Mixing properties for nonautonomous linear dynamics and invariant sets

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Abstract

We study mixing properties (topological mixing and weak mixing of arbitrary order) for nonautonomous linear dynamical systems that are induced by the corresponding dynamics on certain invariant sets. The type of nonautonomous systems considered here can be defined by a sequence $(T_i)_{i\in\mathbb{N}}$ of linear operators $T_i : X \to X$ on a topological vector space X such that there is an invariant set Y for which the dynamics restricted to Y satisfies certain mixing property. We then obtain the corresponding mixing property on the closed linear span of Y. We also prove that the class of nonautonomous linear dynamical systems that are weakly mixing of order n contains strictly the corresponding class with the weak mixing property of order $n+1$.

1 Introduction

Given a sequence of linear and continuous maps (in short, operators) $T_i : X \to X$, $i \in \mathbb{N}$, defined on a topological vector space X (usually a Banach space or a complete and metrizable space), we consider the corresponding *nonautonomous discrete system* (NDS) $(X, T_\infty) = (X, (T_n \circ T_{n-1} \ldots \circ T_1)_{n\in\mathbb{N}})$ and we study the behavior of the orbits $\mathrm{Orb}(x, T_\infty) = \{T^{(k)}x \ ; \ k \geq 0\}$, $x \in X$, where $T^{(k)} := T_k \circ \cdots \circ T_1$, $k \in \mathbb{N}$, $T^{(0)} = Id_X$. More precisely, we are interested in topological mixing and weak mixing properties. We say

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that $Y \subset X$ is an *invariant set* for the NDS (X, T_∞) if $T_n(Y) \subset Y$ for all $n \in \mathbb{N}$. We selected this formulation to be consistent with the concept of NDS, since then $(Y, T_\infty|_Y)$, induced by the sequence $T_i|_Y$, $i \in \mathbb{N}$, is also a NDS. The dynamical properties that we are interested on will be obtained on the closure of the linear span of an invariant subset $Y \subset X$ under general assumptions about dynamics of $T_\infty|_Y$. Also, the class of weakly mixing linear NDS of order n will be shown to contain strictly the corresponding one with the weak mixing property of order $n+1$.

Chaotic behaviour for nonautonomous discrete systems has been studied by several authors [10, 11, 19, 20, 21]. Very recently, Balibrea and Oprocha [1] obtained several results about weak mixing and chaos in nonautonomous discrete systems on compact sets. Some of their results will be used to induce the corresponding dynamical behavior on linear nonautonomous systems. The theory of linear dynamics is well established in the case of iterations of a single operator (autonomous dynamical system). We refer the reader to the recent books of the subject [3, 16]. The case of nonautonomous linear dynamics is not yet developed, although a more general concept of universality of a sequence of operators $(T_n)_{n\in\mathbb{N}}$ where the orbits are defined as $\{T_n x \; ; \; n \in \mathbb{N}\}$, $x \in X$, has been treated by several authors (See, e.g., [4, 6, 8, 18, 22]).

Discretizations of C_0-semigroups of operators on Banach spaces provide a natural source of nonautonomous linear systems, and they deserve a special attention since linear PDEs and infinite systems of linear ODEs involving the time variable usually bring a C_0-semigroup of operators as a solution semigroup. Therefore, the asymptotic behavior of the solutions depends on the behavior of the C_0-semigroup. We refer to [2, 5, 7, 12, 13] for several results about chaotic behaviour of discretizations of C_0-semigroups. Most of these results are collected in section 7.3 of [16]. A type of linear universality which has attracted the attention in recent years is the dynamics of tuples of operators introduced by Feldman [17]. More precisely, given a commuting tuple $(T_1, \dots, T_n)$ of operators defined on a certain topological vector space X, he studied the existence of (somewhere) dense orbits $\{(T_n^{k_n} \circ \cdots \circ T_1^{k_1})x \; ; \; k_i \geq 0\}$. The subsystems that correspond to increasing sequences in $\mathbb{N}^n$, with its natural order, can be written as nonautonomous discrete systems.

We will essentially follow the notation of [1]. A NDS (X, f_∞) is *weakly mixing* of order n if, for any nonempty open sets $U_1, \dots, U_n$, $V_1, \dots, V_n$ and for any $N > 0$ there is $k > N$ such that $f^{(k)}(U_i) \bigcap V_i \neq \emptyset$ for $i = 1, \dots, n$. If (X, f_∞) is weakly mixing of order n for every $n \geq 2$ then we say that it is weakly mixing of all orders. (X, f_∞) is said to be *mixing* if for any nonempty open sets $U, V \subset X$ there exists $N > 0$ such that $f^{(k)}(U) \bigcap V \neq \emptyset$ for all $k \geq N$. These notions can be extended naturally to a system $(X, (f_k)_k)$ of sequences of maps $f_k : X \to X$, $k \in \mathbb{N}$, by substituting $f^{(k)}$ by f_k.

