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A fixed point theorem for generalized contractions involving w-distances on complete quasi-metric spaces

Carmen Alegre, Josefa Marín and Salvador Romaguera*

Dedicated to Professor W. Takahashi on the occasion of his 70th birthday

*Correspondence:
sromague@mat.upv.es
Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Valencia, 46022, Spain

Abstract

We obtain a fixed point theorem for generalized contractions on complete quasi-metric spaces, which involves w-distances and functions of Meir-Keeler and Jachymski type. Our result generalizes in various directions the celebrated fixed point theorems of Boyd and Wong, and Matkowski. Some illustrative examples are also given.

MSC: 47H10; 54H25; 54E50

Keywords: fixed point; generalized contraction; w-distance; complete quasi-metric space

1 Introduction and preliminaries

In their celebrated paper [1], Kada, Suzuki and Takahashi introduced and studied the notion of a w-distance on a metric space. By using that notion they obtained, among other results, generalizations of the nonconvex minimization theorem of Takahashi [2], of Caristi's fixed point theorem [3] and of Ekeland's variational principle [4], as well as a general fixed point theorem that improves fixed point theorems of Subrahmanyam [5], Kannan [6] and Ćirić [7]. This study was continued by Suzuki and Takahashi [8], and by Park [9] who extended several results from [1] to quasi-metric spaces. Park's approach was successful continued by Al-Homidan, Ansari and Yao [10], who obtained, among other interesting results, quasi-metric versions of Caristi-Kirk's fixed point theorem and Nadler's fixed point theorem by using Q-functions (a slight generalization of w-distances). More recently, Latif and Al-Mezel [11], and Marín *et al.* [12–14] have proved some fixed point theorems both for single-valued and multi-valued mappings in complete quasi-metric spaces and preordered quasi-metric spaces by using Q-functions and w-distances, and generalizing in this way well-known fixed point theorems of Mizoguchi and Takahashi [15], Bianchini and Grandolfi [16], and Boyd and Wong [17], respectively.

In this paper we shall obtain a fixed point theorem for generalized contractions with respect to w-distances on complete quasi-metric spaces from which we deduce w-distance versions of Boyd and Wong's fixed point theorem [17] and of Matkowski's fixed point theorem [18]. Our approach uses a kind of functions considered by Jachymski in [19, Corollary of Theorem 2] and that generalizes the notion of a function of Meir-Keeler type.

In the sequel the letters $\mathbb{R}^+$, $\mathbb{N}$ and ω will denote the set of non-negative real numbers, the set of positive integer numbers and the set of non-negative integer numbers, respectively.

By a quasi-metric on a set X we mean a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(i) $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$, and

(ii) $d(x,y) \leq d(x,z) + d(z,y)$.

A quasi-metric space is a pair (X,d) such that X is a set and d is a quasi-metric on X.

Each quasi-metric d on a set X induces a topology τ_d on X which has as a base the family of open balls $\{B_d(x,r) : x \in X, \varepsilon > 0\}$, where $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Given a quasi-metric d on X, the function d^{-1} defined by $d^{-1}(x,y) = d(y,x)$ for all $x, y \in X$, is also a quasi-metric on X, and the function d^s defined by $d^s(x,y) = \max\{d(x,y), d(y,x)\}$ for all $x, y \in X$, is a metric on X.

There exist several different notions of Cauchy sequence and of complete quasi-metric space in the literature (see *e.g.* [20]). In this paper we shall use the following general notion.

A quasi-metric space (X,d) is called complete if every Cauchy sequence $(x_n)_{n\in\omega}$ in the metric space (X,d^s) converges with respect to the topology $\tau_{d^{-1}}$ (*i.e.*, there exists $z \in X$ such that $d(x_n,z) \to 0$).

