ for pairs of Fréchet spaces F and E have been considered by many authors, starting with Vogt's paper [120].

The second problem one encounters refers to the topological coincidence. To mention a concrete example, consider on the complex plane the weights $v_n(z) = e^{-n|z|}$, $z \in \mathbb{C}$. The (LB)-space $VH(\mathbb{C})$ satisfies that its strong dual F is isomorphic to $\mathcal{H}(\mathbb{C})$. Since $VH(\mathbb{C})$ is nuclear, F coincides with the predual of $VH(\mathbb{C})$. For an arbitrary Fréchet space E, we have the isomorphisms

$$\mathcal{L}_b(F,E) \cong \mathcal{L}_b(\mathcal{H}(\mathbb{C}), E) \cong V\mathcal{H}(\mathbb{C})\widehat{\otimes}_\varepsilon E$$

(see [87]). By Vogt [121, Theorem 4.9], this space is barrelled if and only if the space E satisfies the topological invariant (Ω) (cf. [105]). If we take a Fréchet space E which does not satisfy property (Ω), the non-barrelled space $\mathcal{L}_b(F,E)$ cannot coincide topologically with the (LF)-space $VH(\mathbb{C},E)$. Operator representations of the space $V(G,E'_b)$ for Fréchet spaces E, where G is an open subset of $\mathbb{C}^d$, were obtained by Bierstedt and Holtmanns in [29]. The commutativity of inductive limits and tensor products and the topological structure of spaces of type $\mathcal{L}(F,E)$ for pairs of Fréchet spaces F and E have been discussed by many authors after Vogt's work [121]; see e.g. [26].

2.3 Linearization of spaces of vector-valued functions

In Section 2.2 we obtain a linearization of the weighted (LB)-spaces of entire functions $VH(X,E)$, where X and E are Banach spaces. Here we give a more general linearization result which includes the one given for $VH(X,E)$.

Consider Ω a non-empty set and $\mathcal{F}(\Omega) = G'$ a dual Banach space of functions $f : \Omega \to \mathbb{C}$ such that each evaluation map $\delta_x : \mathcal{F}(\Omega) \to \mathbb{C}$, $\delta_x(f) := f(x)$, $x \in \Omega$, belongs to G. By the Hahn-Banach theorem, the linear span of $\{\delta_x,\ x \in \Omega\}$ is norm dense in G. This happens, for instance, if the topology restricted to $B_{\mathcal{F}(\Omega)}$ is the pointwise topology. Now, given a Banach space E, consider the space

$$\mathcal{F}(\Omega, E) := \{f : \Omega \to E,\ e' \circ f \in \mathcal{F}(\Omega) \text{ for every } e' \in E'\}.$$

Given $f \in \mathcal{F}(\Omega, E)$, the map $S_f : E' \to \mathcal{F}(\Omega)$, $S_f(e') := e' \circ f$, $e' \in E'$, is well-defined, linear and weak-star pointwise continuous. By the closed graph theorem S_f is continuous, and therefore we can endow $\mathcal{F}(\Omega, E)$ with the norm $\|f\| := \sup_{\|e'\| \leq 1} \|e' \circ f\|_{\mathcal{F}(\Omega)}$. The map $\Delta : \Omega \to G$, $x \mapsto \delta_x$, $x \in \Omega$, is well-defined, and by [107], $\mathcal{F}(\Omega)$ factors through Δ, that is, there exists an isometric isomorphism $J : \mathcal{F}(\Omega) \to G'_b$, $f \to Jf(u) = u(f)$, such that $Jf \circ \Delta = f$ for every $f \in \mathcal{F}(\Omega)$. From this, we obviously get that Δ belongs to $\mathcal{F}(\Omega, G)$ and $\|\Delta\| = 1$. Following the same steps as in the proof of [44, Lemma 10], we get the next linearization result, where the functions do not need to be holomorphic. For instance, in [57, Example 7] we have linearization in a space of continuous functions.

Lemma 2.3.1 *The space $\mathcal{F}(\Omega, E)$ is isomorphic to the space of linear and continuous operators $\mathcal{L}(G, E)$ in a canonical way. In particular, it is a Banach space.*

Let $Z = ind_n Z_n$ be an inductive limit of Banach spaces such that there exists a Hausdorff locally convex topology τ such that B_{Z_n} is τ-compact for every $n \in \mathbb{N}$. By [102], the space

$$G := \{u \in Z' : u_{|B_{Z_n}} \text{ is } \tau\text{-continuous for every } n \in \mathbb{N}\}$$

is a predual of Z, and by [107], the space

$$G_n := \{u \in Z'_n : u_{|B_{Z_n}} \text{ is } \tau\text{-continuous } \}$$

is a predual of Z_n for every $n \in \mathbb{N}$.

Assume that $Z_n = \mathcal{F}_n(\Omega)$ is a space of functions and that the evaluation maps $\{\delta_x,\ x \in \Omega\}$ belong to the predual G_n of Z_n, for every $n \in \mathbb{N}$. Let E be a Banach space and consider

$$\mathcal{F}_n(\Omega, E) := \{f : \Omega \to E,\ u \circ f \in \mathcal{F}_n(\Omega) \text{ for every } u \in E'\}.$$

By the Hahn Banach theorem, the linear span of $\{\delta_x,\ x \in \Omega\}$ is norm dense in G_n for every $n \in \mathbb{N}$, hence the projective limit $proj_n G_n$ is reduced. Therefore, $G = proj_n G_n$. This fact yields the next general linearization result for weighted (LB)-spaces of vector-valued functions:

Theorem 2.3.2 *The space $ind_n \mathcal{F}_n(\Omega, E)$ is isomorphic to the space of linear and continuous operators $\mathcal{L}_i(G, E)$ in a canonical way, when we consider in $\mathcal{L}(G, E)$ the inductive limit topology.*

Proof. The conclusion comes easily from Lemma 2.3.1 and the fact that $proj_n G_n$ is a reduced projective limit which coincides with G. Indeed,

$$ind_n \mathcal{F}_n(U, E) \cong ind_n \mathcal{L}(G_n, E) \cong \mathcal{L}_i(projG_n, E) \cong \mathcal{L}_i(G, E)$$

(see Definition 0.1.9). □

A version of the last linearization result can be found in Theorem 2.2.4 for the weighted (LB)-space of holomorphic functions $VH(X, E)$. Using Theorem 2.3.2, it is enough to consider $\mathcal{F}_n(\Omega) = H_{v_n}(X)$ for every $n \in \mathbb{N}$, since the fact that weakly holomorphic functions are holomorphic and weakly bounded sets are bounded implies that $\mathcal{F}_n(\Omega, E)$ coincides with the natural definition of the space $H_{v_n}(X, E)$ given in (3.1).

By Proposition 0.1.6 we get:

Theorem 2.3.3 *In the case $Z = proj_n \mathcal{F}_n(\Omega)$ is a projective limit of Banach spaces such that there exists a Hausdorff topology τ with $B_{\mathcal{F}_n(\Omega)}$ τ-compact for every $n \in \mathbb{N}$, and such that the inductive limit of the preduals $ind_n G_n$, $G_n' = \mathcal{F}_n(\Omega)$, is regular, we get:*

$$proj_n \mathcal{F}_n(\Omega, E) \cong proj_n \mathcal{L}_b(G_n, E) = \mathcal{L}_b(ind_n G_n, E).$$

Corollary 2.3.4 *Let X be a Banach space and let $W = \{w_n\}_n$ be an increasing sequence of weights. For every Banach space E, $HW(X, E) \cong \mathcal{L}_b(GW(X), E)$, where*

$$GW(X) = \{u \in HW(X)^* :\ u|_B \text{ is } \tau_{co}\text{-continuous } \forall \text{ bounded set } B \subseteq HW(X)\}$$

is the predual of $HW(X)$.

Proof. In [115, Section 2.2.2] Rueda obtains that $HW(X) = GW(X)_b'$. Moreover, by [115, Teorema 2.2.34 and Nota 2.2.36], $GW(X) = ind_n G_{w_n}(X)$ is a regular inductive limit, where

$$G_{w_n}(X) = \{u \in H_{w_n}(X)^* :\ u|_{B_n} \text{ is } \tau_{co}\text{-continuous}\}$$

is the predual of $H_{w_n}(X)$. The conclusion follows now by Theorem 2.3.3. □

2.4 Extensions of functions in weighted spaces of holomorphic functions

We finish the chapter with the next general question: given two Banach spaces X and E, consider $A \subseteq X$, $H \subseteq E'$, and $f : A \to E$ such that for every $u \in H$ the function $u \circ f : A \to \mathbb{C}$ has an extension in $VH(X)(HW(X))$. When does this imply that there is an extension F of f in the weighted space of vector-valued holomorphic functions $VH(X,E)(HW(X,E))$? This problem is motivated by the fact that a continuous function $f : X \to E$ belongs to $VH(X,E)(HW(X,E))$ if and only if $u \circ f : X \to \mathbb{C}$ belongs to $VH(X)(HW(X))$ for every $u \in E'$. Despite the weak and the formal definition do not coincide in general for $VH_0(X,E)(HW_0(X,E))$, we also give conditions in order to obtain extension results using weak extensions on these spaces.

2.4.1 Extensions of functions in inductive and projective limits of dual Banach spaces

Let U be a connected open subset of a Banach space X. The space of all the holomorphic functions on U is denoted by $\mathcal{H}(U)$. A subset $A \subseteq U$ is called *U-bounded* if it is bounded and the distance of A to the complementary of U is positive. For $U = X$, U-bounded means simply bounded. If E is a Banach space, the space of E-valued holomorphic functions on U is denoted by $\mathcal{H}(U,E)$. A weight $v : U \to \,]0,\infty[$ is a continuous function which is strictly positive, and bounded below in each U-bounded subset A of U. The weighted Banach spaces of holomorphic functions are defined by

$$H_v(U) := \{f \in \mathcal{H}(U) : \sup_{x \in U} v(x)|f(x)| < \infty\}$$

and

$$H_v^0(U) := \{f \in \mathcal{H}(U) : vf \text{ vanishes at infinity on } U\text{-bounded sets}\}.$$

A function $g : U \to \mathbb{R}$ is said to vanish at infinity on U-bounded sets when for each $\varepsilon > 0$ there exists an U-bounded subset A such that $|g(x)| < \varepsilon$ for $x \in U \setminus A$. $H_v(U)$ is continuously included in the space $\mathcal{H}_b(U)$ endowed with the topology of uniform convergence on U-bounded sets.

Analogously, for a Banach space E we define the weighted Banach spaces of vector-valued holomorphic functions

$$H_v(U,E) := \{f \in \mathcal{H}(U,E) : \sup_{x\in U} v(x)\|f(x)\| < \infty\} \tag{4.3}$$

and

$$H_v^0(U,E) := \{f \in \mathcal{H}(U,E) : v\|f\| \text{ vanishes at infinity on } U\text{-bounded sets}\}.$$

Let $A_v(U) \subseteq H_v(U)$ be a closed subspace with compact closed unit ball for τ_{co}. Notice that this condition implies that $A_v(U)$ is norm closed. We define the Banach space of vector valued functions in a *weak sense*:

$$A_v(U,E) := \{f : U \to E : u \circ f \in A_v(U) \text{ for all } u \in E'\}. \tag{4.4}$$

Since weakly holomorphic functions are holomorphic and weakly bounded sets are bounded, it follows that for $A_v(U) = H_v(U)$, definition (4.4) agrees with the strong definition (4.3). By Lemma 2.3.1 we get that $A_v(U,E)$ can be identified with $\mathcal{L}(G_{A_v}, E)$, being G_{A_v} the predual of $A_v(U)$.

In what follows we use some results given by Jordá in [85], which are extensions of those obtained by Frerick, Jordá and Wengenroth in [67] for spaces of bounded holomorphic and harmonic functions on open subsets of finite dimensional subspaces with values in locally convex spaces. They allow us to obtain the analogous for weighted spaces of holomorphic functions.

A subset H of E' is said to *determine boundedness* whenever all the $\sigma(E,H)$-bounded subsets of E are bounded. The linear span of such sets is $\sigma(E',E)$-dense.

A subset $A \subseteq U$ is called a *set of uniqueness* for $A_v(U)$ if each $f \in A_v(U)$ which vanishes on A is identically null. For the case of one variable holomorphic functions it is clearly enough to have an accumulation point in A. By standard duality arguments, A is a set of uniqueness if and only if the linear span of $\{\delta_x : x \in A\}$ is $\sigma(A_v(U)', A_v(U))$-dense in $A_v(U)'$. Given a decreasing (increasing) sequence of weights V (W), we say that $A \subseteq X$ is a set of uniqueness for $VH(X)$ ($HW(X)$) if each function $f \in VH(X)$ ($HW(X)$) which vanishes on A vanishes on the whole X. Obviously, A is a set of uniqueness of $VH(X)$ if and only if it is a set of uniqueness of $H_{v_n}(X)$ for every $n \in \mathbb{N}$.

Theorem 2.4.1 ([85, Theorem 10]) *Let v be a weight on U, let $A_v(U)$ be a subspace of $H_v(U)$ with τ_{co}-compact closed unit ball, let A be a set of uniqueness for $A_v(U)$, let E be a Banach space and let $H \subseteq E'$ be a subspace which determines boundedness in E. If $f : A \to E$ is a function such that $u \circ f$ admits an extension $f_u \in A_v(U)$ for each $u \in H$, then f admits a unique extension $F \in A_v(U,E)$.*

Corollary 2.4.2 *Let V be a decreasing sequence of weights on X, let A be a set of uniqueness for $VH(X)$, let E be a Banach space and let H be a closed subspace of E' which determines boundedness in E. If $f : A \to E$ is a function such that $u \circ f$ admits an extension $f_u \in VH(X)$ for each $u \in H$, then f admits a unique extension $F \in VH(X, E)$.*

Proof. Consider the map $T_f : H \to VH(X)$ defined by $u \to f_u$. Since it is linear and continuous if we consider in $VH(X)$ the pointwise topology given by the elements of A, by the closed graph theorem we obtain that T_f is continuous. Hence, by Grothendieck's factorization theorem, we obtain that there exists some $n \in \mathbb{N}$ such that $f_u \in H_{v_n}(X)$ for every $u \in H$. Therefore, applying Theorem 2.4.1, we get that f admits a unique extension $F \in H_{v_n}(X, E) \subseteq VH(X, E)$. □

Corollary 2.4.3 *Let W be an increasing sequence of weights on X, let A be a set of uniqueness for $H_{w_n}(X)$ for every $n \in \mathbb{N}$, let E be a Banach space and let $H \subseteq E'$ be a subspace which determines boundedness in E. If $f : A \to E$ is a function such that $u \circ f$ admits an extension $f_u \in HW(X)$ for each $u \in H$, then f admits a unique extension $F \in HW(X, E)$.*

Proof. By hypothesis, for every $n \in \mathbb{N}$, $u \circ f$ admits an extension $f_u \in H_{w_n}(X)$ for each $u \in H$. Thus, Theorem 2.4.1 implies that there exists a unique extension $F_n \in H_{w_n}(X)$ of f. Since $F_n \in H_{w_1}(X)$ for every $n \in \mathbb{N}$, the uniqueness yields that $F_1 = F_n$ for every $n \in \mathbb{N}$, and thus, the conclusion holds. □

We study now the problem of extending functions which admit extensions for functionals in a subspace H of E which we assume only to be $\sigma(E', E)$-dense. In this case we require that A is quite large.

A subset $A \subseteq U$ is said to be *sampling* for $A_v(U)$ if there exists some constant $C \geq 1$ such that, for every $f \in A_v(U)$,

$$\sup_{x \in X} v(x)|f(x)| \leq C \sup_{a \in A} v(a)|f(a)|.$$

Theorem 2.4.4 ([85, Theorem 12]) *Let v be a weight on U, let $A_v(U)$ be a subspace of $H_v(U)$ with τ_{co}-compact unit ball, and let A be a sampling set for $A_v(U)$. Let E be a Banach space and let H be a $\sigma(E', E)$-dense subspace of E'. If $f : A \to E$ is a function such that $\sup_{a \in A} v(a)\|f(a)\| < \infty$ and such that $u \circ f$ admits an extension $f_u \in A_v(U)$ for each $u \in H$, then there exists a unique extension $F \in A_v(U, E)$ of f.*

Given V a decreasing sequence of weights on X, we say that A is *sampling* for $VH(X)$ if for every $n \in \mathbb{N}$ there exists some constant $C_n \geq 1$ such that

$$\sup_{x \in X} v_n(x)|f(x)| \leq C_n \sup_{a \in A} v_n(a)|f(a)|$$

for every $f \in VH(X)$. Observe that A sampling on $VH(X)$ yields A sampling on $H_{v_n}(X)$ for every $n \in \mathbb{N}$.

Corollary 2.4.5 *Let V be a decreasing sequence of weights on X, let A be a sampling set for $VH(X)$, let E be a Banach space and let H be a $\sigma(E', E)$-dense subspace of E'. If $f : A \to E$ is a function such that there exists $m \in \mathbb{N}$ with $\sup_{a \in A} v_m(a)\|f(a)\| < \infty$ and such that $u \circ f$ admits an extension $f_u \in VH(X)$ for each $u \in H$, then there exists a unique extension $F \in VH(X, E)$ of f.*

Proof. Since $u \circ f \in VH(X)$ for every $u \in H$ and A is sampling on $VH(X)$, there exists some $C > 0$ such that

$$\sup_{x \in X} v_m(x)|(u \circ f)(x)| \leq C \sup_{a \in A} v_m(a)|(u \circ f)(a)| \leq C\|u\| \sup_{a \in A} v_m(a)\|f(a)\| < \infty.$$

Therefore, since A is sampling on $H_{v_m}(X)$, Theorem 2.4.4 yields the conclusion.
□

Proceeding as in Corollary 2.4.3, we get the next corollary:

Corollary 2.4.6 *Let W be an increasing sequence of weights on X, let A be a sampling set for every $H_{w_n}(X)$, $n \in \mathbb{N}$, let E be a Banach space and let H be a $\sigma(E', E)$-dense subspace of E'. If $f : A \to E$ is a function such that $\sup_{a \in A} w_m(a)\|f(a)\| < \infty$ for every $m \in \mathbb{N}$ and such that $u \circ f$ admits an extension $f_u \in HW(X)$ for each $u \in H$, then there exists a unique extension $F \in HW(X, E)$ of f.*

In [85, Remark 13] it is shown that the conditions on the set A where the functions are defined and in the subspace H cannot be simultaneously relaxed. The condition of boundedness in the extensions cannot be dropped either.

For arbitrary Banach spaces in $H_v(U)$ with no assumption on the unit ball, Theorem 2.4.4 is no longer true. In fact, the equivalence between the weak and the strong definitions does not hold in general. Jordá proves in [85, Example 15] that for X finite dimensional and $H_v^0(U)$ infinite dimensional, the space

$$H_v^0(U, E)_w := \{f : U \to E : u \circ f \in H_v^0(U) \text{ for all } u \in E'\},$$

when $E = c_0$ satisfies $H_v^0(U, c_0) \subsetneq H_v^0(U, c_0)_w$. The equality $H_v^0(U, E)_w = H_v^0(U, E)$ holds if we add some conditions, for instance, if E is a Banach space satisfying the

Schur property, that is, if every sequence $\{x_n\}_n$ in E which is weakly convergent is also norm convergent [85, Proposition 14]. The well known theorem of Schur asserts that ℓ_1 satisfies this property. But the Schur property is not enough in order to have Theorems 2.4.1 and 2.4.4 for $H_{v_0}(U,E)$ (see [85, Example 17]). To have such an extension we need relatively compact range, as we show in the next section.

2.4.2 Extensions of functions in inductive and projective limits of general Banach spaces

In this section we study extensions of functions in general Banach spaces, not necessarily having a predual, considering the natural extension to the weighted case of the vector-valued compact holomorphic functions introduced by Aron and Schottenloher in [8] by means of the weak definition. Most of these results are published by Jordá in [85].

Given two locally convex spaces F and E, we denote by $F\varepsilon E$ its *ε-product of Schwartz*, that is, the space of all linear and continuous mappings $\mathcal{L}_e(F'_{co}, E)$, endowed with the topology of uniform convergence on the equicontinuous subsets of F'. F'_{co} is F' endowed with the topology τ_{co} of uniform convergence on the convex compact subsets of F. The ε-product is symmetric by means of the transpose mapping [87, §43.3(3)]. In case E and F are Banach spaces, $T : F' \to E$ belongs to $F\varepsilon E$ if and only if T is a compact operator which is weak*-weak continuous [87, §43.3.(2)]. Let Ω be a non-void set and let $v : \Omega \to (0,\infty)$ be a bounded continuous function. If E is a Banach space, consider

$$Bv(\Omega, E) := \{f : \Omega \to E,\ (vf)(\Omega) \text{ bounded in } E\},$$

$$Bv^c(\Omega, E_t) := \{f : \Omega \to E, (vf)(\Omega) \text{ relatively compact in } (E,t)\},$$

where t can be the norm, weak, or weak-star (in the case E is a dual Banach space) topology on E. $Bv(\Omega,E)$ is a Banach space under the norm given by $\sup_{x\in\Omega} v(x)\|f(x)\|$. In the case $E = \mathbb{C}$, put $Bv(\Omega) := Bv(\Omega,\mathbb{C}) = Bv^c(\Omega,\mathbb{C})$.

Given a closed subspace $\mathcal{F}(\Omega)$ of $Bv(\Omega)$, and a Banach space E, consider

$$\mathcal{F}(\Omega, E_t) := \{f \in Bv(\Omega, E) : u \circ f \in \mathcal{F}(\Omega) \text{ for every } u \in E'\}.$$

$$\mathcal{F}^c(\Omega, E_t) := \{f \in Bv^c(\Omega, E_t) : u \circ f \in \mathcal{F}(\Omega) \text{ for every } u \in E'\}.$$

As above, we say that a set $A \subseteq \Omega$ is *sampling* for $\mathcal{F}(\Omega)$ if there exists some constant $C \geq 1$ such that, for every $f \in \mathcal{F}(\Omega)$,

$$\sup_{x\in\Omega} v(x)|f(x)| \leq C \sup_{a\in A} v(a)|f(a)|.$$

The key to obtain a linearization of general Banach spaces of functions is the existence of an identification between each Banach space E and the linear functionals on E' that are weak-star continuous [65, Corollary 3.94]. This yields that the evaluation mapping

$$J : \mathcal{F}(\Omega) \to \{T \in \mathcal{F}(\Omega)'' : T \text{ weak-star continuous}\}, \ f \mapsto Jf(u) := u(f),$$

is an isometric isomorphism. Since $\delta_x \in \mathcal{F}(\Omega)'$ for every $x \in \Omega$, the map $\Delta : \Omega \to \mathcal{F}(\Omega)'$, $\Delta(x) := \delta_x$, $x \in \Omega$, is well-defined. Note also that each function $f \in \mathcal{F}(\Omega)$ linearizes through Δ, since $f = Jf \circ \Delta$, and J defined above is an isometry. Observe that this is analogous to [107] for general Banach spaces. In fact, the map $\Delta : \Omega \to \mathcal{F}(\Omega)', x \mapsto \delta_x$, satisfies that $u \circ \Delta \in \mathcal{F}(\Omega)$ for every $u \in \mathcal{F}(\Omega)$.

Theorem 2.4.7 ([85, Theorem 19]) *Let $A \subseteq \Omega$ be a sampling set for $\mathcal{F}(\Omega)$, let E be a Banach space and let H be a weak-star dense subspace of the dual of $E_t := (E, t)$, where t can be the norm, weak, or weak-star (in the case E is a dual Banach space) topology on E. The following are equivalent:*

- *(i) $f : A \to E$ satisfies that $vf(A)$ is relatively compact in E_t and $u \circ f$ admits an extension $f_u \in \mathcal{F}(\Omega)$ for each $u \in H$.*
- *(ii) The linear mapping $T_f : H \to \mathcal{F}(\Omega)$, $u \to f_u$ admits an extension $\hat{T} \in \mathcal{F}(\Omega)\varepsilon E_t$.*
- *(iii) f can be extended to $F \in \mathcal{F}^c(\Omega, E_t)$.*

Observe that, setting $A = \Omega$ and $H = E'$ in Theorem 2.4.7 we have a linearization of the space $\mathcal{F}^c(\Omega, E_t)$. In fact, we get that for each Banach space E and each function $f \in \mathcal{F}^c(\Omega, E_t)$, there is a unique operator $T_f \in \mathcal{F}(\Omega)\varepsilon E_t$ such that $T_f \circ \Delta = f$. This fact can be obtained as a consequence of the much more general linearization result given by Bierstedt in [23, Bemerkung 3.1].

Corollary 2.4.8 *If E is a Banach space, then*

$$\mathcal{F}^c(\Omega, E_t) = \mathcal{F}(\Omega)\varepsilon E_t = \{T \in \mathcal{L}(\mathcal{F}(\Omega)', E),\ T|_{B_{\mathcal{F}(\Omega)'}} \text{ is weak}^*\text{-}t \text{ continuous}\},$$

where t can be the norm or the weak topology on E. For a dual Banach space,

$$\{f \in Bv^c(\Omega, E'_{w*}) : u \circ f \in \mathcal{F}(\Omega) \text{ for every } u \in E\} = \mathcal{F}(\Omega, E'_{w*}) = \mathcal{L}(E, \mathcal{F}(\Omega)),$$

where w^* denotes the weak-star topology on E'.