2 Mixing properties on linear NDS induced by invariant sets

The purpose of this section is, given a linear NDS (X, T_∞), where X is a topological vector space, with an invariant set $Y \subset X$, to obtain mixing properties on the closure of $\text{span}(Y)$, the linear span of Y, induced by the corresponding ones in $(Y, T_\infty|_Y)$. Actually, the main result in this section will be given for sequences of operators, so that we will obtain as a consequence the results for linear NDS and for tuples of operators.

Theorem 1. *Let X be a topological vector space and let the system $(X, (T_n)_n)$, where $\{T_n : X \to X \; ; \; n \in \mathbb{N}\}$ is a sequence of operators such that $T_n(Y) \subset Y$ for every $n \in \mathbb{N}$ and for certain $Y \subset X$ with $0 \in Y$. We consider $Z := \overline{\text{span}(Y)}$.*

(1) If $(Y, (T_n|_Y)_n)$ is weakly mixing of all orders then $(Z, (T_n|_Z)_n)$ is also weakly mixing of all orders.

(2) If $(Y, (T_n|_Y)_n)$ is mixing then $(Z, (T_n|_Z)_n)$ is also mixing.

Proof. We will show *(1)*. *(2)* follows an easier argument.

Suppose then that $(Y, (T_n|_Y)_n)$ is weakly mixing of all orders. Given any $m \in \mathbb{N}$, we have to show that $(Z, (T_n|_Z)_n)$ is weakly mixing of order m. Let $U_j, V_j \subset Z$ be nonempty open sets, $j = 1, \dots m$. We find $n \in \mathbb{N}$, $\alpha_{i,j}, \beta_{i,j} \subset \mathbb{K}$ and $u_{i,j}, v_{i,j} \in Y$, $i = 1, \dots, n$, $j = 1, \dots, m$, such that $u_j := \sum_{i=1}^n \alpha_{i,j} u_{i,j} \in U_j$ and $v_j := \sum_{i=1}^n \beta_{i,j} v_{i,j} \in V_j$, $j = 1, \dots m$. There are nonempty open sets $U_{i,j}, V_{i,j}, W \subset Y$ with $0 \in W$, $u_{i,j} \in U_{i,j}$, $v_{i,j} \in V_{i,j}$, $i = 1, \dots, n$, $j = 1, \dots, m$, such that

$$\sum_{i=1}^n \alpha_{i,j} U_{i,j} + \sum_{i=1}^n \gamma_i W \subset U_j, \qquad \sum_{i=1}^n \beta_{i,j} V_{i,j} + \sum_{i=1}^n \gamma_i W \subset V_j$$

for any $\gamma_i \in \{\alpha_{i,j}, i = 1, \dots, n, j = 1, \dots, m\} \cup \{\beta_{i,j}, i = 1, \dots, n, j = 1, \dots, m\}$, $i = 1, \dots, n$, $j = 1, \dots, m$. Since $(Y, (T_n|_Y)_n)$ is weakly mixing of all orders there are $y_{i,j} \in U_{i,j}$, $w_{i,j} \in W$, and $k \in \mathbb{N}$ such that $T_k(y_{i,j}) \in W$ and $T_k(w_{i,j}) \in V_{i,j}$, $i = 1, \dots, n$, $j = 1, \dots, m$. The above conditions yield that $y_j := \sum_{i=1}^{n} (\alpha_{i,j} y_{i,j} + \beta_{i,j} w_{i,j}) \in U_j$ and $T_k y_j \in V_j$, $j = 1, \dots, m$. □

The result for linear NDS follows now from Theorem 1.

Corollary 2. *Let X be a topological vector space and let (X, T_∞) be a linear NDS with an invariant set $Y \subset X$ such that $0 \in Y$. We consider $Z := \overline{\operatorname{span}(Y)}$.*

1. *If $(Y, T_\infty|_Y)$ is weakly mixing of all orders then $(Z, T_\infty|_Z)$ is also weakly mixing of all orders.*
2. *If $(Y, T_\infty|_Y)$ is mixing then $(Z, T_\infty|_Z)$ is also mixing.*

In [17] examples were given of somewhere dense orbits for tuples of operators that are not dense, and sufficient conditions under which a somewhere dense orbit under a tuple of operators must be everywhere dense were obtained. The type of sufficient conditions of Feldman were "operator-theoretic". We can also derive conditions implying that, when there is a somewhere dense orbit, it must be everywhere dense. We recall that, for linear autonomous systems, no extra assumptions are needed to show that somewhere dense orbits are everywhere dense, as shown by Bourdon and Feldman [9] answering a question in [24] (see also [14] for the corresponding version for C_0-semigroups).