Definition 1 ([9, 10]) A w-distance on a quasi-metric space (X,d) is a function $q : X \times X \to \mathbb{R}^+$ satisfying the following three conditions:

(W1) $q(x,y) \leq q(x,z) + q(z,y)$ for all $x, y, z \in X$;

(W2) $q(x,\cdot) : X \to \mathbb{R}^+$ is lower semicontinuous on $(X, \tau_{d^{-1}})$ for all $x \in X$;

(W3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $q(x,y) \leq \delta$ and $q(x,z) \leq \delta$ imply $d(y,z) \leq \varepsilon$.

Several examples of w-distances on quasi-metric spaces may be found in [9–12].

Note that if d is a metric on X then it is a w-distance on (X,d). Unfortunately, this does not hold for quasi-metric spaces, in general. Indeed, in [12, Lemma 2.2] there was observed the following.

Lemma 1 *If q is a w-distance on a quasi-metric space (X,d), then for each $\varepsilon > 0$ there exists $\delta > 0$ such that $q(x,y) \leq \delta$ and $q(x,z) \leq \delta$ imply $d^s(y,z) \leq \varepsilon$.*

It follows from Lemma 1 (see [12, Proposition 2.3]) that if a quasi-metric d on X is also a w-distance on (X,d), then the topologies induced by d and by the metric d^s coincide, so (X,τ_d) is a metrizable topological space.

2 Results and examples

Meir and Keeler proved in [21] that if f is a self-map of a complete metric space (X,d) satisfying the condition that for each $\varepsilon > 0$ there is $\delta > 0$ such that, for any $x, y \in X$, with $\varepsilon \leq d(x,y) < \varepsilon + \delta$ we have $d(fx,fy) < \varepsilon$, then f has a unique fixed point $z \in X$ and $f^n x \to z$ for all $x \in X$.

This well-known result suggests the notion of a Meir-Keeler function:

A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a Meir-Keeler function if $\phi(0) = 0$, and satisfies the following condition:

(MK) For each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq t < \varepsilon + \delta \quad \text{implies} \quad \phi(t) < \varepsilon, \quad \text{for all } t \in \mathbb{R}^+.$$

Remark 1 It is obvious that if ϕ is a Meir-Keeler function then $\phi(t) < t$ for all $t > 0$.

Later on, Jachymski proved in [19] the following interesting result and showed that both Boyd and Wong's fixed point theorem and Matkowski's fixed point theorem are easy consequences of it.

Theorem 1 ([19, Corollary of Theorem 2]) *Let f be a self-map of a complete metric space (X,d) such that $d(fx,fy) < d(x,y)$ for $x \neq y$, and $d(fx,fy) \leq \phi(d(x,y))$ for all $x,y \in X$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the condition*

(Ja) *for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $t \in \mathbb{R}^+$,*

$$\varepsilon < t < \varepsilon + \delta \quad \textit{implies} \quad \phi(t) \leq \varepsilon.$$

Then f has a unique fixed point $z \in X$ and $f^n x \to z$ for all $x \in X$.

Theorem 1 suggests the following notion:

A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a Jachymski function if $\phi(0) = 0$ and it satisfies condition (Ja) of Theorem 1.

Remark 2 Obviously, each Meir-Keeler function is a Jachymski function. However, the converse does not follow even in the case that $\phi(t) < t$ for all $t > 0$: Indeed, let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined as $\phi(t) = 0$ for all $t \in [0,1]$ and $\phi(t) = 1$ otherwise. Clearly ϕ is a Jachymski function such that $\phi(t) < t$ for all $t > 0$. Finally, for $\varepsilon = 1$ and any $\delta > 0$ we have $\phi(\varepsilon + \delta/2) = \varepsilon$, so ϕ is not a Meir-Keeler function.

Now we establish the main result of this paper.

Theorem 2 *Let f be a self-map of a complete quasi-metric space (X,d). If there exist a w-*distance *q on (X,d) and a Jachymski function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for all $t > 0$, and*

$$q(fx,fy) \leq \phi\bigl(q(x,y)\bigr), \tag{1}$$

for all $x,y \in X$, then f has a unique fixed point $z \in X$. Moreover $q(z,z) = 0$.