Proof. The case t is the norm or the weak topology on E follows easily from Theorem 2.4.7 and [87, §43.3.(2)]. If t is the weak-star topology, then the first equality follows by Alaoglu-Bourbaki theorem and from the identification between E and the linear functionals on E' that are weak-star continuous. In fact, $\mathcal{F}(\Omega, E'_{w^*}) = \mathcal{F}^c(\Omega, E'_{w^*})$. By Theorem 2.4.7 and [87, §43.3.(2)] we get $\mathcal{F}(\Omega, E'_{w^*}) = \mathcal{F}(\Omega)\varepsilon E'_{w^*} = \mathcal{L}(F(\Omega)'_{w*}, E'_{w*})$, and this is equal to $\mathcal{L}(E_w, F(\Omega)_w) = \mathcal{L}(E, \mathcal{F}(\Omega))$ ([84, page 161]). □

Observe that if we apply Corollary 2.4.8 to a Banach space $\mathcal{F}(\Omega)$ with a predual G, we get that

$$\mathcal{F}^c(\Omega, E_t) = G'\varepsilon E_t = \mathcal{K}(G, E_t).$$

We have seen in Lemma 2.3.1 that in this case we do not need compactness to have a linearization.

By [85, (36)], the spaces

$$H_v^{0,c}(U,E)_w := \{f \in H_v(U,E) : \ (vf)(U) \text{ is r.c. and } u \circ f \in H_v^0(U) \ \forall u \in E'\}$$

and

$$H_v^{0,c}(U,E) = \{f \in H_v^0(U,E) : \ (vf)(U) \text{ is relatively compact}\}$$

coincide, where r.c. means relatively compact.

In case X is finite dimensional, the space $H_v^0(U,E)$ is the space of holomorphic functions such that f is continuous in the Alexandroff compactification $U \cup \{\infty\}$ of U and $f(\infty) = 0$. Hence $H_v^0(U,E) = H_v^{0,c}(U,E)$ in this case. If X is infinite dimensional the inclusion $H_v^{0,c}(U,E) \subseteq H_v^0(U,E)$ is strict in general. Observe that if U is the unit ball and v vanishes at ∞ on U then $I|_U \in H_v^0(U,X) \setminus H_v^{0,c}(U,E)$.

So, if we consider $\mathcal{F}(\Omega) = H_v^0(U)$, the next extensions hold:

Proposition 2.4.9 *If $A \subseteq U$ is a sampling set for $H_v^0(U)$, E a Banach space and $f : A \to E$ is a function such that $vf(A)$ is relatively compact and $u \circ f$ admits an extension $f_u \in H_v^0(U)$ for each $u \in H \subseteq E'$, H a weak-star dense subspace, then f admits an extension $F \in H_v^{0,c}(U,E)$.*

Proceeding as in the proof of Corollaries 2.4.5 and 2.4.3, for the spaces $VH_0^c(U,E) := ind_n H_{v_n}^{0,c}(U,E)$ and $HW_0^c(U,E) := proj_n H_{w_n}^{0,c}(U,E)$ we get:

Corollary 2.4.10 *Let A be a sampling set for every $H^0_{v_n}(U)$, $n \in \mathbb{N}$, let E be a Banach space and let H be a $\sigma(E', E)$-dense subspace of E'. If $f : A \to E$ is a function such that there exists $m \in \mathbb{N}$ with $v_m f(A)$ relatively compact and such that $u \circ f$ admits an extension $f_u \in VH_0(U)$ for each $u \in H$, then there exists a unique extension $F \in VH_0^c(U, E)$ of f.*

Corollary 2.4.11 *Let A be a sampling set for every $H^0_{w_n}(U)$, $n \in \mathbb{N}$, let E be a Banach space and let H be a $\sigma(E', E)$-dense subspace of E'. If $f : A \to E$ is a function such that $w_n f(A)$ relatively compact for every $n \in \mathbb{N}$, and such that $u \circ f$ admits an extension $f_u \in HW_0(U)$ for each $u \in H$, then there exists a unique extension $F \in HW_0^c(U, E)$ of f.*

Chapter 3

Dynamics of differentiation and integration operators on weighted spaces of entire functions

In this chapter we are concerned with the dynamical behaviour of the following three operators on weighted Banach spaces of entire functions defined by supremum or integral norms: the differentiation operator $Df(z) = f'(z)$, the integration operator $Jf(z) = \int_0^z f(\zeta)d\zeta$ and the Hardy operator $Hf(z) = \frac{1}{z}\int_0^z f(\zeta)d\zeta$, $z \in \mathbb{C}$. In particular we analyze when they are hypercyclic, chaotic, power bounded, and (uniformly) mean ergodic. Moreover, for weights satisfying some conditions, we estimate the norm of the operators and study their spectrum. For differential operators $\phi(D)$, ϕ an entire function, we study hypercyclicity and chaos, and we include an example of a hypercyclic and mean ergodic operator given by Peris.

The results obtained on the weighted Banach spaces of entire functions $H_v(\mathbb{C})$ and $H_v^0(\mathbb{C})$ are published by Bonet, Fernández and the author in [17], and their generalization to weighted Banach spaces of entire functions defined by means of integral norms are included by Beltrán in [15].

3.1 Notation and Preliminaries

It is easy to see that the three operators, D, J and H, are continuous on $\mathcal{H}(\mathbb{C})$. In 1952 MacLane proved that the differentiation operator D is hypercyclic on $\mathcal{H}(\mathbb{C})$. He used a construction to find an entire function with dense orbit in [96]. Now it can be proved very easily using the Hypercyclicity Criterion, since the polynomials are dense in $\mathcal{H}(\mathbb{C})$. Moreover, using Runge's Theorem (see [58, page 202, Theorem 2.2]), Shapiro characterized simple connectivity of a domain $G \subseteq \mathbb{C}$ in terms of the hypercyclicity (even chaoticity) of the differentiation operator D acting on $\mathcal{H}(G)$. More precisely, he proved that simply connectivity of G is equivalent to D being hypercyclic or chaotic on $\mathcal{H}(G)$.

For $r \geq 0$ and $f \in \mathcal{H}(\mathbb{C})$, consider

$$M_p(f,r) := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p} \quad \text{for } 1 \leq p < \infty$$

and

$$M_\infty(f,r) := \sup_{|z|=r} |f(z)|.$$

By the classical Hardy convexity theorem and the Maximum Modulus Theorem, the mapping $r \to M_p(f,r)$ is increasing and logarithmically convex.

As defined in Section 0.2, a weight v on $\mathbb{C}$ is a strictly positive continuous function on $\mathbb{C}$ which is radial, non-increasing and rapidly decreasing. For such a weight, $1 \leq p \leq \infty$ and $q \in \{0, \infty\}$, the *generalized weighted Bergman spaces of entire functions* are defined by

$$B_{p,\infty}(v) := \{f \in \mathcal{H}(\mathbb{C}) : |||f|||_{p,v} := \sup_{r>0} v(r) M_p(f,r) < \infty\}$$

and

$$B_{p,0}(v) := \{f \in \mathcal{H}(\mathbb{C}) : \lim_{r\to\infty} v(r) M_p(f,r) = 0\}.$$

Both are Banach spaces under the norm given by $|||f|||_{p,v}$. In case $p = \infty$ these spaces are usually denoted by $H_v(\mathbb{C})$ and $H_v^0(\mathbb{C})$ (see [27, 28, 35, 68, 95]). The inclusions $B_{p,0} \subseteq B_{p,\infty} \subseteq B_{1,\infty} \subseteq \mathcal{H}(\mathbb{C})$ are continuous for every $1 \leq p \leq \infty$. As in [42], take $r > 0$, select $R_0 > r$, fix $|z| \leq r$ and apply the Cauchy formula, integrating around the circle of center 0 and radius R_0, to get

$$\frac{R_0 - r}{R_0}|f(z)| \leq M_1(f,R_0) \leq M_p(f,R_0) \leq M_\infty(f,R_0).$$

This implies

$$\sup_{|z|\le r}|f(z)| \le \frac{R_0}{(R_0-r)v(R_0)}v(R_0)M_p(f,R_0) \le \frac{R_0}{(R_0-r)v(R_0)}|||f|||_{p,v}. \tag{1.1}$$

Then, for every $1 \le p \le \infty$, the closed unit ball of $B_{p,\infty}(v)$, denoted by $C_{p,\infty}$, is bounded on $\mathcal{H}(\mathbb{C})$ and τ_{co}-closed, since for $r>0$ the mapping $\delta_{p,r}: \mathcal{H}(\mathbb{C}) \to \mathbb{C}$, $f \mapsto M_p(f,r)$ is continuous and

$$C_{p,\infty} = \bigcap_{r\ge 0} \delta_{p,r}^{-1}([0, \frac{1}{v(r)}]).$$

As $\mathcal{H}(\mathbb{C})$ is Montel, then $C_{p,\infty}$ is τ_{co}-compact.

For $1 \le p \le \infty$ and $1 \le q < \infty$, we consider the space

$$B_{p,q}(v) := \left\{ f \in \mathcal{H}(\mathbb{C}) : \ \|f\|_{p,q,v} := \left(2\pi \int_0^\infty rv(r)^q M_p(f,r)^q dr\right)^{1/q} < \infty \right\}.$$

Given a compact set K and $z \in K$, we get by the mean value formula that

$$|f(z)| \le \frac{1}{\pi}\int_{\overline{D(z,1)}} |f(\lambda)|d\lambda \le \frac{1}{\pi}\int_{\overline{D(0,R)}} |f(\lambda)|d\lambda$$

for every $f \in \mathcal{H}(\mathbb{C})$, where $R>0$ is such that $z \in K \subseteq \cup_{z\in K}\overline{D(z,1)} \subseteq \overline{D(0,R)}$.

$$|f(z)| \le \frac{1}{\pi}\int_0^R r\int_0^{2\pi} |f(re^{i\theta})|d\theta dr = 2\int_0^R rM_1(f,r)dr \le 2\int_0^R rM_p(f,r)dr,$$

so, applying Hölder's inequality, we obtain that for every $z \in K$,

$$|f(z)| \le 2R^{1-\frac{1}{q}}\left(\int_0^R r^q M_p(f,r)^q dr\right)^{\frac{1}{q}} \le \frac{2R^{2-\frac{2}{q}}}{v(R)}\left(\int_0^\infty rv(r)^q M_p(f,r)^q dr\right)^{\frac{1}{q}}. \tag{1.2}$$

This implies that convergence in $B_{p,q}(v)$ implies the uniform convergence on the compact subsets of $\mathbb{C}$. Thus, $B_{p,q}(v)$ is a closed subset of the Banach space

$$\left\{ f:\mathbb{C}\to\mathbb{C} \text{ measurable: } \|f\| := \int_0^\infty rv(r)^q \left(\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{q/p} dr < \infty \right\},$$

and therefore, a Banach space. Observe that for $p=q$ the last measurable space is usually denoted by $L^p_v(\mathbb{C})$, the Banach space of all complex functions f on $\mathbb{C}$ such that $fv \in L^p(\mathbb{C}, d\lambda)$, where λ is the Lebesgue measure on $\mathbb{R}^2$. When $p=2$ this is a Hilbert space. For these spaces, we simply write $B^p_v := B_{p,p}(v)$ and denote the norm by $\| \ \|_{p,v}$. Spaces of this type appear in the study of growth conditions of analytic functions and have been investigated in various articles, see e.g. [27, 28, 35, 93, 94] and the references therein.

By (1.2), the closed unit ball of $B_{p,q}(v)$, denoted by $C_{p,q}$, is bounded on $\mathcal{H}(\mathbb{C})$. It is τ_{co}-closed since for every $r_0 > 0$ the mapping $\delta_{p,q,r_0} : \mathcal{H}(\mathbb{C}) \to \mathbb{C}$, $f \mapsto \int_0^{r_0} rv(r)^q M_p(f,r)^q dr$ is continuous and

$$C_{p,q} = \bigcap_{r_0 \geq 0} \delta^{-1}_{p,q,r_0}([0, \frac{1}{2\pi}]).$$

So, also $C_{p,q}$ is τ_{co}-compact.

Since the weights are rapidly decreasing and $\int_{r_0}^{\infty} r^j v(r)dr = \int_{r_0}^{\infty} r^{j+2} v(r) \frac{1}{r^2} dr$ for every $r_0 > 0$, the polynomials are included in $B_{p,q}(v)$ for all $1 \leq p \leq \infty$ and $q = 0$, $1 \leq q \leq \infty$. By [94, Theorem 2.1] (see also [93, Proposition 2.1]), the polynomials are even dense whenever $q \neq \infty$. In particular, $B_{p,q}(v)$ is separable. For $1 < p < \infty$ and $1 \leq q < \infty$ or $q = 0$, the monomials are a Schauder basis of $B_{p,q}(v)$, but this is not satisfied in general for $p \in \{1, \infty\}$ [94, Theorem 2.3].

Throughout the chapter, $B_{p,q}(a, \alpha)$ shall denote the space associated to the following weight: $v_{a,\alpha}(r) = e^{-\alpha}$, $r \in [0, 1[$, $v_{a,\alpha}(r) = r^a e^{-\alpha r}$, $r \geq 1$, if $a < 0$, and $v_{a,\alpha}(r) = (a/\alpha)^a e^{-a}$, $r \in [0, a/\alpha[$, $v_{a,\alpha}(r) = r^a e^{-\alpha r}$, $r \geq a/\alpha$, if $a > 0$. Clearly, changing the value of v on a compact interval does not change the spaces and gives an equivalent norm. Moreover, we can assume without loss of generality that the weight is differentiable. In case $a = 0$, we simply write $B_{p,q}(\alpha)$. The norms will be denoted by $|| \cdot ||_{p,q,a,\alpha}$ and $|| \cdot ||_{p,q,\alpha}$, respectively. In case $q = \infty$, we simply write $||| \cdot |||_{p,a,\alpha}$ and $||| \cdot |||_{p,\alpha}$. If, in addition, $p = \infty$, then the spaces are denoted by $H_{a,\alpha}(\mathbb{C})$, $H^0_{a,\alpha}(\mathbb{C})$, $H_\alpha(\mathbb{C})$ and $H^0_\alpha(\mathbb{C})$, and the norms by $|| \cdot ||_{a,\alpha}$ and $|| \cdot ||_\alpha$. For $p = \infty$ consider $1/p := 0$ and for $q \in \{0, \infty\}$, consider $1/q := 0$.

In the next lemmata, we show some inclusions that are satisfied between these spaces:

Lemma 3.1.1 *Given a weight v, consider $v_s := v^s$ for $s > 0$. For $1 \leq p \leq \infty$, $q = 0$ or $1 \leq q \leq \infty$, we get*

$$H_{v_a}(\mathbb{C}) \hookrightarrow B_{p,\infty}(v_a) \hookrightarrow B_{p,q}(v_b) \hookrightarrow H^0_{v_c}(\mathbb{C})$$

continuously if $0 < a < b < c$.

Proof. $H_{v_a}(\mathbb{C}) \subseteq B_{p,\infty}(v_a)$ follows easily since $M_p(f,r) \leq M_\infty(f,r)$ for every $f \in \mathcal{H}(\mathbb{C})$ and $r \geq 0$. Given $f \in B_{p,\infty}(v_a)$,

$$\int_0^\infty r v_b(r)^q M_p(f,r)^q dr = \int_0^\infty r(v_a(r)M_p(f,r))^q v_{b-a}(r)^q dr$$
$$\leq |||f|||_{p,v_a}^q \int_0^\infty r v_{b-a}(r)^q dr < \infty$$

if $1 \leq q < \infty$ and

$$v_b(r)M_p(f,r) = v_a(r)M_p(f,r)v_{b-a}(r) \leq |||f|||_{p,v_a} v_{b-a}(r),$$

so $B_{p,\infty}(v_a) \hookrightarrow B_{p,q}(v_b)$. The last inclusion follows from (1.2), since this implies that

$$v_c(R)|f(z)| \leq 2R^{2-2/q} v(R)^{c-b} \|f\|_{p,q,v_b}$$

for every $z \in \mathbb{C}$, $|z| \leq R$ and the weight is rapidly decreasing. □

Lemma 3.1.2 *Given a weight v, consider $1 \leq p \leq \infty$. If there exists some $q_1 \geq 1$ such that*

$$v(R)^{q_1} = O\left(\int_R^\infty r v(r)^{q_1} dr\right) \text{ as } R \to \infty, \tag{1.3}$$

then:

(i) $B_{p,q_1}(v) \hookrightarrow B_{p,0}(v)$ *continuously,*

(ii) $B_{p,q_1}(v) \hookrightarrow B_{p,q_2}(v)$ *continuously for* $q_1 \leq q_2 \leq \infty$.

Proof. (i) This follows easily, since given $f \in B_{p,q_1}(v)$, then

$$(M_p(f,R)v(R))^{q_1} = O\left(M_p(f,R)^{q_1} \int_R^\infty r v(r)^{q_1} dr\right).$$

(ii) By (i), $B_{p,q_1}(v) \subseteq B_{p,0}(v)$, then, given $f \in B_{p,q_1}(v)$ there exists a constant $C > 0$ such that $\sup_{r \geq 0} v(r)^{q_2-q_1} M_p(f,r)^{q_2-q_1} \leq C$, and thus,

$$\int_0^\infty r v(r)^{q_2} M_p(f,r)^{q_2} dr \leq C \int_0^\infty r v(r)^{q_1} M_p(f,r)^{q_1} dr.$$

□

Remark 3.1.3 Condition (1.3) in Lemma 3.1.2 is satisfied when:

(i) v is a weight such that $\sup_{r \geq 0} \frac{v(r)}{v(r+1)} < \infty$.

(ii) $v(r) = r^a e^{-r^b}$ for $1 \leq b \leq 2$ and $a \in \mathbb{R}$.

Proof. (i) follows easily, since for $R \geq 1$,

$$v(R)^q = O(v(R+1)^q) = O\left(\int_R^{R+1} v(r)^q dr\right) = O\left(\int_R^{R+1} rv(r)^q dr\right).$$

(ii) Observe that in this case,

$$\int_R^\infty rv(r)^q dr = \int_R^\infty r^{b-1} e^{-qr^b} r^{aq+2-b} dr \geq \int_R^\infty r^{b-1} e^{-qr^b} r^{aq} dr,$$

therefore

$$\int_R^\infty rv(r)^q dr \geq \frac{1}{qb} e^{-qR^b} R^{aq} + \frac{a}{b} \int_R^\infty e^{-qr^b} r^{aq-1} dr.$$

In case $a \geq 0$ we are done. If $a < 0$,

$$\frac{1}{qb} e^{-qR^b} R^{aq} \leq \int_R^\infty rv(r)^q dr + \frac{|a|}{b} \int_R^\infty e^{-qr^b} r^{aq-1} dr.$$

As the second integral is smaller than the first one, we conclude. □

For the estimates of the norms of the operators on these spaces, we use the Stirling formulas

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \text{ and } \Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x, \ x > 0,$$

where Γ denotes the Gamma function. Recall that $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$, $z \in \mathbb{C}$, and $\Gamma(n) = (n-1)!$ for every $n \in \mathbb{N}$.

The next Lemma is an extension of [42, Lemma 2.2]:

Lemma 3.1.4 *Given a weight v, $a > 0$, $1 \leq p \leq \infty$, and $q = 0$ or $1 \leq q \leq \infty$, the following are equivalent:*

(i) $\{e^{a\theta z} : |\theta| = 1\} \subseteq B_{p,q}(v)$,

(ii) there is $\theta \in \mathbb{C}$, $|\theta| = 1$, such that $e^{a\theta z} \in B_{p,q}(v)$,

(iii) $\lim_{r\to\infty} v(r) \frac{e^{ar}}{r^{\frac{1}{2p}}} = 0$ *if* $q = 0$, $\sup_{r\geq 0} v(r) \frac{e^{ar}}{r^{\frac{1}{2p}}} < \infty$ *if* $q = \infty$, *or* $r^{\frac{1}{q}-\frac{1}{2p}} e^{ar} \in L_v^q([r_0, \infty[)$ *for some* $r_0 > 0$ *if* $1 \leq q < \infty$.

Proof. Observe that for each $\theta \in \mathbb{C}$ with $|\theta| = 1$ we have $\|e^{a\theta z}\|_{p,q,v} = \|e^{az}\|_{p,q,v}$. (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) is proved in [42, Lemma 2.2] for the case $q = 0$. We include the proof here for the sake of completeness. Consider $f(z) = e^{az}$, $z \in \mathbb{C}$, and write $z = r(\cos t + i \sin t)$. Now, apply the Laplace methods for integrals as in formula (2.31) in page 33 in [106] to conclude, for $r > 0$,

$$2\pi M_p(f, r)^p = \int_0^{2\pi} e^{arp\cos t} dt = \left(\frac{\pi}{2arp}\right)^{1/2} e^{arp} + e^{arp} O\left(\frac{1}{rp}\right).$$

This yields, for a certain constant $c_p > 0$ depending on p,

$$M_p(f, r) = c_p \frac{e^{ar}}{r^{\frac{1}{2p}}} + e^{ar} O\left(\frac{1}{r^{\frac{1}{p}}}\right).$$

This implies that for each $1 \leq p < \infty$ there are $d_p, D_p > 0$ and $r_0 > 0$ such that, for each $|\theta| = 1$ and each $r > r_0$,

$$d_p \frac{e^{ar}}{r^{\frac{1}{2p}}} \leq M_p(e^{a\theta z}, r) \leq D_p \frac{e^{ar}}{r^{\frac{1}{2p}}}. \tag{1.4}$$

Now the equivalence follows easily, even for $1 \leq q < \infty$, since for every $s > r_0$,

$$\begin{aligned} d_p^q \int_s^\infty r^{1-\frac{q}{2p}} v(r)^q e^{arq} dr &\leq \int_s^\infty r v(r)^q M_p(e^{a\theta z}, r)^q dr \\ &\leq D_p^q \int_s^\infty r^{1-\frac{q}{2p}} v(r)^q e^{arq} dr. \end{aligned} \tag{1.5}$$

□

From (1.4) we get the next corollary:

Corollary 3.1.5 *If we consider the exponential weight $v(r) = e^{-\alpha r}$, $\alpha > 0$, then $e^{\lambda z} \in B_{p,\infty}(\alpha)$ if and only if $e^{\lambda z} \in B_{p,0}(\alpha)$ for every $1 \leq p < \infty$ and $\lambda \in \mathbb{C}$. This is not satisfied for $p = \infty$, since $e^{\alpha z} \in H_v(\mathbb{C}) \setminus H_v^0(\mathbb{C})$.*

Lemma 3.1.6 *For every $1 \leq p \leq \infty$, the unit ball $C_{p,0}$ is τ_{co}-dense in $C_{p,\infty}$.*

Proof. Given $f \in C_{p,\infty}$, let $f_r(z) := f(rz)$, $r \in (0, 1)$. Fix r and $\varepsilon > 0$ and get $n \in \mathbb{N}$ such that $r^{n+1} < \varepsilon/4$. If $f(z) = \sum_{k=0}^\infty a_k z^k$ is the Taylor series representation of f at 0, then the Taylor polynomial $P_n(z) := \sum_{k=0}^n a_k z^k \in B_{p,0}(v)$, so there exists $R > 0$ such that $v(s) M_p(P_n, s) < \min(\frac{\varepsilon}{2}, 1)$ for all $s > R$. Since $r \to M_p(f, r)$ is increasing for every $f \in \mathcal{H}(\mathbb{C})$, for all $s > R$,

$$v(s) M_p((P_n)_r, s) = v(s) M_p(P_n, rs) \leq v(s) M_p(P_n, s) < \varepsilon/2. \tag{1.6}$$

Besides, if we consider $g := f - P_n$ and $h(z) := \sum_{k=n+1}^{\infty} a_k z^{k-(n+1)}$, $z \in \mathbb{C}$, then $g(z) = z^{n+1}h(z)$. For $s > R$, by Minkowski's inequality, we get

$$\begin{aligned} v(s)M_p(g, rs) &= v(s)M_p(z^{n+1}h, rs) = r^{n+1}s^{n+1}v(s)M_p(h, rs) \\ &\leq r^{n+1}s^{n+1}v(s)M_p(h, s) = r^{n+1}v(s)M_p(g, s) \\ &= r^{n+1}v(s)M_p(f - P_n, s) \leq r^{n+1}v(s)M_p(f, s) \\ &\quad + r^{n+1}v(s)M_p(P_n, s) \leq r^{n+1}(|||f|||_{p,v} + \min(\varepsilon/2, 1)) \\ &\leq 2r^{n+1} \leq \varepsilon/2. \end{aligned} \tag{1.7}$$

By (1.6) and (1.7) we obtain

$$\begin{aligned} v(s)M_p(f_r, s) &= v(s)M_p(g_r + (P_n)_r, s) \\ &\leq v(s)M_p(g_r, s) + v(s)M_p((P_n)_r, s) \leq \varepsilon \end{aligned}$$

for all $s > R$, which implies that $f_r \in B_{p,0}(v)$. Moreover, $f_r \in C_{p,0}$ since

$$|||f_r|||_{p,v} = \sup_{s \geq 0} v(s)M_p(f, rs) \leq |||f|||_{p,v} \leq 1,$$

and it is easy to see that f_r converges to f in τ_{co} as $r \to 1$ since f is uniformly continuous on the compact subsets of $\mathbb{C}$. □

The next lemma is inspired by [41, Proposition 1.1].