Corollary 3. *Let $T = (T_1, \dots, T_n)$ be a commuting tuple of operators defined on a topological vector space X. Let $x \in X$ such that $\operatorname{Orb}(x, T) := \{(T_n^{k_n} \circ \dots \circ T_1^{k_1})x \; ; \; k_i \geq 0$ for all $i\}$ is somewhere dense in X. Let $(R_n)_{n \in \mathbb{N}}$ be an enumeration of $\{T_n^{k_n} \circ \dots \circ T_1^{k_1} \; ; \; k_i \geq 0$ for all $i\}$ and let $Y := \overline{\operatorname{Orb}(x, T)}$. If $(Y, (R_n|_Y)_n)$ is weakly mixing of all orders then $\operatorname{Orb}(x, T)$ is everywhere dense.*

Proof. By Theorem 1, $(X, (R_n)_n)$ is weakly mixing of all orders since $\operatorname{span}(Y) = X$ because Y contains a non-empty open set. In particular, given an arbitrary non-empty open set $V \subset X$ and a non-empty open set $U \subset \overline{\operatorname{Orb}(x, T)}$, there exists $k \in \mathbb{N}$ such that $R_k(U) \cap V \neq \emptyset$. By continuity, we find a non-empty open set $\tilde{U} \subset U$ such that $R_k(\tilde{U}) \subset V$. Let $j_1, \dots, j_n \geq 0$ such that $(T_n^{j_n} \circ \dots \circ T_1^{j_1})x \in \tilde{U}$, and $j'_1, \dots, j'_n \geq 0$ with $R_k = T_n^{j'_n} \circ \dots \circ T_1^{j'_1}$. For $k_i := j_i + j'_i$, $i = 1, \dots, n$, we get $(T_n^{k_n} \circ \dots \circ T_1^{k_1})x \in V$, so $\operatorname{Orb}(x, T)$ is everywhere dense. □

The following example links the nonlinear dynamics in dimension 1 with the linear infinite-dimensional dynamics. The corresponding autonomous version was obtained in [23]. The idea follows what is called Carleman linearization, and it is inspired in [25].

Example 4. Let $\{p_n : I \to I \ ; \ n \in \mathbb{N}\}$ be a sequence of polynomials on an interval I that contains 0 such that $p_n(0) = 0$, $n \in \mathbb{N}$, and the corresponding generated NDS (I, p_∞) is weakly mixing of order 3. By [1, Thm 11] we know that (I, p_∞) is weakly mixing of all orders. We will embed (I, p_∞) in a linear NDS (X, T_∞) via a map ϕ such that $T_n \circ \phi = \phi \circ p_n$ for every $n \in \mathbb{N}$ and $\overline{\text{span}(\phi(I))} = X$. To do so we set

$$X = \{(x_i)_i \in \mathbb{C}^{\mathbb{N}} \ ; \ \exists r > 0 \text{ such that } \sup_i |x_i| \, r^i < \infty\}.$$

Actually, we can identify via Taylor expansion at 0 $X = \mathcal{H}(0)$, the space of holomorphic germs at 0. That is, X consists of the functions that are (defined and) holomorphic on a neighbourhood of 0. X is endowed with the natural topology as inductive limit. We refer the reader to, e.g., [15] for the details.

We define the embedding $\phi : I \to X$ as $\phi(x) = (x, x^2, x^3, \dots)$. Given $n \in \mathbb{N}$, we set the operator $T_n : X \to X$ such that the k-th coordinate of $T_n x$ is

$$T_n(x_1, x_2, \dots)_k = \sum_{j=k}^{km_n} \alpha_{k,j} x_j, \quad k \in \mathbb{N}, \quad x = (x_1, x_2, \dots) \in X,$$

where $m_n = \deg(p_n)$ and $p_n(x)^k = \sum_{j=k}^{km_n} \alpha_{k,j} x^j$. The selection of the sequence space X easily gives that T_n is a well-defined operator on X. Also, a simple computation shows that $T_n \circ \phi = \phi \circ p_n$. Let $Y := \phi(I)$. We observe that $\text{span}(Y)$ is dense in X by the Hahn-Banach theorem. Indeed, since the dual of X is

$$X' = \{(y_i)_i \in \mathbb{C}^{\mathbb{N}} \ ; \ \sum_{i=1}^{\infty} |y_i| \, R^i < \infty \text{ for all } R > 0\},$$

which can be identified with the space of entire functions, we have that $\langle \phi(x), (y_i)_i \rangle = \sum_i y_i x^i = 0$ for some $(y_i)_i \in X'$ and for all $x \in I$, implies $y_i = 0$ for every $i \in \mathbb{N}$ because an entire function that is annihilated on a set with accumulation points should be identically 0. The hypothesis of Corollary 2 are satisfied, and (X, T_∞) is weakly mixing of all orders.