Proof Fix $x_0 \in X$. For each $n \in \omega$ let $x_n = f^n x_0$. Then

$$q(x_{n+1},x_{n+2}) \leq \phi\bigl(q(x_n,x_{n+1})\bigr), \tag{2}$$

for all $n \in \omega$.

First, we shall prove that $\{x_n\}_{n\in\omega}$ is a Cauchy sequence in (X,d^s).

To this end put $r_n = q(x_n,x_{n+1})$ for all $n \in \omega$.

If there is $n_0 \in \omega$ such that $r_{n_0} = 0$, then $r_n = 0$ for all $n \geq n_0$ by (2) and our assumption that $\phi(0) = 0$. Therefore $q(x_n, x_m) = 0$ whenever $m > n \geq n_0$ by condition (W1), and consequently, $d^s(x_n, x_m) = 0$ by Lemma 1. Thus $x_n = x_{n_0+1}$ for all $n \geq n_0 + 1$.

Otherwise, we assume, without loss of generality, that $r_{n+1} < r_n$ for all $n \in \omega$. Then $\{r_n\}_{n\in\omega}$ converges to some $r \in \mathbb{R}^+$. Of course, $r < r_n$ for all $n \in \omega$.

If $r > 0$ there exists $\delta = \delta(r)$ such that

$$r < t < r + \delta \quad \Rightarrow \quad \phi(t) \leq r.$$

Take $n_\delta \in \mathbb{N}$ such that $r_n < r + \delta$ for all $n \geq n_\delta$. Therefore $\phi(r_n) \leq r$, so by condition (2), $r_{n+1} \leq r$ for all $n \geq n_\delta$, a contradiction. Consequently $r = 0$.

Now choose an arbitrary $\varepsilon > 0$. There exists $\delta = \delta(\varepsilon)$, with $\delta \in (0, \varepsilon)$, for which conditions (W3) and (Ja) hold. Similarly, for $\delta/2$ there exists $\mu = \mu(\delta/2)$, with $\mu \in (0, \delta/2)$ for which conditions (W3) and (Ja) also hold, *i.e.*,

$q(x,y) \leq \mu$ and $q(x,z) \leq \mu$, imply $d(y,z) \leq \delta/2$, and for any $t > 0$, $\delta/2 < t < \delta/2 + \mu$ implies $\phi(t) \leq \delta/2$.

Since $r_n \to 0$, there exists $k_0 \in \mathbb{N}$ such that $r_n < \mu$ for all $n \geq k_0$.

By using a similar technique to the one given by Jachymski in [19, Theorem 2] we shall prove, by induction, that for each $k \geq k_0$ and each $n \in \mathbb{N}$, we have

$$q(x_k, x_{n+k}) < \frac{\delta}{2} + \mu. \tag{3}$$

Indeed, fix $k \geq k_0$. Since $q(x_k, x_{k+1}) < \mu$, condition (3) follows for $n = 1$.

Assume that (3) holds for some $n \in \mathbb{N}$. We shall distinguish two cases.

- Case 1: $q(x_k, x_{n+k}) > \delta/2$. Then we deduce from the induction hypothesis and condition (Ja) that

$$\phi\big(q(x_k, x_{n+k})\big) \leq \delta/2,$$

so by (1), $q(x_{k+1}, x_{n+k+1}) \leq \delta/2$. Therefore

$$q(x_k, x_{n+k+1}) \leq q(x_k, x_{k+1}) + q(x_{k+1}, x_{n+k+1}) < \mu + \frac{\delta}{2}.$$

- Case 2: $q(x_k, x_{n+k}) \leq \delta/2$.