Lemma 3.1.7 *Let $T : (\mathcal{H}(\mathbb{C}), \tau_{co}) \to (\mathcal{H}(\mathbb{C}), \tau_{co})$ be a continuous linear operator such that $T(\mathcal{P}) \subseteq \mathcal{P}$, let v be a weight and $1 \leq p \leq \infty$. The following conditions are equivalent:*

(i) $T(B_{p,\infty}(v)) \subseteq B_{p,\infty}(v)$,

(ii) $T : B_{p,\infty}(v) \to B_{p,\infty}(v)$ *is continuous,*

(iii) $T(B_{p,0}(v)) \subseteq B_{p,0}(v)$,

(iv) $T : B_{p,0}(v) \to B_{p,0}(v)$ *is continuous.*

If (i)-(iv) hold, then $||T||_{\mathcal{L}(B_{p,\infty}(v))} = ||T||_{\mathcal{L}(B_{p,0}(v))}$.

Proof. The equivalences (i)⇔(ii) and (iii)⇔(iv) follow from the closed graph theorem, since $B_{p,\infty}(v) \hookrightarrow \mathcal{H}(\mathbb{C})$ continuously and T is continuous on $(\mathcal{H}(\mathbb{C}), \tau_{co})$. (ii)⇒(iii) comes easily from the fact that the polynomials are dense in $B_{p,0}(v)$, $T(\mathcal{P}) \subseteq \mathcal{P}$ and $B_{p,0}(v)$ is closed in $B_{p,\infty}(v)$. Clearly $||T||_{\mathcal{L}(B_{p,0}(v))} \leq ||T||_{\mathcal{L}(B_{p,\infty}(v))}$.

(iv)⇒(i) By Lemma 3.1.6, the unit ball of $B_{p,0}(v)$ is τ_{co}-dense in the unit ball of $B_{p,\infty}(v)$, so given f in $C_{p,\infty}$ there exists $\{f_\alpha\}_\alpha$ in $C_{p,0}$ such that $\{f_\alpha\}_\alpha$ converges to

f in τ_{co} and $|||Tf_\alpha|||_{p,v} \leq \|T\|_{\mathcal{L}(B_{p,0}(v))}$. Since T is τ_{co}-continuous, Tf_α converges to Tf in τ_{co}, and since the unit ball of $B_{p,\infty}(v)$ is τ_{co}-closed, we have $|||Tf|||_{p,v} \leq \|T\|_{\mathcal{L}(B_{p,0}(v))}$. Since this happens for every f in the unit ball of $B_{p,\infty}(v)$, the operator $T : B_{p,\infty}(v) \to B_{p,\infty}(v)$ is continuous with $||T||_{\mathcal{L}(B_{p,\infty}(v))} \leq ||T||_{\mathcal{L}(B_{p,0}(v))}$, and thus, the norms coincide. □

In what follows we write $|||T|||_{p,v}$ instead of $||T||_{\mathcal{L}(B_{p,\infty}(v))} = ||T||_{\mathcal{L}(B_{p,0}(v))}$ for $1 \leq p \leq \infty$ and $||T||_{p,v}$ instead of $||T||_{\mathcal{L}(B_v^p)}$. For $1 \leq q < \infty$ we use the notation $||T||_{p,q,v}$. Moreover, $|||T|||_{p,a,\alpha}$, $||T||_{p,a,\alpha}$ and $||T||_{p,q,a,\alpha}$ refer to the norm of the operator acting on the respective spaces associated to the weight $v_{a,\alpha}$. For $v(r) = e^{-\alpha r}$, $r \geq 0$, we omit the a.

Using Lemma 3.1.7 we get the next proposition. In fact, following the proof we even get that if J is mean ergodic on $B_{p,\infty}(v)$ or on $B_{p,0}(v)$, then $\lim_N \frac{(J+\cdots+J^N)(f)}{N} = 0$ for every f in the corresponding space. Observe also that as the polynomials are dense in $B_{p,0}(v)$, the operator D is mean ergodic on $B_{p,0}(v)$ if and only if it is Cesàro power bounded. In this case, $P(f) = 0$ for every $f \in B_{p,0}(v)$.

Proposition 3.1.8 *Let $T = D$ or $T = J$ and assume that T is continuous on $B_{p,\infty}(v)$, and equivalently, on $B_{p,0}(v)$. The following conditions are equivalent:*

(i) $T : B_{p,\infty}(v) \to B_{p,\infty}(v)$ *is uniformly mean ergodic,*

(ii) $T : B_{p,0}(v) \to B_{p,0}(v)$ *is uniformly mean ergodic,*

(iii) $\lim_{N\to\infty} \frac{|||T+\cdots+T^N|||_{p,v}}{N} = 0$.

Proof.

(i)⇒(ii) and (iii)⇒(i) are clear. We show (ii)⇒(iii).

Suppose first that $T = D$ is uniformly mean ergodic on $B_{p,0}(v)$. Since the polynomials are dense and the sequence $\{\frac{1}{N}\sum_{j=1}^N D^j\}_N$ converges pointwise to zero on $\mathcal{P}$, we have that

$$\lim_{N\to\infty} \frac{1}{N} |||\sum_{j=1}^N D^j|||_{p,v} = 0.$$

For $T = J$, we only have to prove that $\lim_N \frac{(J+\cdots+J^N)(f)}{N} = 0$ for each $f \in B_{p,0}(v)$. By assumption, the limit $\lim_N \frac{(J+\cdots+J^N)(f)}{N}$ exists. Moreover,

$$J\left(\lim_N \frac{(J+\cdots+J^N)(f)}{N}\right) = \lim_N \left(\frac{(J+\cdots+J^{N+1})(f)}{N+1}\frac{N+1}{N} - \frac{Jf}{N}\right)$$
$$= \lim_N \frac{(J+\cdots+J^N)(f)}{N}.$$

Since J has no fixed points different from zero, the conclusion follows. □

Proposition 3.1.9 *If $T \in \mathcal{L}(E)$ is a uniformly mean ergodic operator satisfying that the limit* $\lim_{N\to\infty} \frac{||T+\cdots+T^N||}{N} = 0$, *then* $1 \notin \sigma(T)$.

Proof. If $\lim_{N\to\infty} \frac{||T+\cdots+T^N||}{N} = 0$, for N big enough the operator $I - \frac{1}{N}\sum_{j=1}^N T^j$ is invertible, i.e., $N \notin \sigma(q(T))$ for $q(z) = \sum_{j=1}^N z^j$, which, by the spectral mapping theorem, coincides with $q(\sigma(T))$. Thus, $1 \notin \sigma(T)$. □

For every $1 \leq p \leq \infty$ and $n \in \mathbb{N}$, $M_p(z^n, r) = r^n$, and thus, $\|z^n\|_{p,q,v} = \|z^n\|_{\infty,q,v}$. In what follows, we denote it simply by $\|z^n\|_{q,v}$. As in [41], it is important to estimate the norms of the monomials. In fact, from the inequalities $\|1\|_{q,v} n! = \|D^n(z^n)\|_{q,v} \leq ||D^n||_{p,q,v}||z^n||_{q,v}$ and $\frac{||z^n||_{q,v}}{n!} \leq \|J^n(1)\|_{q,v} \leq ||J^n||_{p,q,v}\|1\|_{q,v}$ we get the next lemma:

Lemma 3.1.10 *Let v be a weight such that the differentiation operator D and the integration operator J are continuous on $B_{p,q}(v)$, $1 \leq p \leq \infty$, $q = 0$ or $1 \leq q \leq \infty$.*

(i) If D is power bounded (resp. uniformly mean ergodic), then $\inf_n \frac{||z^n||_{q,v}}{n!} > 0$ *(resp. $\{\frac{||z^n||_{q,v}}{(n-1)!}\}_n$ tends to infinity).*

(ii) If J is power bounded (resp. mean ergodic), then $\{\frac{||z^n||_{q,v}}{n!}\}_n$ is bounded (resp. $\{\frac{||z^n||_{q,v}}{n!n}\}_n$ tends to zero).

For weights $v(r) = r^a e^{-\alpha r}$ ($\alpha > 0$, $a \in \mathbb{R}$) for $r \geq r_0$ for some $r_0 \geq 0$ we have

$$||z^n||_v \approx \left(\frac{n+a}{e\alpha}\right)^{n+a}, \tag{1.8}$$

with equality for $v(r) = e^{-\alpha r}$, $r \geq 0$. It is enough to estimate the maximum of the function $g(r) = r^{n+a}e^{-\alpha r}$ and to have in mind that the symbol $\approx$ appears as a consequence of the fact that the value of a given weight v can be changed on a compact interval in order to satisfy some required conditions without changing the spaces and giving an equivalent norm.

For $1 \leq q < \infty$, the Stirling formula yields:

$$\begin{aligned} \|z^n\|_{q,a,\alpha} &\approx \left(2\pi \int_0^\infty r^{(a+n)q+1} e^{-\alpha r q} dr\right)^{1/q} = \left(2\pi \frac{\Gamma((a+n)q+2)}{(\alpha q)^{(a+n)q+2}}\right)^{1/q} \\ &\sim \left(\frac{(a+n)q+1}{e\alpha q}\right)^{a+n+\frac{3}{2q}}. \end{aligned} \tag{1.9}$$

Observe that (1.9) tends to (1.8) as $q \to \infty$. Applying again the Stirling formula,

$$\frac{\|z^n\|_{q,a,\alpha}}{n!} \approx \frac{n^{a+\frac{3}{2q}-\frac{1}{2}}}{\alpha^n}. \tag{1.10}$$

3.2 The integration operator

Proposition 3.2.1 *The operator J is never hypercyclic on $\mathcal{H}(\mathbb{C})$, nor on $B_{p,q}(v)$, $1 \leq p \leq \infty$ and $q = 0$ or $1 \leq q < \infty$, provided it is continuous, $J - \lambda I$ is injective on $\mathcal{H}(\mathbb{C})$ for all $\lambda \in \mathbb{C}$, and J has no periodic points different from 0 on $\mathcal{H}(\mathbb{C})$.*

Proof. $Jf(0) = 0$ for every $f \in \mathcal{H}(\mathbb{C})$, thus $\mathrm{Im}(J)$, and the orbit of an element, cannot be dense. We even get that $J^m f$ tends to zero in the compact open topology for every $f \in \mathcal{H}(\mathbb{C})$. Indeed, given $f(z) = \sum_{k=0}^\infty a_k z^k \in \mathcal{H}(\mathbb{C})$, $J^m f(z) = \sum_{k=0}^\infty a_k z^{k+m} \frac{k!}{(k+m)!}$, then

$$|J^m f(z)| \leq R^m \sum_{k=0}^\infty |a_k| R^k \frac{k!}{(k+m)!} \leq \frac{R^m}{m!} \sum_{k=0}^\infty |a_k| R^k$$

for every $z \in \mathbb{C}$, $|z| \leq R$. Thus, $J^m f$ tends to zero in the compact open topology.

If $\lambda = 0$, J is injective, since $Jf = 0$ implies $f = DJf = 0$. If $\lambda \neq 0$ and $Jf - \lambda f = 0$, then $f - \lambda Df = 0$, so $f(z) = Ce^{\frac{1}{\lambda} z}$ for some $C \in \mathbb{C}$. But $f(0) = \frac{1}{\lambda} Jf(0) = 0$, which implies $0 = f(0) = C$, and thus, $f = 0$.

Now suppose that $J^m f = f$ for some $f \neq 0$ and some $m \in \mathbb{N}$. Using the trivial decomposition $J^m - I = (J - \theta_1 I) \ldots (J - \theta_m I)$, $\theta_j^m = 1$, $j = 1, \ldots, m$, we conclude that there is $g \in \mathcal{H}(\mathbb{C})$, $g \neq 0$, and $\theta \in \mathbb{C}$, $\theta^m = 1$, such that $(J - \theta I)g = 0$. But $J - \theta I$ is injective, so we get a contradiction. $\square$

Proposition 3.2.2 *Let v be a weight such that J is continuous on $B_{p,q}(v)$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ or $q = 0$ and assume that $v(r)e^{\alpha r}$ is non decreasing for some $\alpha > 0$. Then, $\sigma(J) \supseteq (1/\alpha)\mathbb{D}$.*

Proof. To see that $(1/\alpha)\mathbb{D} \subseteq \sigma(J)$ we show that $J - \lambda I$ is not surjective on $B_{p,q}(v)$ for $|\lambda| < \frac{1}{\alpha}$. For $\lambda = 0$, J is not surjective on any $B_{p,q}(v)$ (without any additional assumption) since $Jf(0) = 0$ for each f, hence the equation $Jf = 1$ has no solution. Now assume that $\lambda \neq 0$ and that there is $f \in B_{p,q}(v)$ such that $Jf - \lambda f = 1$. Then, $f - \lambda f' = 0$ and, as by Lemma 3.1.4, $e^{z/\lambda} \notin B_{p,q}(v)$, we have $f \equiv 0$, and thus, $Jf - \lambda f \neq 1$. □

Following [10] we define, for every $\lambda \in \mathbb{C}$, an integral operator K_λ on $\mathcal{H}(\mathbb{C})$ by

$$K_\lambda f(z) = e^{\lambda z} \int_0^z e^{-\lambda\zeta} f(\zeta) d\zeta, \;\; f \in \mathcal{H}(\mathbb{C}), \;\; z \in \mathbb{C}.$$

It maps $\mathcal{H}(\mathbb{C})$ into itself continuously and it is a right inverse of the operator $D - \lambda I$. Integrating along the segment that joins 0 to z, we obtain, for every $f \in \mathcal{H}(\mathbb{C})$,

$$K_\lambda f(z) = z \int_0^1 e^{\lambda z(1-t)} f(zt) dt, \;\; z \in \mathbb{C}. \tag{2.11}$$

Observe that for $\lambda = 0$, it is just the integration operator J.

Proposition 3.2.3 *Let v be a weight such that $v(r)e^{\alpha r}$ is non increasing for some $\alpha > 0$ and let $1 \leq p \leq \infty$. If $|\lambda| < \alpha$, then the operator K_λ is continuous on $B_{p,\infty}(v)$ and on $B_{p,0}(v)$ with $|||K_\lambda|||_{p,v} \leq \frac{1}{\alpha - |\lambda|}$. As a consequence, J is continuous on $B_{p,\infty}(v)$ with $|||J|||_{p,v} \leq 1/\alpha$. In particular, $\sigma(J) \subseteq (1/\alpha)\overline{\mathbb{D}}$. Moreover, $|||J^m|||_{p,a,\alpha} \approx 1/\alpha^m$ for all $m \in \mathbb{N}_0$ and $a \leq 0$, with equality for $a = 0$.*

Proof. Given $f \in B_{p,\infty}(v)$, we have

$$\begin{aligned} M_p(K_\lambda f, r) &= \left(\frac{1}{2\pi} \int_0^{2\pi} |K_\lambda f(re^{i\theta})|^p d\theta \right)^{1/p} \\ &= r \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^1 e^{\lambda r e^{i\theta}(1-t)} f(tre^{i\theta}) dt \right|^p d\theta \right)^{1/p}. \end{aligned}$$

So, applying the Minkowski integral inequality we obtain

$$\begin{aligned} M_p(K_\lambda f, r) &\le r\int_0^1 \left(\frac{1}{2\pi}\int_0^{2\pi} e^{|\lambda| r(1-t)p} \left|f(tre^{i\theta})\right|^p d\theta\right)^{1/p} dt \\ &= r\int_0^1 e^{|\lambda|r(1-t)} M_p(f, rt)dt. \end{aligned}$$

Thus, by hypothesis, for $|\lambda| < \alpha$,

$$v(r)M_p(K_\lambda f, r) \le r\int_0^1 v(tr)M_p(f, rt)e^{r(t-1)(\alpha-|\lambda|)}dt \le \frac{|||f|||_{v,p}}{\alpha - |\lambda|}$$

and $K_\lambda : B_{p,\infty}(v) \to B_{p,\infty}(v)$ is continuous with $|||K_\lambda|||_{p,v} \le \frac{1}{\alpha-|\lambda|}$. Let us see now that $K_\lambda(B_{p,0}(v)) \subseteq B_{p,0}(v)$. Since

$$K_\lambda(1) = -\frac{1}{\lambda} + \frac{1}{\lambda}e^{\lambda z} \in B_{p,0}(v),$$

and integrating by parts,

$$K_\lambda(z^n) = -\frac{1}{\lambda}z^n + \frac{n}{\lambda}K_\lambda(z^{n-1}),\ n \in \mathbb{N},$$

we get

$$K_\lambda(\mathcal{P}) \subseteq \mathcal{P} \oplus \operatorname{span}\{e^{\lambda z}\} \subseteq B_{p,0}(v).$$

Since the polynomials are dense in $B_{p,0}(v)$, $K_\lambda : B_{p,0}(v) \to B_{p,0}(v)$ is continuous.

If we consider $\lambda = 0$, we get $|||J|||_{p,v} \le 1/\alpha$, and the spectral radius formula yields $\sigma(J) \subseteq (1/\alpha)\overline{\mathbb{D}}$. As a consequence, $|||J|||_{p,a,\alpha} \lesssim 1/\alpha$ for $a \le 0$.

The lower estimate is satisfied for a general $1 \le q \le \infty$, whenever J is continuous, using equation (1.10):

$$\begin{aligned} ||J^m||_{p,q,a,\alpha} &\ge \sup_k \frac{||J^m(z^k)||_{p,q,a,\alpha}}{||z^k||_{q,a,\alpha}} = \sup_k \frac{||z^{k+m}||_{q,a,\alpha}}{||z^k||_{q,a,\alpha}} \frac{k!}{(k+m)!} \\ &\gtrsim \lim_k \frac{1}{\alpha^m}\left(1 + \frac{m}{k}\right)^{a+\frac{3}{2q}-\frac{1}{2}} = \frac{1}{\alpha^m}. \end{aligned} \tag{2.12}$$

□

Proposition 3.2.4 *Let v be a weight such that $v(r)e^{\alpha r}$ is non increasing for some $\alpha > 0$ and let $1 \le p < \infty$, $p > \frac{1}{\alpha}$. Then, the operator K_λ is continuous on B_v^p if $|\lambda| < \alpha$ and J is continuous on B_v^p with $||J^m||_{p,v} \lesssim \left(\frac{p}{\alpha p-1}\right)^m$ for every $m \in \mathbb{N}$. In particular, $\sigma(J) \subseteq \frac{p}{\alpha p-1}\overline{\mathbb{D}}$. Moreover, $||J^m||_{p,a,\alpha} \gtrsim 1/\alpha^m$ for all $m \in \mathbb{N}_0$.*

Proof. The continuity is proved in [10, Theorem 4] for weights of the type $v(z) = \exp(-\varphi(|z|))$, $z \in \mathbb{C}$, where φ is a non-negative concave function on $\mathbb{R}_+$ such that $\varphi(0) = 0$ and $\lim_{t\to+\infty} \frac{\varphi(t)}{\log t} = +\infty$. We include the proof here with the proper changes needed for our weights, and we get an estimate for the norm.

$K_\lambda : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ is continuous, so it would be enough to show that K_λ is bounded on the space $L^p_v(\mathbb{C})$. But this is not the case for $1 \leq p < \infty$. Simple examples show that the operator is even not defined on this space for $1 \leq p \leq 2$. However, for $1 \leq p < \infty$, the measure $v(z)^p d\lambda(z)$ can be replaced by another positive Borel one μ on $\mathbb{C}$ such that the space $L^p(\mathbb{C}, d\mu)$ includes B^p_v, the restriction of its norm N_p to B^p_v is equivalent to the $L^p_v(\mathbb{C})$ norm, and K_λ maps $L^p(\mathbb{C}, d\mu)$ continuously into itself. Fix $1 \leq p < \infty$, denote by χ the characteristic function of the unit disc $\mathbb{D}$ and consider the positive Borel measure μ on $\mathbb{C}$ defined by

$$d\mu(z) = v(z)^p[|z|^{-1}\chi(z) + (1 - \chi(z))]d\lambda(z). \tag{2.13}$$

Let N_p denote the norm on the Banach space $L^p(\mathbb{C}, d\mu)$. Note that

$$N^p_p(f) = \int_0^{2\pi} \int_0^1 |f(re^{i\theta})|^p v(r)^p dr d\theta + \int_0^{2\pi} \int_1^{+\infty} |f(re^{i\theta})|^p v(r)^p r dr d\theta.$$

It is clear that $L^p(\mathbb{C}, d\mu) \subseteq L^p_v(\mathbb{C})$, and for f in $L^p(\mathbb{C}, d\mu)$

$$\|f\|_{p,v} \leq N_p(f).$$

Now by (1.2), there exists a positive constant C such that for all $f \in B^p_v$

$$|f(z)| \leq C\|f\|_{p,v}, \ |z| \leq 1.$$

Thus setting

$$C_1 = \left(2\pi C^p \int_0^1 v(r)^p dr + 1\right)^{1/p}$$

we get that for all $f \in B^p_v$

$$\|f\|_{p,v} \leq N_p(f) \leq C_1\|f\|_{p,v}.$$

So, on B^p_v, the $L^p_v(\mathbb{C})$ and $L^p(\mathbb{C}, d\mu)$ norms are equivalent.

We show now that for $|\lambda| < \alpha$, the operator K_λ maps $L^p(\mathbb{C}, d\mu)$ continuously into itself. Since $v(r)e^{\alpha r}$ is non increasing, observe that the function $\rho(r) := v(r)$ for $0 \leq r \leq 1$, and $\rho(r) := v(r)r^{1/p}$ for $r > 1$, satisfies $\frac{\rho'(r)}{\rho(r)} = \frac{v'(r)}{v(r)} < -\alpha$ if $r \leq 1$ and $\frac{\rho'(r)}{\rho(r)} = \frac{v'(r)}{v(r)} + \frac{1}{rp} < -\alpha + \frac{1}{p}$ if $r > 1$. So, applying the mean value theorem to the function $\log \rho$, we get

$$\rho(t) \leq \rho(x) e^{(\frac{1}{p}-\alpha)(t-x)}, \ 0 < x < t, \tag{2.14}$$

which yields

$$\int_x^{+\infty} \rho(t)^p dt \leq \frac{\rho(x)^p}{\alpha p - 1}, \ x \in \mathbb{R}_+. \tag{2.15}$$

Let us see that the linear transformation V_γ defined on the vector space $L^1_{loc}(\mathbb{R}_+)$ (of locally integrable functions on $\mathbb{R}_+$) by

$$V_\gamma f(x) = e^{\gamma x} \int_0^x e^{-\gamma t} f(t) dt, \ x \in \mathbb{R}_+,$$

is bounded on $L^p_\rho(\mathbb{R}_+)$, the Banach space of all complex functions f on $\mathbb{R}_+$ such that the function $f\rho$ is in $L^p(\mathbb{R}_+)$, equipped with the norm $\|f\|_{p,\rho} = \|f\rho\|_{L^p(\mathbb{R}_+)}$. Since the linear transformation M_γ defined on $L^1_{loc}(\mathbb{R}_+)$ by $M_\gamma f(x) = e^{\gamma x} f(x)$, $x \in \mathbb{R}_+$, maps $L^p_{M_\gamma \rho}(\mathbb{R}_+)$ isometrically onto $L^p_\rho(\mathbb{R}_+)$, and $V_\gamma = M_\gamma V_0 M_{-\gamma}$, it follows that if the operator V_0 is bounded on $L^p_{M_\gamma \rho}(\mathbb{R}_+)$, then the operator V_γ is bounded on $L^p_\rho(\mathbb{R}_+)$. This shows that it suffices to prove the continuity for $\gamma = 0$. Since for every $f \in L^1_{loc}(\mathbb{R}_+)$,

$$|V_0 f(x)| \leq (V_0 |f|)(x), \ x \in \mathbb{R}_+,$$

it suffices to show that there exists a constant $b > 0$ such that if f is a non-negative continuous function on $\mathbb{R}_+$ with compact support then $\|V_0 f\|_{p,\rho} \leq b \|f\|_{p,\rho}$, since the continuous functions on $\mathbb{R}_+$ with compact support are dense in $L^p_\rho(\mathbb{R}_+)$. If f is such a function, and h is the function on $\mathbb{R}_+$ defined by

$$h(x) = \int_x^{+\infty} \rho(t)^p dt, \ x \in \mathbb{R}_+,$$

then $(V_0 f(0))^p h(0) = 0$, and since $\lim_{t \to \infty} h(t) = 0$ and $V_0 f$ is bounded, we have that $\lim_{t \to +\infty} (V_0 f(t))^p h(t) = 0$ and therefore, integrating by parts, we get

$$\|V_0 f\|^p_{p,\rho} = p \int_0^\infty (V_0 f(t))^{p-1} f(t) h(t) dt.$$

Thus using (2.15), we obtain that

$$\|V_0 f\|^p_{p,\rho} \leq \frac{p}{\alpha p - 1} \int_0^\infty (V_0 f(t))^{p-1} f(t) (\rho(t))^p dt.$$

Applying to the above integral Hölder's inequality (with respect to the measure $\rho(t)^p dt$), we get

$$\|V_0 f\|_{p,\rho}^p \leq \frac{p}{\alpha p - 1} \|V_0 f\|_{p,\rho}^{p-1} \|f\|_{p,\rho},$$

thus,

$$\|V_0 f\|_{p,\rho} \leq \frac{p}{\alpha p - 1} \|f\|_{p,\rho},$$

as we wanted to see. We show next that for all $f \in L^p(\mathbb{C}, d\mu)$

$$N_p(K_\lambda f) \leq \|V_{|\lambda|}\| N_p(f),$$

where $\|V_{|\lambda|}\|$ denotes the norm of $V_{|\lambda|}$ as an operator on $L^p_\rho(\mathbb{R}_+)$. Since the continuous functions on $\mathbb{C}$ with compact support are dense in $L^p(\mathbb{C}, d\mu)$, it suffices to establish the inequality for such functions. Let f be a continuous function on $\mathbb{C}$ with compact support. For every $\theta \in [0, 2\pi]$, denote by f_θ the continuous function on $\mathbb{R}_+$ with compact support defined by $f_\theta(r) = f(re^{i\theta}),\ r \in \mathbb{R}_+$. It follows from (2.11) that

$$|K_\lambda f(re^{i\theta})| \leq V_{|\lambda|} |f_\theta|(r),\ r \in \mathbb{R}_+,\ \theta \in [0, 2\pi],$$

and therefore

$$\begin{aligned} N_p^p(K_\lambda f) &= \int_0^{2\pi} \int_0^\infty |K_\lambda f(re^{i\theta})|^p \rho^p(r) dr d\theta \leq \int_0^{2\pi} \|V_{|\lambda|} |f_\theta|\|_{p,\rho}^p d\theta \\ &\leq \|V_{|\lambda|}\|^p \int_0^{2\pi} \|f_\theta\|_{p,\rho}^p d\theta = \|V_{|\lambda|}\|^p N_p^p(f). \end{aligned}$$

So, in the case $\lambda = 0$ we get $N_p(J) \leq \|V_0\| \leq \frac{p}{\alpha p - 1}$. Since the norms N_p and $\| \ \|_{p,v}$ are equivalent on B_v^p, we conclude $||J^m||_{p,v} \lesssim \left(\frac{p}{\alpha p - 1}\right)^m$ for every $m \in \mathbb{N}$. Therefore, the conclusion about the norm and the spectrum follows. The lower estimate is calculated in equation (2.12). □

Corollary 3.2.5 *The spectrum of* $J : B_{p,q}(a, \alpha) \to B_{p,q}(a, \alpha)$ *satisfies*

$$\sigma(J) = (1/\alpha)\overline{\mathbb{D}}$$

for $1 \leq p \leq \infty$, $q \in \{0, \infty\}$ *and*

$$(1/\alpha)\overline{\mathbb{D}} \subseteq \sigma(J) \subseteq \frac{p}{\alpha p - 1}\overline{\mathbb{D}}$$

for $1 \leq p < \infty$, $p > \frac{1}{\alpha}$, $p = q$.