3 Weak mixing property of different orders

We will prove that it is possible to obtain examples of linear NDS which show the strict inclusion of the different orders for the weak mixing property. This fact contrasts with the case of nonautonomous interval maps, where it was shown that there are examples which are weakly mixing of order 2 which are not weakly mixing of order 3 [1, Thm 9], but once an interval NDS is weakly mixing of order 3, then it follows immediately that it is of arbitrary order $n \geq 2$ [1, Thm 11].

Theorem 5. *Given any $n \geq 2$ there is a linear NDS (ℓ^2, T_∞) defined on the Hilbert space ℓ^2 which is weakly mixing of order n, but it is not weakly mixing of order $n+1$.*

Proof. We consider an arbitrary mixing operator on ℓ^2 like, for instance, the weighted backward shift $T := 2B$, $T(x_1, x_2, \dots) = (2x_2, 2x_3, \dots)$. Since every mixing map is weakly mixing of all orders, given $n \in \mathbb{N}$, let $(w_1, \dots, w_n) \in \ell^2 \times \cdots \times \ell^2$ be a vector whose orbit is dense on $\ell^2 \times \cdots \times \ell^2$ for the operator $T \times \cdots \times T$. For any $k \geq 0$, let $X_k := \operatorname{span}\{T^k w_1, \dots, T^k w_n\}$ and let $P_k : \ell^2 \to X_k$ be the corresponding orthogonal projection. We observe that $\dim(X_k) = n$ for every $k \geq 0$ since, otherwise, there would be $k_0 \geq 0$ such that $\dim(X_k) \leq \dim(X_{k_0}) < n$ for all $k \geq k_0$, which avoids the fact that $\{(T^k w_1, \dots, T^k w_n) \; ; \; k \geq k_0\}$ is dense in the n-product of ℓ^2. We set $T_1 = P_0$ and $T_{k+1} = P_k \circ T$, $k \in \mathbb{N}$.

The linear NDS (ℓ^2, T_∞) is clearly not weakly mixing of order $n+1$. Indeed, let $V_1, \dots, V_{n+1}$ be non-empty open sets of ℓ^2 such that any $n+1$-tuple $(v_1, \dots, v_{n+1}) \in V_1 \times \cdots \times V_{n+1}$ is linearly independent. Then, since $(T_k \circ \cdots \circ T_1)(\ell^2)$ is n-dimensional, it cannot intersect all the V_i's, $i = 1, \dots, n+1$, and therefore (ℓ^2, T_∞) is not weakly mixing of order $n+1$.

On the other hand, given any collection $U_i, V_i \subset \ell^2$ of non-empty open sets, $i = 1, \dots, n$, since P_0 is an orthogonal projection, thus an open mapping, we find vectors $u_i \in U_i$, $i = 1, \dots, n$, such that

$$\{T_1 u_1, \dots, T_1 u_n\} = \{P_0 u_1, \dots, P_0 u_n\}$$

is linearly independent. Let $P_0 u_i = \sum_{j=1}^n \alpha_{i,j} w_j$, $i = 1, \dots, n$. By definition of the T_k's we obtain

$$(T_{k+1} \circ \cdots \circ T_1) u_i = \sum_{j=1}^n \alpha_{i,j} T^k w_j, \quad i = 1, \dots, n,$$

for all $k \in \mathbb{N}$. We consider the matrix $A := (\alpha_{i,j})_{i,j}$, which is invertible since the $P_0 u_i$'s are linearly independent. Let $B = A^{-1} = (\beta_{i,j})_{i,j}$. We fix $v_i \in V_i$, $i = 1, \dots, n$, and a 0-neighbourhood W such that

$$v_i + \sum_{j=1}^{n} \alpha_{i,j} W \subset V_i, \quad i = 1, \dots, n.$$

By the selection of the w_i's, there is $k \in \mathbb{N}$ such that $T^k w_i \in \sum_{j=1}^{n} \beta_{i,j} v_j + W$, $i = 1, \dots, n$. Therefore, since $B = A^{-1}$, we have

$$T_1^{(k+1)} u_i = \sum_{j=1}^{n} \alpha_{i,j} T^k w_j \in v_i + \sum_{j=1}^{n} \alpha_{i,j} W \subset V_i, \quad i = 1, \dots, n,$$

and we conclude that (ℓ^2, T_∞) is weakly mixing of order n. □

Remark 6. It is known that a system $(T_n)_n$ of commuting operators that is weakly mixing of order 2 is necessarily weakly mixing of all orders [6, 8]. In particular, sequences of operators generated by commuting tuples of operators [17] and discretizations of C_0-semigroups of operators, which are weakly mixing of order 2, immediately happen to be weakly mixing of all orders. This means that there is no hope to find examples like the one in Theorem 5 within this framework. We do not know whether it is possible to obtain this type of counterexamples for non artificially constructed linear NDS.

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