If $q(x_k, x_{n+k}) = 0$, we deduce that $q(x_{k+1}, x_{n+k+1}) = 0$ by (1). So, by (W1),

$$q(x_k, x_{n+k+1}) \leq q(x_k, x_{k+1}) < \mu < \mu + \frac{\delta}{2}.$$

If $q(x_k, x_{n+k}) > 0$, we deduce that $\phi(q(x_k, x_{n+k})) < q(x_k, x_{n+k}) \leq \delta/2$, so

$$\begin{aligned} q(x_k, x_{n+k+1}) &\leq q(x_k, x_{k+1}) + q(x_{k+1}, x_{n+k+1}) \\ &\leq q(x_k, x_{k+1}) + \phi\big(q(x_k, x_{n+k})\big) < \mu + \frac{\delta}{2}. \end{aligned}$$

Now take $i, j \in \mathbb{N}$ with $i, j > k$. Then $i = n + k$ and $j = m + k$ for some $n, m \in \mathbb{N}$. Hence, by (3),

$$q(x_k, x_i) = q(x_k, x_{n+k}) < \frac{\delta}{2} + \mu < \delta \quad \text{and} \quad q(x_k, x_j) = q(x_k, x_{m+k}) < \frac{\delta}{2} + \mu < \delta.$$

Now, from Lemma 1 it follows that $d^s(x_i,x_j) \leq \varepsilon$ whenever $i,j > k$. We conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d^s).

Since (X,d) is complete, there exists $z \in X$ such that $d(x_n,z) \to 0$.

Next we show that $q(x_n,z) \to 0$: Indeed, choose an arbitrary $\varepsilon > 0$. We have proved (see (3)) that there is $k_0 \in \mathbb{N}$ such that $q(x_k,x_{n+k}) < \varepsilon$ for all $k \geq k_0$ and $n \in \mathbb{N}$. Fix $k \geq k_0$. Since $d(x_n,z) \to 0$ it follows from condition (W2) that, for n sufficiently large,

$$q(x_k,z) < q(x_k,x_{n+k}) + \varepsilon.$$

Hence $q(x_k,z) < 2\varepsilon$ for all $k \geq k_0$. We deduce that $q(x_n,z) \to 0$.

From (1) it follows that $q(x_{n+1},fz) \to 0$. So $d^s(z,fz) = 0$ by Lemma 1. Consequently $z = fz$, *i.e.*, is a fixed point of f. Furthermore $q(z,z) = 0$. In fact, otherwise we have

$$q(z,z) = q(fz,fz) \leq \phi\big(q(z,z)\big) < q(z,z),$$

a contradiction.

Finally, let $u \in X$ such that $u = fu$ and $u \neq z$. If $q(u,z) > 0$ we deduce that

$$q(u,z) = q(fu,fz) \leq \phi\big(q(u,z)\big) < q(u,z),$$

a contradiction. So $q(u,z) = 0$. Similarly we check that $q(u,u) = 0$. Since $q(z,z) = 0$, we deduce from Lemma 1 that $d^s(u,z) = 0$, *i.e.*, $u = z$. We conclude that z is the unique fixed point of f. □

Corollary 1 *Let f be a self-map of a complete metric space (X,d). If there exist a w-distance q on (X,d) and a Jachymski function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for all $t > 0$, and*

$$q(fx,fy) \leq \phi\big(q(x,y)\big),$$

for all $x,y \in X$, then f has a unique fixed point $z \in X$. Moreover $q(z,z) = 0$.

Corollary 2 *Let f be a self-map of a complete quasi-metric space (X,d). If there exist a w-distance q on (X,d) and a Meir-Keeler function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$q(fx,fy) \leq \phi\big(q(x,y)\big),$$

for all $x,y \in X$, then f has a unique fixed point $z \in X$. Moreover $q(z,z) = 0$.

Proof Apply Remarks 1 and 2, and Theorem 2. □

Corollary 3 [13] *Let f be a self-map of a complete quasi-metric space (X,d). If there exist a w-distance q on (X,d) and a right upper semicontinuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(0) = 0$, $\phi(t) < t$ for all $t > 0$, and*

$$q(fx,fy) \leq \phi\big(q(x,y)\big),$$

for all $x,y \in X$, then f has a unique fixed point $z \in X$. Moreover $q(z,z) = 0$.