Proof. For each $\beta < \alpha$, the function $v_{a,\alpha}(r)e^{\beta r}$ is decreasing in $[r_0, \infty[$ for some $r_0 > 0$. Therefore, by Propositions 3.2.3 and 3.2.4, the integration operator J is continuous on $B_{p,q}(a,\alpha)$, and for an equivalent norm, $||J^m|| \leq \frac{1}{\beta^m}$ on $B_{p,\infty}(v)$ and $||J^m|| \leq \left(\frac{p}{\beta p-1}\right)^m$ on B_v^p for $1 \leq p < \infty$. Thus, the spectral radius $r(J)$ of J satisfies $r(J) \leq \frac{1}{\beta}$ and $r(J) \leq \frac{p}{\beta p-1}$, respectively. Since $\beta < \alpha$ is arbitrary, $\sigma(J) \subseteq (1/\alpha)\overline{\mathbb{D}}$ and $\sigma(J) \subseteq \frac{p}{\alpha p-1}\overline{\mathbb{D}}$ holds. On the other hand, $v_{a,\alpha}(r)e^{\gamma r}$ is non decreasing for every $\gamma > \alpha$. By Proposition 3.2.2, $\sigma(J) \supseteq (1/\gamma)\mathbb{D}$ for every $\gamma > \alpha$, and thus, $\sigma(J) \supseteq (1/\alpha)\mathbb{D}$. □

Theorem 3.2.6 *(a) Let $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ or $q = 0$, and assume $J : B_{p,q}(v) \to B_{p,q}(v)$ is continuous. The following are satisfied:*

(i) If $r^a e^{-\alpha r} = O(v(r))$ for $\alpha < 1$, $a \in \mathbb{R}$ or $\alpha = 1$, $a > \frac{1}{2} - \frac{3}{2q}$, then J is not power bounded on $B_{p,q}(v)$.

(ii) J is not uniformly mean ergodic on $B_{p,q}(v)$ if for all $\beta > 1$, $v(r)e^{\beta r}$ is non decreasing. In particular J is not uniformly mean ergodic on $B_{p,q}(a,1)$ for all $a \in \mathbb{R}$.

(iii) If $r^{\frac{3}{2}-\frac{3}{2q}}e^{-r} = O(v(r))$, then J is not mean ergodic on $B_{p,q}(v)$. In particular, it is not mean ergodic on $B_{p,q}(a,\alpha)$ when $\alpha < 1$, $a \in \mathbb{R}$.

(b) For $1 \leq p \leq \infty$ and $q \in \{0, p, \infty\}$, we get:

(iv) J is power bounded on $B_{p,q}(v)$ for $q \in \{0,p,\infty\}$ and mean ergodic for $q \in \{0,p\}$ provided that $v(r)e^{(1+\frac{1}{q})r}$ is non increasing. In particular, this condition is satisfied for the weight $v_{a,1+\frac{1}{q}}$ for every $a \leq 0$.

(v) J is uniformly mean ergodic on $B_{p,q}(v)$ if for some $\alpha > 1+\frac{1}{q}$, $v(r)e^{\alpha r}$ is non increasing.

Proof.

(i) $\frac{||z^n||_{q,a,\alpha}}{n!} = O(\frac{||z^n||_v}{n!})$ and (1.10) implies that the sequence $\{\frac{||z^n||_{q,a,\alpha}}{n!}\}_n$ is unbounded for $\alpha < 1$, $a \in \mathbb{R}$, or $\alpha = 1$, $a > \frac{1}{2} - \frac{3}{2q}$. So, by Lemma 3.1.10(ii) J is not power bounded.

(ii) If for all $\beta > 1$, $v(r)e^{\beta r}$ is non decreasing in some interval $[r_0, \infty[$, $\sigma(J) \supseteq \overline{\mathbb{D}}$. Since $1 \in \sigma(J)$, Lemma 3.1.9 yields the conclusion.

(iii) By (1.10), the sequence $\left\{\frac{||z^n||_{q,\frac{3}{2}-\frac{3}{2q},1}}{n!n}\right\}_n$ does not tend to zero and

$$||z^n||_{q,\frac{3}{2}-\frac{3}{2q},1} = O(||z^n||_{p,q,v}).$$

By Lemma 3.1.10 (ii), J is not mean ergodic on $B_{p,q}(v)$.

(iv) The first statement follows from the estimates of the norm of J^m in Propositions 3.2.3 and 3.2.4. Moreover, for each $k \in \mathbb{N}$,

$$||J^m(z^k)||_{p,q,v} = \frac{k!}{(m+k)!}||z^{m+k}||_{q,v} \lesssim \frac{k!}{(m+k)!}||z^{m+k}||_{q,1+\frac{1}{q}}.$$

So, by (1.10), the successive iterates tend to zero on the polynomials. As J is power bounded and the polynomials are a dense subset, $\{J^m f\}_m$ converges to zero for each $f \in B_{p,q}(v)$, and thus, $\frac{1}{m}\sum_{j=1}^m J^j f$ also converges to 0.

(v) $\{||J^n||_{p,q,v}\}_n$ tends to zero by Propositions 3.2.3 and 3.2.4, therefore

$$\left\|\frac{1}{m}\sum_{j=1}^{m} J^j\right\|_{p,q,v} \leq \frac{1}{m}\sum_{j=1}^{m} ||J^j||_{p,q,v} \to 0.$$

□

Corollary 3.2.7 *Let $1 \leq p \leq \infty$. The integration operator J is uniformly mean ergodic on $B_{p,q}(\alpha)$, $q \in \{0, p, \infty\}$, if $\alpha > 1 + \frac{1}{q}$, and it is not mean ergodic on these spaces if $\frac{1}{q} < \alpha < 1$. J is power bounded and mean ergodic on $B_{p,q}(1 + \frac{1}{q})$, $q \in \{0, p\}$, not uniformly mean ergodic on $B_{p,q}(1)$, $q \in \{0, p, \infty\}$, and not mean ergodic on $H_1(\mathbb{C})$.*

Proof. All the statements but one follow from Theorem 3.2.6. It only remains to show that J is not mean ergodic on $H_1(\mathbb{C})$. The space $H_1(\mathbb{C})$ is a Grothendieck Banach space with the Dunford-Pettis property since it is isomorphic to ℓ_∞ by [95]. As $||J^n||_1/n \to 0$, we can apply Theorem 0.5.6 to conclude that J is not mean ergodic in $H_1(\mathbb{C})$ because it is not uniformly mean ergodic by Theorem 3.2.6 (ii) and Proposition 3.1.8. □

3.3 The differentiation operator

The results of the first part of this section are inspired by [41] and [42].

Proposition 3.3.1 *Let v be a weight such that $C := \sup_{r>0} \frac{v(r)}{v(r+1)} < \infty$. Then the differentiation operator $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous for every $1 \leq p \leq \infty$, and $q = 0$ or $1 \leq q \leq \infty$.*

Proof. The case $q \in \{0, \infty\}$ is proved in [42, Proposition 2.1], where it is shown that $M_p(f', r) \leq \frac{r+1}{2r+1} M_p(f, r+1)$ for every $f \in \mathcal{H}(\mathbb{C})$, $r > 0$ and $1 \leq p \leq \infty$. Therefore,

$$\begin{aligned} \|Df\|_{p,q,v}^q &= 2\pi \int_0^\infty rv(r)^q M_p(f', r)^q dr \\ &\leq C^q 2\pi \int_0^\infty (r+1)v(r+1)^q M_p(f, r+1)^q dr \\ &\leq C^q \|f\|_{p,q,v}^q, \end{aligned}$$

and D is continuous. □

Proceeding as in [41, Theorem 2.3], we get:

Theorem 3.3.2 *Let $1 \leq p \leq \infty$ and $1 \leq q < \infty$ or $q = 0$. Assume that the differentiation operator $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous. The following conditions are equivalent:*

(i) D satisfies the hypercyclicity criterion.

(ii) D is hypercyclic.

(iii) $\liminf_{n\to\infty} \frac{\|z^n\|_{q,v}}{n!} = 0$.

Proof. (i)⇒(ii) is trivial. Assume now that D is hypercyclic on $B_{p,q}(v)$. By Lemma 0.5.11(i), the sequence $\{(D')^n(\delta_0)\}_n$ is unbounded in $B_{p,q}(v)'$ where $\delta_0 : B_{p,q}(v) \to \mathbb{C}$, $\delta_0(f) = f(0)$, hence, by the Banach Steinhaus Theorem [114, 5.8], there is $f \in B_{p,q}(v)$ such that $\{f^{(n)}(0)\}_n$ is unbounded in $\mathbb{C}$. Fix $n \in \mathbb{N}$. By the Cauchy inequalities, for each $r > 0$,

$$r^n \frac{|f^{(n)}(0)|}{n!} = \frac{r^n}{2\pi} \left| \int_{|w|=r} \frac{f(w)}{w^{n+1}} dw \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| d\theta = M_1(f, r) \leq M_p(f, r),$$

which yields $|f^{(n)}(0)| \frac{\|z^n\|_{q,v}}{n!} \leq \|f\|_{p,q,v}$ for every $n \in \mathbb{N}$. Since $\{f^{(n)}(0)\}_n$ is unbounded, there exists an increasing sequence $\{n_k\}_k$ such that $\lim_{k\to\infty} |f^{(n_k)}(0)| =$

∞. Hence, $\liminf_{n\to\infty} \frac{\|z^n\|_{q,v}}{n!} = 0$.
(iii)$\Rightarrow$(i): since D is continuous, there is $C \geq 1$ such that $\|f^{(j)}\|_{p,q,v} \leq C^j \|f\|_{p,q,v}$ for each $f \in B_{p,q}(v)$ and each $j \in \mathbb{N}$. Set $n_0 = 0$ and use (iii) inductively to find $n_k \in \mathbb{N}$ with $n_{k+1} > n_k + k + 1$ and

$$\frac{\|z^{n_k+k+1}\|_{q,v}}{(n_k+k+1)!} < \frac{1}{kC^k}.$$

This is the increasing sequence of natural numbers required on the hypercyclicity criterion. Take $V = W$ as the set of all polynomials, which is dense in $B_{p,q}(v)$. Define $S_{n_k} := S^{n_k}$ on W, with S the integration map defined on the monomials by $S(z^n) = z^{n+1}/(n+1)$. Since $D \circ S(P) = P$ for each polynomial P and for each P of degree less or equal to N, $D^n P = 0$ for $n \geq N+1$, it only remains to show that $\lim_{k\to\infty} S^{n_k} P = 0$ in $B_{p,q}(v)$ for each polynomial P.

In order to see this, fix $s \in \mathbb{N} \cup \{0\}$ and take $k \geq s$. Observe that

$$S^{n_k}(z^s) = \frac{s!}{(n_k+s)!} z^{n_k+s}$$

and

$$D^{k+1-s}(z^{n_k+k+1}) = \frac{(n_k+k+1)!}{(n_k+s)!} z^{n_k+s}.$$

This implies

$$\begin{aligned} \|S^{n_k}(z^s)\|_{q,v} &= \frac{s!}{(n_k+s)!}\|z^{n_k+s}\|_{q,v} = \frac{s!}{(n_k+k+1)!}\|D^{k+1-s}(z^{n_k+k+1})\|_{q,v} \\ &\leq s! C^{k+1-s} \frac{\|z^{n_k+k+1}\|_{q,v}}{(n_k+k+1)!} < s! C^{k+1-s}/(kC^k) = \frac{s!}{k} C^{1-s}. \end{aligned}$$

Hence, by linearity, $\lim_{k\to\infty} S^{n_k} P = 0$ in $B_{p,q}(v)$ for each polynomial P, and D satisfies the criterion. □

If we consider the stronger assumption $\lim_{n\to\infty} \frac{\|z^n\|_{q,v}}{n!} = 0$, then we obtain the following equivalence, analogous to [41, Theorem 2.4].

Theorem 3.3.3 *Assume that the differentiation operator $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous. The following conditions are equivalent:*

(i) D is topologically mixing.

(ii) $\lim_{n\to\infty} \frac{\|z^n\|_{q,v}}{n!} = 0$.

Proof. (i)⇒(ii) Assume that D is topologically mixing. Then, by Lemma 0.5.11(ii),

$$\lim_{n\to\infty} \|\delta_0 \circ D^n\| = \lim_{n\to\infty} \|(D')^n(\delta_0)\| = \infty. \tag{3.16}$$

Proceeding as in the proof of Theorem 3.3.2, for each $f \in B_{p,q}(v)$ with $\|f\|_{p,q,v} \leq 1$ and each $n \in \mathbb{N}$,

$$|\delta_0 \circ D^n(f)| \frac{\|z^n\|_{q,v}}{n!} = |f^{(n)}(0)| \frac{\|z^n\|_{q,v}}{n!} \leq \|f\|_{p,q,v}.$$

So, (ii) holds. Since the polynomials are dense in $B_{p,q}(v)$, (ii) implies that D satisfies the assumptions of the criterion of Kitai-Gethner-Shapiro, and thus, D is topologically mixing. □

The following lemma is the analogous of [42, Theorem 2.3] for general weighted spaces of entire functions.

Lemma 3.3.4 *Let $A \subseteq \overline{\alpha\mathbb{D}}$, $\alpha > 0$, be a subset with at least one accumulation point in $\alpha\mathbb{D}$ or such that $A \cap \delta(\alpha\mathbb{D})$ is dense in $\delta(\alpha\mathbb{D}) := \{z \in \mathbb{C},\ |z| = \alpha\}$. If for $1 \leq p \leq \infty$, $\lim_{r\to\infty} v(r)\frac{e^{\alpha r}}{r^{\frac{1}{2p}}} = 0$, then the set $Y := \operatorname{span}\{e_a :\ a \in A\}$ is dense in $B_{p,0}(v)$, where $e_\omega(z) := e^{\omega z}$ with $z, \omega \in \mathbb{C}$. If for some $r_0 > 0$, $r^{\frac{1}{q}-\frac{1}{2p}} e^{\alpha r} \in L^q_v([r_0, \infty[)$ for $1 \leq p \leq \infty$ and $1 \leq q < \infty$, then Y is dense in $B_{p,q}(v)$. Under these assumptions, $z^n e_\zeta(z) \in B_{p,q}(v)$ for every $n \in \mathbb{N}$ and $\zeta \in \mathbb{C}$, $|\zeta| \leq \alpha$.*

Proof. Let u be a continuous functional on $B_{p,q}(v)$, and assume that $u(f) = 0$ for each $f \in Y$. Consider the function $S : \overline{\alpha\mathbb{D}} \to B_{p,q}(v)$, $\zeta \mapsto e_\zeta$ and define $\widetilde{u} := u \circ S : \overline{\alpha\mathbb{D}} \to \mathbb{C}$, $\widetilde{u}(\zeta) = u(e_\zeta)$, $\zeta \in \overline{\alpha\mathbb{D}}$. By Lemma 3.1.4, S is well defined and bounded. Indeed, by equation (1.4), for $1 \leq p \leq \infty$, there exists some constant $C > 0$ such that for each $\zeta \in \overline{\alpha\mathbb{D}}$,

$$|||S(\zeta)|||_{p,v} = |||e^{\zeta z}|||_{p,v} = \sup_{r\geq 0} v(r) M_p(e^{\zeta z}, r) \leq D \sup_{r \geq r_0} v(r) \frac{e^{\alpha r}}{r^{\frac{1}{2p}}}$$

in case $q = 0$, whereas for $1 \leq q < \infty$, there exists some constant $D > 0$ such that

$$\|S(\zeta)\|_{p,q,v} = \|e^{\zeta z}\|_{p,q,v} \leq C \left(2\pi \int_{r_0}^{\infty} r^{1-\frac{q}{2p}} v(r)^q e^{\alpha r q} dr\right)^{1/q} := M.$$

By [79, Theorem 1], since S is locally bounded (even bounded), in order to show that S is holomorphic on $\alpha\mathbb{D}$, it is enough to find a $\sigma(B_{p,q}(v)', B_{p,q}(v))$-dense subset G of $B_{p,q}(v)'$ such that $u \circ S : \alpha\mathbb{D} \to \mathbb{C}$ is holomorphic for each $u \in G$. Since the compact open topology is coarser than the norm topology, consider

$$G := \{u \in B_{p,q}(v)' \text{ which are continuous for } \tau_{co}\}.$$

G is $\sigma(B_{p,q}(v)', B_{p,q}(v))$-dense: fix $f \in B_{p,q}(v)$. If $u(f) = 0$ for all $u \in G$, we have that given $\lambda \in \mathcal{H}(\mathbb{C})'$ we can define $\widetilde{u} := \lambda|_{B_{p,q}(v)} \in G$. By hypothesis we get $\lambda(f) = \widetilde{u}(f) = 0$. Thus, since $f \in B_{p,q}(v) \subseteq \mathcal{H}(\mathbb{C})$ and $\langle \mathcal{H}(\mathbb{C})', \mathcal{H}(\mathbb{C})\rangle$ is a dual pair, $f = 0$. Applying now the Hahn-Banach Theorem, we have that G is $\sigma(B_{p,q}(v)', B_{p,q}(v))$-dense. Observe now that $u \circ S : \alpha\mathbb{D} \to \mathbb{C}$ is holomorphic for each $u \in G$: we know that $S : \mathbb{C} \to (\mathcal{H}(\mathbb{C}), \tau_{co})$, $\zeta \mapsto e^{\zeta z}$ is holomorphic, hence, its restriction to $\alpha\mathbb{D}$ is also holomorphic in τ_{co}. Let $u \in G$ and consider $u \circ S : \alpha\mathbb{D} \xrightarrow{S} B_{p,q}(v) \xrightarrow{u} \mathbb{C}$. For a fixed $\zeta_0 \in \alpha\mathbb{D}$,

$$\lim_{\zeta\in\alpha\mathbb{D},\zeta\to\zeta_0} \frac{(u \circ S)(\zeta) - (u \circ S)(\zeta_0)}{\zeta - \zeta_0} = \lim_{\zeta\in\alpha\mathbb{D},\zeta\to\zeta_0} u\left(\frac{S(\zeta) - S(\zeta_0)}{\zeta - \zeta_0}\right) =$$

$$= u\left(\lim_{\zeta\in\alpha\mathbb{D},\zeta\to z_0} \frac{S(\zeta) - S(\zeta_0)}{\zeta - \zeta_0}\right) = u(S'(\zeta_0)) \in \mathbb{C}$$

since u is τ_{co}-continuous and S is τ_{co}-holomorphic. Hence, for every $\zeta_0 \in \alpha\mathbb{D}$, $(u \circ S)'(\zeta_0) = u(S'(\zeta_0)) \in \mathbb{C}$, where $S'(\zeta_0)$ is the derivative of S with respect to τ_{co}, and so, $u \circ S : \alpha\mathbb{D} \to \mathbb{C}$ is holomorphic. Therefore, S is holomorphic on $\alpha\mathbb{D}$ with $z^n e_\zeta(z) = S^{(n)}(\zeta) \in B_{p,q}(v)$.

Let us see now that $S : \overline{\alpha\mathbb{D}} \to B_{p,q}(v)$ is continuous. Since S is holomorphic on $\alpha\mathbb{D}$, it is enough to prove the continuity at each ζ_0 in the boundary of $\alpha\mathbb{D}$. Fix a sequence $\{\zeta_j\}_j \in \overline{\alpha\mathbb{D}}$ converging to ζ_0. The case $q = 0$ can be found in the proof of [42, Theorem 2.3]. We include the proof here for the sake of completeness. Fix $\varepsilon > 0$. Since $\lim_{r\to\infty} v(r)\frac{e^{\alpha r}}{r^{\frac{1}{2p}}} = 0$, there is $r_1 > r_0$ such that $v(r)\frac{e^{\alpha r}}{r^{\frac{1}{2p}}} < \varepsilon/(4D_p)$, with $D_p > 0$ and $r_0 > 0$ as in (1.4). We can apply the second inequality in (1.4) to conclude

$$\sup_{r>r_1} v(r)M_p(e^{\zeta_j z} - e^{\zeta_0 z}, r) < \varepsilon/2.$$

Since the map $\mathbb{C} \to \mathcal{H}(\mathbb{C})$, $\zeta \to e^{\zeta z}$, is continuous, we find $\delta > 0$ such that $|\zeta - \zeta_0| < \delta$ implies $\sup_{|z|\le r_1} |e^{\zeta z} - e^{\zeta_0 z}| < \frac{\varepsilon}{2v(0)}$. Find $j_0 \in \mathbb{N}$ with $|\zeta_j - \zeta_0| < \delta$ for $j \geq j_0$. Therefore, for $r \leq r_1$ and $j \geq j_0$, we get

$$v(r)M_p(e^{\zeta_j z} - e^{\zeta_0 z}, r) \leq v(r)M_\infty(e^{\zeta_j z} - e^{\zeta_0 z}, r) < \varepsilon/2.$$

This implies $|||S(\zeta_j) - S(\zeta_0)|||_{p,v} < \varepsilon$, and S is continuous for $q = 0$. For $1 \leq q < \infty$ observe that, given ζ_0 in the boundary of $\alpha\mathbb{D}$ and a sequence $\{\zeta_j\}_j \in \overline{\alpha\mathbb{D}}$ converging to ζ_0, by (1.4) there exist some $C > 0$ and $r_0 > 0$ such that

$$\begin{aligned}\|S(\zeta_j) - S(\zeta_0)\|_{p,q,v}^q &= 2\pi \int_0^\infty rv(r)^q M_p(e^{\zeta_j z} - e^{\zeta_0 z}, r)^q dr \\ &\leq C \int_{r_0}^\infty r^{1-\frac{q}{2p}} v(r)^q e^{\alpha r q} dr.\end{aligned}$$

Given $\varepsilon > 0$, by hypothesis, there exists $r_1 > r_0$ such that $\int_{r_1}^\infty r^{1-\frac{q}{2p}} v(r)^q e^{\alpha r q} dr < \frac{\varepsilon}{2C}$. Since the map $\mathbb{C} \to \mathcal{H}(\mathbb{C})$, $\zeta \mapsto e^{\zeta z}$, is continuous, there exists $j_0 \in \mathbb{N}$ such that

$$\int_{r_0}^{r_1} rv(r)^q M_p(e^{\zeta_j z} - e^{\zeta_0 z}, r)^q dr \leq \int_{r_0}^{r_1} rv(r)^q M_\infty(e^{\zeta_j z} - e^{\zeta_0 z}, r)^q dr < \frac{\varepsilon}{2C}.$$

So, S is continuous. Since $u \circ S$ is holomorphic on $\alpha\mathbb{D}$, continuous at the boundary and vanishes in A, it is zero in $\alpha\mathbb{D}$. In particular, $0 = (u \circ S)^{(n)}(0) = u(S^{(n)}(0)) = u(z^n)$ for each $n \in \mathbb{N}_0$. As the polynomials are dense in $B_{p,q}(v)$, then $u = 0$. By the Hahn-Banach theorem we conclude that Y is dense in $B_{p,q}(v)$. □

Theorem 3.3.5 *Assume that the differentiation operator $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous. If $\lim_{r\to\infty} v(r)\frac{e^r}{r^{\frac{1}{2p}}} = 0$ if $q = 0$ or $r^{\frac{1}{q}-\frac{1}{2p}} e^r \in L_v^q([r_0, \infty[)$ for some $r_0 > 0$, if $1 \leq q < \infty$, then D is frequently hypercyclic, and thus, hypercyclic. Moreover, it is topologically mixing on $B_{p,q}(v)$ for $1 \leq p \leq \infty$ when $q = 0$ and for $1 < p \leq \infty$ when $1 \leq q < \infty$.*

Proof. By Theorem 0.5.3, in order to prove that D is frequently hypercyclic, it is enough to show that D has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues. As a probability measure we consider the normalized Lebesgue measure on $\mathbb{T}$. If a subset A of $\mathbb{T}$ has Lebesgue measure 1, then A is dense in $\mathbb{T}$. Applying Lemma 3.3.4, span$\{e_a : a \in A\}$ is dense in $B_{p,q}(v)$, and the condition is satisfied.

By Theorem 3.3.3, to prove the assertion about the topologically mixing property, it is enough to show that the limit $\lim_n \frac{\|z^n\|_{q,v}}{n!}$ is equal to zero. For $q = 0$, the hypothesis yields that given $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that $v(r) \leq \varepsilon r^{\frac{1}{2p}} e^{-r}$ for every $r \geq r_\varepsilon$. If we consider r_n a global maximum of the function $r \mapsto v(r)r^n$, by [82, Lemma 1.2], r_n tends to ∞ as n tends to ∞, so, there exists some n_ε such that, for $n \in \mathbb{N}$, $n \geq n_\varepsilon$,

$$\frac{\|z^n\|_v}{n!} = \sup_{r \geq r_\varepsilon} v(r)\frac{r^n}{n!} \leq \varepsilon \sup_{r \geq r_\varepsilon} r^{\frac{1}{2p}} e^{-r} \frac{r^n}{n!} \leq \varepsilon \frac{\|z^n\|_{\frac{1}{2p},1}}{n!}. \tag{3.17}$$

By (1.10), $\frac{\|z^n\|_{\frac{1}{2p},1}}{n!}$ converges to 0 for $1 < p \leq \infty$ and to 1 for $p = 1$. Therefore, since (3.17) holds for every $\varepsilon > 0$, $\lim_n \frac{\|z^n\|_v}{n!} = 0$. For $1 \leq q < \infty$,

$$\begin{aligned} \frac{\|z^n\|_{q,v}^q}{n!^q} &\lesssim \int_{r_0}^{\infty} rv(r)^q \frac{r^{nq}}{n!^q} dr = \int_{r_0}^{\infty} \frac{r^{nq+\frac{q}{2p}} e^{-rq}}{n!^q} v(r)^q r^{1-\frac{q}{2p}} e^{rq} dr \\ &\leq \frac{\|z^n\|_{\frac{1}{2p},1}^q}{n!^q} \int_{r_0}^{\infty} v(r)^q r^{1-\frac{q}{2p}} e^{rq} dr. \end{aligned}$$

Applying again that $\frac{\|z^n\|_{\frac{1}{2p},1}}{n!}$ converges to 0 for $1 < p \leq \infty$, we get $\lim_n \frac{\|z^n\|_{q,v}}{n!} = 0$.
□

Theorem 3.3.6 *Assume that the differentiation operator $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous for some $1 \leq p \leq \infty$, $q = 0$ or $1 \leq q < \infty$. The following conditions are equivalent:*

(i) D is chaotic.

(ii) D has a periodic point different from 0.

(iii) $\lim_{r\to\infty} v(r) \frac{e^r}{r^{\frac{1}{2p}}} = 0$ if $q = 0$ and $r^{\frac{1}{q}-\frac{1}{2p}} e^r \in L^q_v([r_0, \infty[)$ for some $r_0 > 0$, if $1 \leq q < \infty$.

Proof. Clearly (i) implies (ii). Let us see (ii)⇒(iii). By hypothesis, there exists a function $0 \neq f \in B_{p,q}(v)$ such that, for some $n \in \mathbb{N}$, $D^n f = f$. Using the trivial decomposition $D^n - I = (D - \theta_1 I) \dots (D - \theta_n I)$, $\theta_j^n = 1$, $j = 1, \dots, n$, we conclude that there is $\theta \in \mathbb{C}$, $|\theta| = 1$, and $g \in B_{p,q}(v)$, $g \neq 0$, such that $(D - \theta I)g = 0$. This yields $e^{\theta z} \in B_{p,q}(v)$. Using Lemma 3.1.4, we obtain (iii).

(iii)⇒(i) Denote by P the linear span of the functions $e^{\theta z}$, $\theta \in \mathbb{C}$, $\theta^n = 1$ for some $n \in \mathbb{N}$. Obviously, P is formed by periodic points and, by Lemma 3.3.4, it is dense in $B_{p,q}(v)$. On the other hand, since D is hypercyclic by Theorem 3.3.5, it is chaotic. □

Observe that Theorems 3.3.6 and 3.3.5 yield that any chaotic continuous differentiation operator $D : B_{p,q}(v) \to B_{p,q}(v)$ is frequently hypercyclic, even topologically mixing on $B_{p,q}(v)$ for $1 \leq p \leq \infty$ when $q = 0$ and for $1 < p \leq \infty$ when $1 \leq q < \infty$.

In [42, Corollary 2.6, Corollary 2.7 and Corollary 2.10], some examples of weights for which the differentiation operator on $B_{p,q}(v)$, $1 \leq p \leq \infty$, $q = 0$, is topologically mixing, chaotic, or none of them are shown. We present here some examples for the case $1 \leq q < \infty$.

Corollary 3.3.7 *Consider the weight $v_{a,\alpha}$, $a \in \mathbb{R}$, $\alpha > 0$, $1 \le p \le \infty$ and $q = 0$ or $1 \le q < \infty$.*

(a) If $\alpha < 1$, then D is neither hypercyclic nor chaotic on $B_{p,q}(v)$.

(b) If $\alpha > 1$ then D is topologically mixing and chaotic on $B_{p,q}(v)$.

(c) If $\alpha = 1$, D is hypercyclic (even topologically mixing) if and only if $a < \frac{1}{2} - \frac{3}{2q}$ and D is chaotic if and only if $a < \frac{1}{2p} - \frac{2}{q}$.

Proof. Theorem 3.3.6 yields the conclusion about chaos since $v_{a,\alpha}(r)e^r r^{-\frac{1}{2p}} = e^{r(1-\alpha)}r^{a-\frac{1}{2p}}$ tends to zero as $r \to \infty$ if and only if $\alpha > 1$, or $\alpha = 1$ and $a < \frac{1}{2p}$, and $\int_{r_0}^{\infty} r^{1+aq-\frac{q}{2p}}e^{-rq(\alpha-1)}dr < \infty$ if and only if $\alpha > 1$, or $\alpha = 1$ and $a < \frac{1}{2p} - \frac{2}{q}$. Theorem 3.3.3 and (1.10) yield the conclusion about hypercyclicity. □

Corollary 3.3.8 *Assume that $1 \le p \le \infty$ and $q = 0$ or $1 \le q < \infty$.*

(a) If $v(r) = \frac{r^{\frac{1}{2p}-\frac{1}{q}}e^{-r}}{\varphi(r)}$ for r large enough, where $\varphi(r)$ is a positive increasing continuous function such that $\sup_{r>0}\frac{\varphi(r+1)}{\varphi(r)} < \infty$, $\lim_{r\to\infty}\varphi(r) = \infty$ if $q = 0$ or $\frac{1}{\varphi(r)} \in L^q([r_0, \infty[)$ for $1 \le q < \infty$, then $D : B_{p,q}(v) \to B_{p,q}(v)$ is chaotic.

(b) If $v(r) = r^{\frac{1}{2p}-\frac{1}{q}}e^{-r}$ for r large enough, then $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous but it is hypercyclic (even topologically mixing) if and only if $\frac{1}{p} + \frac{1}{q} < 1$. D is never chaotic. Observe that for $q = 0$ and $p > 1$ it is always topologically mixing, but not chaotic.

Proof. (a) is trivial from Proposition 3.3.1 and Theorem 3.3.6. (b) follows from Corollary 3.3.7 considering $a = \frac{1}{2p} - \frac{1}{q}$. □

From now on we restrict our attention to the spaces $B_{p,q}(a, \alpha)$, $a \in \mathbb{R}$, $\alpha > 0$.

Proposition 3.3.9 *Let $1 \le p \le \infty$ and $1 \le q \le \infty$. If $a > 0$, then*

$$||D^n||_{p,q,a,\alpha} = O\left(n!\left(\frac{e\alpha}{n}\right)^n\right).$$

If $a \le 0$ and $n > |a|$, then

$$||D^n||_{p,q,a,\alpha} = O\left(n!\left(\frac{e\alpha}{n+a}\right)^{n+a}\right).$$

If $1 \leq q < \infty$,

$$n!\left(\frac{e\alpha q}{(a+n)q+1}\right)^{n+a+\frac{3}{2q}} = O(||D^n||_{p,q,a,\alpha}),$$

and for $q = \infty$,

$$n!\left(\frac{e\alpha}{n+a}\right)^{n+a} = O(|||D^n|||_{p,a,\alpha}),$$

with equality for $a = 0$.

Proof. For the lower estimate we use

$$\|D^n\|_{p,q,a,\alpha} \geq \|D^n(\frac{z^n}{\|z^n\|_{q,a,\alpha}})\|_{p,q,a,\alpha} = \frac{n!\|1\|_{q,a,\alpha}}{\|z^n\|_{q,a,\alpha}}$$

and (1.9). Applying Jensen's inequality and Fubini's Theorem as in [42, Proposition 2.1], we get

$$\begin{aligned}
M_p(f^{(n)}, r) &= \left(\frac{1}{2\pi}\int_0^{2\pi} |f^{(n)}(re^{i\theta})|^p d\theta\right)^{1/p} \\
&= \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{n!}{2\pi i}\int_0^{2\pi}\frac{f(Re^{i\varphi})iRe^{i\varphi}}{(Re^{i\varphi}-re^{i\theta})^{n+1}}d\varphi\right|^p d\theta\right)^{1/p} \\
&\leq n!\left(\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{|f(Re^{i\varphi})|R}{|Re^{i\varphi}-re^{i\theta}|^{n+1}}d\varphi\right)^p d\theta\right)^{1/p} \\
&= \frac{n!R}{R^2-r^2}\left(\frac{1}{2\pi}\int_0^{2\pi}(\frac{1}{2\pi}\int_0^{2\pi}\frac{|f(Re^{i\varphi})|}{|Re^{i\varphi}-re^{i\theta}|^{n-1}}P_{\frac{r}{R}}(\theta-\varphi)d\varphi)^p d\theta\right)^{1/p} \\
&\leq \frac{n!R}{(R^2-r^2)(R-r)^{n-1}}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\int_0^{2\pi}|f(Re^{i\varphi})|^p P_{\frac{r}{R}}(\theta-\varphi)d\varphi d\theta\right)^{1/p} \\
&= \frac{n!R}{(R^2-r^2)(R-r)^{n-1}}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(Re^{i\varphi})|^p\frac{1}{2\pi}\int_0^{2\pi}P_{\frac{r}{R}}(\theta-\varphi)d\theta d\varphi\right)^{1/p} \\
&= \frac{n!R}{(R^2-r^2)(R-r)^{n-1}}M_p(f,R)
\end{aligned}$$

for every $R > r$, where $P_s(t) = \frac{1-s^2}{1-2s\cos t+s^2}$, $0 \leq s < 1$ is the Poisson Kernel for the unit disc. Then, if we consider $R = r+\varepsilon$ for some $\varepsilon > 0$, we get

$$M_p(f^{(n)}, r) \leq \frac{n!}{\varepsilon^{n-1}}\frac{r+\varepsilon}{\varepsilon^2+2r\varepsilon}M_p(f, r+\varepsilon) \leq \frac{n!}{\varepsilon^n}M_p(f, r+\varepsilon).$$

If $a > 0$, then $\frac{v_{a,\alpha}(r)}{v_{a,\alpha}(r+\varepsilon)} = \frac{r^a e^{-\alpha r}}{(r+\varepsilon)^a e^{-\alpha(r+\varepsilon)}} \leq e^{\alpha\varepsilon}$ for r big enough. Thus,

$$v_{a,\alpha}(r)M_p(f^{(n)}, r) \leq \frac{n!}{\varepsilon^n} e^{\alpha\varepsilon} v_{a,\alpha}(r+\varepsilon)M_p(f, r+\varepsilon) \tag{3.18}$$

for r big enough. This yields that there exists a constant $A > 0$ such that $\|D^n\|_{p,q,a,\alpha} \leq A\frac{n!}{\varepsilon^n}e^{\alpha\varepsilon}$ for every $\varepsilon > 0$. If we take $\varepsilon = \frac{n}{\alpha}$, which minimizes $\frac{e^{\alpha\varepsilon}}{\varepsilon^n}$, we get

$$||D^n||_{p,q,a,\alpha} \leq An! \left(\frac{e\alpha}{n}\right)^n .$$

If $a \leq 0$, then there exists a constant $B > 0$ such that $\frac{v_{a,\alpha}(r)}{v_{a,\alpha}(r+\varepsilon)} = \frac{r^a e^{\alpha\varepsilon}}{(r+\varepsilon)^a} \leq B\frac{e^{\alpha\varepsilon}}{\varepsilon^a}$ for r big enough and $\varepsilon > \varepsilon_0$, for some $\varepsilon_0 > 0$. Thus,

$$v_{a,\alpha}(r)M_p(f^{(n)}, r) \leq B\frac{n!e^{\alpha\varepsilon}}{\varepsilon^{n+a}} v_{a,\alpha}(r+\varepsilon)M_p(f, r+\varepsilon) \tag{3.19}$$

for r, ε, big enough. Therefore, if we take $\varepsilon = \frac{n+a}{\alpha} \geq \varepsilon_0$, we obtain that there exists some $D_2 > 0$ such that

$$||D^n||_{p,q,a,\alpha} \leq D_2 n! \left(\frac{e\alpha}{n+a}\right)^{n+a}$$

for every $n \in \mathbb{N}$. □

Proposition 3.3.10 *The spectrum of $D : B_{p,q}(a,\alpha) \to B_{p,q}(a,\alpha)$ satisfies*

$$\sigma(D) = \alpha\overline{\mathbb{D}}$$

for $1 \leq p \leq \infty$ and $q = 0$ or $1 \leq q \leq \infty$.

Proof. If $|\lambda| < \alpha$, the function $e_\lambda(z) := e^{\lambda z}$ belongs to $B_{p,q}(a,\alpha)$ by Lemma 3.1.4 and satisfies $De_\lambda = \lambda e_\lambda$. Therefore, $\alpha\mathbb{D} \subseteq \sigma(D)$. On the other hand, the spectral radius of D satisfies $r(D) = \lim_n ||D^n||^{1/n}_{p,q,a,\alpha}$. Using the Stirling formula and the upper estimates for the norms in Proposition 3.3.9, $r(D) \leq \alpha$. □

Lemma 3.3.11 ([10, Proposition 4]) *Let T be a bounded operator on a Banach space X. If λ belongs to the boundary of $\sigma(T)$, then $T - \lambda I$ is not surjective.*

Lemma 3.3.11 implies that $D - \lambda I$ is not surjective on $B_{p,q}(a,\alpha)$ for $|\lambda| = \alpha$. $D - \lambda I$ is injective if and only if $e^{\lambda z} \notin B_{p,q}(a,\alpha)$. So, by Lemma 3.1.4, the next proposition holds:

Proposition 3.3.12 *For the weight $v_{a,\alpha}(r) = r^a e^{-\alpha r}$, r big enough, and $1 \leq p \leq \infty$, $D - \lambda I$ is injective on $B_{p,q}(a, \alpha)$ if and only if $|\lambda| > \alpha$ or $|\lambda| = \alpha$ and*

(i) $a \geq \frac{1}{2p}$ *when* $q = 0$,

(ii) $a > \frac{1}{2p}$ *when* $q = \infty$,

(iii) $a \geq \frac{1}{2p} - \frac{2}{q}$ *if* $1 \leq q < \infty$.

By Propositions 3.2.3 and 3.2.4, we get the following:

Proposition 3.3.13 *Let v be a weight such that D is continuous on $B_{p,q}(v)$, $1 \leq p \leq \infty$, $q \in \{0, p, \infty\}$, and $v(r)e^{\alpha r}$ is non increasing. If $|\lambda| < \alpha$, the operator $D - \lambda I$ is surjective on $B_{p,q}(v)$ and it even has*

$$K_\lambda f(z) = z \int_0^1 e^{\lambda z(1-t)} f(zt)dt, \ \ z \in \mathbb{C},$$

as a continuous linear right inverse. In particular, this is satisfied for the weight $v_{a,\alpha}$, $a \leq 0$, $\alpha > 0$.

Theorem 3.3.14 *Given $1 \leq p \leq \infty$, $q = 0$ or $1 \leq q \leq \infty$:*

(i) For $\alpha > 1$ or $\alpha = 1$ and $a < \frac{1}{2} - \frac{3}{2q}$, D is not power bounded on $B_{p,q}(a, \alpha)$.

(ii) If D is chaotic, then D is not mean ergodic on $B_{p,q}(a, \alpha)$. Consequently, D is not mean ergodic on $B_{p,q}(a, \alpha)$ if $\alpha > 1$ or $\alpha = 1$ and $a < \frac{1}{2p} - \frac{2}{q}$.

(iii) For $\alpha < 1$, D is power bounded and uniformly mean ergodic on $B_{p,q}(a, \alpha)$.

(iv) D is not uniformly mean ergodic on $B_{p,q}(a, 1)$, $a \in \mathbb{R}$.

Proof.

(i) By (1.10),

$$\|D^n\|_{p,q,a,\alpha} \geq \frac{n! \|1\|_{q,a,\alpha}}{\|z^n\|_{q,a,\alpha}} \gtrsim \frac{\alpha^n}{n^{a + \frac{3}{2q} - \frac{1}{2}}}$$

and this limit tends to infinity for the values of α in the hypothesis.

(ii) If D is mean ergodic, for each $f \in B_{p,q}(a, \alpha)$, $\frac{f' + f'' + \cdots f^{(N)}}{N} \to 0$, which is not the case if D is chaotic, since $e^z \in B_{p,q}(a, \alpha)$.

(iii) Since

$$n!\left(\frac{e\alpha}{n}\right)^n \leq n!\left(\frac{e\alpha}{n-a}\right)^{n-a}$$

for every $a > 0$ and n big enough, Proposition 3.3.9 yields

$$||D^n||_{p,q,a,\alpha} = O\left(n!\left(\frac{e\alpha}{n-|a|}\right)^{n-|a|}\right).$$

Applying the Stirling's formula we get

$$||D^n||_{p,q,a,\alpha} = O\left(\left(\frac{n}{n-|a|}\right)^{n-|a|} n^{|a|+1/2}\alpha^{n-|a|}\right).$$

Therefore, for $\alpha < 1$, $\lim_{n\to\infty} ||D^n||_{p,q,a,\alpha} = 0$, and thus,

$$\lim_{m\to\infty}\left\|\frac{1}{m}\sum_{j=1}^{m} D^j\right\|_{p,q,a,\alpha} \leq \lim_{m\to\infty}\frac{1}{m}\sum_{j=1}^{m}||D^j||_{p,q,a,\alpha} = 0.$$

(iv) Since $1 \in \sigma(D)$, the conclusion follows from Proposition 3.1.9. □

Corollary 3.3.15 *Given* $v_\alpha(r) = e^{-\alpha r}$, $\alpha > 0$, $1 \leq p \leq \infty$, $q = 0$ *or* $1 \leq q \leq \infty$, *we have the following:*

(i) If $\alpha > 1$, *then* D *is not mean ergodic on* $B_{p,q}(v)$.

(ii) If $\alpha < 1$, *then* D *is uniformly mean ergodic on* $B_{p,q}(v)$.

(iii) If $\alpha = 1$, *then* D *is not uniformly mean ergodic on* $B_{p,q}(v)$. *It is not mean ergodic for* $p = q = \infty$ *and for* $1 \leq p < \infty$ *if* $\frac{2}{q} < \frac{1}{2p}$, $q \neq \infty$.

Proof. All the statements but one follow by Proposition 3.3.14. Let us see that D is mean ergodic on $H_1(\mathbb{C})$. Since $H_1(\mathbb{C})$ is a Grothendieck Banach space with the Dunford-Pettis property (in fact it is isomorphic to ℓ_∞ by Galbis [68] or Lusky [95]) and $||D^n||_1/n \to 0$, we can apply Theorem 0.5.6 to conclude that D is not mean ergodic on $H_1(\mathbb{C})$ because it is not uniformly mean ergodic by Propositions 3.3.14 (iv) and 3.1.8. □

We do not know if the differentiation operator is mean ergodic on the space $B_{p,q}(1)$ for $q = 0$ and $p = \infty$, for $q = \infty$ and $1 \leq p < \infty$, and for $1 \leq q < \infty$ and $q \leq 4p$. Related partial results can be seen in [35].

3.4 The Hardy operator

Whereas the behavior of the iterates of the differentiation and the integration operators depends heavily on the weights, the Hardy operator is power bounded and uniformly mean ergodic in all cases.

Theorem 3.4.1 *Given a weight* v, $1 \leq p \leq \infty$ *and* $q = 0$ *or* $1 \leq q \leq \infty$, *the Hardy operator* $H : B_{p,q}(v) \to B_{p,q}(v)$, $Hf(z) = \frac{1}{z}\int_0^z f(\zeta)d\zeta$, $z \in \mathbb{C}$, *is well defined and continuous with norm* $\|H\| = 1$. *Moreover,* H^2 *is compact and* $H^2(B_{p,\infty}(v)) \subseteq B_{p,0}(v)$. *If the integration operator* $J : B_{p,q}(v) \to B_{p,q}(v)$ *is continuous, then* H *is compact. Moreover,* $H(B_{p,\infty}(v)) \subseteq B_{p,0}(v)$.

Proof. For every $f \in \mathcal{H}(\mathbb{C})$ and $r \geq 0$ we have

$$M_p(Hf,r)^p = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{re^{i\theta}}\int_0^{re^{i\theta}} f(\omega)d\omega\right|^p d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^1 f(tre^{i\theta})dt\right|^p d\theta$$
$$\leq \int_0^1 \frac{1}{2\pi}\int_0^{2\pi} |f(tre^{i\theta})|^p d\theta dt \leq M_p(f,r)^p.$$

Hence, for every $f \in B_{p,q}(v)$ we have $\|Hf\|_{p,q,v} \leq \|f\|_{p,q,v}$ and $\|H\| := \|H\|_{p,q,v} \leq 1$. On the other hand, since $H(c) = c$ for every $c \in \mathbb{C}$, taking $g := c/\|c\|_{q,v} \in B_{p,q}(v)$, we obtain $\|H\| = 1$.

Given $f = \sum_{k=0}^\infty a_k z^k \in B_{p,q}(v)$, the Cauchy and Jensen inequalities imply

$$|a_k| = \left|\frac{1}{2\pi i}\int_{|\omega|=R}\frac{f(\omega)}{\omega^{k+1}}d\omega\right| \leq \frac{1}{R^k}M_1(f,R) \leq \frac{1}{R^k}M_p(f,R) \tag{4.20}$$

for every $R > 0$, then, $|a_k|\|z^k\|_{p,q,v} \leq \|f\|_{p,q,v}$ for every $k \in \mathbb{N}_0$. As $H^2 f(z) = \sum_{k=0}^\infty \frac{a_k}{(k+1)^2} z^k$, one has

$$\left\|H^2 f - \sum_{k=0}^N \frac{a_k}{(k+1)^2} z^k\right\|_{p,q,v} \leq \sum_{k=N+1}^\infty \frac{|a_k|\|z^k\|_{q,v}}{(k+1)^2} \leq \|f\|_{p,q,v}\sum_{k=N+1}^\infty \frac{1}{(k+1)^2}, \tag{4.21}$$

which shows that the finite rank operators $H_N^2(\sum_{k=0}^\infty a_k z^k) := \sum_{k=0}^N \frac{a_k}{(k+1)^2} z^k$ are bounded on $B_{p,q}(v)$ and that

$$||H^2 - H_N^2||_{p,q,v} \leq \sum_{k=N+1}^\infty \frac{1}{(k+1)^2},$$

from where the compactness of H^2 follows. Since $H^2 f$ belongs to the closure of the polynomials, $H^2 f$ belongs to $B_{p,0}(v)$ if $f \in B_{p,\infty}(v)$.