Proof It suffices to show that ϕ is a Meir-Keeler function. Assume the contrary. Then there exist $\varepsilon > 0$ and a sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $\varepsilon \le t_n < \varepsilon + 1/n$ but $\phi(t_n) \ge \varepsilon$ for all $n \in \mathbb{N}$. Since $\varepsilon - \phi(\varepsilon) > 0$, it follows from right upper semicontinuity of ϕ that $\phi(t_n) - \phi(\varepsilon) < \varepsilon - \phi(\varepsilon)$ eventually, *i.e.*, $\phi(t_n) < \varepsilon$, a contradiction. We conclude that f has a unique fixed point by Corollary 2. □

Corollary 4 *Let f be a self-map of a complete quasi-metric space (X,d). If there exist a w-distance q on (X,d) and a non-decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(0) = 0$, $\phi^n(t) \to 0$ for all $t > 0$, and*

$$q(fx,fy) \le \phi\big(q(x,y)\big), \tag{4}$$

for all $x,y \in X$, then f has a unique fixed point $z \in X$. Moreover $q(z,z) = 0$.

Proof Again it suffices to show that ϕ is a Meir-Keeler function. Assume the contrary. Then there exist $\varepsilon > 0$ and a sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $\varepsilon \le t_n < \varepsilon + 1/n$ but $\phi(t_n) \ge \varepsilon$ for all $n \in \mathbb{N}$. Since ϕ is non-decreasing we deduce that $\phi(t) \ge \varepsilon$ whenever $t \ge \varepsilon$. Hence $\phi^n(t) \ge \varepsilon$ whenever $t \ge \varepsilon$, which contradicts the hypothesis that $\phi^n(t) \to 0$ for all $t > 0$. We conclude that f has a unique fixed point by Corollary 2. □

Remark 3 In [22] the authors proved Corollary 2 for the case that (X,d) is a complete metric space. Note also that Boyd and Wong's fixed point theorem [17] and Matkowski's fixed point theorem [18] are special cases of Corollaries 3 and 4, respectively, when (X,d) is a complete metric space and q is the metric d.

We conclude the paper with some examples that illustrate and validate the obtained results.

The first example shows that condition '$\phi(t) < t$ for all $t > 0$' in Theorem 2 cannot be omitted.

Example 1 Let $X = \{0,1\}$ and let d be the discrete metric on X, *i.e.*, $d(x,x) = 0$ for all $x \in X$ and $d(x,y) = 1$ whenever $x \neq y$. Let $f : X \to X$ defined as $f0 = 1$ and $f1 = 0$, and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined as $\phi(1) = 1$ and $\phi(t) = 0$ for all $x \in \mathbb{R}^+ \backslash \{1\}$. It is clear that ϕ is a Jachysmki function such that

$$d(fx,fy) \le \phi\big(d(x,y)\big),$$

for all $x,y \in X$. However, f has no fixed point.

The next is an example where we can apply Theorem 2 for an appropriate w-distance q on a complete quasi-metric space (X,d) but not for d. Moreover, Corollary 1 cannot be applied for any w-distance on the metric space (X,d^s).

Example 2 Let $X = \omega$ and let d be the quasi-metric on X defined as

$$d(x,x) = 0 \quad \text{for all } x \in X;$$

$$d(n,0) = 1/n \quad \text{for all } n \in \mathbb{N};$$

$$d(0,n) = 1 \quad \text{for all } n \in \mathbb{N};$$

$$d(n,m) = |1/n - 1/m| \quad \text{for all } n,m \in \mathbb{N}.$$

Clearly (X,d) is complete (observe that $\{n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d^s) with $d(n,0) \to 0$).

Let q be the w-distance on (X,d) given by $q(x,y) = y$ for all $x,y \in X$.