Finally, suppose that the integration operator $J : B_{p,q}(v) \to B_{p,q}(v)$ is continuous. Since for every $r \geq 0$, $M_p(Hf, r) = \frac{1}{r} M_p(Jf, r)$, then the Hardy operator $H : B_{p,\infty}(v) \to B_{p,0}(v)$ is well defined, as for every $r \geq 0$,

$$v(r)M_p(Hf, r) = v(r)\frac{1}{r}M_p(Jf, r) \leq \frac{|||J|||_{p,v}}{r}|||f|||_{p,v}.$$

Take a sequence $\{f_n\}_n$ in the unit ball of $B_{p,q}(v)$. As it is compact with respect to the compact open topology τ_{co}, there exists a subsequence $\{n_k\}_k$ such that f_{n_k} tends to some f in the unit ball of $B_{p,q}(v)$ in τ_{co}. Given $\varepsilon > 0$, take $R > 0$ such that $R > \frac{2}{\varepsilon}\|J\|_{p,q,v}$ in order to get

$$v(r)M_p(Hf_{n_k} - Hf, r) \leq v(r)\frac{1}{r}M_p(Jf_{n_k} - Jf, r) \leq \frac{2}{R}\|J\|_{p,q,v} \leq \varepsilon$$

for $r \geq R$ and

$$\begin{aligned} 2\pi \int_R^\infty & rv(r)^q M_p(Hf_{n_k} - Hf, r)^q dr \\ & \leq 2\pi \int_R^\infty r^{1-q} v(r)^q M_p(Jf_{n_k} - Jf, r)^q dr \\ & \leq \frac{1}{R^q}\|Jf_{n_k} - Jf\|_{p,q,v}^q \leq \frac{2^q}{R^q}\|J\|_{p,q,v}^q < \varepsilon^q. \end{aligned}$$

Since the Hardy operator $H : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ is continuous, we get that there exists k_0 such that, for $k \geq k_0$, $\|Hf_{n_k} - Hf\|_{p,q,v} \leq \varepsilon$, and therefore, H is compact. □

An operator T is said to be *quasi-compact* if T^m is compact for some $m \in \mathbb{N}$. Quasi-compact operators share some properties of compact operators, in particular its spectrum $\sigma(T)$ reduces to its eigenvalues and $\{0\}$.

Corollary 3.4.2 *The Hardy operator H is power bounded and uniformly mean ergodic on $B_{p,q}(v)$ for $1 \leq p \leq \infty$ and $q = 0$ or $1 \leq q \leq \infty$. Moreover, its spectrum is $\sigma(H) = \{\frac{1}{n}\}_n \cup \{0\}$.*

Proof. As H is quasi-compact, $\sigma(H) = \overline{\{\lambda : \lambda \text{ is an eigenvalue of } H\}}$, and the eigenvalues of H are $\{\frac{1}{n} : n \in \mathbb{N}\}$. Clearly H is power bounded. The compactness of H^2 implies that $\mathrm{Im}(I - H^2) = \mathrm{Im}(I - H)(I + H) = \mathrm{Im}(I - H)$ is closed, since $-1 \notin \sigma(H)$. Now the conclusion follows from a criterion due to Lin (see Theorem 0.5.5). □

Observe that contrary to what happens for the operators of integration J and of differentiation D, the Hardy operator H is uniformly mean ergodic and 1 belongs to the spectrum of H on the space $B_{p,q}(v)$. In this case, the Cesàro means of the iterates of H do not converge to zero on the polynomials. Being power bounded,

H cannot be hypercyclic on $B_{p,q}(v)$. In fact, since $\delta_0(H^n f) = f(0)$ for each $f \in \mathcal{H}(\mathbb{C})$, H is not hypercyclic on $\mathcal{H}(\mathbb{C})$. Moreover, it is not difficult to show that the spectrum of $H : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ reduces to its eigenvalues $\{\frac{1}{n}\}_n$, since by the Cauchy-Hadamard theorem, $H - \lambda I$ is surjective for $\lambda \notin \{\frac{1}{n}\}_n$.

3.5 Differential operators

This section is devoted to study the dynamics of differential operators $\phi(D) : B_{p,q}(v) \to B_{p,q}(v)$ whenever $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous and ϕ is an entire function. Godefroy and Shapiro proved that if $T : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$, $T \neq \lambda I$, commutes with D, that is, $TD = DT$, it can be expressed as a differential operator $\phi(D)$ for an entire function ϕ of exponential type [73]. Moreover, they proved that T is chaotic. MacLane also considered the question about what are the possible rates of growth of D-hypercyclic functions. He showed that there exists a D-hypercyclic entire function f of exponential type 1, that is, for all $\varepsilon > 0$ there is $M > 0$ with $|f(z)| \leq Me^{(1+\varepsilon)|z|}$. Bernal and Bonilla [21] have attacked the same problem for general T following the idea of Chan and Shapiro (1991) of replace $\mathcal{H}(\mathbb{C})$ by a space of entire functions of restricted growth. We continue their work focusing this problem on the weighted spaces of entire functions $B_{p,q}(v)$, $1 \leq p \leq \infty$, $q = 0$ or $1 \leq q < \infty$. See [80, 4.2] for more references and background on this topic.

An entire function ϕ is said to be of *exponential type* if there are constants $A, R > 0$ such that

$$|\phi(z)| \leq Ae^{R|z|} \text{ for all } z \in \mathbb{C}.$$

Equivalently, an entire function $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ is of exponential type if and only if there are $A, B > 0$ such that, for $n \geq 0$, $|a_n| \leq A\frac{B^n}{n!}$ (see [80, Lemma 4.18]).

Given $\phi(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{C})$, the *formal differential operator* $\phi(D)$ is defined by

$$\phi(D)f = \sum_{n=0}^{\infty} a_n f^{(n)}, \;\; f \in \mathcal{H}(\mathbb{C}).$$

In [80, Proposition 4.19] it is shown that given an entire function of exponential type $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\phi(D)f = \sum_{n=0}^{\infty} a_n D^n f$ converges in $\mathcal{H}(\mathbb{C})$ for every entire function f and defines a continuous operator on $\mathcal{H}(\mathbb{C})$.

Our aim now is to extend the results given in [21], where it is shown that if $T : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$, $T \neq \lambda I$, commutes with D and $T = \phi(D)$ is its representation with an entire function ϕ, then for any $\tau > \min\{|z| : |\phi(z)| = 1\}$ there is a T-hypercyclic entire function with $|f(z)| \leq Me^{\tau|z|}$, $z \in \mathbb{C}$. This seems to be the best growth result known for general operators T commuting with D. For individual operators, much better results are available. For $T = D$, since $\phi(z) = z$, and $\min\{|z| : |\phi(z)| = 1\} = 1$, for any $\varepsilon > 0$ there is some D-hypercyclic function f with

$$|f(z)| \leq Me^{(1+\varepsilon)|z|}, \ z \in \mathbb{C}.$$

MacLane had already shown the better result that type 1 is possible. MacLane's growth condition can be improved, and one can even determine the least possible rate of growth.

Theorem 3.5.1 ([75]) *If $\varphi :]0, \infty[\to [1, \infty[$ is any function with $\varphi(r) \to \infty$ as $r \to \infty$ then there is a D-hypercyclic entire function f with*

$$|f(z)| \leq \varphi(|z|) \frac{e^{|z|}}{\sqrt{|z|}} \text{ for } |z| \text{ sufficiently large,}$$

but there is no D-hypercyclic entire function f that satisfies

$$|f(z)| \leq M \frac{e^{|z|}}{\sqrt{|z|}} \text{ for } |z| \neq 0$$

with some $M > 0$.

For the translation operator $T_{z_0} f(z) = f(z + z_0)$, $z_0 \neq 0$, we have that $T_{z_0} = e^{z_0 D}$ since the Taylor series of f centred at $z \in \mathbb{C}$ gives

$$f(z + z_0) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z) z_0^k}{k!}.$$

Hence, $T_{z_0} = \phi(D)$, $\phi(z) = e^{z_0 z}$ and $\min\{|z| : |\phi(z)| = 1\} = 0$. This yields that for any $\varepsilon > 0$ there is some hypercyclic function with $|f(z)| \leq Me^{\varepsilon|z|}$, $z \in \mathbb{C}$. But Duyos-Ruiz (1983) had already shown that there are hypercyclic functions of arbitrarily slow transcendental growth.

Given an operator T defined on a Banach space X and an entire function $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$, the expression $\phi(T) = \sum_{n=0}^{\infty} a_n T^n$ defines also an operator on X. In fact, $\|\phi(T)\| \leq \sum_{n=0}^{\infty} |a_n| \|T\|^n < \infty$. In the next theorem we study the hypercyclicity and chaos of the differential operator $\phi(D)$ on the weighted Banach spaces of entire functions $B_{p,q}(v)$, $1 \leq p \leq \infty$, $q = 0$ or $1 \leq q \leq \infty$, defined by a weight v in which the operator $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous. We thank L. Bernal for suggesting the author the ideas for the proof:

Theorem 3.5.2 *Let ϕ be a nonconstant entire function and $\alpha > \min\{|z| : |\phi(z)| = 1\}$. Let $1 \leq p \leq \infty$ and suppose v is a weight such that $\lim_{r\to\infty} v(r)\frac{e^{\alpha r}}{r^{\frac{1}{2p}}} = 0$ for $q = 0$ or $r^{\frac{1}{q}-\frac{1}{2p}}e^{\alpha r} \in L_v^q([r_0,\infty[)$ for some $r_0 > 0$ and $1 \leq q < \infty$. If $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous, then the operator $\phi(D) : B_{p,q}(v) \to B_{p,q}(v)$ is topologically mixing. Moreover, $\phi(D)$ is chaotic and not mean ergodic.*

Proof. We prove that $\phi(D)$ verifies the Hypercyclicity Criterion for the entire sequence of positive integers. Consider the sets

$$V := \text{span}\{e_a : |a| < \alpha, |\phi(a)| < 1\},$$

$$W := \text{span}\{e_a : |a| < \alpha, |\phi(a)| > 1\},$$

where $e_a(z) := e^{az}$, $a \in \mathbb{C}$, $z \in \mathbb{C}$. Since ϕ is non constant and open, $\phi(\alpha\mathbb{D})$ is an open set, which together with $\phi(\alpha\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ implies that $\alpha\mathbb{D} \cap \phi^{-1}(\mathbb{D})$ and $\alpha\mathbb{D} \cap \phi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}})$ are non-empty open sets of $\mathbb{C}$. Thus, they have an accumulation point in $\alpha\mathbb{D}$. On the other hand, by Lemma 3.1.4, $e_a \in B_{p,q}(v)$ for all $a \in \alpha\mathbb{D}$, which implies by Lemma 3.3.4 that V and W are dense subsets of $B_{p,q}(v)$. Furthermore, $\phi(D)e_a = \phi(a)e_a$ for all $n \in \mathbb{N}$, then, $\phi(D)^n e_a = \phi(a)^n e_a$ for each $n \in \mathbb{N}$, $a \in \mathbb{C}$. Hence, by linearity, $\phi(D)^n f \to 0$ as n tends to infinity for all $f \in V$. Define $S : W \to B_{p,q}(v)$ as $S(e_a) = \frac{1}{\phi(a)}e_a \in W$ and extend it by linearity to W. Then, $S^n(e_a) = \frac{1}{\phi(a)^n}e_a \to 0$ if n tends to infinity, so $S^n f$ converges to 0 for every $f \in W$. As $\phi(D)^n S^n f = f$ for all $f \in W$, the Hypercyclicity Criterion holds. Consider now the set

$$P := \text{span}\{e_a : |a| < \alpha,\ \phi(a)^n = 1,\ n \in \mathbb{N}\} \subseteq B_{p,q}(v).$$

As $\phi(D)^n e_a = \phi(a)^n e_a$, P is formed by periodic points. There exists $z_0 \in \alpha\mathbb{D}$ such that $|\phi(z_0)| = 1$. Take an open set U containing z_0 and such that $\overline{U} \subseteq \alpha\mathbb{D}$ As $\phi(U)$ is open, $\phi(U) \cap \mathbb{T}$ contains a dense set formed by roots of the unity. Then, the preimages by ϕ of this set contain a sequence in $\overline{U}$, and thus, the set $\{a \in \mathbb{C} : |a| < \alpha,\ \phi(a)^n = 1,\ n \in \mathbb{N}\}$ has an accumulation point in $\alpha\mathbb{D}$. Therefore, again by Lemma 3.3.4, P is a dense set of periodic points, and hence, $\phi(D)$ is chaotic. Finally, given $a \in \mathbb{C}$, $|a| < \alpha$, with $|\phi(a)| > 1$, $\frac{\|\phi(D)^n(e_a)\|_{p,q,v}}{n} = \frac{|\phi(a)|^n}{n}\|e_a\|_{p,q,v}$ does not tend to zero. Therefore, $\phi(D)$ cannot be mean ergodic. □

Corollary 3.5.3 *Let $1 \leq p \leq \infty$ and suppose v is a weight such that there exists $\alpha > 0$ with $\lim_{r\to\infty} v(r)\frac{e^{\alpha r}}{r^{\frac{1}{2p}}} = 0$ for $q = 0$, or $r^{\frac{1}{q}-\frac{1}{2p}}e^{\alpha r} \in L_v^q([r_0,\infty[)$ for some $r_0 > 0$ and $1 \leq q < \infty$. If the differentiation operator $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous, then the translation operator $T_{z_0} : B_{p,q}(v) \to B_{p,q}(v)$, $T_{z_0}f(z) = f(z+z_0)$, $z \in \mathbb{C}$, $z_0 \neq 0$, is topologically mixing, chaotic and not mean ergodic.*

Proof. We have seen above that the translation operator is a differential operator given by the entire function of exponential type $\phi(z) = e^{z_0 z}$, $z \in \mathbb{C}$. Since the hypothesis of Proposition 3.5.2 are satisfied, it is topologically mixing and chaotic. □

3.6 Dynamics of differentiation and integration operators on $H_v^0(\mathbb{C})$ and $H_v(\mathbb{C})$

In order to simplify the exposition, in this section we summarize the results obtained in this chapter about the dynamics of the differentiation, the integration and the Hardy operators acting on the weighted Banach spaces of entire functions $H_v^0(\mathbb{C})$ and $H_v(\mathbb{C})$. We omit the proofs here, since they are included in the last sections. This content is published by Bonet, Fernández and the author in [17].

Lemma 3.6.1 *Let $T : (\mathcal{H}(\mathbb{C}), \tau_{co}) \to (\mathcal{H}(\mathbb{C}), \tau_{co})$ be a continuous linear operator such that $T(\mathcal{P}) \subseteq \mathcal{P}$. Given a weight v, the following conditions are equivalent:*

(i) $T(H_v^\infty(\mathbb{C})) \subseteq H_v^\infty(\mathbb{C})$,

(ii) $T : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ *is continuous,*

(iii) $T(H_v^0(\mathbb{C})) \subseteq H_v^0(\mathbb{C})$,

(iv) $T : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C})$ *is continuous.*

Moreover, if (i)-(iv) hold, then $||T||_{L(H_v^\infty(\mathbb{C}))} = ||T||_{L(H_v^0(\mathbb{C}))}$ *and* $\sigma_{H_v^\infty(\mathbb{C})}(T) = \sigma_{H_v^0(\mathbb{C})}(T)$.

Proof. All the statements but the one about the spectrum are proved in Lemma 3.1.7. This follows easily since the reflexivity of $\mathcal{H}(\mathbb{C})$ yields $T'' = T$ and the bidual of $H_v^0(\mathbb{C})$ is $H_v(\mathbb{C})$ by [27]. So, the bitranspose of $T : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C})$ is $T : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$, and it is well-known that $\sigma(T) = \sigma(T'')$. □

In the case of the integration operator J we get the following:

Proposition 3.6.2 *The operator J is never hypercyclic on $H_v^0(\mathbb{C})$.*

Proposition 3.6.3 *(i) Let v be a weight such that $v(r)e^{\alpha r}$ is non increasing for some $\alpha > 0$. Then, J is continuous with $||J||_v \leq 1/\alpha$. In particular, $\sigma_v(J) \subseteq (1/\alpha)\overline{\mathbb{D}}$.*

(ii) If J is continuous on $H_v(\mathbb{C})$ and $v(r)e^{\alpha r}$ is increasing for some $\alpha > 0$, then $\sigma_v(J) \supseteq (1/\alpha)\mathbb{D}$.

(iii) $||J^n||_{a,\alpha} \approx 1/\alpha^n$ *for all* $n \in \mathbb{N}_0$, $\alpha > 0$ *and* $a \leq 0$, *with equality for* $a = 0$, *and* $\sigma_{a,\alpha}(J) = (1/\alpha)\overline{\mathbb{D}}$.

Theorem 3.6.4 *(a) If* $J : H_v(\mathbb{C}) \to H_v(\mathbb{C})$ *is continuous, then:*

- *(i) If* $r^a e^{-r} = O(v(r))$, *with* $a > 1/2$, *then* J *is not power bounded on* $H_v(\mathbb{C})$.
- *(ii)* J *is not uniformly mean ergodic on* $H_v^0(\mathbb{C})$ *if for all* $\beta > 1$, $v(r)e^{\beta r}$ *is increasing in some interval* $[r_0, \infty[$. *In particular* J *is not uniformly mean ergodic on* $H_{a,1}^0(\mathbb{C})$ *for all* $a \in \mathbb{R}$.
- *(iii) If* $r^{3/2}e^{-r} = O(v(r))$, *then* J *is not mean ergodic on* $H_v^0(\mathbb{C})$. *In particular, it is not mean ergodic in* $H_{a,\alpha}^0(\mathbb{C})$ *when* $\alpha < 1$, $a \in \mathbb{R}$.

(b)

- *(iv)* J *is power bounded on* $H_v(\mathbb{C})$ *and mean ergodic on* $H_v^0(\mathbb{C})$ *provided that* $v(r)e^r$ *is non increasing in some interval* $[r_0, \infty[$. *In particular, it is mean ergodic on* $H_{a,1}^0(\mathbb{C})$ *for every* $a \leq 0$.
- *(v)* J *is uniformly mean ergodic on* $H_v(\mathbb{C})$ *if for some* $\alpha > 1$, $v(r)e^{\alpha r}$ *is non increasing.*

For the differentiation operator we get:

Proposition 3.6.5 *The following holds for* $a > 0$*:*

$$||D^n||_{a,\alpha} = O\left(n!\left(\frac{e\alpha}{n}\right)^n\right)$$

and

$$n!\left(\frac{e\alpha}{n+a}\right)^{n+a} = O(||D^n||_{a,\alpha}).$$

For $a \leq 0$ *and* $n > |a|$,

$$||D^n||_{a,\alpha} \approx n!\left(\frac{e\alpha}{n+a}\right)^{n+a},$$

with equality for $a = 0$. *Moreover,* $\sigma_{a,\alpha}(D) = \alpha\overline{\mathbb{D}}$ *for every* $a \in \mathbb{R}$, $\alpha > 0$.

By [10, Proposition 4], $D - \lambda I$ *is not surjective on* $H_{a,\alpha}(\mathbb{C})$ *or on* $H_{a,\alpha}^0(\mathbb{C})$ *for* $|\lambda| = \alpha$. On the other hand, we get the following:

Proposition 3.6.6 *Let v be a weight such that D is continuous on $H_v(\mathbb{C})$ and that $v(r)e^{\alpha r}$ is non increasing. If $|\lambda| < \alpha$, then the operator $D - \lambda I$ is surjective on $H_v(\mathbb{C})$ and it even has a continuous and linear right inverse. The same holds on $H_v^0(\mathbb{C})$. In particular, if $|\lambda| < \alpha$, $D - \lambda I$ has a continuous and linear right inverse on $H_{a,\alpha}(\mathbb{C})$.*

Proposition 3.6.7 *(i) For $\alpha > 1$ or $\alpha = 1$ and $a < 1/2$, D is not power bounded on $H_{a,\alpha}(\mathbb{C})$.*

(ii) If D is chaotic, then D is not mean ergodic on $H_v^0(\mathbb{C})$. Consequently, D is not mean ergodic on $H_{a,\alpha}^0(\mathbb{C})$ if $\alpha > 1$ or if $\alpha = 1$ and $a < 0$.

(iii) For $\alpha < 1$, D is power bounded and uniformly mean ergodic on $H_{a,\alpha}(\mathbb{C})$.

(iv) D is not uniformly mean ergodic on $H_{a,1}^0(\mathbb{C})$, $a \in \mathbb{R}$ and not mean ergodic on $H_1(\mathbb{C})$.

We do not know if the differentiation operator is mean ergodic on the space $H_1^0(\mathbb{C})$. Related partial results can be seen in [35].

Corollary 3.6.8 *Consider the weight $v_{a,\alpha}$, $a \in \mathbb{R}$, $\alpha > 0$.*

(a) If $\alpha < 1$, then D is neither hypercyclic nor chaotic on $H_v^0(\mathbb{C})$.

(b) If $\alpha > 1$ then D is topologically mixing and chaotic on $H_v^0(\mathbb{C})$.

(c) If $\alpha = 1$, D is hypercyclic (even topologically mixing) if and only if $a < \frac{1}{2}$ and D is chaotic if and only if $a < 0$.

To finish, we look at the Hardy operator $H : H_v(\mathbb{C}) \to H_v(\mathbb{C})$:

Theorem 3.6.9 *Let v be an arbitrary weight. The Hardy operator $H : H_v(\mathbb{C}) \to H_v(\mathbb{C})$ is well defined and continuous with $\|H\|_v = 1$. Moreover, $H^2(H_v(\mathbb{C})) \subseteq H_v^0(\mathbb{C})$ and H^2 is compact. If the integration operator $J : H_v(\mathbb{C}) \to H_v(\mathbb{C})$ is continuous, then H is compact. So, H is power bounded and uniformly mean ergodic on $H_v(\mathbb{C})$. Moreover, its spectrum is $\sigma(H) = \{\frac{1}{n}\}_n \cup \{0\}$.*

The operator H can be compact even if J is not continuous. In fact, by the work of Harutyunyan and Lusky [82], the integration operator J is not continuous on $H_v(\mathbb{C})$ for $v(r) = \exp(-(\log r)^2)$. Moreover, by Lusky [94, Theorem 2.5.], the monomials constitute a basis of the space $H_v^0(\mathbb{C})$ and the norm of $H_v(\mathbb{C})$ is equivalent to $||\sum_{k=0}^{\infty} a_k z^k||_v = \sup_k |a_k| ||z^k||_v$. Moreover, $H_v^0(\mathbb{C})$ is isomorphic to c_0. In this example the Hardy Operator H maps $H_v(\mathbb{C})$ into $H_v^0(\mathbb{C})$ (just look at the Taylor expansion of the function). As $H_v(\mathbb{C})$ is canonically isometric to the bidual of $H_v^0(\mathbb{C})$ by [27], H is weakly compact as an operator on both spaces $H_v(\mathbb{C})$ and $H_v^0(\mathbb{C})$. Since $H_v^0(\mathbb{C})$ is isomorphic to c_0, H is compact on $H_v^0(\mathbb{C})$ (see e.g. [84, Corollary 17.2.6]). As H on $H_v(\mathbb{C})$ coincides with the bitranspose, it follows that it is also compact.

The next theorem summarizes our results for the spaces $H_\alpha(\mathbb{C})$ and $H_\alpha^0(\mathbb{C})$:

Theorem 3.6.10 *(i) The differentiation operator D satisfies $||D^n||_\alpha = n!\left(\frac{e\alpha}{n}\right)^n$ for each $n \in \mathbb{N}$, hence it is power bounded if and only if $\alpha < 1$. The spectrum of D is the closed disc of radius α. It is uniformly mean ergodic on $H_\alpha(\mathbb{C})$ and $H_\alpha^0(\mathbb{C})$ if $\alpha < 1$, not mean ergodic if $\alpha > 1$, and it is not mean ergodic on $H_1(\mathbb{C})$ and not uniformly mean ergodic on $H_1^0(\mathbb{C})$.*

(ii) The integration operator J is never hypercyclic on $H_\alpha^0(\mathbb{C})$ and it satisfies $||J^n||_\alpha = 1/\alpha^n$ for each $n \in \mathbb{N}$. Hence, it is power bounded if and only if $\alpha \geq 1$. The spectrum of J is the closed disc of radius $1/\alpha$. If $\alpha > 1$, then J is uniformly mean ergodic on $H_\alpha(\mathbb{C})$ and $H_\alpha^0(\mathbb{C})$ and it is not mean ergodic on these spaces if $\alpha < 1$. If $\alpha = 1$, then J is not mean ergodic on $H_1(\mathbb{C})$, and mean ergodic but not uniformly mean ergodic on $H_1^0(\mathbb{C})$.

Theorem 3.6.11 ([41, Corollary 2.6]) *The differentiation operator on $H_\alpha^0(\mathbb{C})$ is not hypercyclic and has no periodic point different from 0 if $\alpha < 1$, it is hypercyclic and has a dense set of periodic points if $\alpha > 1$ and it is hypercyclic but has no periodic point different from 0 if $\alpha = 1$.*

3.7 An example of a topologically mixing and mean ergodic operator

The author thanks A. Peris for providing her with the following theorem, where an example of a topologically mixing uniformly mean ergodic operator is given. The example is published by Martínez-Giménez, Oprocha and Peris in [97, Theorem 2.1], and gives an example of a topologically mixing not distributionally chaotic operator. Peris uses the same example in order to prove the existence of topologically mixing mean ergodic operators. Examples of operators being mean ergodic and hypercyclic at the same time seem to be unknown until now.

The example consists on the *backward shift operator B* acting on the weighted ℓ_p-space

$$\ell_p(v) = \left\{ \{x_n\}_n \in \mathbb{C}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p v_n < \infty \right\}, \ 1 \leq p < \infty,$$

where $v = \{v_n\}_n$ is a positive discrete weight sequence and $B(\{x_n\}_n) = \{x_{n+1}\}_n$, $\{x_n\}_n \in \ell_p(v)$. By [59] and [80, Example 4.4(a)], B is continuous on $\ell_p(v)$ if and only if $\sup_n \frac{v_n}{v_{n+1}} < \infty$ and it is hypercyclic if and only if $\inf_n v_n = 0$. First, some definitions are needed:

Given a subset $A \subseteq \mathbb{N}$, its *upper density* is the number

$$\overline{dens}(A) = \limsup_{n\to\infty} \frac{1}{n}|\{i < n : \ i \in A\}|,$$

where $|S|$ denotes the cardinality of the set S. Using this notation, distributional chaos can be defined as follows:

Definition 3.7.1 Let T be a continuous self map on a metric space (X, d). If there exists an uncountable set $D \subseteq X$ and $\varepsilon > 0$ such that for every $t > 0$ and every distinct $x, y \in D$ the following conditions hold:

$$\overline{dens}\{i \in \mathbb{N} : d(T^i(x), T^i(y)) \geq \varepsilon\} = 1,$$

$$\overline{dens}\{i \in \mathbb{N} : d(T^i(x), T^i(y)) < t\} = 1,$$

then we say that T exhibits uniform distributional chaos. In this case we say that the operator T is *distributionally chaotic* and that the set D is a *distributionally ε-scrambled* set.

Theorem 3.7.2 ([97, Theorem 2.1]) *Let $n_k := (k!)^3$, $k \in \mathbb{N}$, and let $v = \{v_j\}_j$ be the sequence of discrete weights given by $v_j = k^{-1}$ for $n_k \leq j < n_{k+1}$, $k \in \mathbb{N}$. Then the operator B is topologically mixing on $X := \ell^p(v)$, $1 \leq p < \infty$, but it is not distributionally chaotic.*

Proof. The fact that B is mixing can be deduced from [59] (see also Chapter 4 in [80] for more details), since $\lim_j v_j = 0$.

We will show that, for each $x \in X$ and for every $\varepsilon \in]0, 1[$,

$$\lim_{n\to\infty} \frac{|\{j \leq n : \ \|B^j(x)\| < \varepsilon\}|}{n} = 1,$$

i.e.,

$$\lim_{n\to\infty} \frac{|\{j \leq n : \ \|B^j(x)\| \geq \varepsilon\}|}{n} = 0, \tag{7.22}$$

that excludes the possibility of existence of distributionally chaotic pairs. In fact, if we take $x, y \in X$, then, for every $\varepsilon > 0$, the set $\{j \leq n : d(B^j(x), B^j(y)) \geq \varepsilon\}$ is included in the union

$$\{k \le n : \|B^k(x)\| \ge \varepsilon/2\} \cup \{l \le n : \|B^l(y)\| \ge \varepsilon/2\}.$$

So, (7.22) yields

$$\lim_{n\to\infty} \frac{|\{j \le n : d(B^j(x), B^j(y)) \ge \varepsilon\}|}{n} = 0,$$

and the operator cannot be distributionally chaotic.

First, fix an integer $k_0 > 6$ satisfying

$$\sum_{j \ge n_{k_0}} |x_j|^p v_j < \varepsilon/4 \quad \text{and} \quad k_0^{-1} < \varepsilon/4.$$

If $n \ge n_{k_0+1}$, let $k \ge k_0$ with $n_{k+1} \le n < n_{k+2}$. We can write $n = Nn_k + m$ with $m, N \in \mathbb{N}$, $m \le n_k$, $N > k^3$. Since

$$\sum_{i=1}^{N-1} \left(\sum_{j=in_k}^{(i+1)n_k-1} |x_j|^p v_j \right) \le \sum_{j \ge n_k} |x_j|^p v_j < \varepsilon/4 < 1,$$

then, for

$$I := \left\{ i < N : \sum_{j=in_k}^{(i+1)n_k-1} |x_j|^p v_j \ge k^{-2} \right\},$$

we have $|I| \le k^2$. Thus,

$$|\{1, \dots, N-1\} \setminus I| \ge N - 1 - k^2.$$

If $i \in J := \{1, \dots, N-1\} \setminus I$, then

$$\sum_{j=in_k}^{(i+1)n_k-1} |x_j|^p \le (k+1) \sum_{j=in_k}^{(i+1)n_k-1} |x_j|^p v_j < \frac{k+1}{k} \frac{1}{k_0} < \varepsilon/2$$

by the definition of I, and since $v_j \ge (k+1)^{-1}$ for $j \le (i+1)n_k - 1 \le Nn_k - 1 < n < n_{k+2}$. This implies that, if we fix $i \in J$ and $j \in [in_k, (i+1)n_k - n_{k-1}]$, then

$$\begin{aligned} \|B^{j-1}x\|^p &= \sum_{l=j}^{(i+1)n_k-1} |x_l|^p v_{l-j+1} + \sum_{l \ge (i+1)n_k} |x_l|^p v_{l-j+1} \\ &\le \sum_{l=in_k}^{(i+1)n_k-1} |x_l|^p + \sum_{l \ge (i+1)n_k} |x_l|^p v_l \frac{v_{l-j+1}}{v_l} < \frac{\varepsilon}{2} + \frac{3\varepsilon}{8} < \varepsilon, \end{aligned} \tag{7.23}$$

since $j < n_{k+2}$ and $l - j + 1 \geq n_{k-1}$ whenever $l \geq (i+1)n_k$, and by taking into account that

$$v_s/v_r \leq \frac{v_{n_{k-1}}}{v_{n_{k-1}+n_{k+2}}} = \frac{v_{n_{k-1}}}{v_{n_{k+2}}} = (k+2)/(k-1) \leq 3/2 \tag{7.24}$$

if $r > s \geq n_{k-1}$ and $r - s < n_{k+2}$, since $n_{k-1} + n_{k+2} < n_{k+3}$. Therefore, by (7.23),

$$\frac{|\{j \leq n : \ \|B^j x\|^p < \varepsilon\}|}{n} \geq \frac{\sum_{i \in J}(n_k - n_{k-1})}{n} \geq \frac{(N-1-k^2)(n_k - n_{k-1})}{n}$$

$$\geq \left(\frac{N-1-k^2}{N+1}\right)\left(\frac{n_k - n_{k-1}}{n_k}\right) > \left(1 - \frac{k^2+2}{k^3+1}\right)\left(1 - \frac{1}{k^3}\right) \underset{k\to\infty}{\longrightarrow} 1,$$

since $n \leq n_k(N+1)$, $|J| \geq N - 1 - k^2$ and $N > k^3$. So, as $n_{k+1} \leq n < n_{k+2}$, B is not distributionally chaotic. □

Theorem 3.7.3 (Peris) *Let $n_k := (k!)^3$, $k \in \mathbb{N}$, and let $v = \{v_j\}_j$ be the sequence of discrete weights given by $v_j = k^{-1}$ for $n_k \leq j < n_{k+1}$, $k \in \mathbb{N}$. Then the operator B is topologically mixing on $X := \ell^p(v)$, $1 \leq p < \infty$, and uniformly mean ergodic.*

Proof. Fix $x \in X$ and $\varepsilon > 0$. In the proof of Theorem 3.7.2 we have seen that there exists $M > 0$ such that

$$\frac{|\{j \leq n : \ \|B^j x\| \geq \varepsilon\}|}{n} \leq 1 - \left(1 - \frac{k^2+2}{k^3+1}\right)\left(1 - \frac{1}{k^3}\right) \leq \frac{M}{k}$$

for all $n \in [n_{k+1}, n_{k+2}[$, for all $k \in \mathbb{N}$. By the selection of the weight sequence, $\|B^l\| \leq k^{1/p}$ if $l < n_k$. Indeed, proceeding as in (7.24),

$$\|B^l(x)\|^p = \sum_{j \geq l+1} |x_j|^p v_{j-l} \leq \sup_{j \geq l+1} \frac{v_{j-l}}{v_j} \sum_{j \geq l+1} |x_j|^p v_j \leq \frac{v_1}{v_{n_k}} \|x\|^p = k\|x\|^p.$$

Let $k_0 \in \mathbb{N}$ satisfying

$$\sum_{i \geq n_{k_0}} |x_i|^p v_i < \frac{\varepsilon^p}{M^p}.$$

If $n \geq n_{k_0+1}$, let $k \geq k_0$ with $n_{k+1} \leq n < n_{k+2}$. We have, for $J := \{j \leq n : \|B^j x\| \geq \varepsilon\}$, that

$$\begin{aligned}\frac{\sum_{j\leq n}\|B^j x\|}{n} &\leq \varepsilon + \frac{\sum_{j\in J}\|B^j x\|}{n} \leq \varepsilon + \frac{\sum_{j<n_k}\|B^j x\|}{n} + \frac{\sum_{j\in J\cap[n_k,n]}\|B^j x\|}{n} \\ &\leq \varepsilon + \frac{n_k k^{1/p}\|x\|}{n_{k+1}} + \frac{\sum_{j\in J\cap[n_k,n]}\left(\sum_{i>j}\frac{v_{i-j}}{v_i}|x_i|^p v_i\right)^{1/p}}{n} \\ &\leq \varepsilon + \frac{\|x\|}{k^2} + \frac{\sum_{j\in J\cap[n_k,n]}\left(\sum_{i>j}|x_i|^p v_i\right)^{1/p}\left(\sup_{i>j}\frac{v_{i-j}}{v_i}\right)^{1/p}}{n} \\ &\leq \varepsilon + \frac{\|x\|}{k^2} + \frac{|J|}{n}(k+2)^{1/p}\frac{\varepsilon}{M} < 3\varepsilon + \frac{\|x\|}{k^2},\end{aligned}$$

since, proceeding as in (7.24), $\sup_{i>j}\frac{v_{i-j}}{v_i} \leq \frac{v_1}{v_{n_{k+2}}} = k+2$.

As $\varepsilon > 0$ was arbitrary, we conclude that the operator B is uniformly mean ergodic.
□

Chapter 4

Classical operators on the Hörmander algebras

We conclude the thesis with a chapter devoted to the study of the dynamics of the differentiation operator $Df(z) = f'(z)$ and the integration operator $Jf(z) = \int_0^z f(\zeta)d\zeta$, $z \in \mathbb{C}$, on weighted inductive and projective limits of spaces of entire functions, continuing the research in [39]. Most of our results concerning this topic are included by Bonet, Fernández and the author in [16].

4.1 Notation and Preliminaries

In Chapter 3, given a weight v on $\mathbb{C}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ or $q = 0$, we studied the dynamics of D, J and H on the weighted spaces of entire functions $B_{p,q}(v)$, known as the generalized weighted Bergman spaces when $q \in \{0, \infty\}$, and as the weighted Banach spaces of entire functions $H_v(\mathbb{C})$ and $H_v^0(\mathbb{C})$ when, in addition, $p = \infty$. In this chapter we consider inductive and projective limits of these spaces and we study how the operators behave on them.

In Lemma 3.1.1 it is shown that given a weight v, if we consider $v_s := v^s$ for $s > 0$, then, for $0 < a < b < c$,

$$H_{v_a}(\mathbb{C}) \hookrightarrow B_{p,\infty}(v_a) \hookrightarrow B_{p,q}(v_b) \hookrightarrow H^0_{v_c}(\mathbb{C})$$

continuously for every $1 \leq p \leq \infty$, $q = 0$ or $1 \leq q \leq \infty$. This implies that given a weight $v \leq 1$, if we consider the decreasing sequence of weights $V = \{v_n\}_n$, $v_n = v^n$, the inductive limits $ind_n B_{p,q}(v_n)$, $VH(\mathbb{C}) = ind_n H_{v_n}(\mathbb{C})$ and

$VH_0(\mathbb{C}) = ind_n H^0_{v_n}(\mathbb{C})$ coincide for every $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ or $q = 0$. Analogously, for the increasing sequence of weights $W = \{w_n\}_n$, $w_n = v^{\frac{1}{n}}$, the weighted projective limits $proj_n B_{p,q}(v_n)$, $HW(\mathbb{C}) = proj_n H_{w_n}(\mathbb{C})$ and $HW_0(\mathbb{C}) = proj_n H^0_{w_n}(\mathbb{C})$ coincide. See Section 0.3 for an introduction to the spaces $VH(\mathbb{C})$ and $HW(\mathbb{C})$.

In Section 1.2, given a growth condition $p : [0,\infty[\to [0,\infty[$ we consider the weight $v(z) = e^{-p(|z|)}$, $z \in \mathbb{C}$, the decreasing sequence of weights $V = \{v_n\}_n$, $v_n = v^n$, and we define the weighted spaces of entire functions known as Hörmander algebras (see e.g. [19], [18]):

$$A_p(\mathbb{C}) := \left\{ f \in \mathcal{H}(\mathbb{C}) | \text{ there is } n \geq 1 : \sup_{z\in\mathbb{C}} |f(z)| \exp(-np(z)) < \infty \right\},$$

that is, $A_p(\mathbb{C}) = VH_0(\mathbb{C})$, endowed with the inductive limit topology, for which it is a (DFN)-algebra (cf. [99]). Analogously, if we consider the increasing sequence of weights $W = \{w_n\}_n$, $w_n = v^{1/n}$, we define

$$A^0_p(\mathbb{C}) := \left\{ f \in \mathcal{H}(\mathbb{C}) | \text{ for all } n \in \mathbb{N} : \sup_{z\in\mathbb{C}} |f(z)| \exp\left(-\frac{p(z)}{n}\right) < \infty \right\},$$

that is, $A^0_p(\mathbb{C}) = HW_0(\mathbb{C})$, endowed with the projective limit topology, for which it is a nuclear Fréchet algebra (cf. [100]). Along this section, we do not assume that the growth condition satisfies condition (α) in the definition, that is, the function $\varphi : r \to p(e^r)$ is not needed to be convex.

Clearly $A^0_p(\mathbb{C}) \subseteq A_p(\mathbb{C})$. In Section 1.2 it is shown that the weight $v(z) = e^{-p(|z|)}$, $z \in \mathbb{C}$, is rapidly decreasing, consequently, the polynomials are contained and dense in $A_p(\mathbb{C})$ and in $A^0_p(\mathbb{C})$.

Weighted algebras of entire functions of this type have been considered since the work of Berenstein and Taylor [19] by many authors; see e.g. [18] and the references therein. Braun, Meise and Taylor studied in [51], [99] and [100] the structure of (complemented) ideals in these algebras. As an example, when $p(z) = |z|^a$, then $A_p(\mathbb{C})$ consists of all entire functions of order a and finite type or order less than a, and $A^0_p(\mathbb{C})$ is the space of all entire functions of order at most a and type 0. For $a = 1$, $A_p(\mathbb{C})$ is the space of all entire functions of exponential type, and $A^0_p(\mathbb{C})$ is the space of entire functions of infraexponential type.

In what follows we study the dynamics of the operators D, J and H on the Hörmander algebras $A_p(\mathbb{C})$ and $A^0_p(\mathbb{C})$. According to Proposition 0.5.4 in Section 0.5.1 (see also [2, Proposition 2.4]), since $A_p(\mathbb{C})$ and $A^0_p(\mathbb{C})$ are complete and Montel, they are uniformly mean ergodic, that is, each power bounded operator is automatically uniformly mean ergodic.

In Section 3.4 it is shown that the Hardy operator $(Hf)(z) = \frac{1}{z}\int_0^z f(\zeta)d\zeta$, $z \in \mathbb{C}$, is continuous and power bounded on $H_v^0(\mathbb{C})$ for every weight v. Hence, by 0.5.7, it is power bounded and uniformly mean ergodic on $A_p(\mathbb{C})$ and on $A_p^0(\mathbb{C})$. Thus, we restrict our attention to the differentiation and the integration operators.

Proposition 4.1.1 ([39, Proposition 8]) *Given a decreasing sequence of weights $V = \{v_n\}_n$, the following is satisfied:*

(i) If for each n there are m and $C > 0$ such that $v_m(r) \leq Cv_n(r+1)$ if $r \geq 1$, then the differentiation operator $D : VH(\mathbb{C}) \to VH(\mathbb{C})$ is continuous.

(ii) If for each k there are l and $C > 0$ with $rv_l(r) \leq Cv_k(r)$ if $r \geq 1$, then the integration operator $J : VH(\mathbb{C}) \to VH(\mathbb{C})$ is continuous.

The next proposition is the analogous for the projective limits $HW(\mathbb{C})$:

Proposition 4.1.2 *Given an increasing sequence of weights $W = \{w_n\}_n$, the following is satisfied:*

(i) If for each m there are n and $C > 0$ such that $w_m(r) \leq Cw_n(r+1)$ if $r \geq 1$, then the differentiation operator $D : HW(\mathbb{C}) \to HW(\mathbb{C})$ is continuous.

(ii) If for each l there are k and $C > 0$ with $rw_l(r) \leq Cw_k(r)$ if $r \geq 1$, then the integration operator $J : HW(\mathbb{C}) \to HW(\mathbb{C})$ is continuous

Proof. If the inequalities are satisfied for $r \geq 1$, they are also satisfied for every $r \geq 0$. So, by the Cauchy inequalities, (i) implies that for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ and $C > 0$ such that

$$\|Df\|_{w_m} = \sup_{z\in\mathbb{C}} w_m(|z|)|f'(z)| \leq \sup_{z\in\mathbb{C}} Cw_n(|z|+1) \sup_{|w|=1+|z|} |f(w)| \leq C\|f\|_{w_n}.$$

Hence, we get the continuity. On the other hand, (ii) yields that for every $l \in \mathbb{N}$ there exists $k \in \mathbb{N}$ and $C > 0$ such that

$$w_l(|z|)|Jf(z)| \leq w_l(|z|)|z| \sup_{|\lambda|=|z|} |f(\lambda)| \leq Cw_k(|z|) \sup_{|\lambda|=|z|} |f(\lambda)| = C\|f\|_{w_k},$$

which yields the continuity of the integration operator. □

Corollary 4.1.3 *The differentiation operators $D : A_p(\mathbb{C}) \to A_p(\mathbb{C})$ and $D : A_p^0(\mathbb{C}) \to A_p^0(\mathbb{C})$ and the integration operators $J : A_p(\mathbb{C}) \to A_p(\mathbb{C})$ and $J : A_p^0(\mathbb{C}) \to A_p^0(\mathbb{C})$ are continuous.*

Proof. Conditions (i) and (ii) in Propositions 4.1.1 and 4.1.2 are satisfied by the sequence of weights defining $A_p(\mathbb{C})$ and $A_p^0(\mathbb{C})$. Indeed, by condition (γ) in the definition of the growth condition p (see Definition 1.2.1), there exist constants $A, B > 0$ such that

$$\begin{aligned}\sup_{r\geq 1} \exp(bp(r+1) - ap(r)) &\leq \sup_{r\geq 1} \exp(bp(2r) - ap(r)) \\ &\leq B \sup_{r\geq 1} \exp(p(r)(Ab - a)) < \infty,\end{aligned}$$

for $a > Ab$. By condition (β) in the definition of p (see Definition 1.2.1),

$$\sup_{r\geq 1} r \exp(p(r)(b-a)) \leq \sup_{r\geq 1} \frac{r}{(1+r^2)^{(a-b)}} < \infty,$$

for $a > b$. □

In [18, page 110] it is shown that both spaces are stable under differentiation by the doubling condition (γ) in the definition of p and the Cauchy integral formula.

4.2 The differentiation operator

In this section we consider the action of the differentiation operator on the Hörmander algebras $A_p(\mathbb{C})$ and $A_p^0(\mathbb{C})$. The next lemma provides us with a sufficient condition for the power boundedness of D:

Lemma 4.2.1 *Let u and v be two weights on $\mathbb{C}$ such that there are $0 < \alpha < 1$ and $C > 0$ such that $u(r)e^{\alpha r} \leq Cv(R)e^{\alpha R}$ for each $0 \leq r \leq R$. Then for each $n \in \mathbb{N}$, the operator*

$$D^n : H_v(\mathbb{C}) \to H_u(\mathbb{C})$$

is continuous. Moreover, for each $f \in H_v(\mathbb{C})$ the sequence $\{D^n(f)\}_n$ converges to 0 in $H_u(\mathbb{C})$.

Proof. Fix $f \in H_v(\mathbb{C})$ and $n \in \mathbb{N}$. Given $z \in \mathbb{C}$ and $\varepsilon > 0$, by the Cauchy integral formula we have, since v is decreasing,

$$|f^{(n)}(z)| \leq \frac{n!}{\varepsilon^n} \sup_{|z-\omega|=\varepsilon} |f(\omega)| \leq \frac{n!}{\varepsilon^n} ||f||_v \frac{1}{v(|z|+\varepsilon)}.$$

Thus

$$u(|z|)|f^{(n)}(z)| \leq \frac{n!}{\varepsilon^n} ||f||_v \frac{u(|z|)}{v(|z|+\varepsilon)} \leq C||f||_v \frac{n!}{\varepsilon^n} e^{\alpha\varepsilon}.$$

This implies, by the Stirling formula,

$$||f^{(n)}||_u \leq C||f||_v n! \inf_{\varepsilon>0}(\frac{e^{\alpha\varepsilon}}{\varepsilon^n}) = C||f||_v n! \frac{\alpha^n e^n}{n^n} \leq C'||f||_v \alpha^n \sqrt{2\pi n},$$

which converges to 0 as $n \to \infty$, since $0 < \alpha < 1$. □

As an immediate consequence, if we consider just one weight v, we get the next corollary. See also [82].

Corollary 4.2.2 *If v is a weight such that for some $0 < \alpha < 1$ the function $v(r)e^{\alpha r}$ is increasing, then $D : H_v(\mathbb{C}) \to H_v(\mathbb{C})$ is continuous and the orbit of each $f \in H_v(\mathbb{C})$ converges to 0 in norm.*

As an immediate consequence of Lemma 4.2.1 we get the next corollary:

Corollary 4.2.3 *(a) Let $\{v_k\}_k$ be a decreasing sequence of weights such that for every m there are $k \geq m$, $C > 0$ and $0 < \alpha < 1$ such that $v_k(r)e^{\alpha r} \leq C v_m(R)e^{\alpha R}$ if $0 \leq r \leq R$. Then $D : VH(\mathbb{C}) \to VH(\mathbb{C})$ is continuous and $\{D^n(f)\}_n$ converges to 0 in $VH(\mathbb{C})$ for each $f \in VH(\mathbb{C})$. Therefore, it is power bounded, and thus, not hypercyclic.*

(b) Let $\{w_k\}_k$ be an increasing sequence of weights such that for each m there are $k \geq m$, $C > 0$ and $0 < \alpha < 1$ such that $w_m(r)e^{\alpha r} \leq C w_k(R)e^{\alpha R}$ if $0 \leq r \leq R$. Then the differentiation operator $D : HW(\mathbb{C}) \to HW(\mathbb{C})$ is continuous and $\{D^n(f)\}_n$ converges to 0 in $HW(\mathbb{C})$ for each $f \in HW(\mathbb{C})$. Therefore, it is power bounded, and thus, not hypercyclic.