Now define $f : X \to X$ as $f0 = 0$ and $fn = n-1$ for all $n \in \mathbb{N}$, and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\phi(0) = 0$ and $\phi(t) = n-1$ where $t \in (n-1,n]$, $n \in \mathbb{N}$.

It is routine to check that ϕ is a Jachymski function satisfying $\phi(t) < t$ for all $t > 0$ (in fact, it is a Meir-Keeler function).

Since $q(fx,f0) = 0$ for all $x \in X$, and for each $n,m \in X$ with $m \neq 0$, we have

$$q(fn,fm) = fm = m-1 = \phi(m) = \phi\bigl(q(n,m)\bigr),$$

it follows that all conditions of Theorem 2 are satisfied. In fact $z = 0$ is the unique fixed point of f.

However, the contraction condition (1) is not satisfied for d. Indeed, for any $n > 1$ we have

$$d(f0,fn) = d(0,n-1) = 1 > 0 = \phi(1) = \phi\bigl(d(0,n)\bigr).$$

Finally, note that we cannot apply Corollary 1 because (X,d^s) is not complete (observe that $\{n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d^s) that does not converge in (X,d^s)).

We conclude with an example where we can apply Corollary 2 but not Corollaries 3 and 4.

Example 3 Let d be the quasi-metric on $\mathbb{R}^+$ given by $d(x,y) = \max\{y-x,0\}$ for all $x,y \in \mathbb{R}^+$. Since d^s is the usual metric on $\mathbb{R}^+$ it immediately follows that $(\mathbb{R}^+,d)$ is complete.

Define $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ as $q(x,y) = y$. It is clear that q is a w-distance on $(\mathbb{R}^+,d)$.

Now let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, defined by $\phi(t) = t/2$ if $t \in (1,2]$, and $\phi(t) = 0$ otherwise.

Then ϕ is a Meir-Keeler function: Indeed, we first note that $\phi(0) = 0$. Now, given $\varepsilon > 0$ we distinguish the following cases:

(1) if $0 < \varepsilon < 1$, we take $\delta = 1-\varepsilon$, and thus, from $\varepsilon \leq t < \varepsilon + \delta = 1$, it follows $\phi(t) = 0 < \varepsilon$;
(2) if $\varepsilon = 1$, we take $\delta = 1/2$, and thus, from $1 < t < 3/2$, it follows $\phi(t) = t/2 < 3/4 < \varepsilon$, whereas $\phi(1) = 0 < \varepsilon$;
(3) if $1 < \varepsilon < 2$, we take $\delta = 2-\varepsilon$, and thus, from $\varepsilon \leq t < \varepsilon + \delta = 2$, it follows $\phi(t) = t/2 < 1 < \varepsilon$;
(4) if $\varepsilon \geq 2$, we fix $\delta > 0$, and thus, from $\varepsilon \leq t < \varepsilon + \delta$, it follows $\phi(t) < \varepsilon$ because $\phi(2) = 1$ and $\phi(t) = 0$ for $t > 2$.

Finally, taking $f = \phi$, we obtain $q(fx,fy) \leq \phi(q(x,y))$ for all $x,y \in X$, because

$$q(fx,fy) = fy = \phi(y) = \phi\bigl(q(x,y)\bigr).$$

Therefore, all conditions of Corollary 2 are satisfied. In fact, $z = 0$ is the unique fixed point of f.

However, ϕ is not right upper semicontinuous at $t = 1$, so we cannot apply Corollary 3. Similarly, we cannot apply Corollary 4 because ϕ is not a non-decreasing function.

Observe also that the w-distance q cannot be replaced by the quasi-metric d because for $1 < y \leq 2$ we have

$$d(f1, fy) = d\left(0, \frac{y}{2}\right) = \frac{y}{2} > 0 = \phi(y-1) = \phi\big(d(1,y)\big).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The three authors contributed equally in writing this article. They read and approved the final manuscript.

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