Lemma 4.2.4 *Let E be a locally convex space of entire functions continuously included in $\mathcal{H}(\mathbb{C})$ and assume that there is $a > 1$ such that $e^{az} \in E$. If $D : E \to E$ is continuous, then it is not mean ergodic.*

Proof. Set $e_a(z) := e^{az}$, $z \in \mathbb{C}$. If D is continuous on E and mean ergodic, then $\frac{1}{n}\sum_{m=1}^{n} D^m e_a$ converges in E. Since E is continuously included $\mathcal{H}(\mathbb{C})$, the sequence

$$\left\{\frac{1}{n}\sum_{m=1}^{n} D^m e_a(0)\right\}_n = \left\{\frac{1}{n}\sum_{m=1}^{n} a^m\right\}_n$$

converges. This is impossible for $a > 1$. □

Theorem 4.2.5 *(i) If $r = O(p(r))$ as $r \to \infty$, then D is not mean ergodic on $A_p(\mathbb{C})$.*

(ii) *If $r = o(p(r))$ as $r \to \infty$, then D is not mean ergodic on $A_p^0(\mathbb{C})$.*

(iii) *If $p(r) = o(r)$ as $r \to \infty$, then D is power bounded, hence uniformly mean ergodic and not hypercyclic on $A_p(\mathbb{C})$ and on $A_p^0(\mathbb{C})$.*

Proof. Statements (i) and (ii) follow from Lemma 4.2.4. In fact, by hypothesis, there exists $C > 0$ such that

$$\lim_{r\to\infty} e^{ar}e^{-bp(r)} \leq \lim_{r\to\infty} e^{r(a-b/C)} \tag{2.1}$$

for every $a, b > 0$. So, for $a > 1$ and $b \in \mathbb{N}$, $b > aC$, the limit (2.1) tends to zero, and $e^{az} \in A_p(\mathbb{C})$. For $A_p^0(\mathbb{C})$, fix $a > 1$. Given $1/b := n \in \mathbb{N}$, take $C := \frac{1}{2na}$ so that $\lim_{r\to\infty} e^{ar}e^{-p(r)/n}$ tends to zero for every $n \in \mathbb{N}$. The election of C_n is possible, since $r = o(p(r))$. We conclude (iii) from Corollary 4.2.3. By condition (γ) in the definition of p, there exists $A > 0$ such that $p(r+s) \leq A(1+p(r)+p(s))$ for each r, $s > 0$. As $p(r) = o(r)$, for every $a > 0$ there exists $C > 0$ such that $Aap(s) < C + \frac{1}{2}s$ for each $s \geq 0$. If $b > Aa$ and $0 \leq r \leq R$ we have, for $s = R - r$,

$$\begin{aligned} ap(R) = ap(r + R - r) &\leq aA(1 + p(r)) + aAp(R - r) \\ &< aA(1 + p(r)) + C + \frac{R-r}{2} \leq C_1 + bp(r) + \frac{R}{2} - \frac{r}{2} \end{aligned}$$

for some $C_1 > 0$. This yields

$$e^{-bp(r)}e^{r/2} \leq e^{C_1}e^{-ap(R)}e^{R/2}$$

for $0 \leq r \leq R$. So, Corollary 4.2.3(a) is satisfied if we take $a = m \in \mathbb{N}$ and $b = k \in \mathbb{N}$, and Corollary 4.2.3(b) is satisfied if we take $b = 1/m \in \mathbb{N}$ and $a = 1/k \in \mathbb{N}$. □

Theorem 4.2.6 *(i) If $p(r) = o(r - \frac{1}{2}\log(r))$ as $r \to \infty$, then D is not hypercyclic on $A_p(\mathbb{C})$.*

(ii) *If $r = O(p(r))$ as $r \to \infty$, then D is topologically mixing and has a dense set of periodic points on $A_p(\mathbb{C})$.*

(iii) *If $r = o(p(r))$ as $r \to \infty$, then D is topologically mixing and has a dense set of periodic points on $A_p^0(\mathbb{C})$.*

Proof. Statements (i) and (ii) were proved in [39]. In fact, (ii) and (iii) follow from Bonet [41], since for $\alpha > 1$ the weighted Banach space of entire functions

$$H_\alpha^0(\mathbb{C}) := \{f \in \mathcal{H}(\mathbb{C}) : \sup_{z\in\mathbb{C}}|f(z)|e^{-\alpha|z|} < \infty\}$$

is continuously and densely included in $A_p(\mathbb{C})$ if $r = O(p(r))$ $(r \to \infty)$ and in $A_p^0(\mathbb{C})$ if $r = o(p(r))$ $(r \to \infty)$ (they include the polynomials) and the differentiation operator is topologically mixing and chaotic on this space by Corollary 3.6.8. The conclusions now follow by the comparison principle in Lemma 0.5.10. □

Corollary 4.2.7 *Let* $p_a(r) = r^a,\ a > 0$:

(i) If $a > 1$, *then* D *is topologically mixing, chaotic and not mean ergodic on* $A_{p_a}(\mathbb{C})$ *and on* $A^0_{p_a}(\mathbb{C})$.

(ii) If $a < 1$, *then* D *is power bounded, hence uniformly mean ergodic and not hypercyclic on* $A_{p_a}(\mathbb{C})$ *and on* $A^0_{p_a}(\mathbb{C})$.

(iii) If $a = 1$, *then* D *is topologically mixing, chaotic and not mean ergodic on* $A_{p_1}(\mathbb{C})$ *and it is power bounded, hence uniformly mean ergodic and not hypercyclic on* $A^0_{p_1}(\mathbb{C})$.

Proof. All the statements but (iii) for $A^0_{p_1}(\mathbb{C})$ $(p(r) = r)$ follow by Theorems 4.2.5 and 4.2.6. $A^0_{p_1}(\mathbb{C})$ is the intersection of the spaces $H^0_{v_n}(\mathbb{C})$ for $v_n(r) = e^{-\frac{r}{n+1}}$ and the differentiation operator D is power bounded on each $H^0_{v_n}(\mathbb{C})$ by Theorem 3.6.10. So, by Proposition 0.5.7, D is power bounded on $A^0_{p_1}(\mathbb{C})$. □

It is possible to extend some of these results to the weighted Fréchet spaces $HW_0(\mathbb{C})$. See also [41, Theorems 2.3, 2.4].

Theorem 4.2.8 *If the differentiation operator* $D : HW_0(\mathbb{C}) \to HW_0(\mathbb{C})$ *is continuous, then the following are equivalent:*

(i) D *satisfies the hypercyclicity criterion.*

(ii) D *is hypercyclic.*

(iii) there exists a sequence $\{k_s\}_s$ *such that* $\lim_{s\to\infty} \frac{\|z^{k_s}\|_n}{k_s!} = 0$ *for every* $n \in \mathbb{N}$.

Proof. (i)⇒(ii) is clear. (ii)⇒(iii): If D is hypercyclic and f is a hypercyclic function, we have that $\{f^{(k)}(0) : k \in \mathbb{N}\}$ is dense, and therefore, unbounded in $\mathbb{C}$. So, there exists a subsequence $\{k_s\}_s$ such that $\lim_{s\to\infty} |f^{(k_s)}(0)| = \infty$. By the Cauchy inequalities, for each $n \in \mathbb{N}$,

$$w_n(r)\frac{|f^{(k)}(0)|}{k!}r^k = w_n(r)\left|\frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)}{z^{k+1}}dz\right| r^k \le w_n(r)\max_{|z|=r}|f(z)| \le \|f\|_n, \tag{2.2}$$

so we get $|f^{(k_s)}(0)|\frac{\|z^{k_s}\|_n}{k_s!} \le \|f\|_n$, and thus, $\lim_{s\to\infty}\frac{\|z^{k_s}\|_n}{k_s!} = 0$, which yields (iii).

Let us see (iii)⇒(i). Take $Y = Y_0$ as the set of all polynomials, which is dense in $HW_0(\mathbb{C})$. Define $S_j := S^j$ on Y, $j \in \mathbb{N}$, with S the integration map defined on the monomials by $S(z^k) = z^{k+1}/(k+1)$. Since $D \circ S(q) = q$ for each polynomial q, and for each q of degree less or equal to M, $D^k q = 0$ for $k \geq M+1$, it only remains to show that there exists some sequence $\{N_j\}_j$ such that $\lim_{j\to\infty} S^{N_j} q = 0$ in $HW_0(\mathbb{C})$ for each polynomial q.

Since D is continuous, given $m \in \mathbb{N}$ and $j \in \mathbb{N}$ there exists $n_{m,j} \in \mathbb{N}$ and $C_{m,j} > 1$ such that $\|D^l f\|_m \leq C_{m,j} \|f\|_{n_{m,j}}$ for every $l \leq j$ and each $f \in HW_0(\mathbb{C})$.

Given $j \in \mathbb{N}$, consider the weight $w_{n_{j,j}}$. Since $\lim_{s\to\infty} \frac{\|z^{k_s}\|_{n_{j,j}}}{k_s!} = 0$, set $k_{s_0} = 0$ and find k_{s_j} such that $\frac{\|z^{k_{s_j}}\|_{n_{j,j}}}{k_{s_j}!} \leq \frac{1}{jC_{j,j}}$ and $k_{s_{j+1}} > k_{s_j} + j + 2$. Consider $N_j := k_{s_j} - j - 1$, $j \in \mathbb{N}$. Notice that $N_{j+1} > N_j + j + 1$. Since

$$S^{N_j}(z^k) = \frac{k!}{(N_j+k)!} z^{N_j+k}$$

and

$$D^{j+1-k}(z^{N_j+j+1}) = \frac{(N_j+j+1)!}{(N_j+k)!} z^{N_j+k}$$

for $k \leq j+1$, $k \in \mathbb{N}_0$, given m, for $j \geq m$ we get

$$\begin{aligned} \|S^{N_j}(z^k)\|_m &\leq \frac{k!}{(N_j+k)!}\|z^{N_j+k}\|_j = \frac{k!}{(N_j+j+1)!}\|D^{j+1-k}(z^{N_j+j+1})\|_j \\ &\leq k! C_{j,j} \frac{\|z^{N_j+j+1}\|_{n_{j,j}}}{(N_j+j+1)!} \leq \frac{k!}{j}. \end{aligned}$$

Thus, for every $k \in \mathbb{N}$ and each $m \in \mathbb{N}$, we get $\lim_{j\to\infty} \|S^{N_j}(z^k)\|_m = 0$. Therefore, S^{N_l} tends to zero on the polynomials, and the hypercyclicity criterion is satisfied. □

Remark 4.2.9 (ii)⇔(iii) in Theorem 4.2.8 is a special case of a result by Grosse-Erdmann in [77, Theorem 7] in the case the monomials are a basis of $HW_0(\mathbb{C})$. Indeed, in this case $HW_0(\mathbb{C})$ is a sequence space and the differentiation operator D becomes the weighted backward shift operator $B_w : (x_1, x_2, x_3, \dots) \to (w_2 x_2, w_3 x_3, \dots)$, where $w = \{w_n\}_n$, $w_n = n$, $n \in \mathbb{N}$. Since $\frac{|f^k(0)|}{k!}\|z^k\|_n \leq \|f\|_n$ for every $f \in H_{v_n}(\mathbb{C})$ by (2.2), the hypothesis in [77, Theorem 7] are satisfied. The monomials are a basis of the spaces $A_p(\mathbb{C})$ and $A_p^0(\mathbb{C})$ (see Corollary 1.2.3 in Chapter 1, [71, Theorem 11] or [99] and [100]), but in general this is not satisfied (see [94]).

Theorem 4.2.10 *If the differentiation operator $D : HW_0(\mathbb{C}) \to HW_0(\mathbb{C})$ is continuous, then the following are equivalent:*

(i) D is topologically mixing.

(ii) $\lim_{k\to\infty}\frac{\|z^k\|_n}{k!}=0$ *for every* $n\in\mathbb{N}$.

Proof. (i)⇒(ii): Suppose that for some $n\in\mathbb{N}$ the sequence $\{\frac{\|z^k\|_n}{k!}\}_n$ does not converge to zero. Then we find $M>0$ and a subsequence $\{k_s\}_s$ such that $\frac{\|z^{k_s}\|_n}{k_s!}>M$. As $\{(D')^{k_s}(\delta_0):s\in\mathbb{N}\}$ is unbounded in $(HW_0(\mathbb{C}))'$ by Lemma 0.5.11, there is $f\in HW_0(\mathbb{C})$ such that the set $\{f^{(k_s)}(0):s\in\mathbb{N}\}$ is unbounded, and therefore there is a subsequence that we still denote by $\{k_s\}_s$ such that $\lim_{s\to\infty}|f^{(k_s)}(0)|=\infty$. Now,

$$w_n(r)|f^{(k)}(0)|\frac{r^k}{k!}\leq w_n(r)\max_{|z|=r}|f(z)|\leq\|f\|_n.$$

Then, for $n,s\in\mathbb{N}$ and $r>0$,

$$w_n(r)\frac{r^{k_s}}{k_s!}\leq\frac{\|f\|_n}{|f^{(k_s)}(0)|},$$

which implies

$$M<\frac{\|z^{k_s}\|_n}{k_s!}\underset{s\to\infty}{\to}0,$$

a contradiction.

(ii)⇒(i). It is enough to show that D satisfies the assumptions of the criterion of Kitai-Gethner-Shapiro [80, Theorem 3.4]. As in the proof of Theorem 4.2.8, we take $Y_0=Y_1$ the set of all polynomials and denote by S the operator of integration in the set of polynomials. Clearly $\{D^j\}_j$ tends pointwise to 0 in the set of polynomials and $D\circ S$ coincides with the identity on this set. So, it only remains to prove that $\{S^k(g)\}_k$ converges to 0 in $HW_0(\mathbb{C})$ for all polynomial g. Since $S^j(z^k)=k!z^{k+j}/(k+j)!$ for each $k\in\mathbb{N}$, it is enough to show that $\{\|\frac{z^k}{k!}\|_n\}_k$ converges to 0 for every $n\in\mathbb{N}$, which holds by condition (ii). □

Theorem 4.2.11 *If the differentiation operator $D:HW_0(\mathbb{C})\to HW_0(\mathbb{C})$ is continuous, then the following are equivalent:*

(i) D is chaotic.

(ii) D has a periodic point different from 0.

(iii) $\lim_{r\to\infty}w_n(r)e^r=0$ *for every* $n\in\mathbb{N}$.

Proof. Clearly (i) implies (ii). Let us see (ii)⇒(iii). By hypothesis, there exists a function $0 \neq f \in HW_0(\mathbb{C})$ such that, for some $n \in \mathbb{N}$, $D^n f = f$. Using the trivial decomposition $D^n - I = (D - \theta_1 I) \dots (D - \theta_n I)$, $\theta_j^n = 1$, $j = 1, \dots, n$, we conclude that there is $\theta \in \mathbb{C}$, $|\theta| = 1$, and $g \in HW_0(\mathbb{C})$, $g \neq 0$, such that $(D - \theta I)g = 0$. This yields $e^{\theta z} \in HW_0(\mathbb{C})$, and thus, (iii) is satisfied.

(iii)⇒(i) Denote by P the linear span of the functions $e^{\theta z}$, $\theta \in \mathbb{C}$, $\theta^n = 1$ for some $n \in \mathbb{N}$. Obviously, P is formed by periodic points and, by Lemma 3.3.4, it is dense in $H^0_{w_n}(\mathbb{C})$ for every $n \in \mathbb{N}$, and thus, on $HW_0(\mathbb{C})$. On the other hand, since in the proof of Theorem 3.3.5 it is shown that $\lim_{r\to\infty} w_n(r)e^r = 0$ implies $\lim_{k\to\infty} \frac{\|z^k\|_n}{k!} = 0$, D is topologically mixing by Theorem 4.2.10, and thus, chaotic. □

4.3 The integration operator

In this section we consider the action of the integration operator J on the Hörmander algebras $A_p(\mathbb{C})$ and $A_p^0(\mathbb{C})$. It is worth mentioning that J is not continuous on $H_v(\mathbb{C})$ for $v(r) = e^{-\alpha r^a}$, $a < 1$, $\alpha > 0$. In fact, an easy computation gives that

$$||z^j||_v = \left(\frac{j}{e\alpha a}\right)^{\frac{j}{a}},$$

since $\sup_{r\geq 0} r^j e^{-\alpha r^a}$ is attained in $r = \left(\frac{j}{a\alpha}\right)^{1/a}$. Hence,

$$\frac{\|J(z^j)\|_v}{\|z^j\|_v} = \frac{\|z^{j+1}\|_v}{(j+1)\|z^j\|_v} = \left(\frac{1}{e\alpha a}\right)^{\frac{1}{a}} \left(\frac{j+1}{j}\right)^{\frac{j}{a}} (j+1)^{\frac{1}{a}-1}.$$

As $a < 1$, we get that J is not continuous on $H_v(\mathbb{C})$, since the right hand side diverges to infinity as $j \to \infty$.

The next lemma gives a sufficient condition for which J is power bounded:

Lemma 4.3.1 *Let u, v be two weights and assume that for some $\alpha \geq 1$ and $C > 0$, $u(R)e^{\alpha R} \leq Cv(r)e^{\alpha r}$ for all $0 \leq r \leq R$. Then, for every $n \in \mathbb{N}$ the operator*

$$J^n : H_v(\mathbb{C}) \to H_u(\mathbb{C})$$

is continuous and for each $f \in H_v(\mathbb{C})$ the sequence $\{J^n f\}_n$ is bounded in $H_u(\mathbb{C})$, and in case $\alpha > 1$, it converges to 0.

Proof. Observe that

$$|J^n f(z)| \leq \int_0^1 |z||J^{n-1}f(t_1 z)|dt_1 = \int_0^1 |z| \int_0^1 t_1|z||J^{n-2}f(t_2t_1z)|dt_2dt_1 \leq \dots$$
$$\leq \int_0^1 |z| \int_0^1 t_1|z| \int_0^1 t_2t_1|z| \int_0^1 \dots \int_0^1 t_{n-1}\dots t_1|z||f(t_n\dots t_1 z)|dt_n\dots dt_1.$$

By hypothesis, there exist $\alpha \geq 1$ and $C > 0$ such that

$$u(|z|) \leq Cv(t_1\dots t_n|z|)e^{\alpha t_1\dots t_n|z|}e^{-\alpha|z|}$$
$$= Cv(t_1\dots t_n|z|)e^{\alpha|z|(t_1-1)}e^{\alpha t_1|z|(t_2-1)}e^{\alpha t_1t_2|z|(t_3-1)}\dots e^{\alpha|z|t_1\dots t_{n-1}(t_n-1)}.$$

Then,

$$u(|z|)|J^n f(z)| \leq$$
$$\leq C\|f\|_v \int_0^1 |z|e^{\alpha|z|(t_1-1)} \int_0^1 \dots \int_0^1 t_{n-1}\dots t_1|z|e^{\alpha|z|t_1\dots t_{n-1}(t_n-1)}dt_n\dots dt_1$$

which yields

$$||J^n f||_u \leq ||f||_v \frac{C}{\alpha^n}.$$

□

If we consider just one weight v, we get the next corollary. See also [82].

Corollary 4.3.2 *If v is a weight such that for some $\alpha > 1$ the function $v(r)e^{\alpha r}$ is decreasing, then $J : H_v(\mathbb{C}) \to H_v(\mathbb{C})$ is continuous and the orbit of each $f \in H_v(\mathbb{C})$ converges to 0 in norm.*

As an immediate consequence of Lemma 4.3.1 we get:

Corollary 4.3.3 *(a) Let $\{v_k\}_k$ be a decreasing sequence of weights such that for every m there are $k \geq m$, $C > 0$ and $\alpha \geq 1$ such that $v_k(R)e^{\alpha R} \leq Cv_m(r)e^{\alpha r}$ if $0 \leq r \leq R$. Then $J : VH(\mathbb{C}) \to VH(\mathbb{C})$ is continuous and $\{J^n f\}_n$ is bounded in $VH(\mathbb{C})$ for each $f \in VH(\mathbb{C})$. If $\alpha > 1$, then $\{J^n f\}_n$ converges to 0 in $VH(\mathbb{C})$ for each $f \in VH(\mathbb{C})$. Therefore, it is power bounded, and thus, not hypercyclic.*

(b) Let $\{w_k\}_k$ be an increasing sequence of weights such that for each m there are $k \geq m$, $C > 0$ and $\alpha \geq 1$ such that $w_m(R)e^{\alpha R} \leq Cw_k(r)e^{\alpha r}$ if $0 \leq r \leq R$. Then $J : HW(\mathbb{C}) \to HW(\mathbb{C})$ is continuous and $\{J^n f\}_n$ is bounded in $HW(\mathbb{C})$ for each $f \in HW(\mathbb{C})$. If $\alpha > 1$, then $\{J^n f\}_n$ converges to 0 in

$HW(\mathbb{C})$ for each $f \in HW(\mathbb{C})$. Therefore, it is power bounded, and thus, not hypercyclic.

Proposition 4.3.4 *J is not hypercyclic on the Hörmander algebras $A_p(\mathbb{C})$ nor $A_p^0(\mathbb{C})$.*

Proof. By Proposition 3.2.1, the operator J is power bounded on $\mathcal{H}(\mathbb{C})$, therefore it is not hypercyclic. Consequently, as $A_p(\mathbb{C})$ and $A_p^0(\mathbb{C})$ are densely and continuously included in $\mathcal{H}(\mathbb{C})$, the non-hypercyclicity of J on these spaces follows by the comparison principle in Lemma 0.5.10. □

Theorem 4.3.5 *(i) The operator of integration is power bounded and hence uniformly mean ergodic on $A_p(\mathbb{C})$, provided that $r = O(p(r))$ as $r \to \infty$.*

(ii) If $p(r) = o(r)$ as $r \to \infty$, then J is not mean ergodic on $A_p(\mathbb{C})$.

(iii) J is power bounded and hence uniformly mean ergodic on $A_p^0(\mathbb{C})$ provided that $r = o(p(r))$ as $r \to \infty$.

(iv) If $p(r) = O(r)$ as $r \to \infty$, then J is not mean ergodic on $A_p^0(\mathbb{C})$.

Proof. (i) As $r = O(p(r))$, we may assume without loss of generality that $2r \leq p(r) + c$ for some $c > 0$ and all $r > 0$. Indeed, if $r \leq A(p(r) + 1)$ for some $A > 0$ and every $r \geq 0$, we can consider the growth condition $q(r) = 2Ap(r)$, since $A_p(\mathbb{C}) = A_q(\mathbb{C})$. Put $v_m(r) = e^{-mp(r)}$. Then, for all $0 \leq r \leq R$ we have

$$-(m+1)p(R) + mp(r) = -p(R) + m(p(r) - p(R)) \leq c - 2R \leq c + 2(r - R),$$

that is,

$$v_{m+1}(R)e^{2R} \leq C v_m(r) e^{2r},$$

hence, if $\sup_{z\in\mathbb{C}}|f(z)|v_m(z) < \infty$, the sequence $(J^n f)_n$ converges to 0 in the next step of the inductive limit by Proposition 4.3.1.

(ii) If J is mean ergodic, then for each $f \in A_p(\mathbb{C})$ the sequence $\{\frac{J^n f}{n}\}_n$ tends to zero in $A_p(\mathbb{C})$. Since the sequence of weights is regularly decreasing (see Definition 2.1.5), $A_p(\mathbb{C})$ is boundedly retractive by Proposition 2.1.7, and so, there is m such that $\{||\frac{J^n f}{n}||_m\}_n$ converges to 0. In particular, for $f \equiv 1$ this means that $\{\frac{||z^n||_m}{n!n}\}_n$ converges to zero. But since $p(r) = o(r)$, we have that for some constant $C > 0$, $mp(r) \leq \frac{r}{2} + C$ for all r, hence

$$\frac{||z^n||_m}{n!n} \geq \frac{1}{e^C} \sup_{r\geq 0}(\frac{r^n}{n!n} e^{-r/2}) = \frac{1}{e^C} \frac{||z^n||_{1/2}}{n!n}$$

and the right hand side diverges by (1.10) in Chapter 3.

(iii) Since $r = o(p(r))$, for each m there is $c_m > 0$ such that for all r, $4mr \leq c_m + p(r)$. Then, as before, for $0 \leq r \leq R$ we have

$$-\frac{1}{m}p(R) + \frac{1}{2m}p(r) \leq -\frac{1}{2m}p(R) \leq c_m - 2R \leq c_m + 2(r - R),$$

which by the proof of Lemma 4.3.1 implies that there exists some $C_m > 0$ such that, for all n,

$$\|J^n f\|_{w_m} \leq \frac{C_m}{2^n}\|f\|_{w_{2m}}.$$

(iv) If J is mean ergodic, then for each $f \in A_p^0(\mathbb{C})$ the sequence $\{\frac{J^n f}{n}\}_n$ tends to zero in $A_p^0(\mathbb{C})$, that is, $\{||\frac{J^n f}{n}||_m\}_n$ converges to 0 for every $m \in \mathbb{N}$. In particular, for $f \equiv 1$ this means that $\{\frac{||z^n||_m}{n!n}\}_n$ converges to zero for every $m \in \mathbb{N}$. But since $p(r) = O(r)$, we have that for some constants $C, D > 0$, $p(r) \leq Cr + D$ for all r, hence

$$\frac{||z^n||_m}{n!n} \geq \frac{1}{e^{D/m}} \sup_{r \geq 0}(\frac{r^n}{n!n}e^{-Cr/m}) = \frac{1}{e^{D/m}}\frac{||z^n||_{C/m}}{n!n},$$

and the right hand side diverges for $m > C$ by (1.10) in Chapter 3. □

Remark 4.3.6 The function $p(0) = 0$, $p(r) = 2^n n$ for $2^n \leq r \leq 2^n + 2^{n-1}$, and linear and continuous in $[2^n + 2^{n-1}, 2^{n+1}]$, is increasing, $p(2r) = O(p(r))$ and $r = o(p(r))$ as $r \to \infty$. Hence J is power bounded on $A_p(\mathbb{C})$ as well as on $A_p^0(\mathbb{C})$ by Theorem 4.3.5. However, $e^{-p(r)}e^{\alpha r}$ is not decreasing for each $\alpha > 0$. Hence Theorem 3.2.6 cannot be applied to the operator J on weighted Banach spaces of entire functions defined by weights of the form $e^{-ap(|z|)}$, $a > 0$. Observe that the function p is not convex. However, the convexity of the function $\varphi : r \to p(e^r)$ is not assumed in this chapter to study the behaviour of the differentiation or the integration operators on the algebras $A_p(\mathbb{C})$ and $A_p^0(\mathbb{C})$.

As an immediate consequence of Theorem 4.3.5, we get the next corollary:

Corollary 4.3.7 *Given $p_a(r) = r^a$, $r \geq 0$, then:*

(i) J is power bounded and uniformly mean ergodic on $A_{p_a}(\mathbb{C})$ for $a \geq 1$, and it is not mean ergodic for $a < 1$.

(ii) J is power bounded and uniformly mean ergodic on $A_{p_a}^0(\mathbb{C})$ for $a > 1$ and it is not mean ergodic for $a \leq 1$.

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